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# THE DENSITY OF RATIONAL AND INTEGRAL POINTS ON ALGEBRAIC VARIETIES

By M. L. BROWN

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## 1. Introduction

In this paper we prove the existence of a *Banach density* of the set of rational and integral points on subvarieties of abelian varieties (or more generally, commutative group schemes) over algebraic number fields and algebraic function fields of arbitrary characteristic. We state in this introduction some of the main results for the simplest case where  $k$  is a global field (*i.e.* a finite extension of  $\mathbf{Q}$  or a function field of

transcendence degree 1 over a finite field). We have (*cf.* Theorem 4.5, Corollary 4.6):

**THEOREM 1.1.** — *Let  $A/k$  be a smooth commutative group scheme,  $V/k$  a closed subscheme, and  $\Gamma$  a finitely generated subgroup of the group of  $k$ -rational points  $A(k)$ . Then there are a finite number of translates  $A_i$ , contained in  $V$ , of algebraic subgroups of  $A$  such that*

$$\Gamma \cap V = \bigcup_{i=1}^m (\Gamma \cap A_i) \cup S,$$

where the density  $d_\Gamma(S)$  is zero. In particular, the (Banach) density  $d_\Gamma(\Gamma \cap V)$  of  $\Gamma \cap V$  in  $\Gamma$  exists and is a rational number.

Indeed, we prove an  $O$ -estimate for size of the exceptional set  $S$  of this theorem, which immediately implies zero density. An application of the sharper  $O$ -estimate is the following: with  $A, V, k$  as in Theorem 1.1, assume further that  $A/k$  is an abelian variety and let  $\mathcal{L}$  be an ample line bundle on  $V$ . Define the Dirichlet series

$$Z(s; V, h_\mathcal{L}) = \sum'_{x \in V(k)} h_\mathcal{L}(x)^{-s}$$

where  $h_\mathcal{L}$  is a global (logarithmic) Weil height on  $V$  associated to  $\mathcal{L}$  and the sum runs over the all but finitely many  $x \in V(k)$  with  $h_\mathcal{L}(x) > 0$ . We have (*cf.* Theorem 4.20):

**THEOREM 1.2.** — *The Dirichlet series  $Z(s; V, h_\mathcal{L})$  has a meromorphic continuation to the half plane  $\operatorname{Re} s > \operatorname{rank} A(k)/2 - \varepsilon$ , for some  $\varepsilon > 0$ , such that the only singularity in this half plane is a possible simple pole at  $s = \operatorname{rank} A(k)/2$ . Further, a simple pole occurs at  $s = \operatorname{rank} A(k)/2$  if and only if  $A$  contains an irreducible component of the Zariski closure of  $A(k)$  and  $A(k)$  is infinite.*

Lang ([24]; *cf.* Conjecture 4.1 below) conjectured that if  $k$  has characteristic zero and  $A/k$  is a semi-abelian variety then the exceptional set  $S$  of Theorem 1.1 is empty. This conjecture has been proved in various cases (listed at the beginning of Section 4 below) and supersedes Theorem 1.1 when they apply. This higher dimensional generalisation of Mordell's conjecture would become false over fields of positive characteristic.

Suppose now that  $A/k$  is an abelian variety and  $k'$  is a finite field extension of  $k$ . The group  $A(k')$  is finitely generated, by the Mordell-Weil theorem; hence by Theorem 1.1 the density  $d_{A(k')}(V(k'))$  of the set  $V(k')$ , of  $k'$ -rational points of  $V$ , in  $A(k')$  is a well defined rational number in the interval  $[0, 1]$ . One may ask for the variation of this quantity as the field  $k'$  varies. We prove (*cf.* Theorem 4.12):

**THEOREM 1.3.** — *If  $\operatorname{char} k = 0$ , then the subset of  $\mathbf{Q} \cap [0, 1]$  given by*

$$\{d_{A(k')}(V(k')); k' \text{ a finite extension field of } k\}$$

*omits infinitely many rational numbers in  $\mathbf{Q} \cap [0, 1]$ .*

As for integral points, we have (*cf.* Theorems 5.6 and 5.7):

**THEOREM 1.4.** — *Let  $D$  be an ample divisor on the abelian variety  $A/k$ . If  $I$  is a  $D$ -integral subset of  $A(k)$ , then  $I$  is either finite or of density*

$$d_{A(k)}(I) = 0.$$

Lang conjectured ([24]; *cf.* Conjecture 5.5 below) that if the field  $k$  is a finitely generated extension of  $\mathbf{Q}$  then a  $D$ -integral subset of  $A(k)$  is finite. When  $k$  is a number field, the conjecture has been proved by Faltings ([12], [46]) and this supersedes Theorem 1.4 in characteristic zero. Again in positive characteristic, this conjecture would become false.

The main technique we use is the Masser-Wüstholz zero estimates (*cf.* § 2) of transcendental number theory ([28], [30]), for ground fields of arbitrary characteristic, combined with a combinatorial theorem (§ 3, Theorem 3.1) on “large” subsets of abelian groups. For the question of integral points, we consider the distribution of the values of local Néron heights on abelian varieties (§ 5) by means of these zero estimates as well as special considerations of locally compact valuations.

The methods of this paper can be contrasted with those of Faltings’s paper [12]. Both use some techniques of diophantine approximation; roughly speaking, the main difference is that we consider the “spatial distribution” of rational points and show that they cannot be “dense” whereas Faltings shows, under restrictions, that one cannot have long finite chains of rational points with rapidly increasing heights.

Some of the main results of this paper were announced in [5]. Prof. D. Masser remarked to me that the zero density results could be improved to  $O$ -estimates by slightly sharpening the zero estimates and using the combinatorial Theorem 3.1, in place of a deep theorem of Furstenberg; I thank him very much for his suggestions. It is a pleasure also to thank my colleagues at Orsay for their kind hospitality, with especial thanks to Prof. L. Illusie.

## 2. Zero estimates

We state here a slight extension of the zero estimates of Masser and Wüstholz for commutative algebraic groups (without multiplicities). We refer the reader to [2] for a survey of zero estimates and their applications to transcendental number theory.

Let  $k$  be a field of arbitrary characteristic and  $G/k$  a smooth connected commutative group scheme of finite type. Let  $\Gamma$  be a finitely generated subgroup of  $G(k)$ .

Let  $G \rightarrow \mathbf{P}_k^n$  be an embedding of  $G$ , as a locally closed subscheme, into projective space over  $k$ . Then the degree  $\deg(X)$  of a closed subscheme  $X$  of  $G$  is then defined via this embedding.

**THEOREM 2.1.** — *Let  $\Gamma_1, \dots, \Gamma_a$  be finite subsets of  $G(k)$ , where  $d = \dim G$ . Suppose that the restriction to  $G$  of the section  $s \in \Gamma(\mathbf{P}_k^n, \mathcal{O}(m))$ ,  $m > 0$ , is not identically zero and vanishes on the set of points  $\Gamma_1 + \dots + \Gamma_a$  in  $G(k)$ . Then there is a connected algebraic*

subgroup  $H$  of  $G$ ,  $H \neq G$ , and an integer  $i$  such that

$$|\pi_{G/H}(\Gamma_i)| \cdot \deg(H) \leq \deg(G) \cdot (am)^{\dim G/H},$$

where  $a > 0$  is a constant depending only on  $G$  and its embedding in  $\mathbf{P}_k^n$ , and  $\pi_{G/H}$  is the projection  $G \rightarrow G/H$ .

The proof of Theorem 2.1 (which we omit) is similar to the proof of Théorème 2 of [2]. It differs only in that the field  $k$  has arbitrary characteristic (the hypothesis char.  $k=0$  in [2] is superfluous for the zero estimates without multiplicities, see [8], [5], [36]) and that the section  $s$  vanishes on a set of points of the form  $\Gamma_1 + \dots + \Gamma_d$ .

### 3. A combinatorial theorem

The main result of this section (Theorem 3.1) asserts that a sufficiently large subset of a finitely generated abelian group contains a set of "well-spaced" elements. It is similar to, but effective and much easier to prove than, the following famous theorem of Furstenberg [14]:

**THEOREM (Furstenberg).** — *Let  $S$  be a subset of  $\mathbf{Z}^n$  with positive upper (Banach) density. Let  $\Omega$  be a finite subset of  $\mathbf{Z}^n$ . Then there is an integer  $a \neq 0$  and an element  $b \in \mathbf{Z}^n$  such that*

$$a\Omega + b \subset S.$$

Furstenberg's theorem can be used in place of Theorem 3.1 to obtain many of the zero density results of this paper (*cf.* the remark after Theorem 4.4); but, for the sharper  $O$ -estimates, we require Theorem 3.1.

**3.1. DENSITIES AND RANKS.** — Let  $A$  be a finitely generated abelian group and let  $A_{\text{tors}}$  denote the torsion subgroup of  $A$ . Let  $a_1, \dots, a_r \in A$  be a set of generators of  $A$ .

If  $S$  is a subset of  $A$ , we define for every real number  $X > 0$

$$S(X) = \left\{ s \in S; s = \sum_{i=1}^r \alpha_i a_i, \quad -X \leq \alpha_i \leq X, \alpha_i \in \mathbf{Z}, \text{ for all } i \right\}.$$

The upper (Banach) density  $d^*(S)$  [respectively, the lower density  $d_*(S)$ ] of  $S$  is defined to be

$$d^*(S) = \limsup_{X \rightarrow \infty} |S(X)| / |A(X)|$$

[resp.

$$d_*(S) = \liminf_{X \rightarrow \infty} |S(X)| / |A(X)|].$$

If  $d^*(S) = d_*(S)$ , one calls the common value the density  $d_A(S)$  of  $S$  in  $A$ . If  $A$  is finite, then  $d_A(S)$  exists and is equal to  $|S|/|A|$ . In general, the quantities  $d_\Gamma^* S$ ,  $d_{\Gamma^*} S$ ,  $d_\Gamma S$  depend on the choice of generators of  $\Gamma$ ; but there are important cases where they are independent of the choice: (a) if  $d_\Gamma^* S = 0$ ; (b) if  $S$  is a coset of a subgroup of  $\Gamma$ ; (c) if  $S$  is obtained from a finite number of sets of types (a) or (b) by the operations of union and intersection. This applies particularly to all the density results of this paper; we shall therefore not usually specify any generators.

The rank (or logarithmic upper density)  $\text{rank}(S)$  of a subset  $S$  of  $A$  is defined analogously to be

$$\text{rank}(S) = \limsup_{X \rightarrow \infty} \log |S(X)| / \log X.$$

If  $S$  is a subgroup of  $A$ , then  $\text{rank}(S)$  is equal to the rational rank of  $S$  (hence there is no contradiction in terminology). Clearly, if  $S \subset A$  has  $\text{rank}(S) < \text{rank} A$  then  $d_A(S) = 0$ .

It is convenient to fix a particular set of elements of  $A$ : let  $a_1, \dots, a_k$  be linearly independent elements of  $A$  so that

$$A = (\oplus_{i=1}^k \mathbf{Z} a_i) \oplus A_{\text{tors}}.$$

Let  $N$  be a positive integer. A subset  $S \subset A$  is called  $N$ -special (with respect to  $a_1, \dots, a_k$ ) if there are subsets  $F_i \subset \mathbf{Z}$ ,  $i = 1, \dots, k$ , with

$$\begin{aligned} |F_i| &= N, \quad \text{for all } i, \\ S &= a + \sum_{i=1}^k F_i a_i, \end{aligned}$$

for some  $a \in A$ .

If  $S$  is  $N$ -special, then we have  $|S| = N^k$ . The main property of  $N$ -special sets is their ‘‘rigidity’’ under group homomorphism (cf. Lemma 3.3).

3.2. STATEMENT OF THE THEOREM. — Let  $A, a_1, \dots, a_k$  be as above. Fix a finite set of generators of  $A$  by adjoining torsion elements  $a_{k+1}, \dots$  to the set  $a_1, \dots, a_k$ .

THEOREM 3.1. — For any pair of integers  $N, n \geq 1$ , there is a real number  $\varepsilon > 0$  with the following property. Let  $M > 0$  be any integer and let  $S \subset A$  be a subset satisfying

$$\limsup_{X \rightarrow \infty} |S(X)| / X^{k-\varepsilon} = +\infty$$

[in particular, if  $\text{rank}(S) > k - \varepsilon$ ]. Then there are  $N$ -special subsets  $U_1, \dots, U_n \subset A$  and a subset  $S' \subset S$  with  $|S'| = M$  so that

$$S \supset S' + \sum_{i=1}^n U_i.$$

Remarks. — 1) The proof below shows that we may take  $\varepsilon$  to be

$$\varepsilon(N, n, k) = (N + 2)^{-nk} \min(k, N + 2),$$

but this is unlikely to be best possible.

2) The above theorem suffices for our application but in fact the proof gives the following stronger assertion: Assume  $S$  satisfies (for fixed  $N, n$  and suitable  $\varepsilon > 0$ )

$$\limsup_{X \rightarrow \infty} |S(X)| X^{k-\varepsilon} = +\infty.$$

Given a sufficiently large integer  $X > 0$ , there is a subset  $S' \subset S(X)$  satisfying, as in the theorem above,

$$S \supset S' + \sum_{i=1}^n U_i, \quad \text{and} \quad U_i \text{ is } N\text{-special for all } i,$$

and

$$|S'| \gg X^\delta,$$

where  $\delta > 0$  depends only on  $\varepsilon, N, n, k$ .

3) For  $A = \mathbf{Z}^2, k=2, N=2, n=1$ , Theorem 3.1 has the following geometric interpretation: If a subset  $S \subset \mathbf{Z}^2$  satisfies

$$\limsup_{X \rightarrow \infty} |S(X)|/X^{1.5/8} = +\infty$$

then  $S$  contains a rectangle.

Theorem 3.1 is proved inductively from the following technical lemma.

LEMMA 3.2. — *Let  $N > 0$  be an integer. Then there is a constant  $c_1 > 0$ , depending only on  $A$  and  $N$ , with the following property. Let  $S$  be a subset of  $A$ ; let  $r, \alpha$  be real numbers with  $r > 0, 0 < \alpha < 1$ , and  $X > 0$  an integer satisfying*

- 1)  $|S(X)| > r X^{k-\alpha}$ ,
- 2)  $X > (N 4^{k+1} |A_{\text{tors}}|/r) X^\alpha > 1$ .

*Then there are subsets  $S_1 \subset S, F \subset \mathbf{Z}$  with  $|F| = N$  and*

- (a)  $|S_1(2X)| > r^{N+2} c_1 (2X)^{k-(N+2)\alpha}$ ,
- (b)  $S_1 + F a_1 \subset S$ .

We prove Theorem 3.1 in Section 3.4 and Lemma 3.2 in the next section.

3.3. PROOF OF LEMMA 3.2. — We have the decomposition

$$A = (\oplus_{i=1}^k \mathbf{Z} a_i) \oplus A_{\text{tors}}.$$

Let  $L$  be the real number

$$(3.1) \quad L = (N 4^{k+1} |A_{\text{tors}}|/r) X^\alpha;$$

then

$$(3.2) \quad X > L \geq 1$$

by hypothesis (2) of the lemma. Put

$$(3.3) \quad Y = [X/L],$$

where  $[n]$  denotes the greatest integer  $\leq n$ .

