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COMPOSITUM OF GALOIS EXTENSIONS OF HILBERTIAN FIELDS

BY D. HARAN AND M. JARDEN ⁽¹⁾

Introduction

Hilbert [H] proved in 1892 that for given irreducible polynomials $f_i(T_1, \dots, T_r, X)$, $i=1, \dots, m$, and a nonzero polynomial $g(T_1, \dots, T_r)$ with rational coefficients there exists $(a_1, \dots, a_r) \in \mathbb{Q}^r$ such that $f_1(\mathbf{a}, X), \dots, f_m(\mathbf{a}, X)$ are irreducible in $\mathbb{Q}[X]$ and $g(\mathbf{a}) \neq 0$.

Numerous proofs of Hilbert's irreducibility theorem have since been given. Many of them apply to other fields. So, each field K which satisfies the theorem has been called **Hilbertian**. The sets of $\mathbf{a} \in K^r$ whose substitution in the polynomials leaves them irreducible and nonzero have been called **Hilbert sets**.

The investigation of Hilbertian fields has been extended in the last 98 years since Hilbert's original paper in several directions:

(a) *Study of Hilbert subsets of Hilbertian fields* (e. g. Dörge [D], Geyer [G], Sprindžuk [S], and Fried [F]).

(b) *Search for arithmetical conditions on a field which make it Hilbertian*. Beyond the classical example of fields of rational function over any field (Inaba [I] and Franz [Fr]) two results stand out: "Each ω -free PAC field is Hilbertian" (Roquette [FJ], Cor. 24.38) and "The field of formal power series in at least two variables over any field is Hilbertian" (Weissauer [FJ], Cor. 14.18).

(c) *Infinite algebraic extensions of Hilbertian fields*. The first result in this direction is due to Kuyk [K]: "Every abelian extension of Hilbertian field is Hilbertian" In particular the field \mathbb{Q}_{cycl} obtained from \mathbb{Q} by adjoining all roots of unity is Hilbertian. Uchida [U] extended a result of Kuyk and proved that if an algebraic extension L of a Hilbertian field K is contained in a nilpotent extension and if the supernatural number $[L:K]$ (see [FJ], Section 20.9) is divisible by at least two prime numbers, then L is Hilbertian. The strongest result however in this direction, is again due to Weissauer: "Every finite proper

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extension of a Galois extension of a Hilbertian field is Hilbertian". (See [W], Satz 9.7, for a nonstandard proof and [FJ], Cor. 12.15 for a standard proof.) We make an extensive use of this result and refer to it as **Weissauer's theorem**.

(d) *Realization of finite groups over Hilbertian fields, especially over number fields via Riemann existence theorem (see Matzat's exposition [M]).*

(e) *Properties of almost all e -tuples $(\sigma_1, \dots, \sigma_e)$ of elements of the absolute Galois group of a Hilbertian field K .* For example, the group generated by almost all $(\sigma_1, \dots, \sigma_e)$ is a free profinite group [FJ], Thm. 16.13, and if K is countable, then the fixed field $K_s(\sigma_1, \dots, \sigma_e)$ of $\sigma_1, \dots, \sigma_e$ in the separable closure K_s of K is PAC [FJ], Thm. 16.18.

This note is a contribution to the study of infinite algebraic extensions of Hilbertian fields. Weissauer's theorem implies that the compositum of a Galois extension M_1 of a Hilbertian field K and a finite extension M_2 of K which is not contained in M_1 is Hilbertian. So, it is natural to ask whether the compositum N of two (infinite) linearly disjoint proper Galois extensions M_1 and M_2 of K is Hilbertian. Indeed, this has been stated as Problem 12.18 of [FJ]. However, the question goes back at least to Kuyk [K] (see Remark 2.6) and Weissauer. Kuyk proved that N is Hilbertian if an extra condition holds: "For each finite Galois extension L of K which is contained in N we have $L \cap M_1 \neq K$ or $L \cap M_2 \neq K$." In particular this is the case if the degrees $[N:M_1]$ and $[N:M_2]$ are relatively prime. The main tool in Kuyk's proof is the possibility to realize wreath products over K . Zorn [Z] gave a clearer exposition of Kuyk's proof while strengthening Kuyk's extra condition to: "Each open normal subgroup of an open normal subgroup of $\mathcal{G}(N/K)$ is the direct product $\mathcal{G}(N/M'_1) \times \mathcal{G}(N/M'_2)$, where M'_i is a finite extension of M_i contained in N ."

