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EXISTENCE OF FLAT TORI IN ANALYTIC MANIFOLDS OF NONPOSITIVE CURVATURE

BY V. BANGERT AND V. SCHROEDER

Introduction

A *k-flat* in a complete Riemannian manifold M is a totally geodesic and isometric immersion $F:\mathbb{R}^k \to M$ of euclidean $\mathbb{R}^k$. A k-flat is *closed*, if F is periodic with respect to some cocompact lattice of $\mathbb{R}^k$. Hence a closed flat induces a totally geodesic and isometric immersion of a flat k-torus.

The purpose of the paper is to prove the following result which answers a question raised by Yau ([Y], Problem 65), *see* also [G1], p. 169.

THEOREM. — *Let* M *be a compact real analytic Riemannian manifold with nonpositive sectional curvature. If* M *contains a k-flat, then* M *contains also a closed k-flat.*

We briefly describe the context of the theorem and some of its consequences.

By the flat torus theorem ([GW], [LY]) the existence of a closed k-flat is equivalent to the existence of a subgroup isomorphic to $\mathbb{Z}^k$ in the fundamental group $\pi_1(M)$.

The existence of k-flats in M is closely related to the "hyperbolicity" of M in the sense of Gromov [G2] and to the Tits geometry of the universal covering space $\tilde{M}$ ([BGS], chapt. I). Combining results from [G2], [BGS], [E1] one can see that for a compact manifold of nonpositive curvature the following properties are equivalent:

(1) M contains no 2-flat.

(2) M is hyperbolic in the sense of Gromov.

(3) The Tits geometry of $\tilde{M}$ is degenerate.

(4) M is a visibility manifold (*cf.* [EO]).

We relate these conditions to the Preissmann property:

(P) Every non-trivial abelian subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.

Preissmann [P] proved this property for a compact manifold with strictly negative curvature. Under the assumptions of our theorem, condition (1) is equivalent to the nonexistence of a subgroup $\mathbb{Z}^2$ in $\pi_1(M)$. Since $\pi_1(M)$ is torsion free this is equivalent to (P).

Thus we have:

COROLLARY. — *Under the assumption of the Theorem any of the conditions* (1)-(4) *is equivalent to the Preissmann property* (P).

Our result fits into the program to detect geometric properties of manifolds M of nonpositive curvature which are equivalent to algebraic properties of the fundamental group $\pi_1(M)$. Examples are the existence of flat tori, the existence of a splitting ([GW], [LY]), the visibility property [E2] and, more generally, the existence of a k-flat [AS]. By our theorem the last property is equivalent to the existence of a subgroup $\mathbb{Z}^k \subset \pi_1(M)$ provided the metric is real analytic. Another important result in this context is the equivalence of the geometric and algebraic rank [BE].

In the case that there exists a flat of codimension $\leqq 2$ in M the above theorem was proved in [S1]. For the special case of higher rank manifolds *see* [BBS], section 4. For examples of analytic manifolds containing higher dimensional flats *see* [S1], [S3] and [S4]. In the C^∞-category the theorem has been proved for codimension one flats, *cf.* [B], [S5], but for higher codimension this question is open, *cf.* also the discussion in [G2], p. 135.

The methods developed in this paper may be useful in investigating the structure of the set of flats (or more generally of higher rank subspaces) in analytic manifolds of nonpositive curvature. Such a structure theory exists in dimensions $\leqq 4$ [S2].

We now indicate the main steps of our proof which combines methods from

(i) synthetic geometry of manifolds with nonpositive sectional curvature;

(ii) the theory of subanalytic sets;

(iii) the theory of dynamical systems.

Assume that k is the maximal dimension of a flat in M. By $F_k(M)$ we denote the subset of all k-planes σ in the Grassmannian $G_k(M)$ such that $\exp : \sigma \to M$ is a k-flat.

In the first part of the proof (section 2) we look for flats with an additional structure of singular subspaces. Note that a vector $v \in \sigma$, $\sigma \in F_k(M)$ induces a parallel vectorfield along the flat $\exp : \sigma \to M$. We call v singular, if v has additional parallel vectors outside the flat and regular otherwise. This notion generalizes the corresponding notion for symmetric spaces. We define P-rank(v) to be the dimension of the space of vectors parallel to v. Thus a vector v tangent to a k-flat is regular if P-rank$(v)=k$. Under a certain nonclosing condition which we may assume by induction, *cf.* section 5, we show in Theorem 2.5 that there is a subset of $F_k(M)$ containing flats with an additional structure of singular subspaces. In particular every σ in this subspace contains a flag $\sigma_1 \subset \ldots \subset \sigma_k = \sigma$ of subspaces σ_i with $\dim(\sigma_i) = i$ such that the sequence $m_i := \min_{v \in \sigma_i}$ P-rank(v) is strictly decreasing $m_1 > m_2 > \ldots > m_k = k$. Thus σ_{k-1} is a singular hyperplane in σ and the σ_i for $i < k-1$ are singular subspaces of higher codimension. We call a flat containing a flag with this property well structured. Furthermore we show that a well structured flat can only contain finitely many singular subspaces.

We can consider the set of well structured flats as a subset V_0 of the Stiefelbundle $St_k(M)$ of orthonormal k-frames: A point $(x, v_1, \ldots, v_k) \in V_0$ is a k-frame of vectors

$v_i \in T_x M$ such that the span $\langle v_1, \ldots, v_i \rangle = \sigma_i$ defines a flag as above. Note that parallel translation of the vectors $v_1, \ldots, v_k$ in the corresponding flat gives a natural $\mathbb{R}^k$-operation on V_0.

In section 3 we use the theory of subanalytic sets to prove the existence of a compact $\mathbb{R}^k$-invariant analytic submanifold V of V_0. We construct a direction $w = (w_1, \ldots, w_k) \in S^{k-1}$ such that the map $w: V \to SM$, $w(x, v_1, \ldots, v_k) = \sum w_i v_i$ is an analytic diffeomorphism onto a submanifold $W = w(V)$ of the unit tangent bundle SM. The submanifold W is invariant under the geodesic flow and the sets of parallel vectors in W define a k-dimensional foliation of W. Now the following point is crucial: Choosing V and W such that their dimension is the minimal possible one we can show that there are no parallel and not even affine Jacobifields on W which are normal to the foliation.

In the final part of the proof (section 4) we use this last property of W to find a compact subset G of W which is saturated with respect to our foliation and on which the geodesic flow is normally hyperbolic in the sense of [HPS]. Using the tools from the theory of dynamical systems developed by Hirsch, Pugh and Shub in [HPS], in particular the Shadowing Lemma (7 A. 2), we can then prove the existence of a closed k-flat, *see* section 5.

1. Preliminaries

A. Convexity properties ([BGS], chapt. I, [EO], [BO]). – In this paper M will denote an n-dimensional connected compact real analytic Riemannian manifold of nonpositive sectional curvature ($K \leqq 0$) with universal covering $p: \tilde{M} \to M$. By TM, $T\tilde{M}$ and SM, $S\tilde{M}$ we denote the tangent and the unit tangent bundles of M and $\tilde{M}$. For a tangent vector $v \in T_x M$ let $\gamma_v(t) = \exp_x(tv)$ be the geodesic with initial vector v and let $g^t: SM \to SM$, $g^t v := \dot{\gamma}_v(t)$, be the geodesic flow. By $d(\ ,\ )$ we denote the distance function on M and $\tilde{M}$.

A function $f: \tilde{M} \to \mathbb{R}$ is *convex*, if $f \circ \gamma: \mathbb{R} \to \mathbb{R}$ is convex for every geodesic γ in $\tilde{M}$ The curvature condition $K \leqq 0$ implies the convexity of the following functions:

1. The distance function $d: \tilde{M} \times \tilde{M} \to \mathbb{R}$.
2. The norm $t \mapsto \| Y(t) \|$ of a Jacobifield along a geodesic.
3. The distance function $d(\ , H): \tilde{M} \to \mathbb{R}$, where H is a convex subset of $\tilde{M}$.

For a subset $A \subset \tilde{M}$ let $\mathrm{Tube}_r(A) = \{ x \in \tilde{M} \mid d(x, A) \leqq r \}$ be the r-tube of A. Two complete totally geodesic submanifolds H and H′ of $\tilde{M}$ are called *parallel* ($H \| H'$), if the Hausdorff distance HD between H and H′ is finite, *i.e.* if there exists r such that $H \subset \mathrm{Tube}_r(H')$ and $H' \subset \mathrm{Tube}_r(H)$. Two parallel totally geodesic submanifolds H, H′ bound a convex subset isometric to $H \times [0, r]$, where $r = \mathrm{HD}(H, H')$ (Sandwich Lemma).

In general the set P_H of all points in $\tilde{M}$ which lie on parallels to H is convex and splits isometrically as $P_H = H \times Q$, where Q is a convex subset of $\tilde{M}$. The analyticity implies that P_H is without boundary.

If H is a convex subset of $\tilde{M}$, then there exists an orthogonal projection $\text{proj}_H : \tilde{M} \to H$, which is distance nonincreasing. If H^* is a complete totally geodesic submanifold of $\tilde{M}$ such that $d(\ , H)$ is constant on H^*, then $H' := \text{proj}_H(H^*)$ is also complete and totally geodesic with $H' \| H$. This follows from the proof of [BGS], Lemma 2.3.

B. Geometry of the tangent and frame bundle. — Let TM, $T\tilde{M}$ and SM, $S\tilde{M}$ be the tangent (unit tangent) bundle of M and $\tilde{M}$. All bundle projections will be denoted by π. Let $K : T(TM) \to TM$ be the connection map of the Levi-Civita connection. On TM and SM we use the metric

$$\langle \xi, \eta \rangle^* := \langle \pi_* \xi, \pi_* \eta \rangle + \langle K(\xi), K(\eta) \rangle$$

induced by the metric $\langle \ , \ \rangle$ on M. We make the usual identification:

$$TSM = \{ (w, A, B) \mid w \in SM; A, B \in T_{\pi(w)} M, B \perp w \}$$

where $A = \pi_* (w, A, B)$ and $B = K(w, A, B)$.