Let  $x$  be an element of  $(\bigoplus_{j \neq 1} \mathbf{Z} a_j \oplus A_{\text{tors}})(X)$  and let  $y$  be an integer with  $-Y \leq y \leq Y$ . Define

$$(3.4) \quad I(x; y) = \{ \lambda a_1; \lambda \in [yL, (y+1)L] \} \oplus x \in A.$$

Call the interval  $I(x; y)$  good if it contains at least  $N+1$  elements of  $S$ , bad otherwise. Let  $G(X)$ ,  $B(X)$ , respectively, be the number of good and bad intervals  $I(x; y)$ . Since  $(Y+1)L > X$  [from (3.2)] we see that

$$(3.5) \quad \left\{ \begin{array}{l} A(X) \subset \bigcup_{x, y} I(x; y), \\ S(X) \subset \bigcup_{x, y} (I(x; y) \cap S). \end{array} \right.$$

Every interval  $I(x; y)$  contains  $L+1$  elements; in particular, each of the good intervals contains at most  $L+1$  elements of  $S$ . By definition, each of the bad intervals contains at most  $N$  elements of  $S$ . Comparing cardinalities in (3.5) gives

$$(3.6) \quad |S(X)| \leq (L+1)G(X) + NB(X).$$

The total number of intervals  $I(x; y)$  is at most

$$(3.7) \quad |A_{\text{tors}}| (2X+1)^{k-1} (2Y+1) \leq |A_{\text{tors}}| (2X+1)^{k-1} (2X+L)/L \leq |A_{\text{tors}}| 4^k X^k/L,$$

as  $L < X$ , by (3.2). In particular,  $|A_{\text{tors}}| 4^k X^k/L$  is an upper bound for  $B(X)$ . Hence from (3.6) we have

$$(3.8) \quad G(X) \geq \frac{1}{(L+1)} (|S(X)| - N |A_{\text{tors}}| 4^k X^k/L) \geq \frac{1}{(L+1)} (|S(X)| - r X^{k-\alpha}/4),$$

using (3.1). It follows this and hypothesis (1) of the lemma that

$$(3.9) \quad G(X) \geq (1/(L+1)) (r X^{k-\alpha} - r X^{k-\alpha}/4) \geq (3/(4L+4)) r X^{k-\alpha}.$$

As  $L \geq 1$ , from (3.2), we have

$$(3.10) \quad G(X) \geq (1/8L) r X^{k-\alpha} = c_2 r^2 X^{k-2\alpha},$$

where  $c_2 = 1/(8N4^{k+1}|A_{\text{tors}}|)$ , from (3.1). Roughly speaking, this shows that there are "many" good intervals.

Let  $T$  be the number of subsets with  $N$  elements of the set  $\{x \in \mathbf{Z}; 0 \leq x \leq L\}$  and let  $F_1, \dots, F_T$  be these distinct subsets. One then has

$$(3.11) \quad T = \binom{[L+1]}{N}.$$



For each  $m = 1, \dots, T$  define

$$(3.12) \quad S_m = \{s \in S; s + F_m a_1 \subset S\}.$$

As a good interval  $I(x; y)$  contains at least  $N+1$  elements of  $S$ , there is a subset  $F_j$ ,  $1 \leq j \leq T$ , and an element  $s(x; y) \in I(x; y) \cap S$  with

$$s(x; y) + F_j a_1 \subset I(x; y) \cap S.$$

It follows that each good interval  $I(x; y)$  contains an element  $s(x; y)$  such that [from (3.4)]

$$s(x; y) \in \bigcup_{j=1}^T S_j(X+L) =_{\text{def}} U(X+L).$$

As an element  $s(x; y)$  belongs to at most two distinct good intervals, it follows from (3.10) that

$$|U(X+L)| \geq G(X)/2 \geq c_3 r^2 X^{k-2\alpha}, \quad c_3 = c_2/2 > 0.$$

Hence for some  $m$ ,  $1 \leq m \leq T$ , we have

$$(3.13) \quad |S_m(X+L)| \geq c_3 r^2 X^{k-2\alpha}/T.$$

From (3.11) we have the elementary estimate

$$T \leq 4^N L^N / N!.$$

Hence from (3.1) we obtain

$$(3.14) \quad T \leq (4^N / N!) (N 4^{k+1} |A_{\text{tors}}| X^\alpha / r)^N \leq c_4 r^{-N} X^{N\alpha}$$

where

$$c_4 = N^N 4^{(k+2)N} |A_{\text{tors}}|^N / N! > 0.$$

Combining (3.13) and (3.14) gives that for some suitable  $m$  we have

$$|S_m(X+L)| \geq (c_3/c_4) r^{N+2} X^{k-(N+2)\alpha}.$$

As  $X > L$  [by (3.2)], this gives

$$|S_m(2X)| \geq (c_3/c_4) r^{N+2} X^{k-(N+2)\alpha},$$

and the lemma follows from this and (3.12) by taking

$$c_1 = c_3 / (c_4 2^k) > 0.$$

*Remark.* – The proof of the lemma shows that one can take  $c_1$  to be

$$c_1 = N! 4^{-2N(k+2)-k-3} N^{-N-1} |A_{\text{tors}}|^{-N-1}.$$

3.4. PROOF OF THEOREM 3.1. — Fix integers  $N, n \geq 1$ . Let  $c_1 > 0$  be the constant given by Lemma 3.2 applied to  $A$  and  $N$ . Put

$$c_2 = N 4^{k+1} |A_{\text{tors}}|.$$

Define recursively the functions  $f_i$  of the real variable  $q$  by:

$$(3.15) \quad \begin{cases} f_0(q) = q, \\ f_{i+1}(q) = c_1 (f_i(q))^{N+2}, \end{cases} \quad i=0, 1, \dots$$

Clearly,  $f_i(q) \rightarrow \infty$ , as  $q \rightarrow \infty$ , for all  $i=0, 1, \dots$ . Let  $\varepsilon$  be any real number satisfying

$$(3.16) \quad 0 < \varepsilon \leq (N+2)^{-nk} \min(k, N+2).$$

With  $M > 0$  an integer as in the theorem, select a real number  $q > 0$  so large that

$$(3.17) \quad f_{nk}(q) \geq M.$$

We may then select an integer  $X > 0$  so large that

$$(3.18) \quad 2^i X > (c_2/f_i(q)) (2^i X)^{(N+2)^i \varepsilon} > 1, \quad \text{for all } i=0, 1, \dots, nk-1.$$

This choice of  $X$  is possible because the exponents  $(N+2)^i \varepsilon$  satisfy

$$(3.19) \quad 0 < (N+2)^i \varepsilon < 1, \quad \text{for all } i=0, \dots, nk-1,$$

from (3.16).

Put  $S_0 = S$ . For any integer  $i$ , we may uniquely write  $i = sk + t$ , where  $1 \leq t \leq k$  and  $s, t \in \mathbf{Z}$ .

We may then inductively construct subsets  $S_i \subset S_{i-1}$ ,  $i=1, \dots, nk$ , and subsets  $F_{st} \subset \mathbf{Z}$  with  $|F_{st}| = N$ ,  $1 \leq s \leq n$ ,  $1 \leq t \leq k$ , such that

$$(a)_i \quad |S_i (2^i X)| > f_i(q) (2^i X)^{k - (N+2)^i \varepsilon},$$

$$(b)_i \quad S_i + F_{st} a_t \subset S_{i-1}, \quad i = sk + t, \quad 1 \leq t \leq k.$$

For this, assume that  $(a)_i, (b)_i$  are true and apply Lemma 3.2 to  $S_i$  where  $r = f_i(q)$ ,  $\alpha = (N+2)^i \varepsilon$ . This gives a subset  $S_{i+1}$  satisfying  $(a)_{i+1}$  and  $(b)_{i+1}$  provided the hypotheses of the lemma are satisfied; but hypothesis (1) for  $S_i$  is simply  $(a)_i$  and hypothesis (2) is given by (3.18).

The hypotheses  $(b)_1, \dots, (b)_{nk}$  now give

$$S_{nk} + \sum_{s=1}^n \sum_{t=1}^k F_{st} a_t \subset S_0 = S.$$

Furthermore, the hypothesis  $(a)_{nk}$  gives

$$|S_{nk} (2^{nk} X)| > f_{nk}(q) (2^{nk} X)^{k - (N+2)^{nk} \varepsilon} \geq f_{nk}(q) \geq M,$$

by (3.16) and (3.17). This proves the theorem.

3.5. A LEMMA. — The next lemma is the main property of N-special sets.

LEMMA 3.3. — *Let  $A, a_1, \dots, a_k$  be as above. Let  $S$  be an N-special subset of  $A$  with respect to  $a_1, \dots, a_k$  and let  $H$  be a subgroup of  $A$  of rank  $r$ . Let  $\pi: A \rightarrow A/H$  be the natural surjection. Then we have*

$$|\pi(S)| \geq N^{k-r}.$$

*Proof.* — Let  $S = a + \sum_{i=1}^k F_i a_i$ . As it suffices to prove the lemma for the translate  $\sum_{i=1}^k F_i a_i$  of  $S$ , and as  $a_1, \dots, a_k$  generate a torsion free subgroup of  $A$  which contains this translate, we may reduce to the case where  $A$  itself is free with basis  $a_1, \dots, a_k$ .

As  $\text{rank } A/H = k-r$ , the images of some subset of  $k-r$  elements of  $a_1, \dots, a_k$  generate a free subgroup of  $A/H$ . By reindexing we may suppose these elements are  $a_1, \dots, a_{k-r}$ . Put

$$S' = \sum_{i=1}^{k-r} F_i a_i \subset S.$$

Then the elements of  $S'$  remain distinct under  $\pi$  hence  $|\pi(S')| = N^{k-r}$ , whence the result.

#### 4. Rational points on subvarieties of commutative group schemes

Let  $G/k$  be a commutative group scheme of finite type over a field  $k$ . Let  $\bar{k}$  be an algebraic closure of  $k$ . Let  $\Gamma$  be a subgroup of  $G(\bar{k})$  of finite rank i.e.  $\Gamma \otimes_{\mathbf{Z}} \mathbf{Q}$  is a finite dimensional  $\mathbf{Q}$ -vector space. Lang [24], combining conjectures of Chabauty, Manin and Mumford, proposed the following higher dimensional analogue of Mordell's conjecture.

CONJECTURE 4.1. — *Suppose that  $G/k$  is a semi-abelian variety (i.e. an extension of an abelian variety by a linear torus) and that  $\text{char. } k=0$ . Let  $V/k$  be a closed subscheme of  $G/k$ . Then there are a finite number of translates  $H_i, i=1, \dots, m$ , of algebraic subgroups of  $G$  contained in  $V$  such that*

$$\Gamma \cap V(\bar{k}) = \bigcup_{i=1}^m \Gamma \cap H_i(\bar{k}).$$

This conjecture has been proved in the following cases:

- 1)  $V/k$  is a curve,  $G/k$  is an abelian variety and  $\text{char. } k=0$  (Raynaud [38] and Faltings [11]);
- 2)  $\Gamma$  is the full torsion subgroup of  $G(\bar{k})$ ,  $\text{char. } k=0$  (Raynaud [38], extended by Hindry [18]);
- 3)  $G/k$  is a linear torus,  $\text{char. } k=0$  (Laurent [26]);

4)  $k$  is a function field over a number field  $k_0$ ,  $G/k$  is an abelian variety with  $\text{Tr}_{k|k_0}(G)=0$ , and  $V$  contains no translate of a sub-abelian variety (Raynaud [39]);

5)  $k$  is a number field,  $G/k$  is an abelian variety,  $\Gamma$  is finitely generated, and  $V$  contains no translate of a sub-abelian variety (Faltings [12], [46]).

This conjecture would become false for an arbitrary commutative group scheme; furthermore, it would become false in characteristic  $p>0$  (e.g. the counterexamples to Mordell's conjecture in positive characteristic). Nevertheless, one of the main results of this section (Theorem 4.5) shows that when  $\Gamma$  is finitely generated and  $G$  is any commutative group scheme of finite type over a field  $k$  this conjecture is true up to subsets of  $\Gamma$  of lower rank, in particular, up to subsets of density zero.

In particular, this theorem shows that the density of  $\Gamma \cap V$  in  $\Gamma$  is well-defined (Corollary 4.6). A sharper result (Theorem 4.7) is proved for the special case where  $G$  is a semi-abelian variety, by using the results of Section 4.1 on tori contained in semi-abelian varieties. The variation of the density with the ground field is examined in Section 4.3. Subgroups of finite rank of abelian varieties are considered in Section 4.4 and in fact we prove the Conjecture 4.1 (*cf.* Theorem 4.17) for the case where  $V$  is a curve by a reduction to the Mordell conjecture. In Section 4.6, we give an application to the meromorphic continuation of certain zeta functions.

4.1. TORI IN SEMI-ABELIAN VARIETIES. — We begin by extending some results of Bogomolov [3] to the case of semi-abelian varieties.

Let  $G/k$  be a semi-abelian variety. Let  $H$  be an algebraic subgroup of  $G$ . By a torus of type  $H$  we mean a translate of  $H$  by  $x \in G(\bar{k})$ . The set of tori of type  $H$  is parametrised by the scheme  $G/H$ . The torus lying over the point  $t \in G/H$  (or passing through  $t \in G$ ) is written  $H_t$ .

Let  $X/k$  be a closed subscheme of  $G$ . A torus  $H_t \subset X$  is maximal if it is not properly contained in a torus  $H'_t \subset X$ . Note that more than one maximal torus may pass through a given point of  $X$ . Let

$$p: G \rightarrow G/H$$

be the projection and let  $p'$  be the restriction of  $p$  to  $X$ . Then the subset

$$M_H \subset X(\bar{k})$$

of points of  $X$  having maximal torus of type  $H$  is contained in the closed subscheme  $\tilde{M}_H$  of  $X$  whose set of  $\bar{k}$ -valued points is

$$\tilde{M}_H(\bar{k}) = \{x \in X(\bar{k}); p'^{-1} p'(x) = H_x\}.$$

Let  $\bar{M}_H$  be the Zariski closure of  $M_H$ , which is a closed subscheme of  $\tilde{M}_H$ ; it is clear that if  $\bar{M}_H \neq \emptyset$ , then  $\bar{M}_H$  is a component of  $\tilde{M}_H$ .

PROPOSITION 4.2. — *A closed subscheme  $X/k$  of  $G$  contains only a finite number of types of maximal tori.*

*Proof.* – Let  $G'$  be the Serre compactification of  $G$  (see [7]) i. e.  $G'$  is an irreducible projective  $k$ -scheme containing  $G$  as an open subscheme and equipped with an action by  $G$  as a group scheme of operators

$$G \times G' \rightarrow G'$$

extending that of  $G$  on itself. If  $V$  is a subvariety of  $G$ , we denote by  $V'$  the Zariski closure of  $V$  in  $G'$ . Fix some projective embedding of  $G'$ .

We can choose a set of points  $\alpha_i \in H(\bar{k})$  of cardinality at most  $\dim X + 1$  such that (where  $X'_\alpha$  denotes the translate  $\alpha + X'$  of  $X'$ )

$$\bigcap_{\alpha_i} X'_{\alpha_i} = \tilde{M}'_H.$$

This can be done by using a sufficiently general set of elements  $\alpha_i$ . Hence computing degrees with respect to the fixed projective embedding of  $G'$ , we have

$$\deg \bar{M}'_H \leq \deg \tilde{M}'_H \leq (\deg X')^{\dim X + 1}.$$

Now,  $x \in \bar{M}'_H$  has maximal torus  $H_x$  if and only if  $p'(x)$  has trivial maximal torus in  $p'(\bar{M}'_H)$ . Therefore there exists a set of elements  $\beta_j \in G(\bar{k})$  of cardinality at most  $\dim p'(\bar{M}'_H)$  such that

$$\bigcap_j \bar{M}'_{H, \beta_j}$$

is the union of a finite number of the  $H_i$  and is not empty. As

$$\deg H'_t \leq \deg \left( \bigcap_j \bar{M}'_{H, \beta_j} \right) \leq (\deg \bar{M}'_H)^{\dim p'(\bar{M}'_H)} \leq (\deg X)^{(\dim X + 1) \dim p'(\bar{M}'_H)},$$

we have that the degree of  $H$  is bounded. Hence the image  $H^*$  of  $H$  in the abelian variety  $G/P$ , where  $P$  is the maximal linear connected algebraic subgroup of  $G$ , has bounded degree; therefore, there are only finitely many possibilities for the subgroup  $H^*$  of  $G/P$ . Furthermore, the subgroup  $H \cap P$  has bounded degree in  $P$ ; hence there are only finitely many possibilities for  $H \cap P$ . Hence there are only a finite number of such subgroups  $H$ . This proves the proposition.