We extend here Kuyk's result to a complete affirmative solution of Problem 12.18 of [FJ]. Our proof is an elaboration of Zorn's in the case where $[N:M_1]$ and $[N:M_2]$ are relatively prime. For the case where the degrees are not relatively prime we generalize a lemma of Chatzidakis on normalizers of elements in wreath products [FJ], Lemma 52. Then we apply the setup used in the first case to conclude the proof in the second case.

An application of Weissauer's theorem gives even a sharper result:

THEOREM. — *The composition of two Galois extensions of a Hilbertian field, neither of which is contained in the other, is Hilbertian.*

Of course, the solution of Problem 12.18 of [FJ] immediately supplies an affirmative solution to Problem 12.19 of [FJ]:

COROLLARY. — *The separable closure of a Hilbertian field K cannot be presented as the compositum of two Galois extensions of K , neither of which is contained in the other.*

1. Wreath products

Recall that the wreath product $H = A \text{ wr } G$ of finite groups A and G is the semidirect product $G \rtimes A^G$, where A^G is the group of all functions $f: G \rightarrow A$ with the canonical

multiplication rule, and G acts on A^G by the formula $f^\tau(\sigma) = f(\tau\sigma)$. Thus each element of H is a pair (σ, f) with $\sigma \in G$ and $f \in A^G$. The product and the inverse in H are given by

$$(1) \quad (\sigma, f)(\tau, g) = (\sigma\tau, f^\tau g) \quad \text{and} \quad (\sigma, f)^{-1} = (\sigma^{-1}, f^{-\sigma^{-1}}).$$

Let $\pi: H \rightarrow G$ be the canonical projection. Embed A in A^G by identifying each $a \in A$ with the function which maps 1 to a and σ to 1 for each $\sigma \neq 1$. Then A^G may also be considered as a direct product, $A^G = \prod_{\sigma \in G} A^\sigma$, and each element of A^σ has the form a^σ with $a \in A$.

Our first result generalizes a lemma of Chatzidakis [FJ], Lemma 24.52.

LEMMA 1.1. — *Let G and A be finite groups. For $\sigma_1, \dots, \sigma_e \in G$ and $1 \neq a \in A$ let $G_0 = \langle \sigma_1, \dots, \sigma_e \rangle$ and $H_0 = \langle (\sigma_1, a), \dots, (\sigma_e, a) \rangle$. Then π maps the normalizer $N = N_H(H_0)$ of H_0 in H onto G_0 .*

Proof. — Since $\pi(H_0) = G_0$ it suffices to prove that $\pi(N) \leq G_0$. Consider A^{G_0} as the subgroup of A^G consisting of all functions $f: G \rightarrow A$ for which $f(\tau) = 1$ for each $\tau \in G - G_0$. It follows from (1) that $H_1 = \{(\sigma, f) \mid \sigma \in G_0 \text{ and } f \in A^{G_0}\}$ is a subgroup of H . The main point to be observed here is that if $(\sigma, f), (\tau, g) \in H_1$ and $\rho \in G - G_0$, then $\tau\rho, \sigma^{-1}\rho \notin G_0$ and therefore $(f^\tau g)(\rho) = f(\tau\rho)g(\rho) = 1$ and $f^{-\sigma^{-1}}(\rho) = f(\sigma^{-1}\rho)^{-1} = 1$. As $(\sigma_i, a) \in H_1, i = 1, \dots, e$, we have $H_0 \leq H_1$. In other words

$$(2) \quad (\sigma, f) \in H_0 \quad \text{implies that} \quad \sigma \in G_0 \text{ and } f \in A^{G_0}.$$

Let $(\tau, g) \in N$. Then $(\sigma, f) = (\tau, g)^{-1}(\sigma_1, a)(\tau, g) \in H_0$. By (1) and (2),

$$\sigma = \tau^{-1}\sigma_1\tau \in G_0 \quad \text{and} \quad f = g^{-\sigma}a^\tau g \in A^{G_0}.$$