For $w \in SM$ we denote by $J(w)$ the space of Jacobifields along the geodesic γ_w. Let

$$J^*(w) := \{ Y \in J(w) \mid \langle Y'(t), \dot{\gamma}_w(t) \rangle = 0 \}$$

where Y' is the covariant derivative of Y along γ_w. We have $\dim J(w) = 2n$, $\dim J^*(w) = 2n - 1$. There is a canonical isomorphism

$$\xi \mapsto Y_\xi$$

between $T_w SM$ and $J^*(w)$: for $\xi = (w, A, B)$ let Y_ξ be the Jacobifield along γ_w with $Y_\xi(0) = A$ and $Y'_\xi(0) = B$. For the geodesic flow $g^t : SM \to SM$ we have

$$g^t_* \xi = (g^t w, Y_\xi(t), Y'_\xi(t))$$

We call $Y \in J(w)$ *stable* (*unstable*), if $\| Y(t) \|$ is bounded for $t \to \infty$ ($t \to -\infty$). Let $J^s(w)$, $J^u(w) \subset J(w)$ be the subspaces of stable and unstable fields. For every vector $w \in TM$ there exists a unique stable (unstable) field $Y \in J^s(w)$ ($Z \in J^u(w)$) with $Y(0) = Z(0) = w$. In particular $\dim J^s(w) = \dim J^u(w) = n$.

Let $J^p(w)$ be the subspace of $J(w)$ consisting of parallel Jacobifields along γ_w. The following properties are equivalent (*see* [BBE] 1.4):

(i) $Y \in J^p(w)$;

(ii) $\| Y(t) \|$ is constant on $\mathbb{R}$;

(iii) $\| Y(t) \|$ is bounded on $\mathbb{R}$;

(iv) $R(Y(t), \dot{\gamma}_w(t)) \dot{\gamma}_w(t) \equiv 0$.

In particular $J^p(w) = J^s(w) \cap J^u(w)$.

We define the *strong stable* (*strong unstable*) fields by

$$J^{ss}(w) := \{ Y \in J^s(w) \mid Y(0) \perp Z(0) \text{ for all } Z \in J^p(w) \}$$

$$J^{su}(w) := \{ y \in J^u(w) \mid Y(0) \perp Z(0) \text{ for all } Z \in J^p(w) \}$$

By [BBE], Lemma 3.3, we have: if $Y \in J^s(w)$ ($\in J^u(w)$) and $Y(0) \perp Z(0)$ for $Z \in J^p(w)$, then $Y(t) \perp Z(t)$ for all $t \in \mathbb{R}$. Note that every Jacobifield $Y \in J(w)$ with $\lim_{t \to \infty} Y(t)=0$ [resp. $\lim_{t \to -\infty} Y(t)=0$] is in $J^{ss}(w)$ ($J^{su}(w)$). This is true since $\langle Y(t), Z(t) \rangle$ is linear in t if $Y \in J(w)$ and $Z \in J^p(w)$, *cf.* [BBE], Lemma 3.3.

If $\dim J^p(w)=m$, then $\dim J^{ss}(w)=\dim J^{su}(w)=n-m$. Note that $J^{ss}(w)$, $J^{su}(w)$, $J^p(w)$ are all contained in $J^*(w)$. We call a Jacobifield *affine*, if

$$Y(t)=\sum_{i=1}^{m}(a_i t+b_i) Z_i(t)$$

where $Z_i \in J^p(w)$. Let $J^a(w)$ be the space of affine fields, and

$$J^{*a}(w)=J^a(w) \cap J^*(w)$$

Then $\dim J^a(w)=2m$ and $\dim J^{*a}(w)=2m-1$. If $Y \in J^{ss}(w)$ or $Y \in J^{su}(w)$ and $Z \in J^a(w)$ then $Y(t) \perp Z(t)$ for all $t \in \mathbb{R}$. We have the decomposition

$$J(w)=J^a(w) \oplus J^{ss}(w) \oplus J^{su}(w)$$
$$J^*(w)=J^{*a}(w) \oplus J^{ss}(w) \oplus J^{su}(w).$$

Stiefelbundle. — By $ST_k(M)$ we denote the Stiefelbundle of orthogonal k-frames:

$$St_k(M)=\{(x, v_1, \ldots, v_k) \mid x \in M, v_i \in T_x M, \|v_i\|=1, v_i \perp v_j \textit{ for } i \neq j, 1 \leq i,j \leq k\}$$

Using the connection map K, we describe the space $T_v St_k(M)$, $v=(x, v_1, \ldots, v_k)$ as follows: Represent $\eta \in T_v St_k(M)$ by a path $v(t)=(x(t), v_1(t), \ldots, v_k(t))$ with $v(0)=v$. Let $A=\dot{x}(0) \in T_x M$ and $B_i=K(\dot{v}_i(0)) \in T_x M$. The orthogonality relations of $v_1, \ldots, v_k$ imply

$$(*) \qquad \langle v_i, B_j \rangle + \langle v_j, B_i \rangle = 0 \quad \text{for} \quad 1 \leq i,j \leq k$$

We identify

$$T_v St_k(M)=\{(x, v_1, \ldots, v_k, A, B_1, \ldots, B_k) \mid v_i, A, B_i \in T_x M \text{ and } v_i, B_i \text{ satisfy } (*)\}$$

Thus $\eta \in T_v St_k(M)$ can also be described by Jacobifields Y_i along γ_{v_i} with $Y_i(0)=A$ and $Y_i'(0)=B_i$.

Jacobifields along flats. — Infinitesimal deformations of a flat $F_0 : \mathbb{R}^k \to M$ by flats are described by *Jacobifields along flats*. Suppose $F : \mathbb{R}^k \times (-\varepsilon, \varepsilon) \to M$ is a 1-parameter family of flats $F_t=F(\ ,t)$. Then $Y(z)=\partial/\partial t_{|t=0} F(z,t)$ is a vector field along F_0 such that $Y(as+b)$ is a Jacobifield for every line $s \mapsto as+b$ in $\mathbb{R}^k$. Every vectorfield with this property will be called a Jacobifield along F_0. A Jacobifield Y along F_0 is uniquely determined by its value $Y(z)=A$ at an arbitrarily fixed $z \in \mathbb{R}^k$ and its covariant derivatives $D_i Y(z)=K(Y_{*z}(e_i))=B_i$, $1 \leq i \leq k$ at z. Note that with this notation the curve of frames

$$v(t)=(F(z,t), F_{t*z}(e_i), \ldots, F_{t*z}(e_k))$$

has derivative

$$\dot{v}(0)=(v(0), A, B_1, \ldots, B_k).$$

The Jacobifields $Y=\partial F/\partial t$ coming from a variation F_t as above are special in that the F_t are isometric whereas they need only be affine to define a Jacobifield. This corresponds to the fact that for every $z \in \mathbb{R}^k$ the vectors $v_i=F_{0*z}(e_i)$ and $B_i=D_i Y(z)$, $1 \leqq i \leqq k$, satisfy the relations (*). Since F_0 is totally geodesic the components Y^T and Y^N of Y tangent and normal to F_0 are Jacobifields as well, *cf.* [BBE], Lemma 2.4. Since F_0 is flat Y^T is affine, *i.e.* $Y^T(as+b)$ is affine for every line $s \mapsto as+b$ in $\mathbb{R}^k$. Geometrically the presence of the tangential component Y^T corresponds to the freedom to reparametrize F_0. More precisely we can find an infinitesimal isometry $Rz=Sz+a$ of $\mathbb{R}^k$, *i.e.* $S \in so(k)$ and $a \in \mathbb{R}^k$, such that $Y^T(z)=F_{0*z}(Rz)$. One has to take S and a so that $Y^T(0)=F_{0*0}(a)$ and $D_i Y^T(0)=F_{0*0}(S e_i)$.

The generic nonexistence of flats of dimension $k>1$ in Riemannian manifolds has the following infinitesimal counterpart: if $k>1$ there may not be a Jacobifield Y along F_0 for every (non-tangential) choice of initial values and if Y exists it need not come from a variation F_t of F_0 by flats.

C. Subanalytic sets. — In sections 2 and 3 our assumption that the manifold M and the Riemannian metric $\langle\ ,\ \rangle$ be analytic will be crucial. We shall frequently appeal to the theory of subanalytic sets as described in [T] or [BM]. Since the precise definition of a subanalytic set is a little lengthy we only present a class of examples which is important for us: If $f: M \to N$ is a proper (real) analytic map between (real) analytic manifolds and if $A \subset M$ is analytic, then $f(A)$ is subanalytic in N, *cf.* [T], Theorem 1.2.2 (vi).

A particularly nice property of the set SUB(M) of subanalytic subsets of M is that SUB(M) is closed under finite union and intersection and under set theoretic difference, *cf.* [T], Theorem 1.2.2 (i). Moreover every $A \in \text{SUB}(M)$ can be stratified into analytic submanifolds $A_i \in \text{SUB}(M)$, *see* [T], Theorem 1.2.2 (iv). In particular subanalytic subsets are locally pathwise connected, *cf.* also [BM], Theorem 6.10.

The following theorem due to Tamm is of fundamental importance for us, *cf.* [T], Theorem 2.4.2, or [BM], Theorem 7.2:

For $A \subset M$ and $0 \leqq q \leqq \dim M$ let $r^q(A)$ denote the set of analytic q-regular points of A, *i.e.* the set of $x \in A$ which have a neighborhood U in M such that $U \cap A$ is a q-dimensional analytic submanifold of M. If A is subanalytic in M then so is $r^q(A)$ for every $0 \leqq q \leqq \dim M$. If $A \neq \emptyset$ there exists a maximal q such that $r^q(A) \neq \emptyset$. For this q the set $B := A \setminus r^q(A)$ is subanalytic in M and $\dim B < \dim A$.

2. Flats with additional structure

In this section we prove the existence of flats which have an additional structure of singular subspaces similar to the situation of symmetric spaces. Before we explain this more explicitly (*cf.* the remark before Theorem 2.5) we have to present some basic properties of flats and of their singular subspaces.