Recall that the logarithmic Kodaira dimension  $\bar{\kappa}(V)$  of a separated irreducible and reduced  $k$ -scheme of finite type  $V$ , where  $k$  is a field of characteristic zero, is defined as follows (see [20]). By a theorem of Nagata, there is a complete algebraic variety  $\bar{V}/k$  which contains  $V$  as an open subscheme; further, by Hironaka there is a smooth complete variety  $\bar{V}^*$  and a birational morphism

$$\mu: \bar{V}^* \rightarrow \bar{V}$$

so that

$$\bar{D} = \bar{V}^* - \mu^{-1}(V)$$

is a divisor with normal crossings. Then

$$\bar{\kappa}(V) = \text{tr. deg}_k \bigoplus_{i=0}^{\infty} H^0(\bar{V}^*, (K_{\bar{V}^*} \otimes \mathcal{O}(\bar{D}))^{\otimes i}) - 1$$

where  $K_{\bar{V}^*}$  is the canonical line bundle on  $\bar{V}^*$ . The logarithmic Kodaira dimension  $\bar{\kappa}(V)$  is independent of the choices of compactification and desingularisation of  $V$  (see [20]); if  $V$  is complete, it coincides with the usual Kodaira dimension  $\kappa(V)$  of  $V$ .

**COROLLARY 4.3.** — *Suppose that  $X/k$  is a closed subscheme of  $G/k$ .*

(i) *For any algebraic subgroup  $H$  of  $G$ , the subset  $M_H \subset X(\bar{k})$  is the set of  $k$ -rational points of a locally closed subscheme (also written  $M_H$ ) of  $X$ .*

(ii) *There is a unique decomposition into non-empty locally closed subschemes*

$$X = M_{H_1} \cup M_{H_2} \cup \dots \cup M_{H_m}$$

where the  $H_i$  are distinct algebraic subgroups of  $G$ .

(iii) *Assume that  $X$  is absolutely irreducible and that  $\text{char. } k = 0$ . If  $M_H$  is dense in  $X$ , then*

$$\bar{\kappa}(X) \leq \dim M_H - \dim H.$$

*Equality holds here in the following cases: (a)  $G/k$  is an abelian variety; (b)  $G/k$  is a linear torus; or (c)  $\bar{\kappa}(X) = 0$ .*

*Proof.* — We may immediately reduce to the case where  $X$  is absolutely irreducible.

(i) Proposition 4.2 gives the decompositions

$$\begin{aligned} X(\bar{k}) &= M_{H_1} \cup M_{H_2} \cup \dots \cup M_{H_m}, \\ X &= \bar{M}_{H_1} \cup \bar{M}_{H_2} \cup \dots \cup \bar{M}_{H_m}. \end{aligned}$$

As  $X$  is irreducible,  $X = \bar{M}_{H_i}$  for some  $i$ . It follows that an element  $x \in \bar{M}_{H_i}(\bar{k})$  belongs to  $M_{H_i}$  if and only if there is no  $H_j \supset H_i$ ,  $H_j \neq H_i$  with  $x \in \bar{M}_{H_j}$ . Hence  $M_{H_i}$  is an open subset of  $\bar{M}_{H_i}(\bar{k})$  for the Zariski topology, with corresponding open subscheme also denoted by  $M_{H_i}$ . Put  $X' = X \setminus M_{H_i}$ . Then  $\dim X' < \dim X$  and the proof may now be completed by induction on  $\dim X$ .

(ii) This follows immediately from (i) and Proposition 4.2.

(iii) The torus  $H$  is a subgroup of the automorphism group of  $X$ ; as  $X$  is a closed subscheme of the semi-abelian variety  $G$ , we have  $\bar{\kappa}(X) \geq 0$  [20], Theorem 4. Hence we have

$$\dim H \leq \dim X - \bar{\kappa}(X)$$

by [20], Theorem 7; the first part follows.

For the last part, for each case (a), (b), (c), we have by [20], [21], [22], respectively, that  $X$  is stabilised by a sub-algebraic group of  $G$  of dimension equal to  $\dim X - \bar{\kappa}(X)$ ;

let  $H$  be the connected component of the identity of this stabiliser. We then have

$$\bar{\kappa}(X) = \dim M_H - \dim H,$$

as required.

This completes the proof.

*Remark.* — 1) I understand from Iitaka that one may have strict inequality in Corollary 4.3 (iii) for general semi-abelian varieties.

4.2. RANKS OF SETS OF RATIONAL POINTS. — We combine the combinatorial theorem of Section 3 with the zero estimates of Section 2 to prove the next theorem.

**THEOREM 4.4.** — *Suppose that  $G/k$  is a smooth connected commutative group scheme of dimension  $>0$  over a field  $k$ . Let  $G \rightarrow \mathbf{P}_k^n$  be a projective embedding of  $G$ . Let  $\Gamma$  be a finitely generated subgroup of  $G(k)$  whose Zariski closure in  $G$  is equal to  $G$ . Then there is a real number  $\varepsilon > 0$  such that for any section  $s \in \Gamma(\mathbf{P}_k^n, \mathcal{O}(m))$ ,  $m > 0$ , whose restriction to  $G$  is not identically zero, we have*

$$\text{rank} \{ \gamma \in \Gamma; s(\gamma) = 0 \} \leq \text{rank } \Gamma - \varepsilon.$$

*Proof.* — For any integer  $N \geq 1$ , let  $\varepsilon(N) > 0$  be the real number given by Theorem 3.1 applied to  $A = \Gamma$ , a fixed set of suitable elements  $\gamma_1, \dots, \gamma_k$  of  $\Gamma$ ,  $k = \text{rank } \Gamma$ , and the integer  $n = \dim G$  [one can take

$$\varepsilon(N) = (N+2)^{-nk} \min(k, N+2)$$

by Remark 1 after Theorem 3.1]. Put

$$S = \{ \gamma \in \Gamma; s(\gamma) = 0 \}.$$

Assume that for all integers  $N \geq 1$ , we have

$$(4.1) \quad \text{rank } S > \text{rank } \Gamma - \varepsilon(N).$$

By Theorem 3.1, there are  $N$ -special subsets  $U_1, \dots, U_n \subset \Gamma$  and an element  $u \in S$  with

$$S \supset u + \sum_{i=1}^n U_i.$$

By Theorem 2.1, there is a connected algebraic subgroup  $H \subset G$ ,  $H \neq G$ , an integer  $i$ ,  $1 \leq i \leq n$ , and a constant  $a > 0$ , depending only on  $G$  and the projective embedding, such that

$$|\pi_{G/H}(U_i)| \leq (\deg G) (am)^{\dim G/H}.$$

Lemma 3.3 shows that

$$|\pi_{G/H}(U_i)| \geq N^{\text{rank } \Gamma - \text{rank } (\Gamma \cap H)},$$

hence we have

$$(4.2) \quad N^{\text{rank } \Gamma - \text{rank } (\Gamma \cap H)} \leq (\text{deg } G) (am)^{\dim G - 1}.$$

As the Zariski closure of  $\Gamma$  does not consist of a finite number of translates of  $H$  in  $G$ , we have

$$\text{rank } (\Gamma \cap H) \leq \text{rank } \Gamma - 1.$$

It follows that (4.2) is violated for all

$$N > (\text{deg } G) (am)^{\dim G - 1}.$$

Hence (4.1) is false for all sufficiently large  $N$ , whence the result.

*Remarks.* — 2) In [5], the above Theorem 4.4 is stated with the weaker conclusion that  $d^* \{ \gamma \in \Gamma; s(\gamma) = 0 \} = 0$ ; in place of Theorem 3.1, the proof there used a theorem of Furstenberg [14] (*cf.* the beginning of Section 3).

3) The above proof and Remark 1 of Section 3 show that the real number  $\varepsilon$  in Theorem 4.4 can be taken to be

$$(d(am)^{n-1} + 2)^{-n \text{rank } \Gamma} \min(\text{rank } \Gamma, d(am)^{n-1} + 2),$$

where  $n = \dim G$ ,  $d = \text{deg } G$ , and  $a$  depends only on the projective embedding of  $G$ . In particular, for a fixed  $G$  and fixed projective embedding,  $\varepsilon$  depends only on the rank of  $\Gamma$  and  $m$ .

We now come to the main result of this section:

**THEOREM 4.5.** — *Let  $G/k$  be a commutative group scheme of finite type and  $V/k$  a closed subscheme. Let  $\Gamma$  be a finitely generated subgroup of  $G(k)$  and let  $H$  be the connected component of the identity of the Zariski closure of  $\Gamma$  in  $G$ . Then there is a subset  $S \subset \Gamma \cap V(k)$  and a finite number of translates  $H_i$  of  $H$  such that  $H_i$  is contained in  $V_{\text{red}}$  for all  $i$  and*

$$\Gamma \cap V(k) = \bigcup_{i=1}^n (H_i \cap \Gamma) \cup S, \quad \text{rank } S < \text{rank } \Gamma.$$

*Remark.* — 4) The known instances of Lang's Conjecture 4.1 (listed at the beginning of Section 4) supersede Theorem 4.5 when they apply.

*Proof.* — By [15], VII<sub>A</sub>, Prop. 8.3, there is a finite local (in particular, purely inseparable over  $k$ ) closed characterized subgroup  $F/k$  of  $G$  such that  $G/F$  is a smooth  $k$ -group scheme. By taking the image of  $V$  in  $G/F$  we may reduce to the case where  $G/k$  is smooth. Furthermore, as  $G$  is a finite disjoint union of principal homogeneous spaces over the connected component of the identity of  $G$ , we may reduce to the case where  $G/k$  is connected and smooth. Let  $\Gamma^0 = \Gamma \cap H$ , a subgroup of  $\Gamma$  of finite



index. Let  $\gamma_1, \dots, \gamma_m \in \Gamma$  be coset representatives of  $\Gamma^0$  in  $\Gamma$ . The decomposition

$$V \cap \Gamma = \bigcup_{H-\gamma_i \subset V} \{\Gamma^0 - \gamma_i\} \cup \bigcup_{H-\gamma_i \not\subset V} V \cap \{H - \gamma_i\} \cap \{\Gamma^0 - \gamma_i\}$$

shows that we may reduce to be case where  $H=G$ .

We may assume that  $V \neq G$  else the theorem is obvious. We may then select a sufficiently small open quasi-affine subscheme  $U$  of  $G \setminus V$  such that there is a closed subscheme  $X$  of  $G$ , pure of codimension 1, for which

$$V \subset X \subset G \setminus U.$$

Let  $D$  the divisor given by the sum over all the irreducible components of  $G \setminus U$  of codimension 1 in  $G$ . Then [37], Théorème VI.2.3, the sheaf  $\mathcal{O}(D)$  is ample on  $G$  and has a non-zero section vanishing along  $V$ . Take the projective embedding  $G \rightarrow \mathbf{P}_k^n$  given by  $\mathcal{O}(nD)$ , for some integer  $n > 0$  so large that  $\mathcal{O}(nD)$  is very ample. Then there is a section  $s \in \Gamma(\mathbf{P}_k^n, \mathcal{O}(m))$ , for some  $m > 0$ , whose restriction to  $G$  is not identically zero and which vanishes along  $V$ .

By Theorem 4.4, we then have

$$\text{rank} \{ \Gamma \cap V(k) \} \leq \text{rank} \{ \gamma \in \Gamma; s(\gamma) = 0 \} \leq \text{rank } \Gamma - \varepsilon$$

for some  $\varepsilon > 0$ , and the result follows.

*Remark.* — 5) By Remark 2 above, one can write the conclusion of the theorem above as:  $\text{rank } S \leq \text{rank } \Gamma - \varepsilon$  for some  $\varepsilon > 0$  depending only on the morphism  $V \rightarrow G$ ,  $\text{rank } \Gamma$  and the Zariski closure of  $\Gamma$  in  $G$ .

**COROLLARY 4.6.** — *Let  $G, V, k, \Gamma$  be as in Theorem 4.5. Then the density  $d_\Gamma(\Gamma \cap V(k))$  exists and is the rational number*

$$d_\Gamma(\Gamma \cap V(k)) = \frac{m}{[\Gamma : \Gamma \cap H]}$$

where  $H$  is the connected component of the identity of the Zariski closure of  $\Gamma$  and  $m$  is the number of distinct translates of  $H$  by  $\Gamma$  which lie in  $V$ .

*Proof.* — Take the densities of

$$\Gamma \cap V(k) = \bigcup_{i=1}^n (H_i \cap \Gamma) \cup S, \quad \text{rank } S < \text{rank } \Gamma.$$

We have  $d_\Gamma(S) = 0$  and

$$d_\Gamma\left(\bigcup_{i=1}^n (H_i \cap \Gamma) \cup S\right) = \sum_{i=1}^n d_\Gamma(H_i \cap \Gamma),$$

whence the result.

4.3. SEMI-ABELIAN VARIETIES OVER NUMBER FIELDS. — Let  $k$  be an algebraic number field. We have the following strengthening of Theorem 4.5, which is an immediate consequence of that theorem and Corollary 4.3:

THEOREM 4.7. — *Let  $V/k$  be a closed subscheme of a semi-abelian variety  $A/k$ . Then there are a finite number of semi-abelian sub-varieties  $A_i$  of  $A$  with the following property. Let  $k'/k$  be a finite extension field and  $\Gamma$  a finitely generated subgroup of  $A(k')$ . Then  $V$  contains a finite number of translates  $B_{ij}$  of the  $A_i$ , for all  $i$ , such that*

$$\Gamma \cap V = \bigcup_{ij} (\Gamma \cap B_{ij}) \cup S$$

where

$$\text{rank } S < \text{rank } \Gamma.$$

*Remark.* — 6) The following conjecture is a strengthening of a similar conjecture of Bombieri:

CONJECTURE 4.8 (S. Lang). — *Let  $X/k$  be an irreducible projective  $k$ -scheme of general type. Then there is a closed subvariety  $Z \subset X$ ,  $Z \neq X$ , such that for every finite extension field  $k'$  of  $k$  the open subscheme  $X \setminus Z$  has only finitely many  $k'$ -rational points.*

We consider this conjecture for the case of an irreducible projective  $k$ -scheme  $V$  of general type which is embeddable in some abelian variety  $A/k$ . By Corollary 4.3, there is a non-empty open subscheme  $U$  of  $V$  of the form  $M_H$  for some algebraic subgroup  $H$  of  $A$  defined over some finite extension of  $k$ . By Corollary 4.3 (iii), we have

$$\dim H = \dim V - \kappa(V) = 0.$$

(i) Conjecture 4.1 implies that  $U(k')$  is a finite set for all finite extension fields  $k'$  of  $k$ , verifying Conjecture 4.8 in this case. [Conversely, in [6] it is proved that Conjecture 4.8 implies Conjecture 4.1 for the case where  $\Gamma$  is finitely generated and  $A/k$  is an abelian variety.]