Let $n = \text{ord}(\sigma)$ and act with the powers of σ on f to get

$$f = g^{-\sigma}a^\tau g, \quad f^\sigma = g^{-\sigma^2}a^{\tau\sigma}g^\sigma, \quad \dots, \quad f^{\sigma^{n-1}} = g^{-\sigma^n}a^{\tau\sigma^{n-1}}g^{\sigma^{n-1}}.$$

Hence

$$(3) \quad f^{\sigma^{n-1}} \dots f^\sigma f = (g^{-1}a^{\tau\sigma^{n-1}}g^{\sigma^{n-1}}) \dots (g^{-\sigma^2}a^{\tau\sigma}g^\sigma)(g^{-\sigma}a^\tau g) = g^{-1}a^{\tau\sigma^{n-1}} \dots a^{\tau\sigma}a^\tau g$$

As $\sigma \in G_0$ and $f \in A^{G_0}$, the left hand side of (3) belongs to A^{G_0} . Therefore, so does the right hand side of (3). So if $\tau \notin G_0$, then the value of the right hand side of (3) at τ^{-1} is 1. Thus

$$(4) \quad g(\tau^{-1})^{-1}a(\tau\sigma^{n-1}\tau^{-1}) \dots a(\tau\sigma\tau^{-1})a(1)g(\tau^{-1}) = 1.$$

Finally, note that for j between 1 and $n-1$ we have $\tau\sigma^j\tau^{-1} \neq 1$. Hence (4) reduces to $a = 1$. This contradiction to the choice of a proves that $\tau \in G_0$, as desired. ■

As a result, a certain embedding problem for a direct product of profinite groups cannot be properly solved:

LEMMA 1.2. — *Let C_1, C_2 be nontrivial profinite groups. Let G_1, G_2 be nontrivial finite quotients of C_1, C_2 , respectively, such that either*

- (a) *the orders G_1 and G_2 are not relatively prime, or*
- (b) *the orders of C_1 and C_2 are relatively prime.*

Let $G = G_1 \times G_2$ and let $\rho: C_1 \times C_2 \rightarrow G$ be the product of the quotient maps.

Let A be a nontrivial finite group, $H = A \text{ wr } G$, and $\pi: H \rightarrow G$ the canonical projection. Then there exists no epimorphism $\theta: C_1 \times C_2 \rightarrow H$ such that $\pi \circ \theta = \rho$.

Proof. — Assume that there exists an epimorphism $\theta: C_1 \times C_2 \rightarrow H$ such that $\pi \circ \theta = \rho$. We derive a contradiction in each of the two cases.

Case (a): There exists a prime p and elements $\sigma_i \in G_i$, $i = 1, 2$, of order p . Then the order of $\sigma = \sigma_1 \sigma_2$ is also p . Use Lemma 1.1 for $e = 1$ to find $h \in H$ such that $\pi(h) = \sigma$ and $\pi(N) = \langle \sigma \rangle$, with $N = N_H \langle h \rangle$. Write $h = h_1 h_2$, with $h_i = \theta(c_i)$ and $c_i \in C_i$. Then c_1 commutes with c_2 and therefore $h_i \in N$. Hence $\pi(h_i) = \rho(c_i) \in \langle \sigma \rangle \cap G_i = 1$. It follows that $\sigma = \pi(h) = 1$. This is a contradiction.

Case (b): The orders of C_1 and C_2 are relatively prime. For $i = 1, 2$ put $H_i = \theta(C_i)$. Then $H_i \triangleleft H$, $\pi(H_i) = G_i$ and there exists $h \in H_i$ such that $\sigma = \pi(h) \neq 1$. Thus $h = (\sigma, f)$, where $f \in A^G$. As $A^\sigma \triangleleft A^G$, we have $A^h = (A^\sigma)^f = A^\sigma$. It follows that $A = (A^\sigma)^{h^{-1}} \leq H_i \cdot A^\sigma$, and therefore

$$A^G = A \cdot \prod_{\substack{\tau \in G \\ \tau \neq 1}} A^\tau \leq H_i \cdot \prod_{\substack{\tau \in G \\ \tau \neq 1}} A^\tau.$$

Hence, with $n = |G|$, the order of A^n divides $|H_i| \cdot |A|^{n-1}$, and therefore $|A|$ divides $|H_i|$, for $i = 1, 2$. This is a contradiction, since $|H_1|$ and $|H_2|$ are relatively prime. ■