We assume that $k \geqq 2$ is the maximal dimension of a flat in M. For $1 \leqq m \leqq n$ let $G_m(M)$ and $G_m(\tilde{M})$ be the Grassmannbundle of m-planes with bundle projections $\pi : G_m(M) \to M$ and $\pi : G_m(\tilde{M}) \to \tilde{M}$. By $d^m(,)$ we denote the induced distance functions on $G_m(M)$ and $G_m(\tilde{M})$.

For $1 \leqq m \leqq k$ let $F_m(M)$ [resp. $F_m(\tilde{M})$] be the subset of all $\sigma \in G_m(M)$ ($\sigma \in G_m(\tilde{M})$) such that $\exp : \sigma \to M$ ($\exp : \sigma \to \tilde{M}$) is an m-flat in M (resp. $\tilde{M}$).

We call $\tau, \tau' \in F_m(\tilde{M})$ *parallel* ($\tau \| \tau'$), if the flats $\exp(\tau)$ and $\exp(\tau')$ are parallel as subsets of $\tilde{M}$ (section 1.A). Let

$$\mathbf{P}_\tau := \{ \tau' \in F_m(\tilde{M}) \mid \tau' \| \tau \}$$

By 1.A the projection $P_\tau = \pi(\mathbf{P}_\tau) \subset \tilde{M}$ is a complete totally geodesic submanifold which splits isometrically as $\mathbb{R}^m \times Q$. For $x \in P_\tau$ let $\tau(x)$ be the (unique) m-plane in $T_x \tilde{M}$ with $\tau(x) \| \tau$. Then $\tau(x)$ is tangent to the $\mathbb{R}^m$-factor of P_τ. We define

$$\text{P-rank}(\tau) := \dim \mathbf{P}_\tau = \dim P_\tau$$
$$F_m^q(\tilde{M}) := \{ \tau \in F_m(\tilde{M}) \mid \text{P-rank}(\tau) = q \}$$

We define parallelism in the quotient as follows: let $\sigma_0, \sigma_1 \in F_m(M)$ and c: $[0, 1] \to M$ a path from $\pi(\sigma_0)$ to $\pi(\sigma_1)$. We call σ_0 *parallel to* σ_1 *along* c, if there is a lift $\tilde{c} : [0, 1] \to \tilde{M}$ and $\tau_0, \tau_1 \in G_m(\tilde{M})$ lifts of σ_0, σ_1 with $\pi(\tau_0) = \tilde{c}(0)$, $\pi(\tau_1) = \tilde{c}(1)$ and $\tau_0 \| \tau_1$.

For $\sigma \in F_m(M)$ let $P_\sigma := p(P_\tau)$, where τ is a lift of σ. Then P_σ is an immersed submanifold. We set P-rank $(\sigma) :=$ P-rank (τ) and

$$F_m^q(M) = \{ \sigma \in F_m(M) \mid \text{P-rank}(\sigma) = q \}$$

Clearly the P-rank is semicontinuous, *i.e.* if $\sigma_i \in F_m^q(M)$ and $\sigma_i \to \sigma$, then

$$\text{P-rank}(\sigma) \geqq \limsup \text{P-rank}(\sigma_i)$$

For a vector $v \in SM$ (resp. $S\tilde{M}$), we define $P_v := P_{\langle v \rangle}$ and P-rank$(v) :=$ P-rank$(\langle v \rangle)$ where $\langle v \rangle$ denotes the linear subspace generated by v. We will use the following fact frequently:

Let $\tau \in F_m(\tilde{M})$ and τ' be a linear subspace of τ, then $P_\tau \subset P_{\tau'}$.

Let $\sigma \in F_m(M)$. We call P_σ *closed*, if the set P_σ is compact. Note that P_σ is closed if and only if for a lift $\tau \in F_m(\tilde{M})$ of σ we have P_τ / Γ_τ is compact, where Γ_τ is the group of those decktransformations which leave P_τ invariant.

Let $F \subset \tilde{M}$ be a flat of maximal dimension, *i.e.* $\dim F = k$, let $p \in F$ and $\sigma = T_p F \in F_k(\tilde{M})$. If $\tau \subset \sigma$ is a linear subspace, then $\tau \in F_m(\tilde{M})$ where $m = \dim \tau$ and $P_\tau \supset F$. We call a vector $v \in T_p F$ *regular* if $P_v = F$ and *singular* if $P_v \neq F$. In symmetric spaces the singular vectors are contained in finitely many hyperplanes. We shall show that in our case the situation is similar.

We call a subset P of $\tilde{M}$ a *parallel space of* F, if there is a linear subspace τ of σ such that $P=P_\tau$.

2.1. LEMMA. — *If* P_1 *and* P_2 *are parallel spaces of* F, *then* P_1 *is orthogonal to* P_2 *in the sense that*

$$\mathrm{proj}_{P_1} P_2 = P_1 \cap P_2 = \mathrm{proj}_{P_2} P_1$$

Proof. – Let $P_i = P_{\tau_i}$ with $\tau_i \subset \sigma$ and let $x \in P_2$. Then P_2 contains the flat $\exp(\tau_2(x))$. Since $\exp(\tau_2(x)) \| \exp(\tau_2)$ and the latter space is contained in P_1 we see that $d(\ ,P_1)$ is bounded on $\exp(\tau_2(x))$. By 1.A we see that $\mathrm{proj}_{P_1}(\exp(\tau_2(x)))$ is parallel to $\exp(\tau_2(x))$ and hence is contained in P_2. In particular $\mathrm{proj}_{P_1}(x) \in P_1 \cap P_2$. □

As an easy consequence we obtain:

2.2. LEMMA. — (i) *There exist only finitely many parallel spaces of* F.

(ii) *If* τ *is an m-dimensional subspace of* σ, *then there is a neighborhood* U *of* τ *in* $G_m(\sigma)$ *such that* $\tau_1 \in U$ *implies* $P_{\tau_1} \subset P_\tau$.

For a parallel space P of F with $\sigma = T_P F$ we define

$$\mathrm{kern}_\sigma(P) := \{ v \in \sigma \mid P \subset P_v \}$$

If $v_1, v_2 \in \mathrm{kern}_\sigma(P)$, then P is foliated by parallels to $\exp(\langle v_1 \rangle)$ and to $\exp(\langle v_2 \rangle)$. It follows that P is foliated by parallels to $\exp(\langle v_1, v_2 \rangle)$. Thus $\mathrm{kern}_\sigma(P)$ is a linear subspace of σ and characterized by the property that $P = P_{\mathrm{kern}_\sigma(P)}$. In particular we have $\tau \subset \mathrm{kern}_\sigma P_\tau$. From the definition we have

$$P \subset P' \quad \Rightarrow \quad \mathrm{kern}_\sigma(P') \subset \mathrm{kern}_\sigma(P)$$

2.3. LEMMA:

(i) *If* P *is a parallel space with* $F \overset{\subset}{\neq} P$, *then* $\mathrm{kern}_\sigma(P) \overset{\subset}{\neq} \sigma$.

(ii) $P_1 \overset{\subset}{\neq} P_2 \Rightarrow \mathrm{kern}_\sigma P_2 \overset{\subset}{\neq} \mathrm{kern}_\sigma P_1$.

(iii) *Let* $\tau = \mathrm{kern}_\sigma(P)$ *with* $m = \dim \tau$. *Then there is a neighborhood* U *of* τ *in* $G_m(\sigma)$ *such that* $\tau_1 \in U \setminus \{\tau\}$ *implies* $P_{\tau_1} \overset{\subset}{\neq} P_\tau$.

Proof. – (i) If $\mathrm{kern}_\sigma(P) = \sigma$, then $P = P_\sigma$. Note that P_σ splits isometrically as $\mathbb{R}^k \times Q$ and by the maximality of k, Q is a point. Thus $P = F$.

(ii) If $\tau = \mathrm{kern}_\sigma P_1 = \mathrm{kern}_\sigma P_2$ then $P_1 = P_\tau = P_2$.

(iii) Because of Lemma 2.2(ii) we have $P_{\tau_1} \subset P_\tau$ for suitable U. If $P_{\tau_1} = P_\tau$, then $\mathrm{kern}_\sigma(P_{\tau_1}) = \mathrm{kern}_\sigma(P_\tau) = \tau$. Thus $\tau_1 \subset \tau$ and hence $\tau_1 = \tau$. □

Now we construct flats with a flag $\sigma_1 \subset \ldots \subset \sigma_k = \sigma$, $\sigma_i \in F_i(\tilde{M})$, such that the σ_i are singular subspaces with parallel spaces of maximal dimensions. For $1 \leqq s \leqq k$ let $G_{s,k}(\tilde{M})$ be the bundle of flags

$$(\sigma_s, \sigma_{s+1}, \ldots, \sigma_k) \in G_s(\tilde{M}) \oplus G_{s+1}(\tilde{M}) \oplus \ldots \oplus G_k(\tilde{M})$$

with $\sigma_s \subset \ldots \subset \sigma_k$ and $G_{s,k}(M)$ the corresponding bundle over M. The bundle projections are denoted by π.

Define inductively subsets $E_{s,k}(\tilde{M}) \subset G_{s,k}(\tilde{M})$ and integers m_s, $1 \leqq s \leqq k$, by

1. $E_{k,k}(\tilde{M}) := F_k(\tilde{M})$, $m_k := k$.