(ii) Theorem 4.7 implies that for every finite extension field  $k'/k$ , either  $U(k')$  is finite or it has density zero in  $A(k')$ .

If  $p/q \in \mathbf{Q}$ ,  $p, q \in \mathbf{Z}$ ,  $(p, q) = 1$ , we write

$$\begin{aligned} \text{denom}(p/q) &= |q|, & \text{num}(p/q) &= |p|, \\ \text{height}(p/q) &= \max(|p|, |q|). \end{aligned}$$

For the above purpose, we shall agree to write 0 as 0/1.

THEOREM 4.9. — *Let  $V/k$  be a closed subscheme of a semi-abelian variety  $A/k$ . Let  $m(A, V)$  be the number of types of maximal tori contained in  $V$  (cf. Proposition 4.2). Then there are non-negative real constants  $c_1(A, V, k)$ ,  $c_2(\dim A)$ ,  $c_3(A, V)$  with the following property. For all finite extension fields  $k'$  of  $k$ , and any finitely generated subgroup  $\Gamma$  of*

$A(k')$ , there are non-negative rational numbers  $\alpha_1, \dots, \alpha_m$  such that

$$d_{A(k')}(\mathbf{V}(k')) = \sum_{i=1}^m \alpha_i$$

$$\text{denom}(\alpha_i) \leq c_1 [k' : k]^{c_2}, \quad \text{for all } i,$$

$$\text{num}(\alpha_i) \leq c_3, \quad \text{for all } i.$$

*Remark.* — 7) An optimistic conjecture is that if  $B$  is an abelian variety defined over  $k$  then  $|\mathbf{B}(k')_{\text{tors}}|$  is bounded by a function depending only on  $\dim B$  and  $[k' : k]$ . This conjecture would imply that in Theorem 4.9,  $\text{denom}(\alpha_i)$  is bounded by a function depending only on  $\dim A$  and  $[k' : k]$ .

*Proof.* — (i) Let

$$X = M_{H_1} \cup M_{H_2} \cup \dots \cup M_{H_m}, \quad m = m(A, V),$$

be the decomposition of  $V$  into locally closed subschemes given by Corollary 4.3 (ii), where the  $H_i$  are distinct algebraic subgroups of  $G$ . Each quotient variety  $M_{H_i}/H_i$  is a locally closed subscheme of  $A/H_i$  and which contains only finitely many points of  $(A/H_i)(\bar{k})_{\text{tors}}$ , by Raynaud's theorem (generalised in [18], Théorème 2). Put

$$(4.3) \quad c_3(A, V) = \max_i |M_{H_i}/H_i \cap (A/H_i)(\bar{k})_{\text{tors}}|.$$

By Lemma 4.10 below, there are positive real constants  $c_{1,i}(A/H_i, k)$ ,  $c_{2,i}(\dim A/H_i)$  such that for any finite extension field  $k'$  of  $k$  we have

$$(4.4) \quad |(A/H_i)(k')_{\text{tors}}| \leq c_{1,i} [k' : k]^{c_{2,i}}, \quad \text{for all } i,$$

where if  $A/H_i$  is not defined over  $k'$  then  $(A/H_i)(k')_{\text{tors}}$  is taken to be empty.

Let  $k'$  be a finite extension field of  $k$ . Let  $B$  be the connected component of the identity of the Zariski closure of  $\Gamma \subset A(k')$  and let

$$\{H_i; i \in I\}$$

be the set of those  $H_i$ 's which contain  $B$ . By an elementary combinatorial argument (as  $I$  is finite), we may construct locally closed subschemes  $V_{H_i}$  of  $M_{H_i}$ , for all  $i \in I$ , such that the  $V_{H_i}$  are disjoint and

$$\bigcup_{i \in I} V_{H_i} = \bigcup_{i \in I} M_{H_i}.$$

By Corollary 4.6, we have

$$\begin{aligned} d_{\Gamma}(\Gamma \cap V) &= d_{\Gamma}(\Gamma \cap \bigcup_{i \in I} M_{H_i}) = d_{\Gamma}(\Gamma \cap \bigcup_{i \in I} V_{H_i}) \\ &= \sum_{i \in I} d_{\Gamma}(\Gamma \cap V_{H_i}) = \sum_{i \in I} d_{(\Gamma/\Gamma \cap H_i)}(\Gamma \cap V_{H_i}/H_i). \end{aligned}$$

But each  $\Gamma/\Gamma \cap H_i$  is a finite group, for all  $i$ , hence by (4.3) and (4.4) we have

$$\begin{aligned} \text{denom}(d_{(\Gamma/\Gamma \cap H_i)}(\Gamma \cap V_{H_i}/H_i)) &\leq c_{1,i} [k' : k]^{c_{2,i}}, \quad \text{for all } i, \\ \text{num}(d_{(\Gamma/\Gamma \cap H_i)}(\Gamma \cap V_{H_i}/H_i)) &\leq c_3. \end{aligned}$$

The result follows by taking

$$\begin{aligned} c_1(A, V, k) &= \max_i c_{1,i}, \\ c_2(\dim A) &= \max_i c_{2,i}. \end{aligned}$$

LEMMA 4.10. — *Let  $A/k$  be a semi-abelian variety. There are positive real constants  $c_1(A, k)$ ,  $c_2(\dim A)$  such that for any finite extension field  $k'$  of  $k$  we have*

$$|A(k')_{\text{tors}}| \leq c_1 [k' : k]^{c_2}.$$

*Proof.* — The semi-abelian variety  $A$  is an extension of an abelian variety  $B/k$  by a linear torus  $L/k$ . We have

$$|A(k')_{\text{tors}}| = |B(k')_{\text{tors}}| \cdot |L(k')_{\text{tors}}|.$$

Hence we only need prove the bound of the lemma for the two cases of abelian varieties and linear tori. For the former case, the lemma is a theorem of Serre [42], Théorème 4. For the latter case, the lemma follows from the isotriviality of linear tori and elementary bounds for the degrees of cyclotomic extensions of number fields.

*Remark.* — 8) With reference to Lemma 4.10, the referee has pointed out that Masser [29] has obtained effective bounds of the form  $O([k' : k]^\lambda)$  for the order of a torsion point of  $A(k')$ , when  $A/k$  is an abelian variety.

COROLLARY 4.11. — *Let  $d > 0$  be an integer. There is a set  $R_d$  of at most  $c_4 d^{c_5}$  rational numbers, for some constants  $c_4(A, V, k)$ ,  $c_5(A, V, k) > 0$ , with the following property. If  $k'/k$  is a finite field extension where  $[k' : k] \leq d$  and  $\Gamma$  is a finitely generated subgroup of  $A(k')$  then*

$$d_\Gamma(\Gamma \cap V) \in R_d.$$

*Proof.* — Let  $m$  be the number of types of maximal tori in  $V$ . The number of rational numbers  $\beta \in [0, 1]$  of the form

$$\beta = \sum_{i=1}^m \alpha_i, \quad \alpha_i \geq 0$$

where

$$\begin{aligned} \text{denom}(\alpha_i) &\leq c_1 d^{c_2}, \quad \text{for all } i, \\ \text{num}(\alpha_i) &\leq c_3, \quad \text{for all } i, \end{aligned}$$

is at most  $(c_3 c_1 d^{c_2})^m$ . The result now follows from Theorem 4.9.

Theorem 4.9 gives an arithmetic restriction on the rational numbers which are densities. With  $A$ ,  $V$ , and  $k$  as in Theorem 4.9, let  $S$  be the subset of  $\mathbf{Q} \cap [0, 1]$  given by

$$S = \{d_\Gamma(V \cap \Gamma); k'/k \text{ a finite field extension, } \Gamma \text{ a subgroup of } A(k')\}.$$

For  $X$  a positive real number, write

$$S(X) = \{\alpha \in S; \text{height}(\alpha) \leq X\}.$$

THEOREM 4.12. — 1) *The set  $S$  omits a set of rational numbers of positive density in  $\mathbf{Q} \cap [0, 1]$ .*

2) *Suppose that  $V$  contains at most two types of maximal tori. Then for some constant  $c > 0$ , we have*

$$|S(X)| \ll X^{1+c/\log \log X}, \text{ as } X \rightarrow \infty.$$

*In particular, almost all rational numbers (in the sense of density) in the interval  $[0, 1]$  are not in  $S$ .*

*Proof.* — Let  $p/q \in S$ , where  $p, q \in \mathbf{Z}$ ,  $\gcd(p, q) = 1$ . By Theorem 4.9 we have

$$(4.5) \quad \left\{ \begin{array}{l} p/q = \sum_{i=1}^m \alpha_i/\beta_i, \quad m = \text{number of types of maximal tori in } V, \\ \alpha_i, \beta_i \in \mathbf{Z}, \quad \gcd(\alpha_i, \beta_i) = 1, \quad \alpha_i, \beta_i \geq 0, \quad \text{for } i = 1, \dots, m, \\ 0 \leq \alpha_1, \dots, \alpha_m \leq C, \quad \text{for some integer constant } C = C(A, V). \end{array} \right.$$

1) Reordering the  $\alpha_i/\beta_i$ , we may assume that

$$(4.6) \quad \beta_1 \leq \dots \leq \beta_m.$$

We consider the maximum value of the quantity  $M = \sum_{i=1}^m \alpha_i/\beta_i$  subject to the restrictions (4.5), (4.6) and that

$$M = \sum_{i=1}^m \alpha_i/\beta_i < 1.$$

This extremal problem has no interior critical point. It follows that the values of the  $\alpha_i$  are extremal and therefore

$$\alpha_1 = \dots = \alpha_m = C$$

and the  $\beta_i$ 's are such that

$$\sum_{i=1}^j \alpha_i/\beta_i \text{ is maximum for all } j=1, \dots, m.$$

Taking  $j=1$ , this gives

$$\alpha_1/\beta_1 = C/(C+1), \quad \beta_1 = C+1.$$

Taking  $j=2$ , we have

$$\alpha_2/\beta_2 = C/\beta_2 < 1 - C/(C+1);$$

hence we obtain

$$\beta_2 = C^2 + C + 1.$$

And one checks by induction that

$$\beta_i = f_i(C), \quad i=1, \dots, m,$$

where  $f_i(C)$  is a monic polynomial in  $C$  with positive integer coefficients, constant term 1, and of degree  $2^{i-1}$ . Whence the maximum value of  $M$  is given by

$$M = \sum_{i=1}^m \alpha_i/\beta_i = 1 - C/(f_{m+1}(C) - 1).$$

In conclusion,  $S$  omits the set of all rational numbers in the open interval  $(1 - C/(f_{m+1}(C) - 1), 1)$ , whence the result.

2) We consider the conditions (4.5) with  $m=2$ .

Let  $h = \gcd(\beta_1, \beta_2)$ ,  $\beta'_i = \beta_i/h$ , for  $i=1, 2$ . Then we have

$$\frac{p}{q} = \frac{1}{h} \left\{ \frac{\alpha_1}{\beta'_1} + \frac{\alpha_2}{\beta'_2} \right\} = \frac{1}{h} \frac{\alpha_1 \beta'_2 + \alpha_2 \beta'_1}{\beta'_1 \beta'_2}.$$

where

$$\gcd(\alpha_1 \beta'_2 + \alpha_2 \beta'_1, \beta'_1 \beta'_2) = 1.$$

Hence we have  $\beta'_1 \beta'_2 | q$ . Let

$$h' = \gcd(h, q/\beta'_1 \beta'_2), \quad h = h' h''.$$

Then we have

$$p = \frac{q}{h' \beta'_1 \beta'_2} \frac{\alpha_1 \beta'_2 + \alpha_2 \beta'_1}{h''}, \quad h' \beta'_1 \beta'_2 | q$$

and

$$h'' \mid \alpha_1 \beta'_2 + \alpha_2 \beta'_1.$$

As  $p$  and  $q$  are coprime, we must then have

$$(4.7) \quad \begin{cases} h' \beta'_1 \beta'_2 = q, & 0 \leq \alpha_1, \alpha_2 \leq C, \\ p = \frac{\alpha_1 \beta'_2 + \alpha_2 \beta'_1}{h''}. \end{cases}$$

In particular, for a given integer  $q \geq 0$  the possible numerators  $p$  of  $p/q \in S$  must satisfy the restrictions (4.7). We shall estimate the number of such numerators in terms of  $q$ .

Let  $Z$  be the set of pairs of positive integers  $(\beta'_1, \beta'_2)$  satisfying  $\beta'_1 \beta'_2 \mid q$ . Then we have  $|Z| = D(q)$ , where  $D(n)$  is the arithmetic function

$$D(n) = \sum_{m \mid n} d(m)$$

$d(n)$  being the number of divisors of  $n$ . An elementary estimate for  $d(n)$  is [17], Theorem 317

$$\limsup_{n \rightarrow \infty} \frac{\log d(n) \log \log n}{\log n} = \log 2.$$

Whence for some constant  $c_1 > 0$  we have

$$D(n) \leq d(n)^2 \ll \exp(c_1 \log n / \log \log n).$$

The number  $N(q)$  of possible integers  $p$  satisfying the conditions (4.7), for a fixed  $q$ , is clearly at most

$$\sum_{0 \leq \alpha_1, \alpha_2 \leq C} \sum_{(\beta'_1, \beta'_2) \in Z} d(\alpha_1 \beta'_2 + \alpha_2 \beta'_1).$$

It follows that

$$N(q) \ll D(q) C^2 \exp(c_1 \log 2 C q / \log \log 2 C q) \ll \exp(c_2 \log q / \log \log q).$$

Whence we obtain

$$|S(X)| \leq \sum_{q \leq X} N(q) \ll X \exp(c_2 \log X / \log \log X),$$

as required.

For the last part, note that the number of rational numbers  $\alpha$ ,  $0 \leq \alpha \leq 1$ , with height  $(\alpha) \leq X$  is

$$\sim (3/\pi^2) X^2, \quad \text{as } X \rightarrow \infty.$$

Hence  $S$  omits almost all rational numbers in the interval  $[0, 1]$ , as required.

As to the existence of varieties with given density of rational points, we have (I thank O. Debarre for some help with this construction):

THEOREM 4.13. — *Let  $\alpha \in [0, 1]$  be a rational number. Then there are irreducible projective varieties  $V/\mathbf{Q}$  contained in abelian varieties  $A/\mathbf{Q}$  where*

- (i)  $d_{A(\mathbf{Q})}(V(\mathbf{Q})) = \alpha$ ;
- (ii)  $V/\mathbf{Q}$  is smooth and of general type;
- (iii)  $\text{rank } A(\mathbf{Q}) = 1$ ;
- (iv)  $V$  is of arbitrarily high dimension.

*Remark.* — 9) If  $V$  and  $A$  are such that the denominator of  $d_{A(\mathbf{Q})}(V(\mathbf{Q}))$  is divisible by a prime number  $> 7$  then  $A$  is not  $\mathbf{Q}$ -isogenous to a direct product of elliptic curves.

[For if  $A$  were  $\mathbf{Q}$ -isogenous to a direct product of elliptic curves, each factor in a Jordan-Hölder decomposition of  $A/\mathbf{Q}$  would be an elliptic curve  $E$  over  $\mathbf{Q}$ . The torsion group  $E(\mathbf{Q})_{\text{tors}}$  of such a factor would then have order  $\leq 10$  or  $12$  or  $16$ , by a theorem of Mazur [31], Introduction, Theorem 8. Hence  $|A(\mathbf{Q})_{\text{tors}}|$  would be divisible only by the prime numbers among  $2, 3, 5, 7$  whence  $\text{denom}(d_{A(\mathbf{Q})}(V(\mathbf{Q})))$  would be an integer of the same form (by the proof of Theorem 4.9, for example), as required.]