REMARK 1.3. — *Characterization of wreath products.* Although we shall not use it in the sequel it is interesting to note that wreath products can be characterized by less data than above:

Given an extension of finite groups

$$(5) \quad 1 \rightarrow B \rightarrow H \rightarrow G \rightarrow 1,$$

the lifting of elements of G to elements of H determines a homomorphism $\psi: G \rightarrow \text{Aut}(B)/\text{In}(B)$. The set of all congruence classes of extensions with the same ψ bijectively corresponds to the group $H^2(G, Z(B))$ [Mc], p. 128. In particular let $B = A^G$ and ψ be the homomorphism obtained from the natural action of G on B . Then the G -module $Z(B) = Z(A)^G$ is the induced module $\text{Ind}_1^G Z(A)$. Hence $H^2(G, Z(B))$ is trivial [R], p. 146. It follows that the only extension (5) such that ψ is induced by the natural action of G on $B = A^G$ is the wreath product.

REMARK 1.4. — *Interpretation of wreath products in Galois theory.* Consider a tower of fields $K \subseteq L \subseteq F \subseteq \hat{F}$ where L/K , F/L and \hat{F}/K are finite Galois extensions. Let also K' be a field such that $K' \cap L = K$ and $LK' = \hat{F}$. Put $G = \mathcal{G}(L/K) \cong \mathcal{G}(\hat{F}/K')$ and $A = \mathcal{G}(F/L)$. Suppose that the fields F^σ , $\sigma \in \mathcal{G}(\hat{F}/K')$ are linearly disjoint over L and their compositum is \hat{F} . Then there exists an isomorphism $\varphi: \mathcal{G}(\hat{F}/K) \rightarrow A \text{ wr } G$ which maps $\mathcal{G}(\hat{F}/L)$ onto A^G and induces the identity maps $\mathcal{G}(F/L) = A$ and $\mathcal{G}(L/K) = G$. We say in this set up that the fields L, F, \hat{F} **realize the wreath product $A \text{ wr } G$ over K** .

$K'_0 = \hat{F}_0 \cap K'$, and $A_0 = \mathcal{G}(F_0/L)$. Then $K'_0 \cap L = K$ and $LK'_0 = \hat{F}_0$. Hence L, F_0, \hat{F}_0 realize $A_0 \text{ wr } G$ over K , as above.

2. Main results

We take the crucial step toward the solution to Problem 12.18 of [FJ] in the following lemma. It involves a construction of wreath products over fields of rational functions as in [K], Prop. 1.

LEMMA 2.1. — *Let M_1, M_2 be linearly disjoint infinite Galois extensions of a field K , and let $N = M_1 M_2$. Let $f \in K[T, X]$ be an absolutely irreducible polynomial, monic in X , and Galois over $K(T)$. Then there exists a finite Galois extension L of K contained in N such that for every basis c_1, \dots, c_n of L over K there is a Hilbert subset B of K^n such that for each $(b_1, \dots, b_n) \in B$ the polynomial $f(b_1 c_1 + \dots + b_n c_n, X)$ is irreducible over N .*

Proof. — There are three parts in the proof.

Part A: Construction of L . Let $C_1 = \mathcal{G}(N/M_1)$ and $C_2 = \mathcal{G}(N/M_2)$. Then $\mathcal{G}(N/M) = C_1 \times C_2$. Choose nontrivial finite quotients G_1, G_2 of C_1, C_2 , respectively. If the orders of C_1 and C_2 are not relatively prime, choose G_1 and G_2 with orders having a common prime divisor. Let $\rho: C_1 \times C_2 \rightarrow G_1 \times G_2$ be the product of the quotient maps. Consider the fixed field L of $\text{Ker}(\rho)$ in N . Then $G = \mathcal{G}(L/K) = G_1 \times G_2$. By Lemma 1.2, for no nontrivial finite group A_0 there exist fields $L \subseteq E \subseteq \hat{E} \subseteq N$ such that L, E, \hat{E} realize $A_0 \text{ wr } G$ over K .