2. If $E_{s+1,k}(\tilde{M})$ is defined let m_s be the maximum of all dimensions $\dim P_\sigma$ where $\sigma \in G_s(\tilde{M})$ is such that there exists $(\sigma_{s+1}, \ldots, \sigma_k) \in E_{s+1,k}(\tilde{M})$ with $\sigma \subset \sigma_{s+1}$. Then we set

$$E_{s,k}(\tilde{M}) = \{ (\sigma_s, \ldots, \sigma_k) \in G_{s,k}(\tilde{M}) \mid \sigma_k \in F_k(\tilde{M}), \dim P_{\sigma_i} = m_i \text{ for } s \leqq i \leqq k \}$$

Correspondingly we define $E_{s,k}(M) \subset G_{s,k}(M)$. Thus

$$E_{s,k}(M) = \{ (\sigma_s, \ldots, \sigma_k) \in G_{s,k}(M) \mid \sigma_k \in F_k(M), \dim P_{\sigma_i} = m_i \text{ for } s \leqq i \leqq k \}$$

2.4. Definition. — We call $E_{s,k}(M)$ *well structured*, if $m_s > m_{s+1} > \ldots > m_k = k$.

Remark. — The arguments in section 5 and inductive use of Theorem 2.5 below will allow us to assume that $E_{1,k}(M)$ is well structured. If this is the case a flat $F = \exp(\sigma_k)$ is called well structured if σ_k can be completed to $(\sigma_1, \ldots, \sigma_k) \in E_{1,k}(M)$. The important property of well structured flats $F = \exp(\sigma_k)$ is that (by Lemma 2.8 below) σ_k can be completed to an element of $E_{1,k}(M)$ in only finitely many ways. This implies that a well structured flat carries — up to finite ambiguity — a natural basis. This will be crucial in the proof of Theorem 3.1, *cf.* Lemma 3.5 (ii).

2.5. Theorem. — *Let us assume that* $E_{s,k}(M)$ *is well structured and* $s \geqq 2$. *If there exists a flag* $(\sigma_s, \ldots, \sigma_k) \in E_{s,k}(M)$ *such that* P_{σ_s} *is not closed, then* $E_{s-1,k}(M)$ *is well structured.*

Remark. — For $s = k$, the theorem says that the existence of a nonclosed k-flat implies the existence of a k-flat with a "singular" hyperplane. This was proved in [S1], section 4.

We start with some lemmas.

2.6. Lemma

(i) $F_m(M)$ *is a compact analytic subset of* $G_m(M)$.

(ii) $F_m^q(M)$ *and* $F_m^{\geqq q}(M) = \bigcup_{r \geqq q} F_m^r(M)$ *are subanalytic subsets of* $G_m(M)$.

(iii) $E_{s,k}(M)$ *is a compact subanalytic subset of* $G_{s,k}(M)$.

Proof. — (i) For $\sigma \in G_m(M)$ let $S(\sigma) \subset \sigma$ be the unit sphere. If σ is tangent to an m-flat, then the volume of the immersed sphere $\exp(S(\sigma))$ equals ω_{m-1}, where ω_{m-1} is the volume of the standard $(m-1)$-sphere. In general $\operatorname{vol}(\exp(S(\sigma))) \geqq \omega_{m-1}$ and equality implies that the unit ball in σ is mapped totally geodesically onto an immersed flat ball in M ([BGS], 1.E). By analyticity σ is tangent to a flat.

Thus $F_m(M) = f_m^{-1}(0)$ where

$$f_m(\sigma) := \operatorname{vol}(\exp(S(\sigma))) - \omega_{m-1}$$

Clearly f_m is an analytic function on $G_m(M)$.

(ii) For $(v, w) \in SM \oplus SM$ let $v_w(t)$ denote the parallel vectorfield along $\exp(tw)$ with $v_w(0) = v$. The map $(SM \oplus SM) \times \mathbb{R} \to SM$ defined by $(v, w, t) \mapsto v_w(t)$ is real analytic.

Choose a constant $\varepsilon > 0$ smaller that one third of the injectivity radius of M and define

$$g_\pm : \quad SM \oplus SM \to \mathbb{R}$$

$$g_\pm(v, w) = [d(\exp(\pm \varepsilon v_w(\varepsilon)), \exp(\pm \varepsilon v))]^2 - \varepsilon^2$$

Then the geodesics $\exp(tv)$ and $\exp(tv_w(\varepsilon))$ are parallel, if and only if $g_+(v, w) = g_-(v, w) = 0$. Define $g : SM \oplus SM \to \mathbb{R}$ by

$$g(v, w) = g_+^2(v, w) + g_-^2(v, w)$$

Since the distances involved in the definitions of $g_\pm$ are smaller that the injectivity radius, the function g is real analytic. We have $g(v, w) \geqq 0$ and $g(v, w) = 0$ implies that w is tangent to P_v. Now we define

$$\bar{g} : \quad G_m(M) \oplus G_q(M) \to \mathbb{R}$$

$$\bar{g}(\sigma, \tau) := f_m(\sigma) + \int_{v \in S(\sigma)} \int_{w \in S(\tau)} g(v, w)\, dv\, dw$$

where f_m is the function of (i) and $S(\sigma)$, $S(\tau)$ are the unit spheres in σ and τ. The function $\bar{g}$ is analytic and $\bar{g}(\sigma, \tau) \geqq 0$. Now $\bar{g}(\sigma, \tau) = 0$ first implies $f_m(\sigma) = 0$, *i.e.* $\sigma \in F_m(M)$. Secondly $g(v, w) = 0$ for every $v \in S(\sigma)$ and $w \in S(\tau)$. This implies that τ is tangent to P_v for all $v \in \sigma$, thus τ is tangent to P_σ and in particular $\dim P_\sigma \geqq q$. This computation shows that $F_m^{\geqq q}(M) = p_1(\{\bar{g} = 0\})$ where

$$p_1 : \quad G_m(M) \oplus G_q(M) \to G_m(M)$$

is the canonical projection. Thus $F_m^{\geqq q}(M)$ and $F_m^q(M) = F_m^{\geqq q}(M) \setminus F_m^{\geqq q+1}(M)$ are subanalytic by 1.C.

(iii) $E_{s,k}(M) = \{(\sigma_s, \ldots, \sigma_k) \in G_{s,k}(M) \mid \sigma_k \in F_k(M),\ \dim P_{\sigma_i} = m_i\}$. Since the m_i are choosen to be maximal possible, we see that $E_{s,k}(M)$ is compact by the semicontinuity of the P-rank.

Consider on

$$G_{s,k}(M) \oplus G_{m_s}(M) \oplus \ldots \oplus G_{m_k}(M)$$

the function

$$h((\sigma_s, \ldots, \sigma_k), \tau_s, \ldots, \tau_k) := f_k(\sigma_k) + \sum_{i=s}^{k} \int_{v \in S(\sigma_i)} \int_{w \in S(\tau_i)} g(v, w)\, dv\, dw$$

then we see as in (ii) that $E_{s,k}(M) = p_1(\{h = 0\})$ is subanalytic where p_1 is now the projection onto $G_{s,k}(M)$. $\square$

2.7. LEMMA. — *Let* $\tau \in F^q_m(\tilde{M})$ *and assume that* $p(P_\tau)$ *is not closed. Then there exists a continuous path* $\tau(t) \in F^q_m(\tilde{M})$, $t \in [0, 1]$, *such that* $\tau(0) = \tau$ *and* $P_{\tau(t)} \neq P_\tau$ *for* $t > 0$.

Proof. — We represent $M = \tilde{M}/\Gamma$ with $p: \tilde{M} \to M$. Let

$$P_\sigma = p(P_\tau) \qquad \text{and} \qquad \Gamma_\tau := \{\gamma \in \Gamma \mid \gamma P_\tau = P_\tau\}.$$

CLAIM. — *If* P_σ *is not closed, then for every* $\varepsilon > 0$ *there exists* $\gamma \in \Gamma$ *and* $\tau_1 \in \mathbf{P}_\tau$ *such that* $d^m(\gamma_* \tau_1, \mathbf{P}_\tau) \leqq \varepsilon$ *but* $\gamma_* \tau_1$ *is not contained in* $\mathbf{P}_\tau$.

To prove the claim assume that for some $\varepsilon > 0$ the following holds: $\tau_1 \in \mathbf{P}_\tau$ and $d^m(\gamma_* \tau_1, \mathbf{P}_\tau) \leqq \varepsilon$ implies $\gamma_* \tau_1 \in \mathbf{P}_\tau$. Let D be the $\varepsilon/2$ distance tube of $\mathbf{P}_\tau$ in $G_m(\tilde{M})$. Then $\gamma_* D \cap D \neq \emptyset$ implies $\gamma \in \Gamma_\tau$. Thus D/Γ_τ is injectively embedded in $G_m(M)$ and thus $\mathbf{P}_\tau/\Gamma_\tau$ and hence P_τ/Γ_τ is compact. This proves the claim.

Consider now the set $F^{\geqq q}_m(M)$ which is compact by the semicontinuity of the P-rank and subanalytic by Lemma 2.6. Since subanalytic sets can be stratified, $F^{\geqq q}_m(M)$ is locally pathwise connected, *cf.* 1.C. Thus for every $\varepsilon > 0$ there exists $\delta > 0$ with the property:

If $\sigma_0, \sigma_1 \in F^{\geqq q}_m(M)$ with $d^m(\sigma_0, \sigma_1) \leqq \delta$, then there is a path $\sigma_t \in F^{\geqq q}_m(M)$ of length $< \varepsilon$ joining σ_0 and σ_1.

Choose $\delta > 0$ for ε = injectivity radius of M. By the claim there is $\tau' \in \mathbf{P}_\tau$ and $\gamma \in \Gamma$ with $d^m(\gamma_* \tau', \mathbf{P}_\tau) \leqq \delta$ and $\gamma_* \tau' \notin \mathbf{P}_\tau$. Let $\tau_1 := \gamma_* \tau'$ and $\tau_0 \in \mathbf{P}_\tau$ with $d^m(\tau_0, \tau_1) \leqq \delta$. Let $\sigma_0, \sigma_1 \in F^q_m(M)$ be the projections of τ_0, τ_1. By construction there is a path $\sigma_t \in F^{\geqq q}_m(M)$ joining σ_0, σ_1 with length $< \varepsilon$. In particular the length of the curve $c(t) = \pi(\sigma_t)$ is smaller than the injectivity radius. Lift $c(t)$ to a path in $\tilde{M}$ with initial point $\pi(\tau_0)$ and lift σ_t to a path τ_t in $F^{\geqq q}_m(\tilde{M})$ starting in τ_0. By construction the lift ends in τ_1. Let $t_0 := \max\{t \in [0, 1] \mid \tau_t \parallel \tau\}$, then $t_0 < 1$ since τ_1 is not parallel to τ. By the semicontinuity of the P-rank there exists $\eta > 0$ such that $\tau_t \in F^q_m(\tilde{M})$ for $t \in [t_0, t_0 + \eta]$.