Indeed, the conjecture mentioned in Remark 7 following Theorem 4.9 would imply that the  $\dim A$  (resp. the maximal dimension of a  $\mathbf{Q}$ -simple Jordan-Hölder factor of  $A/\mathbf{Q}$ ) tends to infinity as  $\text{denom}(d_{A(\mathbf{Q})}(V(\mathbf{Q})))$  (resp. the greatest prime factor of  $\text{denom}(d_{A(\mathbf{Q})}(V(\mathbf{Q})))$ ) tends to infinity.

*Proof of Theorem 4.13.* — Let

$$J = J(X_0(N))/\mathbf{Q}$$

be the jacobian of the modular curve  $X_0(N)$ , which classifies the pairs  $(E, C_N)$  where  $E$  is an elliptic curve and  $C_N$  is a cyclic subgroup of  $E$  of order  $N$ . Let  $\tilde{J}$  be the Eisenstein quotient [31] of  $J$ . This is an abelian variety over  $\mathbf{Q}$  with the following properties, in the case where  $N$  is a prime number (at least):

- 1)  $\tilde{J}(\mathbf{Q}) \cong \mathbf{Z}/n\mathbf{Z}$ , where  $n = \text{num}((N-1)/12)$  (cf. [31], Introduction, Theorem 4).
- 2) If  $p|n$ , then  $\tilde{J}$  contains absolutely simple factors of dimension (cf. [31], Chapt. III, Proposition 7.2)

$$\geq \log p/2 (\log(1 + 2^{1/2})).$$

To prove the theorem, let  $M \geq 2$  be a positive integer and let  $\alpha \in [0, 1]$  be a rational number. Write  $\alpha = a/b$ ,  $a, b \in \mathbf{N}$ ,  $(a, b) = 1$ . Select  $p$  to be any prime number such that

$$p \geq (1 + 2^{1/2})^{2M}.$$

Choose, by Dirichlet's theorem, a prime number  $N$  such that

$$N \equiv 1 \pmod{12pb}.$$



It follows from 1) that the Eisenstein quotient  $\tilde{J}$  of  $J(N)$  satisfies

$$\tilde{J}(\mathbf{Q}) \cong \mathbf{Z}/m\mathbf{p}\mathbf{b}\mathbf{Z},$$

for some integer  $m \geq 1$ , and (2) shows that

$$\dim \tilde{J} \geq \log p/2 (\log(1 + 2^{1/2})) \geq M.$$

We may then select a finite set of points

$$S \subset \tilde{J}(\mathbf{Q})$$

such that

$$d_{\tilde{J}(\mathbf{Q})}(S) = a/b.$$

Let  $C/\mathbf{Q}$  be the elliptic curve given by the Weierstrass equation

$$y^2 = x^3 - 6x^2 + 17x.$$

Then

$$C(\mathbf{Q}) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}$$

(see [44], p. 303, Example 4.10). Let  $A/\mathbf{Q}$  be the abelian variety  $C \times \tilde{J}$ . Then  $A$  satisfies (iii) and  $\dim A > M$ .

Let  $Z/\mathbf{Q}$  be the smooth closed subscheme of  $A$  given by

$$Z = \bigcup_{P \in S} C \times \{P\}.$$

As

$$\text{codim}_A Z = \dim \tilde{J} \geq M \geq 2,$$

there is a  $\mathbf{Q}$ -rational ample divisor  $D$  on  $A$  whose support contains  $Z$ . Let

$$i: \tilde{A} \rightarrow A$$

be the blowing up of  $A$  along  $Z$ . Let  $E$  be the exceptional divisor of  $i$ . For  $n \geq 0$ , the linear system

$$|i^*nD - E|$$

has no base points on  $\tilde{A}$  and has dimension  $\geq 2$ . Hence for a suitable  $n \geq 0$ , Bertini's theorem gives a  $\mathbf{Q}$ -rational section  $H$  of  $|i^*nD - E|$  which is smooth, as a closed subscheme, meets every fibre of  $E \rightarrow Z$ , and which does not properly contain any of the finitely many elliptic curves in  $\tilde{A}$  which are the lifts of the  $C \times \{P\}$ , for all  $P \in \tilde{J}(\mathbf{Q}) \setminus S$ .

Let  $V$  the closed subscheme  $i_*H$  of  $A$  [*i.e.* the Zariski closure of  $i(H \setminus E)$  in  $A$ ]. Then  $V$  is a smooth subscheme of  $A$  which contains  $Z$  and such that  $V \cap (C \times \{P\})$ , for all

$P \in \tilde{J}(\mathbf{Q}) \setminus S$ , is at most a finite set. Hence we have

$$V(\mathbf{Q}) = Z(\mathbf{Q}) \cup T$$

where  $T$  is a finite set of points. The canonical line bundle on  $V$  is just the restriction of  $\mathcal{O}(nD)$  to  $V$ ; hence it is ample and  $V$  is of general type. Furthermore, we have

$$\dim V = \dim A - 1 \geq M.$$

As  $\text{rank } A(\mathbf{Q}) = 1$ , we conclude that

$$d_{A(\mathbf{Q})}(V(\mathbf{Q})) = d_{A(\mathbf{Q})}(Z(\mathbf{Q})) = |S|/|\tilde{J}(\mathbf{Q})| = \alpha$$

and the properties (i)-(iv) then hold.

4.4. GROUPS OF POINTS OF FINITE RANK. — Conjecture 4.1 has been proved by Raynaud [38] when  $G/k$  is an abelian variety and  $\Gamma$  is the full torsion subgroup; this was extended to semi-abelian varieties by Hindry [18]. The results of Faltings ([12], [46]) and Raynaud [39] combine to prove the conjecture for  $\Gamma$  of finite rank when  $G/k$  is an abelian variety and  $V$  contains no translate of a sub-abelian variety.

The next theorem is a density version of this conjecture for abelian varieties and groups of finite rank.

THEOREM 4.14. — *Let  $k$  be a number field and  $A/k$  an abelian variety. Let  $\Gamma \subset A(\bar{k})$  be a subgroup of finite rank. Let  $V/k$  be a closed subscheme of  $A$ . Then there are a finite number of sub-abelian varieties  $B_i$ ,  $i=1, \dots, m$ , of  $A$  and subsets  $S_i$ ,  $i=1, \dots, m$ , contained in a finitely generated subgroup of  $\Gamma$  such that*

- 1)  $\Gamma \cap V = \bigcup_{i=1}^m \left( \bigcup_{s \in S_i} (s + B_i) \cap \Gamma \right)$ ;
- 2)  $s + B_i \subset V$ , for all  $s \in S_i$  and all  $i$ ;
- 3)  $\text{rank } S_i < \text{rank } \Gamma$ , for all  $i=1, \dots, m$ .

We shall use the following lemma, which is a consequence of results of Serre, Faltings and Ribet on Galois action on abelian varieties (for the proof, see [18], Théorème 4).

LEMMA 4.15. — *Let  $V, A, k$  be as in Theorem 4.14. Then there is a finitely generated subgroup  $\Gamma' \subset \Gamma$  such that  $\Gamma/\Gamma'$  is a torsion group and the following property holds: if  $P \in \Gamma \cap V(\bar{k})$  then one of the following is true:*

- 1)  $P \in \Gamma' \cap V(\bar{k})$ ;
- 2) there is a sub-abelian variety  $B$  of  $A$  of dimension  $> 0$  such that  $P + B \subset V$ .

*Proof of Theorem 4.14.* — By induction on  $\dim V$ . By an easy reduction, we may assume that  $V$  is reduced and irreducible. Assume that the theorem is true for all sub-varieties of dimension  $< \dim V$  of abelian varieties over number fields. By Corollary 4.3, there is a decomposition

$$V = M_{H_1} \cup M_{H_2} \cup \dots \cup M_{H_m},$$

where  $H_1 \subset \dots \subset H_m$  are sub-abelian varieties of  $A$  and where

$$\dim H_1 = \dim V - \kappa(V).$$

*Case 1.* —  $V$  is of general type.

We have  $\dim H_1 = 0$ . The above Lemma 4.15 gives a finitely generated subgroup  $\Gamma'$  of  $\Gamma$  such that

$$\Gamma \cap M_{H_1} = \Gamma' \cap M_{H_1}.$$

Write

$$Z = V \setminus M_{H_1},$$

which is a closed subscheme of  $V$  with  $\dim Z < \dim V$ . We have

$$\begin{aligned} \Gamma \cap V &= (\Gamma \cap Z) \cup (\Gamma' \cap M_{H_1}) \\ &= (\Gamma \cap Z) \cup (\Gamma' \cap V). \end{aligned}$$

We may now apply Theorem 4.5 to  $\Gamma' \cap V$  and apply the induction hypothesis to  $\Gamma \cap Z$  to complete the proof of the theorem in this case.

*Case 2.* —  $V$  is not of general type.

Hence

$$\dim H_1 = \dim V - \kappa(V) \geq 1$$

and  $H_1$  lies in the stabiliser of  $V$ . Let

$$p: A \rightarrow A/H_1$$

be the natural projection. Then  $p(V)$  is a closed subvariety of  $A/H_1$  of dimension  $< \dim V$ . Applying the induction hypothesis to  $p(\Gamma) \cap p(V)$  gives

$$p(\Gamma) \cap p(V) = \bigcup_{j=1}^t \bigcup_{s \in S_j} \{s + B_i \cap p(\Gamma)\}$$

where  $S_j \subset p(\Gamma)$  are subsets contained in a finitely generated subgroup  $\Delta$  of  $p(\Gamma)$ , the  $B_i$  are sub-abelian varieties of  $A/H_1$  such that

$$s + B_i \subset V/H_1$$

for all  $s \in S_j$  and all  $i$ , and with

$$\text{rank } S_j < \text{rank } p(\Gamma), \quad \text{for all } j.$$

Let  $\Delta'$  be a finitely generated subgroup of  $\Gamma$  such that  $p(\Delta') = \Delta$ . By taking a splitting of the surjection of the finitely generated torsion free abelian groups

$$\Delta'/\Delta'_{\text{tors}} \rightarrow \Delta/\Delta_{\text{tors}},$$

one can find subsets  $S'_j \subset \Delta'$  with

$$p(S'_j) = S_j \quad \text{and} \quad \text{rank } S'_j < \text{rank } \Delta, \quad \text{for all } j.$$

We then have

$$\Gamma \cap V = \bigcup_{j=1}^t \bigcup_{s \in S'_j} \{s + p^{-1}(B_j) \cap \Gamma\},$$

where  $p^{-1}(B_j)$  are sub-abelian varieties of  $A$  such that

$$s + p^{-1}(B_j) \subset V, \quad \text{for all } s \in S'_j \text{ and all } j,$$

and

$$\text{rank } S'_j < \text{rank } \Delta \leq \text{rank } p(\Gamma) \leq \text{rank } \Gamma, \quad \text{for all } j.$$

This proves the theorem.

4.5. CURVES IN COMMUTATIVE GROUP SCHEMES. — Let  $G$  be a commutative group scheme of finite type over a field  $k$  of arbitrary characteristic. Let  $V/k$  be a curve (*i.e.* an irreducible and reduced 1-dimensional  $k$ -scheme) contained in  $G$  and let  $\Gamma$  be a finitely generated subgroup of  $G(k)$ . The next result immediately follows from Theorem 4.4.

THEOREM 4.16. — *Suppose that  $V$  is not  $\bar{k}$ -isomorphic to  $G_a$ ,  $G_m$  or a complete elliptic curve over  $k$ . Then*

$$\text{rank } \Gamma \cap V < \text{rank } \Gamma.$$

*Remarks.* — 10) This result is not entirely superseded by Mordell's Conjecture (proved by Faltings [11], for the case where  $\text{char. } k = 0$ , and Manin-Grauert-Samuel [40], for the case where  $\text{char. } k > 0$ ) because of the phenomenon of lowering of the genus of a curve under a finite extension of the ground field (necessarily of positive characteristic). For non-singular curves  $V/k$  of arithmetic genus  $\geq 2$  but geometric genus  $\leq 1$  the nature of the set  $C(k)$  appears to be unknown (*see* the concluding remarks of [40]).

11) Mumford [32] proved that on a curve of geometric genus  $\geq 2$  over a global field the (logarithmic) heights of the rational points grow at least exponentially rapidly. This result is qualitatively similar to, but quantitatively much stronger than, Theorem 4.16.

We prove Conjecture 4.1 for curves over fields of characteristic zero by a reduction to Mordell's Conjecture:

THEOREM 4.17. — *Let  $G/k$  be a semi-abelian variety over a field  $k$  of characteristic zero and let  $\Gamma$  be a subgroup of  $G(\bar{k})$  of finite rank. Let  $V/k$  be a closed irreducible subscheme of  $G$  of dimension 1. Then  $V \cap \Gamma$  is a finite set unless  $V$  is a translate of an algebraic subgroup of  $G$ .*

Two lemmas are required for the proof of this theorem. The first is well known.

LEMMA 4.18 (Hermité-Minkowski). — *There are only finitely many field extensions of a number field of bounded degree and unramified outside a given set of primes.*

The second is a refinement of the Chevalley-Weil theorem.

LEMMA 4.19. — *Let  $G/k$  be a smooth group scheme over a number field  $k$  and let  $\Gamma \subset G(k)$  be a finitely generated subgroup. Let  $f: X \rightarrow G$  be a finite étale covering of  $k$ -schemes. Then there is a finite field extension  $k'$  of  $k$  such that  $\Gamma \subset f(X(k'))$ . If further,  $f$  is a homomorphism of group schemes over  $k$ , then  $\Gamma \subset f(\Gamma')$  for some finitely generated subgroup  $\Gamma'$  of  $X(k')$ .*

*Proof of Lemma 4.19.* — Let  $R$  be the ring of integers of  $k$ . For any finite subset  $S$  of  $R \setminus \{0\}$ , let  $R_S$  be the ring of  $S$ -integers in  $k$ . We may write  $k$  as a direct limit of subrings  $R_S$  where  $S$  runs over all the finite subsets of  $R \setminus \{0\}$ , ordered by inclusion. General results on the projective limits of schemes of finite presentation [15], IV, § 8 then show that for some sufficiently large  $S$  there is a smooth group scheme  $\bar{G}/R_S$  and a finitely generated subgroup  $\bar{\Gamma} \subset G(R_S)$  whose generic fibres are  $G/k$  and  $\Gamma$  respectively. As  $X/k$  and  $f$  are of finite type, then, possibly by enlarging  $S$ ,  $f$  extends to a finite étale covering of  $R_S$ -schemes of finite type  $\bar{f}: \bar{X} \rightarrow \bar{G}$ . Let  $x \in \Gamma$  and let  $\bar{x} \in \bar{\Gamma}$  be the corresponding element of  $\bar{\Gamma}$ . We construct the pullback diagram

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & \bar{G} \\ \uparrow & & \uparrow \bar{x} \\ \text{Spec } R_S \times_{\bar{G}} \bar{X} & \rightarrow & \text{Spec } R_S \end{array}$$

Now

$$\text{Spec } R_S \times_{\bar{G}} \bar{X}$$

is a finite étale covering of  $\text{Spec } R_S$  of degree  $\leq \deg \bar{f}$ . Hence  $\text{Spec } R_S \times_{\bar{G}} \bar{X}$  is affine, equal to  $\text{Spec } B$ , say, where  $B$  is a finite étale  $R_S$ -algebra. The generic residue fields of  $B$  are then finite field extensions of  $k$  of degree  $\leq \deg \bar{f}$  and unramified outside  $S$ . It follows from the Lemma 4.18 that there is a finite field extension  $k'/k$  which contains all finite field extensions of  $k$  of degree  $\leq \deg \bar{f}$  and unramified outside  $S$ . Hence  $B \otimes_{R_S} k$  is a direct product of field extensions of  $k$  contained in  $k'$ . Therefore  $x \in f(X(k'))$ , for all  $x \in \Gamma$  as required. For the last part, one may take  $\Gamma'$  to be the subgroup of  $X(k')$  generated by a finite set of elements of  $X(k')$  whose images in  $G$  generate  $\Gamma$ .