Part B: Construction of wreath product over a field of rational functions. Choose a set $\{u^\sigma \mid \sigma \in G\}$ of algebraically independent elements over K . For each $\sigma \in G$ let x^σ be a root of $f(u^\sigma, X)$. As f is absolutely irreducible, the field $K(u^\sigma, x^\sigma)$ is a regular extension of K . Hence $L(u^\sigma, x^\sigma)$ is a regular extension of L . As these fields are algebraically independent over L , the field $\hat{Q} = L(u^\sigma, x^\sigma \mid \sigma \in G)$ is a regular extension of L [FJ], p.112. Moreover, the field $Q = L(u^\sigma \mid \sigma \in G)$ is linearly disjoint from $K(u^\sigma, x^\sigma)$ over $K(u^\sigma)$. Hence $Q(x^\sigma)/Q$ is a Galois extension with Galois group isomorphic to $A = \mathcal{G}(f(T, X), K(T))$. The set of all $Q(x^\sigma)$ is linearly disjoint over Q . So, $\mathcal{G}(\hat{Q}/Q) \cong A^G$.

Put $n = [L:K] = |G|$, and let c_1, \dots, c_n be a basis for L/K . Let t_1, \dots, t_n be the unique solution of the following system of linear equations:

$$(1) \quad T_1 c_1^\sigma + \dots + T_n c_n^\sigma = u^\sigma, \quad \sigma \in G$$

As the matrix (c_i^σ) is invertible [L], p. 212, $L(t_1, \dots, t_n) = L(u^\sigma \mid \sigma \in G) = Q$. Since n is the transcendence degree of Q over L , the elements t_1, \dots, t_n are algebraically independent over L and hence also over K .

Extend the action of G on L to an action on \hat{Q} in a natural way: $(u^\sigma)^\tau = u^{\sigma\tau}$ and $(x^\sigma)^\tau = x^{\sigma\tau}$. In particular τ permutes the equations of the system (1). As $(t_1^\tau, \dots, t_n^\tau)$ is also a solution of (1), it coincides with (t_1, \dots, t_n) . Thus τ leaves each element of $P = K(t_1, \dots, t_n)$ element wise fixed. So, the fixed field $Q(G)$ of G in Q contains P . In particular $n \leq [Q:P]$. As $LP = Q$, this implies that $P = Q(G)$ and that $L \cap P = K$.

The subgroup H of $\text{Aut}(\hat{Q})$ generated by G and $\mathcal{G}(\hat{Q}/Q)$ is contained in $\text{Aut}(\hat{Q}/P)$. As \hat{Q}/P is separable, the latter group is finite and therefore so is H . Since P is the fixed field of H , the field \hat{Q} is Galois over P and $H = \mathcal{G}(\hat{Q}/P)$.

Now consider the fixed field $P' = \hat{Q}(G)$. Its intersection with Q is P and their compositum is \hat{Q} . So, $Q, Q(x), \hat{Q}$ realize $A \text{ wr } G$ over P .

Part C: Definition of B and conclusion of the proof. Write \hat{Q} as $P(z)$ with z integral over $K[t_1, \dots, t_n]$ and let $h(t_1, \dots, t_n, Z) = \text{irr}(z, P)$. Then $f(T_1 c_1 + \dots + T_n c_n, X)$ is irreducible over L . Use [FJ], Lemma 12.12 and Cor. 11.7, to find a Hilbert subset B of K^n such that for each $\mathbf{b} \in B$ and for $a = \sum_{i=1}^n b_i c_i$.

$$(2a) \quad \mathcal{G}(h(\mathbf{b}, Z), K) \cong \mathcal{G}(h(\mathbf{t}, Z), P),$$

$$(2b) \quad f(a, X) \text{ is irreducible over } L,$$

and the specialization $\mathbf{t} \mapsto \mathbf{b}$ extends to a place of \hat{Q} over K such that the residue fields of $P, Q, Q(x^\sigma), P', \hat{Q}$, respectively, are $K, L, F^\sigma, K', \hat{F}$, where F^σ is the splitting field of $f(a^\sigma, X)$ over L , for $\sigma \in G$. In particular L, F, \hat{F} realize $A \text{ wr } G$ over K and $[F:L] = \deg(f(a, X))$.