We reparametrize the path τ_t for $t \in [t_0, t_0 + \eta]$ on the interval $[0, 1]$ and call the new path $\tilde{\tau}(t)$, $t \in [0, 1]$. Since $t \mapsto \mathbf{P}_{\tilde{\tau}(t)}$ is continuous in the compact open topology we can find a continuous path $\tau(t) \in F^q_m(\tilde{M})$, $t \in [0, 1]$ such that $\mathbf{P}_{\tau(t)} = \mathbf{P}_{\tilde{\tau}(t)}$ for all t and $\tau(0) = \tau$. The path $\tau(t)$ satisfies the required properties. □

2.8. LEMMA. — *Assume that* $E_{s,k}(M)$ *is well structured. Then*

(i) *If* $(\sigma_s, \ldots, \sigma_k) \in E_{s,k}(\tilde{M})$, *then* $\text{kern}_{\sigma_k}(P_{\sigma_i}) = \sigma_i$.

(ii) *For given* $\sigma_k \in F_k(\tilde{M})$ *there are only finitely many possibilities to complete* σ_k *to a flag* $(\sigma_s, \ldots, \sigma_k) \in E_{s,k}(\tilde{M})$.

(iii) *There exists* $\varepsilon > 0$ *such that the following holds: If* $(\sigma_s, \ldots, \sigma_k) \in E_{s,k}(\tilde{M})$ *and* τ *is an r-dimensional subspace of* σ_k *for* $s \leqq r \leqq k$, *with* $d^r(\tau, \sigma_r) \leqq \varepsilon$, *then* $P_\tau \subset P_{\sigma_r}$ *and equality implies* $\tau = \sigma_v$.

Proof. — (i) Inductively we may assume that $\text{kern}_{\sigma_k}(P_{\sigma_j}) = \sigma_j$ for $i+1 \leqq j \leqq k$. Let $\tau = \text{kern}_{\sigma_k}(P_{\sigma_i})$, then $\tau \supset \sigma_i$. Since $E_{s,k}(M)$ is well structured we have $P_{\sigma_{i+1}} \overset{\subset}{\neq} P_{\sigma_i}$ and hence

$$\tau = \mathrm{kern}_{\sigma_k}(P_{\sigma_i}) \overset{\subset}{\neq} \mathrm{kern}_{\sigma_k}(P_{\sigma_{i+1}}) = \sigma_{i+1}$$

where the first inequality follows from Lemma 2.3 (ii) and the last equality by induction. Thus $\tau = \sigma_i$.

(ii) By Lemma 2.2 the flat $F = \exp(\sigma_k)$ is only contained in finitely many parallel spaces and hence σ_k contains only finitely many kernel spaces.

(ii) Let us consider a sequence $\mu_i = (\sigma_s^i, \ldots, \sigma_k^i) \in E_{s,k}(M)$ and r-dimensional subspaces $\tau_i \subset \sigma_k^i$ such that $d^r(\tau_i, \sigma_r^i) \to 0$. We can assume that $\mu_i \to \mu = (\sigma_s, \ldots, \sigma_k) \in E_{s,k}(M)$. If P_{τ_i} is not contained in $P_{\sigma_r^i}$ for large i then the orthogonality $P_{\tau_i} \perp P_{\sigma_r^i}$ of Lemma 2.1 and the semicontinuity of the P-rank imply P-rank $(\sigma_r) > m_r$, a contradiction. Thus $P_{\tau_i} \subset P_{\sigma_r^i}$. Equality implies

$$\tau_i \subset \mathrm{kern}_{\sigma_k^i}(P_{\tau_i}) = \mathrm{kern}_{\sigma_k^i}(P_{\sigma_r^i}) = \sigma_r^i$$

and hence $\tau_i = \sigma_r^i$.

2.9. Lemma. — *Suppose* $F = \exp(\sigma)$ *is a k-flat and* $F \subset P_\tau$ *for some* $\tau \in F_s(\tilde{M})$. *Then* P_τ *is a parallel space of* F *and there exists* $\tau' \parallel \tau$ *with* $\tau' \subset \sigma$.

Proof. — Let $P_\tau = \mathbb{R}^s \times Q$ be the isometric splitting of P_τ so that $\mathbb{R}^s \times \{q\}$, $q \in Q$, are the flats parallel to $\exp(\tau)$. Let $p_1 : P_\tau \to \mathbb{R}^s$, $p_2 : P_\tau \to Q$ denote the orthogonal projections. It is not difficult to show that $p_1(F) = F_1$ and $p_2(F) = F_2$ are flats. Since $\mathbb{R}^s \times F_2$ is a flat we have $\dim F_2 \leqq k - s$. Since $F \subset F_1 \times F_2$, $\dim F = k$ and $\dim F_1 \leqq s$ we conclude $F = F_1 \times F_2$ and $\dim F_1 = s$, *i.e.* $F_1 = \mathbb{R}^s$. Hence F contains the flat $\mathbb{R}^s \times \{q\}$, $q \in F_2$, parallel to $\exp(\tau)$. □

The following lemma is a crucial ingredient of the proof of Theorem 2.5. It is a purely topological consequence of the discreteness result (2.8) (iii) and of Lemma 2.9.

2.10. Lemma. — *Assume that* $E_{s,k}(M)$ *is well structured. Then there exists* $\varepsilon_0 > 0$ *such that for every* $\varepsilon > 0$ *there exists* $\eta > 0$ *with the following property:*

Let

$$(\sigma_s, \ldots, \sigma_k) \in E_{s,k}(\tilde{M}) \quad \textit{and} \quad \tau \in F_s^{\geqq m_s}(\tilde{M})$$

satisfy $d^s(\tau, \sigma_s) < \varepsilon_0$. *Suppose* $d(\pi(\tau), \pi(\sigma_k)) \leqq \eta$ *and there exists a ball* $B \subset \exp(\sigma_k)$ *of radius one containing* $\pi(\sigma_k)$ *such that* $d(x, P_\tau) \leqq \eta$ *for all* $x \in B$. *Then* $d^s(\tau, \sigma_s) < \varepsilon$.

Remark. — In less precise terms (2.10) says the following: if $\tau \in F_s^{\geqq m_s}(\tilde{M})$ lies sufficiently close to σ_s for some $(\sigma_s, \ldots, \sigma_k) \in E_{s,k}(\tilde{M})$ then τ is very close to σ_s provided σ_k is very close to a subspace of TP_τ.

Proof. — Choose ε_0 according to (2.8) (iii). If our claim does not hold we can find $\varepsilon > 0$ and sequences $(\sigma_s^i, \ldots, \sigma_k^i) \in E_{s,k}(\tilde{M})$, $\tau^i \in F_s^{\geqq m_s}(\tilde{M})$ with $d^s(\tau^i, \sigma_s^i) < \varepsilon_0$ and balls

$B^i \subset \exp(\sigma_k^i)$ of radius one containing $\pi(\sigma_s^i)$ such that the following is true:

$$\lim_{i \to \infty} d(\pi(\tau^i), \pi(\sigma_s^i)) = 0$$

$$\lim_{i \to \infty} (\sup_{x \in B^i} d(x, P_{\tau^i})) = 0$$

and $d^s(\tau^i, \sigma_s^i) \geqq \varepsilon$. If necessary choose a sequence of deck transformations $\gamma_i \in \Gamma$ such that the sequence $\gamma_i(\pi(\tau^i))$ remains bounded and replace τ^i by $\gamma_{i*}(\tau^i)$, σ_τ^i by $\gamma_{i*}(\sigma_r^i)$ and B^i by $\gamma_i(B^i)$. Then we may assume that τ^i, $(\sigma_s^i, \ldots, \sigma_k^i)$ and B^i converge to $\tau \in F_s^{\geqq m_s}(\tilde{M})$, $(\sigma_s, \ldots, \sigma_k) \in E_{s,k}(\tilde{M})$ and $B \subset \exp(\sigma_k)$ respectively. Our assumptions imply that τ and σ_s have the same footpoint, that $\varepsilon \leqq d^s(\tau, \sigma_s) \leqq \varepsilon_0$ and that $B \subset P_\tau$. Since P_τ is totally geodesic we even obtain $\exp(\sigma_k) \subset P_\tau$. Now Lemma 2.9 implies $\tau \subset \sigma_k$. Since $d(\tau, \sigma_s) \leqq \varepsilon_0$ we get $P_\tau \subset P_{\sigma_s}$ from (2.8) (iii). On the other hand $\dim P_\tau \geqq m_s = \dim P_{\sigma_s}$ so that $P_\tau = P_{\sigma_s}$. Hence $\tau = \sigma_s$ by (2.8) (iii). This contradicts $d^s(\tau, \sigma_s) \geqq \varepsilon > 0$ and proves our claim. □

Proof of Theorem 2.5. — We argue in the universal cover $\tilde{M}$. Choose $(\sigma_s, \ldots, \sigma_k) \in E_{s,k}(\tilde{M})$ such that $p(P_{\sigma_s})$ is not closed. By Lemma 2.7 we can find a sequence $\tau_i \in F_s^{m_s}(\tilde{M})$ converging to σ_s such that $P_{\tau_i} \neq P_{\sigma_s}$ for all i. We abbreviate $P_i := P_{\tau_i}$, $P := P_{\sigma_s}$ and $F := \exp(\sigma_k)$.

Now we can give a sketch of the proof. Using the distance function from P_i restricted to $F = \exp(\sigma_k)$ and the accumulation construction first described in [S1], section 5, we find limit spaces $P^0 \neq P^*$ of P_i and P, a limit flat $F^* \subset P^*$ of F and a codimension one subflat D^* of F^* which has a parallel $D^0 \subset P^0$. The crucial point is to show that D^0 is not contained in P^* since this implies $m_{s-1} > m_s$. We prove that D^0 is not contained in P^* if a parameter in the accumulation construction is chosen sufficiently small. Here an application of Lemma 2.10 is the key step.