*Proof of Theorem 4.17.* — First, [18], Appendix 1, Proposition C, shows that it suffices to prove the theorem for the case where  $k$  is a number field and  $\Gamma$  is a finitely generated subgroup of  $G(k)$ .

Let  $g$  be the geometric genus of  $V/k$  i.e. the genus of a smooth complete model  $\tilde{V}/k$  of  $V/k$ . Let  $T$  be a maximal linear torus contained in  $G$ ; thus  $G/T = A$  is an abelian variety over  $k$ .

If  $g \geq 2$  then  $V(k)$  is finite, by Faltings [11], and hence  $V \cap \Gamma$  is finite, as required.

If the image of  $V$  in  $A$  is reduced to a point then by translation we may assume that  $V$  is contained in  $T$ . But Laurent's proof [26] of Lang's Conjecture 4.1 for linear tori completes the proof of the theorem in this case.

Hence we may assume that  $g=1$ , the image of  $V$  in  $A$  is not reduced to a point,  $V/k$  is smooth and is not a translate of an algebraic subgroup of  $G$ . We may also assume that  $V(k)$  is not empty and hence there is a  $k$ -rational open immersion

$$\alpha: V \rightarrow \tilde{V}$$

from  $V$  to its jacobian. We then have a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & T & \rightarrow & G & \rightarrow & A & \rightarrow & 0 \\ & & & & \uparrow i & & \uparrow j & & \\ & & & & V & \xrightarrow{\alpha} & \tilde{V} & & \end{array}$$

where  $i$  is a closed immersion. As  $V$  is not a translate of an algebraic subgroup of  $G$ ,  $T$  is not reduced to a point. By translating  $V$  in  $G$ , we may assume that  $j$  is a non-zero homomorphism of abelian varieties.

Let  $G'$  be the pullback  $G \times_A \tilde{V}$ . Then  $G'/k$  is a semi-abelian variety, an extension of  $\tilde{V}$  by  $T$ , and which contains  $V$  as a closed subscheme. It is a finite étale covering of a subgroup scheme  $G''$  of  $G$  containing  $V$  and so  $V$  is a subgroup of  $G'$  if and only if it is a subgroup of  $G$ . Applying Lemma 4.19 to  $G' \rightarrow G''$  and the subgroup  $\Gamma \cap G''$  of  $G''(k)$ , we may therefore reduce to the case where  $G=G'$  and  $j$  is the identity morphism.

As  $G$  can be regarded as a fibre bundle over  $\tilde{V}$  locally trivial for the Zariski topology and with fibres isomorphic to  $T$ , the function field of  $G$  is the compositum  $k(\tilde{V}).k(T)$  where  $k(\tilde{V}), k(T)$  are the function fields of  $\tilde{V}$  and  $T$ , respectively.

Suppose first that the restrictions to  $V$  of all elements of  $k(T)$ , as a subfield of  $k(G)$ , to  $V$  are constant. Then we must have that  $V$  is complete and hence the map  $\alpha$  is an isomorphism. The inverse of  $\alpha$  then gives a splitting of  $G$  and hence the group scheme  $G$  is a direct product  $T \times \tilde{V}$ ; whence  $V$  is a translate of an algebraic subgroup of  $G$ , which is a contradiction. Therefore there is a non-constant element  $f \in k(T)$  which is a unit on  $G$  and has a non-constant restriction to  $V$ .

Let  $Z_m$  be the normalisation of  $G$  in the field  $k(G)(f^{1/m})$ , and

$$\pi_m: Z_m \rightarrow G$$

the normalisation map. As  $f$  is a unit on  $G$ ,  $\pi_m$  is étale. Hence the pullback  $V_m$  of  $V$  under  $\pi_m$  is a curve, étale over  $V$ . For some sufficiently large  $m \geq 1$ , all the geometric components of  $V_m$  have genus  $\geq 2$ , by the Hurwitz genus formula. By Lemma 4.19, we have

$$\Gamma \subset \pi_m(Z_m(k'))$$

for some finite field extension  $k'/k$ . Hence we have

$$\Gamma \cap V \subset \pi_m(V_m(k')).$$

But  $V_m(k')$  is finite, by Faltings [11], and thus  $\Gamma \cap V$  is finite.

4.6. APPLICATION TO ZETA FUNCTIONS. — Let  $k$  be a global field (*i.e.* an algebraic number field or a function field of transcendence degree 1 over a finite field). Let  $V$  be

an irreducible and reduced projective  $k$ -scheme which is embeddable as a closed subscheme in an abelian variety  $A/k$ . Let  $\mathcal{L}$  be an ample line bundle on  $V$  and let  $h_{\mathcal{L}}$  be a (logarithmic) Weil height on  $V$  associated to  $\mathcal{L}$ , with respect to a proper set of valuations of  $k$  equipped with a product formula (cf. [9], [24], [43]). The function

$$h_{\mathcal{L}}: V(k) \rightarrow \mathbf{R}$$

is determined uniquely by  $\mathcal{L}$  up to the addition of a bounded function.

Define the Dirichlet series

$$(4.8) \quad Z(s; V, h_{\mathcal{L}}) = \sum'_{x \in V(k)} h_{\mathcal{L}}(x)^{-s},$$

where  $s$  is a complex variable. The sign  $\sum'$  means that the sum is over all points  $x \in V(k)$  for which  $h_{\mathcal{L}}(x) > 0$ ; this excludes only finitely many points of  $V(k)$  by Northcott's finiteness theorem.

A related Dirichlet series, with  $h_{\mathcal{L}}$  replaced by the exponential height  $\exp h_{\mathcal{L}}$ , has been considered by Franke, Manin and Tschinkel [13] for Fano varieties. For special choices of heights on homogeneous spaces of parabolic subgroups of semi-simple algebraic groups over number fields (generalised flag manifolds), they showed that their Dirichlet series has a meromorphic continuation to the whole complex plane.

In this section, we prove the following theorem.

**THEOREM 4.20.** — *For some  $\varepsilon > 0$ , the Dirichlet series  $Z(s; V, h_{\mathcal{L}})$  can be continued meromorphically to a half plane*

$$\operatorname{Re} s > r/2 - \varepsilon, \quad \text{where } r = \operatorname{rank} A(k),$$

*such that the only singularity of  $Z(s; V, h_{\mathcal{L}})$  in this half plane is a possible simple pole at  $s = r/2$ . The residue of  $Z(s; V, h_{\mathcal{L}})$  at  $s = r/2$  is equal to*

$$|B_0(k)_{\text{tors}}| \pi^{r/2} \Gamma(r/2)^{-1} \sum_i D_i^{-1/2}$$

*where the sum ranges over the finitely many distinct  $k$ -rational translates  $B_i$  of  $B_0$  contained in  $V$  and where*

$B_0$  = the connected component of the identity of the Zariski closure of  $A(k)$ ;

$D_i$  = the discriminant of the quadratic part of the Néron-Tate height on  $B_i(k)$  associated to the ample line bundle induced on  $B_i$  by  $\mathcal{L}$ .

We first take the case where  $V = A$  and  $h_{\mathcal{L}}$  is the Néron-Tate height on  $A$  associated to  $\mathcal{L}$ .

**LEMMA 4.21.** — *The Dirichlet series  $Z(s; A, h_{\mathcal{L}})$ , where  $h_{\mathcal{L}}$  is the Néron-Tate height on  $A$  associated to  $\mathcal{L}$ , can be continued meromorphically to the whole complex plane. Its only singularity is a possible simple pole at  $s = r/2$ , where  $r = \operatorname{rank} A(k)$ , with residue*

$$|A(k)_{\text{tors}}| \pi^{r/2} D^{-1/2} \Gamma(r/2)^{-1}$$

where  $D$  is the discriminant of the quadratic part of  $h_{\mathcal{L}}$  on  $A(k)/A(k)_{\text{tors}}$ .

*Proof of Lemma 4.21.* — As  $A(k)$  is a finitely generated group, this lemma follows from the meromorphic continuation of the Epstein function of a quadratic form (cf. [25], pp. 250-253, [35]. Theorem, p. xiii and Proposition 23, p. VI-10).

*Proof of Theorem 4.20.* — Fix a symmetric ample line bundle  $\mathcal{L}'$  on  $A/k$  and let  $h_{\mathcal{L}'}$  be the corresponding Néron-Tate on  $A/k$ , which is a quadratic form on  $A(k)/A(k)_{\text{tors}}$ . Then there are real constants

$$\alpha > 0, \quad \beta, \quad \gamma > 0, \quad \delta$$

such that

$$\alpha h_{\mathcal{L}'}|_V + \beta \leq h_{\mathcal{L}} \leq \gamma h_{\mathcal{L}'}|_V + \delta.$$

By comparison with the series for  $Z(s; A, h_{\mathcal{L}'})$ , it easily follows that the Dirichlet series for  $Z(s; V, h_{\mathcal{L}})$  converges absolutely to an analytic function in the half plane  $\text{Re } s > r/2$ .

By Theorem 4.5, we have a disjoint decomposition

$$V(k) = \bigcup_{i=1}^m B_i(k) \cup S$$

where

$$\text{rank } S < \text{rank } A(k)$$

and the  $B_i$  are translates, contained in  $V$ , of the connected component of the identity  $B_0$  of the Zariski closure of  $A(k)$ . Let  $h_i$  be the Néron-Tate height on the translated abelian variety  $B_i$  associated to the induced ample line bundle  $\mathcal{L}|_{B_i}$ ; let  $D_i$  be the discriminant of the quadratic part of  $h_i$  on  $B_i(k)$ .

If  $h'$  and  $h''$  are logarithmic Weil heights associated to the same ample line bundle on  $V$  then  $h' - h''$  is a bounded function on  $V$ ; simple estimates then show that the difference

$$Z(s; V, h') - Z(s; V, h'')$$

extends holomorphically to the half plane

$$\text{Re } s > r/2 - 1.$$

Hence it suffices to prove the theorem for a particular height  $h_{\mathcal{L}}$  in the equivalence class of heights associated to  $\mathcal{L}$ . As the  $B_i$  are disjoint, we may and do choose a height  $h_{\mathcal{L}}$  on  $V$  associated to  $\mathcal{L}$  which restricts to  $h_i$  on  $B_i$  for all  $i$ .

We then have

$$Z(s; V, h_{\mathcal{L}}) = \sum_{i=1}^m Z(s; B_i, h_i) + \sum_{x \in S} h_{\mathcal{L}}(x)^{-s}.$$



By Lemma 4.21,  $Z(s; B_i, h_i)$  is a meromorphic function of  $s$  whose only singularity is a simple pole at

$$s = r/2, \quad \text{where } r = \text{rank } A(k),$$

and with residue

$$|B_0(k)_{\text{tors}}| \pi^{r/2} D_i^{-1/2} \Gamma(r/2)^{-1}.$$

Hence

$$\sum_{i=1}^m Z(s; B_i, h_i)$$

is a meromorphic function of  $s$  whose only singularity is a possible simple pole at  $s = r/2$  and with residue precisely that given in the theorem.

For any real number  $X$ , put

$$f(X) = |\{x \in S; h_{\mathcal{L}}(x) \leq X\}|.$$

As  $\text{rank } S < \text{rank } A(k)$ , we have, by comparing  $h_{\mathcal{L}}$  and  $h_{\mathcal{L}'}$ ,

$$f(X) = O(|\{x \in S; h_{\mathcal{L}'}(x) \leq X\}|) = O(X^{r/2-\varepsilon}), \quad \text{as } X \rightarrow \infty,$$

for some  $\varepsilon > 0$ . The tail of the series

$$\sum'_{x \in S} h_{\mathcal{L}}(x)^{-s}, \quad s = \sigma + it,$$

is then bounded above in absolute value by

$$\begin{aligned} \sum_{n \geq N, n \in \mathbf{Z}} (f(n) - f(n-1))/n^\sigma &= -f(N-1)/N^\sigma + \sum_{n \geq N, n \in \mathbf{Z}} f(n) (1/n^\sigma - 1/(n+1)^\sigma) \\ &= O(N^{r/2-\varepsilon-\sigma}) + \sum_{n \geq N, n \in \mathbf{Z}} \sigma f(n)/n^{\sigma+1} = O(N^{r/2-\varepsilon-\sigma}), \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Hence the series

$$\sum'_{x \in S} h_{\mathcal{L}}(x)^{-s}$$

converges absolutely to an analytic function in the half plane  $\text{Re } s > r/2 - \varepsilon$ , whence the result.

*Remark.* — 12) Let  $k$  be an algebraic number field and let  $A, V, \mathcal{L}$  be as in Theorem 4.20.

Lang's Conjecture 4.1 implies (e.g. by Lemma 4.21) that for some logarithmic height  $h_{\mathcal{L}}$  on  $V$  associated to  $\mathcal{L}$ , the Dirichlet series  $Z(s; V, h_{\mathcal{L}})$  would extend meromorphically

to the whole complex plane and the only singularities would be possible simple poles at

$$s = 1/2, 1, 3/2, 2, \dots, \text{rank } A(k)/2.$$

Indeed, a simple pole would then occur at  $s = n/2$ ,  $n > 0$ , if and only if the Zariski closure of  $V(k)$  has an irreducible component which is an abelian variety  $B/k$  with  $\text{rank } B(k) = n$ .

Over global ground fields  $k$  of positive characteristic, a meromorphic continuation of the zeta function  $Z(s; V, h_\varphi)$  may well have infinitely many simple poles on the line  $\text{Re } s = 0$  e.g. if  $V/k$  is an isotrivial curve of genus  $\geq 2$  with infinitely many  $k$ -rational points.

### 5. Local Néron heights and integral points

We now turn to the distribution of values of local Néron heights on abelian varieties. This has an application to the question of the distribution of integral points on an abelian variety, relative to an ample divisor.

In Section 5.1 some basic facts about local Néron heights are summarised. Then we determine the asymptotic behaviour of the cumulative frequency of the values of local Néron heights for archimedean and non-archimedean valuations in Sections 5.2 and 5.3 respectively. This is applied to the distribution of integral points on abelian varieties in Section 5.4.

5.1 GENERALITIES ON LOCAL NÉRON HEIGHTS. — Let  $A$  be an abelian variety defined over a field  $k$  equipped with a proper absolute value  $v$ . Let  $\text{Div}(A)_k$  be the group of divisors on  $A$  rational over  $k$ . Let  $Z_0(A(k))$  be the group of zero cycles on  $A$  of degree zero generated by the elements of  $A(k)$ . Then there is a pairing

$$\begin{aligned} \text{Div}(A)_k \times Z_0(A(k)) &\rightarrow \mathbf{R} \\ (D, a) &\rightarrow \langle D, a \rangle_v \end{aligned}$$

which is well-defined whenever  $D$  and  $a$  have disjoint supports. The pairing has the following properties:

- (i)  $\langle D, a \rangle_v$  is bilinear;
- (ii) if  $D = (f)$  is a principal divisor, then  $\langle D, a \rangle_v = v(f(a))$ ;
- (iii)  $\langle D, a \rangle_v$  is invariant under translation:  $\langle D, a \rangle_v = \langle D_u, a_u \rangle_v$ , for  $u \in A(k)$ ;
- (iv) let  $x \in A(k) \setminus D$ . Then the map  $x \rightarrow \langle D, (x) - (x_0) \rangle_v$  is bounded on every  $v$ -adically bounded subset of  $A(k) \setminus D$ .