Let $\mathbf{b} \in B$, $a = \sum_{i=1}^n b_i c_i$, and assume that $f(a, X)$ is reducible over N . Then $E = N \cap F$ is a proper Galois extension of L . Extend each $\sigma \in \mathcal{G}(\hat{F}/K')$ to an element σ of the absolute Galois group $G(K)$ of K to observe that $E^\sigma = N \cap F^\sigma$ is contained in N . Let $A_0 = \mathcal{G}(E/L)$ and $\hat{E} = \prod_{\sigma \in \mathcal{G}(F/K')} E^\sigma$. Then $\hat{E} \subseteq N$ and, by Remark 1.4, L, E, \hat{E} realize

$A_0 \text{ wr } G$ over K . This contradiction to Part A proves that $f(a, X)$ is irreducible over N , as desired. ■

LEMMA 2.2. — *Let N be a field, N' a finite Galois extension of N , $f \in N[T, X]$ an irreducible polynomial, which is separable in X , and $g \in N'[T, X]$ a factor of f which is irreducible over N' . Then, for almost all $a \in N$, if $g(a, X)$ is irreducible over N' , then $f(a, X)$ is irreducible over N .*

Proof. — The polynomial f decomposes over N' as $f(T, X) = \prod_{i=1}^m g_i(T, X)$ where each g_i is conjugate to g over N and for $i \neq j$, g_i is not a multiple of g_j by an element of $N'(T)$. Suppose that for $a \in N$ and each $i \neq j$, $g_i(a, X)$ is not a multiple of $g_j(a, X)$ by an element of N' (this happens for almost all $a \in N$) and $g(a, X)$ is irreducible over N' . Then $f(a, X)$ is irreducible over N . Indeed, let $f(a, X) = h_1(X)h_2(X)$ be a decomposition over N . Then $h_1(X)h_2(X) = \prod_{i=1}^m g_i(a, X)$. As $g(a, X)$ is irreducible, it divides, say, $h_1(X)$. Since each $g_i(a, X)$ is conjugate to $g(a, X)$ over N , it also divides $h_1(X)$. As $g_1(a, X), \dots, g_m(a, X)$ are relatively prime, $f(a, X) = \prod_{i=1}^m g_i(a, X)$ divides $h_1(X)$. Conclude that $f(a, X)$ is irreducible over N . ■

PROPOSITION 2.3. — *Let M_1 and M_2 be infinite Galois extensions of Hilbertian field K such that $M_1 \cap M_2 = K$. Then their compositum $N = M_1 M_2$ is Hilbertian. Moreover, given an irreducible polynomial $f \in N[T, X]$, separable in X , there exist $c_1, \dots, c_n \in N$ and a Hilbert subset B of K^n such that for each $(b_1, \dots, b_n) \in B$, and for $a = \sum_{i=1}^n b_i c_i$, the polynomial $f(a, X)$ is irreducible over N .*

Proof. — Note that the second statement means that if K is only separably Hilbertian [FJ], p. 147, then so is N . If K is Hilbertian, as we suppose, then it is imperfect. Hence, the second statement implies in this case that N is Hilbertian [FJ], Prop. 11.16.

To prove the second statement consider a transcendental element t over K . Let \hat{N} be the splitting field of $f(t, X)$ over $N(t)$. Choose a primitive element y for \hat{N} over $N(t)$ such that $h = \text{irr}(y, N(t))$ has coefficients in $N[t]$. Then h is monic and Galois in X . If we find $c_1, \dots, c_n \in N$ and a Hilbert subset B of K^n such that for each $(b_1, \dots, b_n) \in B$ and with $a = \sum_{i=1}^n b_i c_i$, the polynomial $h(a, X)$ is irreducible over N , then K^n has a Hilbert subset B_0 of B such that for $(b_1, \dots, b_n) \in B_0$ the polynomials $f(a, X)$ is also irreducible over N . Indeed, the proof of [FJ], Lemma 12.12, shows that if a is not a zero of a certain nonzero polynomial with coefficients in N and $h(a, X)$ is irreducible, then $\mathcal{G}(f(a, X), N)$ and $\mathcal{G}(f(t, X), N(t))$ are isomorphic as permutation groups of the roots. In particular the former group operates transitively on the roots of $f(a, X)$. This implies that $f(a, X)$ is irreducible. Note that the exclusion of finitely many values a_1, \dots, a_k for a imposes the extra condition

$$\prod_{j=1}^k \left(\sum_{i=1}^n b_i c_i - a_j \right) \neq 0$$

on $(b_1, \dots, b_n) \in B$. This defines B_0 . So, without loss, assume that f is monic and Galois in X .