Before we describe the accumulation construction we choose $\varepsilon_1 > 0$ with the following property:
(*) Suppose $(\sigma_s^*, \ldots, \sigma_k^*) \in E_{s,k}(\tilde{M})$ and $\tau^* \in F_s^{m_s}(\tilde{M})$ have footpoints x and y and $d^{m_s}(T_x(P_{\sigma_s^*}), T_y(P_{\tau^*})) \geqq \pi/4$. Then $d^s(\tau^*, \sigma_s^*) \geqq \varepsilon_1$.
The existence of such an $\varepsilon_1 > 0$ follows from the compactness of $E_{s,k}(M)$ and the semicontinuity of the P-rank.

For the accumulation construction we consider the convex distance functions $f_i = d(\ , P_i)_{|F} : F \to [0, \infty)$. We need that almost all of the f_i are unbounded. First we prove that f_i is not identically zero for i large enough. If $f_i \equiv 0$ then P_i is a parallel space of F by Lemma 2.9. Since $P_i \neq P$, since P_i converge to P by construction and since F has only finitely many parallel spaces by Lemma 2.2 we see that f_i is not identically zero for large enough $i \in \mathbb{N}$. Finally f_i cannot be bounded and different from zero: Otherwise $f_i \equiv a > 0$ is constant by convexity and by 1.A there exists a parallel F' to F in P_i. By the Sandwich Lemma F and F' bound a convex subset isometric to $F \times [0, a]$. By analyticity we even obtain a $(k+1)$-flat $F \times \mathbb{R}$ in $\tilde{M}$ which is impossible. Thus f_i is unbounded for almost all $i \in \mathbb{N}$.

Now set $\varepsilon=(1/2)\min\{\varepsilon_0,\varepsilon_1\}$ where ε_0 is defined in (2.10) and ε_1 in (*) above. Choose $\eta>0$ so that (2.10) holds for this ε. Let $z=\pi(\sigma_k)$. Since P_i converges to P uniformly on compact subsets we have $\lim f_i(z)=0$ and we can assume $f_i(z)\leqq\eta$. Let R_i be the radius of the largest distance ball $B_i=B_{R_i}(z)\subset F$ such that $B_i\subset\{f_i\leqq\eta\}$. Since $P_i\to P$ we see $R_i\to\infty$. On the other hand $R_i\neq\infty$ since f_i is unbounded. Let x_i be a point in ∂B_i with $f_i(x_i)=\eta$ and let $y_i=\mathrm{Proj}_{P_i}(x_i)\in P_i$. In x_i we have the flag $(\sigma_s(x_i),\ldots,\sigma_k(x_i))\in E_{s,k}(\tilde{M})$. Let H_i be the affine hyperplane in F tangent to ∂B_i with $x_i\in H_i$.

By construction H_i is also tangent to $\{f_i=\eta\}$ and thus $f_i\geqq\eta$ on H_i by convexity of f_i. Let D_i be the ball in H_i centered at x_i with radius $r_i:=\sqrt{(R_i+1)^2-R_i^2}$. By Pythagoras' theorem $d(x,\partial B_i)\leqq 1$ for all $x\in D_i$ and hence $f_i(x)\leqq 1+\eta$ by the triangle inequality. Since the decktransformation group Γ operates with compact quotient on $\tilde{M}$, there are isometries $\gamma_i\in\Gamma$ such that the points $\gamma_i(x_i)$ are contained in a fixed compact fundamental domain. By considering subsequences we can assume

$$\gamma_i(x_i)\to x$$
$$\gamma_i(y_i)\to y$$
$$\gamma_i(P)\to P^*$$
$$\gamma_i(F)\to F^* \qquad \text{with} \quad x\in F^*\subset P^*$$
$$\gamma_i(P_i)\to P^0 \qquad \text{with} \quad y\in P^0$$
$$\gamma_i(D_i)\to D^*,$$

where D^* is a hyperplane in F^*

$$(\gamma_{i*}\sigma_s(x_i),\ldots,\gamma_{i*}\sigma_k(x_i))\to(\sigma_s^*,\ldots,\sigma_k^*)\in E_{s,k}(\tilde{M})$$

with $\exp(\sigma_k^*)=F^*$, $\pi(\sigma_k^*)=x$. Since $\eta\leqq f_i\leqq 1+\eta$ on D_i and $f_i(x_i)=\eta$, we see by convexity that every $x^*\in D^*$ has distance η from P^0. By 1.A $\mathrm{proj}_{P^0}(D^*)$ is a parallel D^0 of D^*.

Now we complete the proof of Theorem 2.5 under the assumption that D^0 is not contained in P^*—an assumption that we will prove later. Note that $y\in D^0$. Let $\tau^*\subset\sigma_k^*$ be the hyperplane tangent to D^* and set $\sigma_{s-1}^*:=(\sigma_s^*\cap\tau^*)$. Note that $P_{\sigma_{s-1}^*}$ contains P^* and D^0. Since D^0 is not contained in P^* we have $\dim P_{\sigma_{s-1}^*}>m_s=\dim P^*$. In particular we have $\sigma_{s-1}^*\neq\sigma_s^*$ and hence $\dim\sigma_{s-1}^*=s-1$. This implies $m_{s-1}>m_s$ and thus $E_{s-1,k}(M)$ is well structured.

It remains to show that D^0 is not contained in P^*. We argue by contradiction and assume $D^0\subset P^*$. Recall that $\tau_i(y_i)$ and $\sigma_s(x_i)$ denote the parallels of τ_i and σ_s with footpoints y_i and x_i. We have the following simple

SUBLEMMA. — *If* $D^0\subset P^*$ *then* $d^s(\tau_i(y_i),\sigma_s(x_i))\geqq\varepsilon_1$ *for almost all* $i\in\mathbb{N}$.

Proof. — Since x and $y\in D^0$ are in P^* the unit speed geodesic α from x to y is contained in P^*, hence $\dot{\alpha}(0)\in T_xP^*$. On the other hand $\dot{\alpha}(\eta)\perp T_yP^0$ since

$d(x, P^0) = d(x, y) = \eta$. This implies $d^{m_s}(T_x P^*, T_y P^0) \geqq \pi/2$. Since

$$T_x P^* = \lim_{i \to \infty} \gamma_{i*}(T_{x_i} P)$$

$$T_y P^0 = \lim_{i \to \infty} \gamma_{i*}(T_{y_i} P_i)$$

we conclude

$$d^{m_s}(T_{x_i} P, T_{y_i} P_i) \geqq \frac{\pi}{4}$$

for almost all i. Now our claim follows from (*) since

$$P = P_{\sigma_s(x_i)}, \quad (\sigma_s(x_i), \ldots, \sigma_k(x_i)) \in E_{s,k}(\tilde{M}), \qquad P_i = P_{\tau_i(y_i)}$$

and $\tau_i(y_i) \in F_s^{m_s}(\tilde{M})$. □

The idea for the rest of the proof is as follows: While τ_i and σ_s are very close $\tau_i(y_i)$ and $\sigma_s(x_i)$ are at distance $\geqq \varepsilon_1$. The discreteness expressed in Lemma 2.10 shows that τ_i and σ_s cannot be continuously deformed into $\tau_i(y_i)$ and $\sigma_s(x_i)$ without violating the hypothesis of (2.10). On the other hand we can easily find such a deformation. This contradiction will complete the proof. The details are as follows:

We consider the geodesics $\alpha_i : [0, 1] \to B_i \subset F$ connecting the base point $z = \pi(\sigma_k)$ to $x_i \in F$ and its projection $\beta_i = \text{proj}_{P_i} \circ \alpha_i : [0, 1] \to P_i$. Since $f_i \leqq \eta$ on B_i we have $d(\alpha_i(t), \beta_i(t)) \leqq \eta$ for $t \in [0, 1]$. We consider $\sigma_r^i(t) := \sigma_r(\alpha_i(t))$, $s \leqq r \leqq k$ and $\tau_i(t) = \tau_i(\beta_i(t))$. In particular we have $\sigma_s^i(0) = \sigma_s$, $\tau_i(0) = \tau_i$, $\sigma_s^i(1) = \sigma_s(x_i)$ and $\tau_i(1) = \tau_i(y_i)$. The sublemma implies $d^s(\tau_i(1), \sigma_s^i(1)) \geqq \varepsilon_1$ for almost all i. On the other hand $\lim d^s(\tau_i(0), \sigma_s^i(0)) = 0$. Since $\varepsilon = (1/2) \min \{\varepsilon_0, \varepsilon_1\}$ we can find $t \in [0, 1]$ and $i \in \mathbb{N}$ such that $\varepsilon < d^s(\tau_i(t), \sigma_s^i(t)) < \varepsilon_0$. Note that the footpoints $\beta_i(t)$ of $\tau_i(t)$ and $\alpha_i(t)$ of $\sigma_s^i(t)$ have distance $\leqq \eta$ and that $\alpha_i(t) \in B_i$ and $f_{i \mid B_i} \leqq \eta$. Since the radii R_i of B_i diverge to ∞ we may assume $R_i \geqq 1$. Hence Lemma 2.10 implies $d^s(\tau_i(t), \sigma_s^i(t)) < \varepsilon$. This contradicts the way we chose t and thus completes the proof of Theorem 2.5. □

3. Dynamics of well structured flats

In this section we assume that the set $E_{1,k}(M)$ is well structured [compare (2.4)]. Under this assumption we prove:

3.1. THEOREM. — *There exists a compact analytic submanifold* W *of* SM *which is invariant under the geodesic flow and an analytic operation* $\psi : \mathbb{R}^k \times W \to W$ *of* $\mathbb{R}^k$ *on* W *with the properties:*

(i) *For given* $w \in W$ *the map* $F_w : \mathbb{R}^k \to M$, $F_w(z) := \pi \circ \psi(z, w)$ *is a well structured k-flat and w is a regular vector in* F_w.

(ii) $\psi(z, w)$ *is the result of the parallel translation of w along the geodesic* $t \mapsto F_w(t . z)$, $t \in [0, 1]$. *In particular* $\psi^z = \psi(z, \;)$ *commutes with the geodesic flow*, $\psi^z \circ g^t = g^t \circ \psi^z$.