[If  $A(k)$  is dense in  $A$ , then the pairing  $\langle \ , \ \rangle_v$  is uniquely defined by these properties.]

Let  $\Gamma$  be a finitely generated subgroup of  $A(k)$ . Let  $D$  be an effective divisor on  $A/k$  such that  $0 \notin \text{Supp } D$ . For any real number  $\alpha$ , we write for any subset  $X \subset \Gamma$

$$S_v(\alpha, D, X) = \{ P \in X; \langle D, (P) - (0) \rangle_v \geq \alpha \};$$

for this purpose, we shall agree to define

$$\langle D, (P) - (0) \rangle_v = +\infty, \quad \text{if } P \in \text{Supp } D.$$

Write

$$\mathcal{D}_{\Gamma, v, D}(\alpha) = d_{\Gamma} S_v(\alpha, D, \Gamma).$$

The quantity  $\mathcal{D}_{\Gamma, v, D}(\alpha)$  is a cumulative frequency of the value of  $\langle D, (P) - (0) \rangle_v$  as  $P$  runs over the elements of  $\Gamma$ ; it is, roughly speaking, the density (if it exists) of the set of points  $P \in \Gamma$  whose  $v$ -distance from  $\text{Supp } D$  is at most  $e^{-c(v)\alpha}$  [for some  $c(v) > 0$ ].

5.2. THE CASE OF AN ARCHIMEDIAN VALUATION. — Let  $k$  be a field equipped with an archimedean valuation  $v$ . Denote by

$\Gamma$  a finitely generated subgroup of  $A(k)$ ;

$D$  a  $k$ -rational effective divisor on  $A$  such that  $0 \notin \text{Supp } D$ ;

$\bar{\Gamma}$  the Zariski closure of  $\Gamma$  in  $A$ ;

$\bar{\Gamma}_0$  the connected component of the identity of  $\bar{\Gamma}$ .

The asymptotic estimate below for  $\mathcal{D}_{\Gamma, v, D}(\alpha)$  is founded on Kronecker's theorem on uniform distribution.

**THEOREM 5.1.** — *The quantity  $\mathcal{D}_{\Gamma, v, D}(\alpha)$  exists for all  $\alpha \in \mathbf{R}$  and we have the asymptotic expansion, for constants  $a_{jk} \in \mathbf{R}$ ,*

$$\mathcal{D}_{\Gamma, v, D}(\alpha) \sim \frac{m}{[\bar{\Gamma} : \bar{\Gamma}_0]} + \sum_{j \in \mathbf{N}^*} \sum_{k=0}^{2d-1} a_{jk} \exp(-j\alpha/N) \alpha^k, \quad \text{as } \alpha \rightarrow +\infty,$$

where

$N > 0$  is an integer;

$d = \dim A$ ;

$m = \text{number of irreducible components of } \bar{\Gamma} \text{ contained in } \text{Supp } D.$

*Proof.* — There is an embedding  $k \rightarrow \mathbf{C}$  such that the restriction of  $-\log|x|$ , where  $|\cdot|$  is the usual absolute value on  $\mathbf{C}$ , to  $k$  is just  $v$ . Associated to the embedding  $k \rightarrow \mathbf{C}$  we have a complex analytic isomorphism

$$\psi: \mathbf{C}^d/\Lambda \rightarrow A(\mathbf{C})$$

where  $\Lambda$  is a lattice in  $\mathbf{C}^d$ . There is a normalised theta function  $\theta(z)$  on  $\mathbf{C}^d$ , such that  $\theta(0) = 1$ , whose corresponding divisor is  $\psi^{-1}(D)$ . The functional equation of  $\theta$  is

$$\theta(z + \lambda) = \theta(z) \exp(\pi h(z, \lambda) + \pi/2 h(\lambda, \lambda) + 2\pi i K(\lambda))$$

where  $h(w, z)$  is a hermitian quadratic form on  $\mathbf{C}^d$  and  $K(\lambda)$  is real valued. Then for all  $z \in \mathbf{C}^d$  one has (see [33])

$$\langle D, (\psi(z)) - (0) \rangle_v = -\log|\theta(z)| + \pi/2 h(z, z).$$

The right hand side of this equation depends only on  $z \bmod \Lambda$  *i.e.* only on  $\psi(z)$ , as one sees from the functional equation of  $\theta$ . Hence

$$F(z) = \exp(-2 \langle D, (\psi(z)) - (0) \rangle_v)$$

is a real analytic function on  $\mathbf{C}^d/\Lambda$ .

Let  $T$  be a translate of a real sub-torus of the real torus

$$\mathbf{C}^d/\Lambda = \mathbf{R}^{2d}/\Lambda.$$

For  $s > 0$ , put

$$V_T(s) = \{z \in T; F(z) \leq s\}.$$

Let  $\omega_T$  be a Haar measure (a volume form) on  $T$  normalised so that

$$\int_T \omega_T = 1.$$

Let  $F_s, s \in \mathbf{R}$ , denote the fibre of the map  $F|_T: T \rightarrow \mathbf{R}$  over  $s$ . Then we have

$$\frac{d}{ds} \int_{V_T(s)} \omega_T = \int_{F_s} \omega'$$

for some real analytic form  $\omega'$  on  $T$ .

By a theorem of H. M. Mairé (*see* [27]; for the complex analytic analogue, *see* [1]; both results depend on Hironaka's resolution of singularities),  $\int_{F_s} \omega'$  has an asymptotic expansion as  $s \rightarrow 0$  of the form

$$\int_{F_s} \omega' \sim \sum_{j \in \mathbf{N}^*} \sum_{k=0}^{2d-1} c_{jk} |s|^{-1+j/N_T} (\log |s|)^k,$$

for some constants  $c_{jk}$  and a suitable integer  $N_T > 0$ . It follows that

$$(5.1) \quad \int_{V_T(e^{-2\alpha})} \omega_T \sim \sum_{j \in \mathbf{N}^*} \sum_{k=0}^{2d-1} b_{jk} \exp(-j\alpha/N_T) \alpha^k, \quad \text{as } \alpha \rightarrow \infty.$$

Kronecker's theorem on uniform distribution of sequences implies that there are a finite number of disjoint translates of real sub-tori

$$T_i, \quad i = 1, \dots, r,$$

of the real torus  $\mathbf{R}^{2d}/\Lambda$  such that  $\Gamma$  is dense in  $\bigcup_i T_i$  and  $\Gamma \cap T_i$  is uniformly distributed in  $T_i$  for all  $i$ . It follows from the property of uniform distribution and (5.1) that we

have

$$\begin{aligned} \mathcal{D}_{\Gamma, v, D}(\alpha) &= \sum_{i=1}^r d_{\Gamma} S_v(\alpha, D, \Gamma \cap T_i) = 1/r \sum_{i=1}^r \int_{V_{T_i}(e^{-2\alpha})} \omega_{T_i} \\ &\sim \frac{m}{[\bar{\Gamma} : \bar{\Gamma}_0]} + \sum_{j \in \mathbf{N}^*} \sum_{k=0}^{2d-1} a_{jk} \exp(-j\alpha/N) \alpha^k, \quad \text{as } \alpha \rightarrow \infty, \end{aligned}$$

for a suitable integer  $N > 0$  and where  $m$  is the number of components of  $\bar{\Gamma}$  contained in  $D$ . This is the desired result.

5.3. THE CASE OF A NON-ARCHIMEDIAN VALUATION. — Let  $k$  be a field equipped with a discrete valuation  $v$ . Denote by

$R$  the ring of valuation integers of  $k$ ;

$\pi$  a uniformising parameter of  $R$ ;

$\kappa$  the residue field of  $R$ ;

$A/k$  an abelian variety;

$D$  an effective  $k$ -rational divisor on  $A$  such that  $0 \notin \text{Supp } D$ ;

$\tilde{A}/R$  the Néron model of  $A/k$ ;

$\tilde{A}^0/\kappa$  the closed fibre of  $\tilde{A}/R$ ;

$\tilde{A}_j^0$ ,  $j=0, \dots, t$ , the connected components of  $\tilde{A}^0/\kappa$  (where  $\tilde{A}_0^0$  is the connected component of the identity);

$\tilde{D}$  the *thickening* (cf. [24], p. 287) of  $D$  in  $\tilde{A}$  (defined by linearity after taking the thickening of a prime divisor of  $A$  to be its Zariski closure in  $\tilde{A}$ );

$\tilde{D}^0$  the pullback of  $\tilde{D}$  to the closed fibre  $\tilde{A}^0$ ;

$\Gamma$  a finitely generated subgroup of  $A(k)$ ;

$\Gamma^0$  the subgroup of  $\tilde{A}^0(\kappa)$  induced by  $\Gamma$ ;

$\bar{\Gamma}$ ,  $\bar{\Gamma}^0$  the Zariski closure of  $\Gamma$  in  $A$ , resp.  $\Gamma^0$  in  $\tilde{A}^0$ ;

$\bar{\Gamma}_0$ ,  $\bar{\Gamma}_0^0$  the connected component of the identity of  $\bar{\Gamma}$ , resp.  $\bar{\Gamma}^0$ ;

$[\bar{\Gamma} : \bar{\Gamma}_0]$  = number of distinct irreducible components of  $\bar{\Gamma}$ ;

$m$  = number of irreducible components of  $\bar{\Gamma}$  contained in  $D$ .

The subscript  $v$  added to one of the above symbols denotes that it is relative to the valuation  $v$ , if more than one valuation is being considered.

The local Néron height  $\langle D, (P) - (0) \rangle_v$  can be interpreted as an intersection multiplicity as follows. If  $P \in \Gamma$ , denote by  $P^0$  the induced element of  $\Gamma^0$ . There is a constant  $\gamma_j \in \mathbf{R}$  such that for all  $P \in A(k) \setminus D$ , and  $P^0 \in \tilde{A}_j^0(\kappa)$ , one has (cf. [24], Chap. 11, § 5)

$$(5.2) \quad \langle D, (P) - (0) \rangle_v = i(D, P)_v + \gamma_j,$$

where  $i(D, P)_v$  is the intersection multiplicity of  $P$  and  $D$  at  $v$  (i.e.  $\deg P^* \tilde{D}$ ). Further, we may normalise the local Néron height so that  $\gamma_0 = 0$ .

THEOREM 5.2. — *Suppose that  $\text{rank } \Gamma \geq 1$  and  $\Gamma^0$  does not have an irreducible component contained in  $\tilde{D}^0$ . Then  $\mathcal{D}_{\Gamma, v, D}(\alpha)$  exists for all  $\alpha$ , and is upper semi-continuous and locally constant. Further, we have :*

- (i)  $\mathcal{D}_{\Gamma, v, D}(\alpha) \in [\tilde{A}^0 : \tilde{A}_0^0]^{-1} \mathbf{Z}$ ;
- (ii)  $\text{rank } S_v(\alpha, D, \Gamma) < \text{rank } \Gamma$ , for all  $\alpha \gg 0$ .

*Proof.* — If  $P \in \Gamma$ , write  $P^0$  for its image in  $\Gamma^0$ . Let  $\Gamma(j)$  be the set of  $P \in \Gamma$  such that  $P^0 \in \tilde{A}_j^0(\kappa)$ . Then  $\Gamma(j)$  is a coset of the subgroup  $\Gamma(0)$  or is empty; further, the index  $[\Gamma : \Gamma(0)]$  divides  $[\tilde{A}^0 : \tilde{A}_0^0]$ . For  $P \in \Gamma(j)$ , we have from (5.2), as  $D$  is effective,

$$(5.3) \quad \begin{cases} \langle D, (P) - (0) \rangle_v > \gamma_j, & \text{if } P^0 \in \text{Supp } \tilde{D}^0, \\ = \gamma_j, & \text{if } P^0 \notin \text{Supp } \tilde{D}^0. \end{cases}$$

Applying Theorem 4.5 to  $\Gamma^0$  and  $\tilde{D}^0$ , we have from (5.3)

$$(5.4) \quad \text{rank } \{P \in \Gamma(j); \langle D, (P) - (0) \rangle_v > \gamma_j\} < \text{rank } \Gamma.$$

When  $\Gamma(j)$  is non-empty, we conclude that

$$(5.5) \quad \begin{aligned} d_{\Gamma} S_v(\alpha, D, \Gamma(j)) &= 1/[\Gamma : \Gamma(0)], & \text{if } \alpha \leq \gamma_j, \\ &= 0, & \text{if } \alpha > \gamma_j. \end{aligned}$$

As

$$(5.6) \quad \mathcal{D}_{\Gamma, v, D}(\alpha) = \sum_j d_{\Gamma} S_v(\alpha, D, \Gamma(j)),$$

it follows that  $\mathcal{D}_{\Gamma, v, D}(\alpha)$  exists and is upper semi-continuous, decreasing, and locally constant. Part (ii) follows from (5.4) and part (i) from (5.5) and (5.6).

When  $\kappa$  is a finite field, we have a more precise result:

THEOREM 5.3. — *Suppose that  $\kappa$  is finite. Then  $\mathcal{D}_{\Gamma, v, D}(\alpha)$  exists for all  $\alpha$ .*

- (i) *If char.  $k > 0$  then*

$$\mathcal{D}_{\Gamma, v, D}(\alpha) = m/[\bar{\Gamma} : \bar{\Gamma}_0] + o(1), \quad \text{as } \alpha \rightarrow +\infty.$$

- (ii) *If char.  $k = 0$  then for any integer  $N > 0$ , the Poincaré series (which has only a finite number of negative powers of  $t$ )*

$$P_N(t) = \sum_{j=-\infty}^{+\infty} (1 - \mathcal{D}_{\Gamma, v, D}(j/N)) t^j$$

*is a rational function of  $t$ . Further, for some constant  $c > 0$ , we have*

$$\mathcal{D}_{\Gamma, v, D}(\alpha) = m/[\bar{\Gamma} : \bar{\Gamma}_0] + O(\exp(-c\alpha)), \quad \text{as } \alpha \rightarrow +\infty.$$

*Remark.* — The constants  $\gamma_j$  of (5.2) (cf. [24], Chapter 11, Theorem 5.2, p. 289) have the following rationality property:

$$\gamma_j \in \frac{1}{[\tilde{A}_0 : \tilde{A}_0^0]^2} \mathbf{Z}, \quad \text{for all } j.$$

The locally constant function  $\mathcal{D}_{\Gamma, v, D}(\alpha)$  is therefore recoverable from the Poincaré series  $P_{[\tilde{A}_0 : \tilde{A}_0^0]^2}(t)$ .

*Proof.* — Let  $\hat{k}$  be the  $v$ -adic completion of  $k$ .

*Special case.* —  $\bar{\Gamma} = A$ .

Assume that  $\text{char. } k > 0$ . We only have to show that

$$\mathcal{D}_{\Gamma, v, D}(\alpha) \rightarrow 0, \quad \text{as } \alpha \rightarrow +\infty.$$

This is an immediate consequence of the monotone convergence theorem, where we equip  $A(\hat{k})$  with a  $v$ -adic Haar measure.