Choose an absolutely irreducible factor g of f . Let K'_0 be a finite Galois extension of K which contains the coefficients of g . Let K_1 and K_2 be finite Galois extensions of K contained in M_1 and M_2 , respectively, such that $K'_0 \cap N \subseteq K_1 K_2$. Then $K' = K_1 K_2 K'_0$ satisfies $N \cap K' = K_1 K_2$ and $M_1 K_2 \cap M_2 K_1 = K_1 K_2$ (use the tower property of linear disjointness [FJ], Lemma 9.3).

Let $M'_1 = M_1 K_2 K'$, $M'_2 = M_2 K_1 K'$, $N' = NK'$. Then M'_1, M'_2 are linearly disjoint Galois extensions of K' and $N' = M'_1 M'_2$. By Lemma 2.1 there is a finite Galois extension L' of K' contained in N' such that for every basis c_1, \dots, c_n of L'/K' there is a Hilbert subset B' of $(K')^n$ such that for each $b_1, \dots, b_n \in B'$ the polynomial $g(b_1 c_1 + \dots + b_n c_n, X)$ is irreducible over N' . As $\mathcal{G}(N'/K') = \mathcal{G}(N/K_1 K_2)$, there is a finite Galois extension L of $K_1 K_2$ in N such that $L' = LK'$. A basis c_1, \dots, c_n of $L/K_1 K_2$ is also a basis of L'/K' . By Lemma 2.2 and by [FJ], Cor. 11.7, K^n has a Hilbert subset $B \subseteq B'$ such that $f(b_1 c_1 + \dots + b_n c_n, X)$ is irreducible over N , for every $b_1, \dots, b_n \in B$. ■

We are now ready to solve Problem 12.18 of [FJ] in a much stronger form:

THEOREM 2.4. — *Let M_1 and M_2 be Galois extensions of Hilbertian field K neither of which is contained in the other. Then their compositum $N = M_1 M_2$ is Hilbertian.*

Proof. — If N is a finite extension of M_1 or of M_2 , then it is Hilbertian, by Weissauer's theorem. So, assume that N is an infinite extension of both M_1 and M_2 . In particular $K_1 = M_1 \cap M_2$ has a finite proper Galois extension K' which is contained in M_2 . Let $M'_1 = M_1 K'$. By Weissauer's theorem, K' is Hilbertian. Also, M'_1 and M_2 are infinite extensions of K' whose intersection is K' and whose compositum is N . Conclude from Proposition 2.3 that N is Hilbertian. ■

One of the consequences of Theorem 2.4 is a solution of Problem 12.19 of [FJ]:

COROLLARY 2.5. — *The separable (resp. solvable, p -) closure K_s (resp. K_{solv} , $K^{(p)}$) of a Hilbertian field K is not the compositum of two Galois extensions of K neither of which is equal to K_s (resp., K_{solv} , $K^{(p)}$).*

Proof. — None of the above fields is Hilbertian. So the corollary follows from Theorem 2.4.

Nevertheless, as the separable case was the subject of an open question we sketch a short cut in the above proof in this case.

Assume that M_1 and M_2 are Galois extensions of K which are not separably closed such that $M_1 M_2 = K_s$. Use Weissauer's theorem to replace M_1 , M_2 , and K , if necessary, by algebraic extensions to assume that M_1 , M_2 are Hilbertian and $M_1 \cap M_2 = K$. In particular M_i has a cyclic extension M'_i of degree p , $i = 1, 2$ [FJ], Thm. 24.48.

Let $K_1 = M_1 \cap M'_2$, $K_2 = M_2 \cap M'_1$ and $L = K_1 K_2$. Then

$$G = \mathcal{G}(L/K) = \mathcal{G}(L/K_1) \times \mathcal{G}(L/K_2) \cong \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}.$$

By [FJ], Prop. 24.47, there exists a Galois extension F of K which contains L and there exists an isomorphism $\varphi: (\mathbf{Z}/p\mathbf{Z}) \text{ wr } G \rightarrow \mathcal{G}(F/K)$ such that $\text{res}_L \circ \varphi$ is the canonical projection of the wreath product on G .