(iii) *Let* $\xi \in T_w W$ *and let* $Y_\xi(t)$ *be the corresponding Jacobifield along* $\exp(t.w)$. *Then* Y_ξ *is an affine field if and only if* Y_ξ *is parallel and tangent to the flat* F_w. *In this case* $\xi = \psi_{w*}\zeta$, *where* $\psi_w = \psi(\ ,w)$ *and* $\zeta \in T_0 \mathbb{R}^k$.

The proof of Theorem 3.1 needs some preparation. It turns out to be more convenient to describe the set $E_{1,k}(M)$ of well structured flats as a subset of the Stiefelbundle $St_k(M)$ in the following way:

$$V_0 := \{(x, v_1, \ldots, v_k) \in ST_k(M) \mid \langle v_1, \ldots, v_k \rangle \in F_k(M), \text{P-rank}(\langle v_1, \ldots, v_i \rangle) = m_i\}$$

A point $v \in V_0$ describes a flag

$$(\langle v_1 \rangle, \langle v_1, v_2 \rangle, \ldots, \langle v_1, \ldots, v_k \rangle) \in E_{1,k}(M)$$

On the other hand a given flag in $E_{1,k}(M)$ is represented by exactly 2^k points in V_0. Obviously V_0 is also compact and subanalytic. For a fixed point $z = (z_1, \ldots, z_k) \in \mathbb{R}^k$ we have a diffeomorphism

$$\varphi^z: \quad St_k(M) \to St_k(M)$$

$$\varphi^z(x, v_1, \ldots, v_k) = \left(\exp_x\left(\sum_{i=1}^{k} z_i v_i\right), \text{Par}_z v_1, \ldots, \text{Par}_z v_k\right)$$

where Par_z is the result of the parallel transport along the geodesic $t \mapsto \exp_x(t.(\sum z_i v_i))$, $t \in [0, 1]$.

A point $v = (x, v_1, \ldots, v_k) \in V_0$ is a frame tangent to the flat $F_v : \mathbb{R}^k \to M$, $F_v(z) = \exp_x(\sum z_i v_i)$. Note that V_0 is invariant under φ^z for all $z \in \mathbb{R}^k$ so that we have a map

$$\varphi: \quad \mathbb{R}^k \times V_0 \to V_0$$

$$\varphi(z, v) = \varphi^z(v)$$

For $v \in V_0$, $\text{Par}_z v_i$ is the parallel transport in the flat F_v, thus $\text{Par}_z v_i$ and $\text{Par}_y v_i$ commute for $y, z \in \mathbb{R}^k$. Therefore φ defines an operation of $\mathbb{R}^k$ on V_0.

We will show that the tangent vectors to V_0 (at the points where V_0 is a C^1-smooth submanifold) can be described by Jacobifields along flats, *cf.* 1.B. Let

$$\eta = (x, v_1, \ldots, v_k, A, B_1, \ldots, B_k) \in T_v V_0$$

and represent η by a differentiable path $v(t) = (x(t), v_1(t), \ldots, v_k(t))$ in V_0, *i.e.* $\dot{v}(0) = \eta$. Then $Y_\eta(z) = \partial/\partial t_{|t=0} F_{v(t)}(z)$ is a Jacobifield along the flat $z \mapsto F_v(z)$. Let $\xi := \varphi^z_* \eta \in T_{\varphi^z(v)} V_0$. Then

$$Y_\xi(y) = \frac{\partial}{\partial t_{|t=0}} F_{\varphi^z(v(t))}(y) = \frac{\partial}{\partial t_{|t=0}} F_{v(t)}(y+z) = Y_\eta(y+z)$$

Thus

$$\varphi^z_* \eta = (F_v(z), \text{Par}_z v_1, \ldots, \text{Par}_z v_k, Y_\eta(z), D_1 Y_\eta(z), \ldots, D_k Y_\eta(z))$$

3.2. LEMMA. — *Let* $V' \subset V_0$ *be a nonempty compact subanalytic* $\mathbb{R}^k$*-invariant subset. If* V' *is not an analytic submanifold of* $St_k(M)$ *then there exists a nonempty, compact,* $\mathbb{R}^k$*-invariant, subanalytic subset* $V^* \subset V'$ *with* $\dim V^* < \dim V'$.

Proof. — Let $A \subset V'$ be the subset of points $p \in V'$ such that V' is an analytic submanifold of maximal dimension in a neighborhood of p. Then A is an open subset of V' which is subanalytic by Tamm's Theorem, *cf.* 1.C, and $A \neq V'$ by assumption. Since V' is φ^z-invariant, $z \in \mathbb{R}^k$, and φ^z is an analytic diffeomorphism of $St_k(M)$, also A is φ^z-invariant. Thus $V^* = V' \setminus A$ is nonempty, compact and $\mathbb{R}^k$-invariant. V^* is subanalytic since both V' and A are subanalytic, *see* 1.C. By construction $\dim V^* < \dim V'$. □

Inductively we obtain that each compact $\mathbb{R}^k$-invariant subanalytic subset $V' \subset V$ contains a compact $\mathbb{R}^k$-invariant analytic submanifold.

Let now $V \subset V_0$ be a nonempty connected compact $\mathbb{R}^k$-invariant analytic submanifold of minimal dimension.

For a given point $w \in S^{k-1}$ (the standard sphere in $\mathbb{R}^k$) we define a map

$$w: \quad V \to SM$$
$$w(x, v_1, \ldots, v_k) := \sum_{i=1}^{k} w_i v_i \in T_x M$$

We will prove that for a properly choosen w the set $W := w(V)$ satisfies the properties of Theorem 3.1.

We first study the differential of the map w. For $\eta \in T_v V$ let $\xi := w_* \eta \in T_{w(v)} SM$. Then Y_ξ is the Jacobifield along the geodesic $t \mapsto \exp(t.w(v))$ with $Y_\xi(t) = Y_\eta(t.w)$.

We define a distribution PJF_w ("Parallel Jacobifields in direction w") on V by

$$PJF_w(v) := \{ \eta \in T_v V \mid t \mapsto Y_\eta(t.w) \text{ is parallel} \}$$

3.3. LEMMA. — *For given* $w \in S^{k-1}$ *the dimension of* PJF_w *is constant on* V. *The distribution* PJF_w *is analytic and integrable.*

Proof. — Let $r_w := \max_{v \in V} \dim PJF_w(v)$ and consider the bundle $G_{r_w}(V)$ of r_w-dimensional tangent planes to V. Let $\tau : G_{r_w}(V) \to V$ be the bundle projection. For a tangent vector $\eta \in TV$ let

$$g_w(\eta) := (\| Y_\eta(w) \|^2 - \| Y_\eta(0) \|^2)^2 + (\| Y_\eta(0) \|^2 - \| Y_\eta(-w) \|^2)^2$$

Then $g_w(\eta) \geqq 0$ and the convexity of $\| Y_\eta \|$ and the analyticity of Y_η imply that $g_w(\eta) = 0$ if and only if $t \mapsto Y_\eta(t.w)$ is parallel, *i.e.* if and only if $\eta \in PJF_w$.

Define $\bar{g}_w : G_{r_w}(V) \to \mathbb{R}$ by

$$\bar{g}_w(E) := \int_{SE} g_w(\eta)\, d\eta$$

where $E \in G_{r_w}(V)$ and SE is the unit sphere in E. Let $E_v \in G_{r_w}(V)$, $E_v \subset T_v V$. Then $\bar{g}_w(E_v) \geqq 0$ and $\bar{g}_w(E_v) = 0$ if and only if $E_v = PJF_w(v)$. Thus

$$B_w := \{ v \in V \mid \dim PJF_w(v) = r_w \} = \tau(\{ \bar{g}_w = 0 \})$$

Note that $\tau : \{ \bar{g}_w = 0 \} \to B_w$ is injective. Since $\{ \bar{g}_w = 0 \}$ is a compact analytic subset of $G_{r_w}(V)$, B_w is a subanalytic subset of V. We claim that B_w is $\mathbb{R}^k$-invariant, *i.e.* for $z \in \mathbb{R}^k$ we have $\varphi^z B_w = B_w$. In order to prove this we show that $\{ \bar{g}_w = 0 \}$ is invariant under the differential φ^z_*. Therefore let $v \in B_w$, $\eta \in T_v V$ with $\eta \in PJF_w(v)$. Thus $t \mapsto Y_\eta(t.w)$ is parallel. Then $\varphi^z_* \eta = \xi$ with $Y_\xi(y) = Y_\eta(y+z)$. Since $t \mapsto \| Y_\eta(t.w) \|$ is bounded and $\| Y_\eta \|$ is convex, also $t \mapsto \| Y_\eta(t.w+z) \|$ is bounded for fixed z and hence $\xi \in PJF_w(\varphi^z v)$.

This proves that $\{ \bar{g}_w = 0 \}$ is φ^z_*-invariant. Thus B_w is a compact subanalytic $\mathbb{R}^k$-invariant subset of V and hence $B_w = V$ by the choice of V.

We now prove that $\{ \bar{g}_w = 0 \}$ is an analytic submanifold of $G_{r_w}(V)$. If $\{ \bar{g}_w = 0 \}$ is not an analytic submanifold, then exactly as in Lemma 3.2 we could obtain a proper compact $\mathbb{R}^k$-invariant subanalytic subset $A \overset{\subset}{\neq} \{ \bar{g}_w = 0 \}$. Then $\pi(A)$ is a proper compact subanalytic subset of B_w which is $\mathbb{R}^k$-invariant. This is impossible by the choice of V.

Now let $C \subset \{ \bar{g}_w = 0 \}$ be the set of all points $E_v = E$ such that $\tau_* : T_E \{ \bar{g}_w = 0 \} \to T_v V$ has minimal rank. It is not difficult to show that C is compact, subanalytic and $\mathbb{R}^k$-invariant. As above this implies that $C = \{ \bar{g}_w = 0 \}$ and rank (τ_*) is constant. Since $\tau : \{ \bar{g}_w = 0 \} \to V$ is bijective, this implies that τ is a diffeomorphism and thus the distribution $v \mapsto PJF_w(v)$ is analytic.