Assume that  $\text{char. } k = 0$ . Let  $\hat{R}$  be the ring of  $v$ -adic integers of  $\hat{k}$ . The Cartier divisor  $\tilde{D}$  on  $\tilde{A}$  is represented by a finite set of pairs

$$(U_i, f_i), \quad i \in I,$$

where the  $U_i$  form a Zariski open covering of  $\tilde{A}$  and  $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$  for all  $i$  such that  $f_i f_j^{-1} \in \Gamma(U_i \cap U_j, \mathcal{O}_{U_i \cap U_j}^*)$  for all  $i, j \in I$ .

As  $\tilde{A} \times_{\mathbf{R}} \hat{R}/\hat{R}$  is the Néron model of  $A \times_k \hat{k}$ , we have a bijection

$$\begin{aligned} A(\hat{k}) &\rightarrow A(\hat{R}), \\ P &\mapsto \tilde{P}. \end{aligned}$$

The local Néron height is then given by

$$\langle D, (P) - (0) \rangle_v = i(D, P) + \gamma_j = v(f_i(\tilde{P})) + \gamma_j, \quad P \in A(\hat{k}),$$

where  $i$  is such that  $\tilde{P} \in U_i$ .

The exponential map of the Lie algebra of  $A(\hat{k})$  gives an analytic isomorphism of groups

$$\exp : \hat{R}^d \rightarrow W, \quad d = \dim A,$$

where  $W$  is a  $v$ -adic open subgroup of  $A(\hat{k})$  (cf. [4], Chap. III, §7, No. 2, Prop. 3). As  $\Gamma \cap W$  has finite index in  $\Gamma$ , by covering  $A(\hat{k})$  with finitely many translates of  $W$  we may reduce to the case where  $\Gamma \subset W$ , because of the additivity of the density  $\mathcal{D}_{\Gamma, v, D}$  and of the Poincaré series  $P_N(t)$ .

The pairs  $(\exp^{-1} U_i, f_i \circ \exp)$  form an analytic Cartier divisor  $\exp^* D$  on the analytic space  $\hat{R}^d$ . By refining the open cover  $\{\exp^{-1} U_i\}_{i \in I}$ , we may assume that  $\exp^* D$  is given by a finite set of pairs  $(B_i, f_i \circ \exp)$ , where each  $B_i$  is an open ball in  $\hat{R}^d$  whose image in  $\tilde{A}^0$  is contained in a single connected component. As each  $B_i$  is then homothetic

to  $\hat{\mathbf{R}}^d$ , we may reduce, again by the additivity of the density  $\mathcal{D}_{\Gamma, v, D}$  and the series  $P_N(t)$ , to the case where the divisor  $\text{exp}^* D$  is given by one pair  $(\hat{\mathbf{R}}^d, f)$ .

We may identify  $\Gamma$  with a subgroup of  $\hat{\mathbf{R}}^d$  via the exponential map. The  $v$ -adic closure  $\hat{\Gamma}$  of  $\Gamma$  is then a subgroup of  $\hat{\mathbf{R}}^d$  isomorphic to  $\hat{\mathbf{R}}^{d'}$  for some  $d' \leq d$ . Therefore for an appropriate constant  $\gamma$  we have

$$d_{\Gamma} \{ P \in \Gamma; \langle D, (P) - (0) \rangle_v \geq \alpha - \gamma \} \\ = d_{\Gamma} \{ P \in \Gamma; v(f(P)) \geq \alpha \} = |\{ x \in \hat{\mathbf{R}}^{d'} / \pi^{\alpha} \hat{\mathbf{R}}^{d'}; f(x) \equiv 0 \pmod{\pi^{\alpha}} \}| / q^{\alpha d'}.$$

In particular, this density exists. The theorem now follows from the conclusions of Lemma 5.4 below.

*General case.* — We reduce this to the special case above. The connected component of the identity  $\bar{\Gamma}_0$  of  $\bar{\Gamma}$  is a sub-abelian variety of  $A$  and  $\bar{\Gamma}$  is a finite disjoint union of translates  $\bar{\Gamma}_{0,i}$  of  $\bar{\Gamma}_0$ . If some translate  $\bar{\Gamma}_{0,i}$  lies in  $\text{Supp } D$ , then for all points  $P \in \bar{\Gamma}_{0,i}$  we have  $\langle D, (P) - (0) \rangle_v = +\infty$  and hence

$$S_v(\alpha, D, \Gamma) \supset \bar{\Gamma}_{0,i}, \quad \text{for all } \alpha \in \mathbf{R}.$$

We then have

$$\mathcal{D}_{\Gamma, v, D}(\alpha) = m / [\bar{\Gamma} : \bar{\Gamma}_0] + \sum_{\bar{\Gamma}_{0,i} \not\subset D} d_{\Gamma}(S_v(\alpha, D, \bar{\Gamma}_{0,i} \cap \Gamma));$$

hence to prove the theorem we need only consider those translates  $\bar{\Gamma}_{0,i}$  not contained in  $\text{Supp } D$ . The divisor  $D$  then induces an ample divisor on such a translate; by the functoriality of Néron heights, we then reduce to the case where  $\bar{\Gamma} = A$ : this is the special case above.

LEMMA 5.4. — *Assume that char.  $k=0$  and  $q=|\kappa| < \infty$ . Let  $f \neq 0$  be a convergent power series on  $\hat{\mathbf{R}}^d$ , i. e.  $f \in \hat{\mathbf{R}} \{ \{ z_1, \dots, z_d \} \}$ . Put*

$$Y_n = \{ x \in \hat{\mathbf{R}}^d / \pi^n \hat{\mathbf{R}}^d; f(x) \equiv 0 \pmod{\pi^n} \}.$$

*Then for some  $\varepsilon > 0$ , we have*

$$|Y_n| = O(q^{n(d-\varepsilon)}), \quad \text{as } n \rightarrow +\infty.$$

*Indeed, the Poincaré series  $\sum_{n=0}^{\infty} |Y_n| t^n$  is a rational function of  $t$ .*

*Proof of Lemma 5.4.* — The  $O$ -estimate can be proved in an elementary way by using the Weierstrass preparation theorem and induction on the number  $d$  of variables (we omit the details).

The rationality of the Poincaré series  $\sum |Y_n| t^n$  can be proved as in Igusa [19], who proved the same result for the special case where  $f$  is a polynomial by using Hironaka's resolution of singularities; the latter is applicable in the more general case here of a



convergent power series because the ring  $\hat{R} \{z_1, \dots, z_d\}$  is excellent. (See also [19], [34], [41] for related results.)

5.4. INTEGRAL POINTS. — Let  $k$  be an algebraic number field or a function field over a subfield  $k_0$  (i.e.  $k/k_0$  is a finitely generated field extension in which  $k_0$  is algebraically closed) and let  $R$  be a  $\mathbf{Z}$ -algebra (or  $k_0$ -algebra) of finite type whose field of fractions is  $k$ .

Let  $D$  be a  $k$ -rational ample divisor on a projective variety  $V/k$ . Let  $n > 0$  be an integer so large that  $nD$  is very ample. Let  $x_0 = 1, x_1, \dots, x_m$  be a basis over  $k$  of vector space of global sections  $\Gamma(V, \mathcal{O}(nD))$ . One says that a subset  $I \subset V(k)$  is *D-integral with respect to R* if there is  $d \in R - \{0\}$  such that  $dx_i(P) \in R$  for all  $i = 1, \dots, m$  and all  $P \in I$ . The *D-integrality* of  $I$  is independent of the choice of integer  $n$  and basis  $x_0, \dots, x_m$ .

Let  $A/k$  be an abelian variety and  $D$  be an ample divisor on  $A$ . In the case where  $k/k_0$  is a function field, let  $(B, \tau)$  be a  $k|k_0$ -trace of  $A/k$  i.e.  $B/k_0$  is an abelian variety and  $\tau : B \rightarrow A$  is a homomorphism of  $k_0$ -group schemes with an appropriate universal property (cf. [23], p. 138). Lang [24] conjectured:

CONJECTURE 5.5. — *Let  $k$  be a number field (respectively, a function field over a field  $k_0$  of characteristic zero). Then all D-integral subsets of  $A/k$  with respect to  $R$  are finite (respectively, fall into a finite number of cosets of  $\tau B(k_0)$  in  $A(k)$ ).*

For the case of an elliptic curve over a number field, this is a theorem of Siegel and Mahler and can be proved using Diophantine approximation or Faltings's proof of Mordell's conjecture. The conjecture has been proved for abelian varieties over number fields by Faltings ([12], [46]) and this supersedes Theorem 5.6 below in characteristic zero. Silverman [44] proved that, for abelian surfaces over number fields, a set of *D-integral points* is "widely spaced". For a discussion of this conjecture and its relation with other Diophantine conjectures, see [10].

THEOREM 5.6. — *Suppose that  $k$  is a global field. Let  $D$  be a  $k$ -rational ample divisor on  $A/k$  and let  $I$  be a *D-integral subset* of  $A(k)$ , with respect to the subring  $R$  of  $k$ . Then  $I$  either has density zero in the finitely generated group  $A(k)$  or is finite.*

In the case where  $k$  is a function field over a field  $k_0$ , let  $M_{k/k_0}$  be an associated proper set of valuations on  $k$  satisfying a product formula (cf. [24], [43]).

THEOREM 5.7. — *Suppose that  $k$  is an algebraic function field over a field  $k_0$ . Let  $S$  be a finite subset of  $M_{k/k_0}$  and  $R$  the finitely generated  $k_0$ -algebra which is the ring of *S-integers* of  $k$ . Let  $D$  be a  $k$ -rational ample divisor on  $A/k$ . Assume that:*

- 1)  $\text{Tr}_{k/k_0}(A) = 0$ ;
- 2) For all  $v \in S$ ,  $\tilde{D}_v^0$  contains no irreducible component of the Zariski closure of  $A(k)_v^0$  (in particular, if  $\tilde{D}_v^0$  contains no translate of an algebraic subgroup of  $\tilde{A}_v^0$  for all  $v \in S$ ).

*Let  $I$  be a *D-integral subset* of  $A(k)$ , with respect to  $R$ . Then either  $I$  is finite or we have*

$$\text{rank } I < \text{rank } A(k).$$

*Remark.* — For a global field  $k$  of characteristic  $p > 0$ , Lang's Conjecture 5.5 would become false: Let  $A/k$  be an abelian variety definable over a finite subfield  $k_0$  of  $k$  and let  $D$  be an ample  $k_0$ -rational divisor on  $A$ . Suppose  $P \in A(k)$  is not definable over any finite field. Then the set of iterates of  $P$  under the Frobenius of  $k_0$  is an infinite  $D$ -integral set of points, with respect to a suitable finitely generated  $k_0$ -subalgebra of  $k$ .

*Proof of Theorems 5.6 and 5.7.* — By translation, we may assume that  $0 \notin \text{Supp } D$ . Select an integer  $n > 0$  so large that the divisor  $nD$  is very ample. Then  $\Gamma(A, \mathcal{O}(nD))$  gives a projective embedding  $A \rightarrow \mathbf{P}_k^m$ . We may choose homogeneous coordinates  $X_0, \dots, X_m$  on  $A$ , which are restrictions of homogeneous coordinates on  $\mathbf{P}_k^m$ , such that  $0 \notin \text{Supp } X_i$  for all  $i$  and  $X_0 = 0$  cuts out the divisor  $nD$  on  $A$ ; put

$$x_i = X_i/X_0, \quad \text{for all } i=0, \dots, m.$$

Let  $R$  be a finitely generated  $\mathbf{Z}$ -subalgebra (respectively, finitely generated  $k_0$ -subalgebra, in the function field case) of  $k$  with  $\text{fract}(R) = k$ . Let  $I$  be a  $D$ -integral set of points, with respect to  $R$ , of the finitely generated group  $A(k)$ .

We may select a proper set  $M$  of well-behaved valuations on  $k$  satisfying a product formula (see [24]); for the function field case, we take  $M = M_{k/k_0}$ . For all points  $P \in A(k) \setminus \text{Supp } D$  we have

$$v(x_i(P)) = \langle X_i, (P) - (0) \rangle_v - \langle X_0, (P) - (0) \rangle_v \quad \text{for all } v \in M.$$

Furthermore, the  $D$ -integrality of  $I$  implies that there is a finite subset  $S$  of  $M$ , containing all the archimedean valuations of  $M$ , such that

$$v(x_i(P)) \geq 0, \quad \text{for all } P \in I, \text{ for all } i=1, \dots, m, \text{ and all } v \in M \setminus S.$$

Hence we have

$$(5.7) \quad \sup_{1 \leq i \leq n} -v(x_i(P)) \leq 0, \quad \text{for all } P \in I \text{ and for all } v \in M \setminus S.$$

On the other hand, we have

$$(5.8) \quad \sup_{1 \leq i \leq n} -v(x_i(P)) = \langle X_0, (P) - (0) \rangle_v + \sup_{1 \leq i \leq n} -\langle X_i, (P) - (0) \rangle_v$$

where, in the terminology of Néron (cf. [33] or [9]),

$$\gamma(P, v) = \sup_{1 \leq i \leq n} -\langle X_i, (P) - (0) \rangle_v$$

is a bounded quasi-function. Let  $S'$  be the finite set of valuations

$$S' = \{ v \in M; \text{ there is } P \in A(k) \text{ such that } \gamma(P, v) \neq 0 \}.$$

It follows from (5.7) and (5.8) that

$$(5.9) \quad \langle X_0, (P) - (0) \rangle_v \leq 0, \quad \text{for all } P \in I \text{ and for all } v \in M \setminus (S' \cup S);$$

and there is a constant  $C \geq 0$  such that

$$(5.10) \quad \langle X_0, (P) - (0) \rangle_v \leq C, \quad \text{for all } P \in I \text{ and all } v \in S'.$$

Hence the global Néron-Tate height

$$\langle X_0, (P) - (0) \rangle = \sum_{v \in M} \langle X_0, (P) - (0) \rangle_v$$

satisfies the inequality [by (5.9) and (5.10)]

$$\langle X_0, (P) - (0) \rangle \leq \sum_{v \in S} \langle X_0, (P) - (0) \rangle_v + C|S'|, \quad \text{for all } P \in I.$$

For any real number  $\alpha$ , we then have

$$(5.11) \quad \{P \in I; \langle X_0, (P) - (0) \rangle \geq \alpha\} \subset \bigcup_{v \in S} S_v \left( \frac{\alpha - C|S'|}{|S|}, D, A(k) \right).$$

The group  $A(k)$  is finitely generated, under the hypotheses of either of the Theorems 5.6 and 5.7. Hence if  $k$  is a global field, we have by Theorems 5.1, and 5.3, either that  $A(k)$  is finite or that

$$(5.12) \quad d_{A(k)} \left( \bigcup_{v \in S} S_v(\alpha, D, A(k)) \right) \rightarrow 0, \quad \text{as } \alpha \rightarrow +\infty.$$

If  $k/k_0$  is a function field and the hypotheses of Theorem 5.7 are satisfied, then Theorem 5.2 gives either that  $A(k)$  is finite or that

$$(5.13) \quad \text{rank} \bigcup_{v \in S} S_v(\alpha, D, A(k)) < \text{rank } A(k), \quad \text{for } \alpha \gg 0.$$

The Theorems 5.6 and 5.7 now follow from (5.11), (5.12), and (5.13) as the set

$$\{P \in I; \langle X_0, (P) - (0) \rangle < \alpha\}$$

is finite for all  $\alpha$ , from the non-degeneracy of the global Néron height.

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