Now choose a generator σ_i of $\mathcal{G}(L/K_i)$, $i = 1, 2$ and let $\sigma = \sigma_1 \sigma_2$. Chatzidakis' Lemma [FJ], Lemma 24.52, extends σ to an element τ of $\mathcal{G}(F/K)$ such that restriction to L maps the normalizer of $\langle \tau \rangle$ onto $\langle \sigma \rangle$. This gives a group theoretic contradiction as in Lemma 1.2.

Note that this proof actually works for each normal extension N of K which admits no p -extensions. In particular it works also for K_{solv} and $K^{(p)}$. ■

REMARK 2.6. — Kuyk [K], p. 120, states, contrary to Theorem 2.4, that the compositum of linearly disjoint Galois extensions of a Hilbertian field need not be Hilbertian. He adjoins p -th roots of all elements of \mathbb{Q} to $K = \mathbb{Q}(\zeta_p)$ to get a Galois extension $\mathbb{Q}^{(p)}$ of K . Then $\mathcal{G}(\mathbb{Q}^{(p)}/K)$ is isomorphic to the direct product of infinitely many cyclic extensions of order p . Kuyk claims, without a proof, that $\mathbb{Q}^{(p)}$ is not Hilbertian. However, as $\mathbb{Q}^{(p)}$ is the compositum of a linearly disjoint finite Galois extension and an infinite Galois extension, already Weissauer's theorem implies that $\mathbb{Q}^{(p)}$ is Hilbertian, contrary to Kuyk's statement.

REFERENCES

- [D] K. DÖRGE, *Einfacher Beweis des Hilbertschen Irreduzibilitätssatzes* (*Math. Ann.*, Vol. 96, 1927, pp. 176-182).
- [F] M. FRIED, *Irreducibility Results for Separated Variables Equations* (*J. Pure Appl. Alg.*, Vol. 48, 1987, pp. 9-22).
- [FJ] M. D. FRIED and M. JARDEN, *Field Arithmetic* (*Ergebnisse der Mathematik III*, Vol. 11, Springer, Heidelberg, 1986).
- [Fr] W. FRANZ, *Untersuchungen zum Hilbertschen Irreduzibilitätssatz* (*Math. Z.*, Vol. 33, 1931, pp. 275-293).
- [G] W.-D. GEYER, *Galois Groups of Intersections of Local Fields* (*Israel J. Math.*, Vol. 30, 1978, pp. 382-396).
- [H] D. HILBERT, *Über die Irreduzibilität ganzer rationaler Funktionen mit ganzzahligen Koeffizienten* (*J. reine angew. Math.*, Vol. 110, 1892, pp. 104-129).
- [I] E. INABA, *Über den Hilbertschen Irreduzibilitätssatz* (*Jap. J. Math.*, Vol. 19, 1944, pp. 1-25).
- [K] W. KUYK, *Extensions de corps hilbertiens* (*J. Alg.*, Vol. 14, 1970, pp. 112-124).
- [L] S. LANG, *Algebra*, Addison-Wesley, Reading, 1970.
- [M] B. H. MATZAT, *Über das Umkehrproblem der Galoisschen Theorie* (*Jahresbericht Deutschen Mathematiker-Vereinigung*, Vol. 90, 1988, pp. 155-183).
- [Mc] S. MACLANE, *Homology*, Springer, Berlin, 1963.
- [R] L. RIBES, *Introduction to Profinite Groups and Galois Cohomology* (*Queen's Papers Pure Appl. Math.*, No. 24, Queen's University, Kingston, 1970).
- [S] V. G. SPRINDŽUK, *Reducibility of Polynomials and Rational Points on Algebraic Curves* (*Seminar on Number Theory*, Paris, 1979-1980).
- [U] K. UCHIDA, *Separably Hilbertian Fields* (*Kodai Math. J.*, Vol. 3, 1980, pp. 83-95).
- [W] R. WEISSAUER, *Der Hilbertsche Irreduzibilitätssatz* (*J. reine angew. Math.*, Vol. 334, 1982, pp. 203-220).

- [Z] P. ZORN, *Der Hilbertsche Irreduzibilitätssatz*, Zulassungsarbeit zur wissenschaftlichen Prüfung für das Höhere Lehramt, written under the supervision of W.-D. GEYER, Erlangen, 1975.

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