To prove the integrability, let $v : (-\varepsilon, \varepsilon) \to V$ be a smooth curve tangent to PJF_w. Consider the curve $w(v(t))$ in SM and the geodesic variation $\alpha_t(s) = \exp(s.w(v(t)))$. Then

$$\frac{\partial}{\partial t} \alpha_t(s) = Y_{\dot{v}(t)}(s.w(v(t)))$$

Since $v(t)$ is tangent to PJF_w, $s \mapsto (\partial/\partial t)\alpha_t(s)$ is a parallel field. By integration we see as in [BBE], Lemma 2.2, that α_0 and α_t bound a flat strip, *i.e.* $w(v(t))$ consists of parallel vectors. As in [BBE] we conclude that the distribution is integrable. □

3.4. LEMMA. — *Assume there is a point* $v \in V$ *such that* $w(v)$ *is a regular vector of the flat* F_v. *Then* $\dim PJF_w \equiv k$ *and the integral manifold* Z *with* $v \in Z$ *is equal to* $\{ \varphi^z v \mid z \in \mathbb{R}^k \}$. *In particular if* $\eta \in PJF_w(v)$ *then* Y_η *is parallel and tangent to* F_v.

Proof. — Since $\mathbb{R}^k$ operates on V by parallel translation in the flat F_v we see $\{ \varphi^z v \mid z \in \mathbb{R}^k \} \subset Z$ and $\dim PJF_w \geqq k$. Since $w(\varphi^z v)$ is obtained from $w(v)$ by parallel translation in F_v also $w(\varphi^z v)$ is a regular vector of F_v. Let $u(t)$ be a smooth curve in Z with $u(0) = \varphi^z v$. By the last part of the proof of Lemma 3.3 $w(u(t))$ is a path of parallel vectors. Since $w(u(0))$ is a regular vector tangent to F_v also $w(u(t))$ is a regular tangent vector to F_v for all t. Since $w(u(t))$ is regular, it is only contained in the k-flat F_v and hence $u(t) = (x(t), u_1(t), \ldots, u_k(t))$ is a path of frames tangent to F_v. Now

Lemma 2.8 (iii) implies that $\langle u_1(t), \ldots, u_i(t) \rangle$ is parallel to $\langle u_1(0), \ldots, u_i(0) \rangle$ for $i=1, \ldots, k$ and hence $u_i(t) \| u_i(0)$.

Thus $u(t)=\varphi^{z(t)} v$ for a path $z(t) \in \mathbb{R}^k$ with $z(0)=z$. This implies $Z \subset \{ \varphi^z v \mid z \in \mathbb{R}^k \}$. □

We now look for a vector $w \in S^{k-1}$ such that the regularity condition for $w(v)$ is satisfied for all $v \in V$. Let therefore ε_0 be the constant which exists for the well structured flags $E_{1,k}(M)$ according to Lemma 2.8 (iii), *i.e.* if $(\sigma_1, \ldots, \sigma_k) \in E_{1,k}(M)$, if $\tau \subset \sigma_k$, $\dim \tau = r$, and if $d^r(\tau, \sigma_r) \leqq \varepsilon_0$ then $P_\tau \subset P_{\sigma_r}$ and equality implies $\tau = \sigma_r$. Now we choose $w \in S^{k-1}$ with the properties:

(*a*) w is not contained in $\langle e_1, \ldots, e_{k-1} \rangle$

(*b*) for $1 \leqq i \leqq k-1$ we have

$$d^i(\langle e_1, \ldots, e_{i-1}, w \rangle, \langle e_1, \ldots, e_i \rangle) < \min\left(\frac{\varepsilon_0}{2}, \frac{\pi}{2}\right)$$

It is elementary to construct w with these properties.

From now on we will work with this fixed vector w. By abuse of notation the symbol w will denote (i) the vector $w \in S^{k-1} \subset \mathbb{R}^k$, (ii) the corresponding map $w: V \to W$, (iii) arbitrary elements $w = w(v) \in W$.

3.5. Lemma:

(i) *For all* $v \in V$ *the vector* $w(v)$ *is regular in* F_v.

(ii) *If* $v, v' \in V$ *and* $w(v) = w(v')$, *then* $v = v'$.

(iii) *If* $W := w(V) \subset SM$, *then* $w: V \to W$ *is an analytic diffeomorphism.*

Proof. – (i) Let $v=(x, v_1, \ldots, v_k) \in V$ and let $L := \operatorname{kern}(P_{w(v)}) \subset \langle v_1, \ldots, v_k \rangle$. We show that $L = \langle v_1, \ldots, v_k \rangle$ which implies that $w(v)$ is regular. Since $d^1(\langle w \rangle, \langle v_1 \rangle) < \varepsilon_0/2 < \varepsilon_0$ by (*b*) we have by Lemma 2.8 (iii) that $P_{w(v)} \subset P_{v_1}$ and hence $\langle v_1 \rangle \subset L$.

Assume inductively that

$$\langle v_1, \ldots, v_{i-1} \rangle \subset L$$

Clearly $w(v) \in L$, hence $\langle v_1, \ldots, v_{i-1}, w(v) \rangle \subset L$. Since

$$d^i(\langle v_1, \ldots, v_{i-1}, w(v) \rangle, \langle v_1, \ldots, v_i \rangle) < \varepsilon_0$$

we have [by (2.8) (iii)]

$$P_{\langle v_1, \ldots, v_{i-1}, w(v) \rangle} \subset P_{\langle v_1, \ldots, v_i \rangle}$$

and thus $\langle v_1, \ldots, v_i \rangle \subset L$. Hence $\langle v_1, \ldots, v_k \rangle \subset L$ and this proves (i).

(ii) Let $v=(x, v_1, \ldots, v_k)$, $v'=(x', v'_1, \ldots, v'_k)$ with $w(v)=w(v')$. Then $x=x'$ and

$$d^1(\langle v_1 \rangle, \langle v'_1 \rangle) \leqq d^1(\langle v_1 \rangle, \langle w(v) \rangle) + d^1(\langle w(v') \rangle, \langle v'_1 \rangle) < \varepsilon_0$$

Then Lemma 2.8 (iii) implies $P_{v_1} \subset P_{v'_1}$ and $P_{v'_1} \subset P_{v_1}$ and hence $\langle v_1 \rangle = \langle v'_1 \rangle$ by 2.8 (i). By (*b*) we obtain $v_1 = v'_1$. Assume inductively $v_j = v'_j$ for $1 \leqq j \leqq i-1$. Then

$$\begin{aligned} d^i(\langle v_1, \ldots, v_i \rangle, \langle v'_1, \ldots, v'_i \rangle) & \\ \leqq d^i(\langle v_1, \ldots, v_i \rangle, \langle v_1, \ldots, v_{i-1}, w(v) \rangle) & \\ + d^i(\langle v'_1, \ldots, v'_{i-1}, w(v) \rangle, \langle v'_1, \ldots, v'_i, w(v) \rangle) < \varepsilon_0 & \end{aligned}$$

and hence $P_{\langle v_1, \ldots, v_i \rangle} = P_{\langle v'_1, \ldots, v'_i \rangle}$. Thus $v_i = v'_i$ by 2.8 (i) and we obtain $v = v'$.

(iii) The map $w: V \to SM$ is analytic and injective by (ii). Thus it suffices to prove that w has maximal rank everywhere. Let $\eta \in T_v V$ and $\xi = w_* \eta \in T_{w(v)} SM$. We saw already $Y_\xi(t) = Y_\eta(t.w)$. Assume $\xi = 0$. Then $t \mapsto Y_\eta(t.w)$ is the zero field and in particular parallel. Thus $\eta \in PJF_w(v)$. Since $w(v)$ is regular by (i) we have by Lemma 3.4 that $\dim PJF_w(v) = k$ and that Y_η is a parallel field tangent to F_v. Since Y_η vanishes on the line $t \mapsto t.w$ it vanishes everywhere. Therefore $\xi = 0$ implies $\eta = 0$. Thus w has maximal rank. □

In order to prove Theorem 3.1 we define $W := w(V) \subset SM$ and the operation

$$\psi: \quad \mathbb{R}^k \times W \to W$$
$$\psi(z, w(v)) = w(\varphi^z v)$$

We show that W satisfies properties (i) and (ii) of Theorem 3.1:

(i) Let $w = w(v)$ with $v = (x, v_1, \ldots, v_k)$. Then

$$F_w(z) := \pi \circ \psi(z, w) = \exp_x\left(\sum_{i=1}^{k} z_i v_i\right) = F_v(z)$$

Thus F_w is a will structured flat and by Lemma 3.5 (i) w is a regular vector in this flat.

(ii) Let $v = (x, v_1, \ldots, v_k)$. Then

$$\psi(z, w(v)) = w(F_{w(v)}(z), \mathrm{Par}_z v_1, \ldots, \mathrm{Par}_z v_k)$$

and hence $\psi(z, w(v))$ is the result of the parallel translation of $w(v)$ along $t \mapsto F_{w(v)}(t.z)$, $t \in [0, 1]$.

In order to prove (iii) let $\xi \in T_{w(v)} W$ be such that Y_ξ is affine. Then $\xi = w_* \eta$ for some $\eta \in T_v V$ and $Y_\xi(t) = Y_\eta(t.w)$. Thus it remains to prove:

3.6. Lemma. — *Let* $\eta \in T_v V$. *Assume that* $t \mapsto Y_\eta(t.w)$ *is an affine field, then* Y_η *is parallel and tangent to* F_v.

Proof. — We split

$$Y_\eta(z) = Y_\eta^T(z) + Y_\eta^N(z)$$

where $Y_\eta^T (Y_\eta^N)$ is the tangent (normal) component to the flat F_v. By [BBE] (2.4) Y_η^T and Y_η^N are Jacobifields along F_v. Since F_v is flat and totally geodesic Y_η^T is an affine Jacobifield along any line in $\mathbb{R}^k$. If $t \mapsto Y_\eta(t.w)$ is affine, then also $Y_\eta^N(t.w)$ is affine.

Since $Y_\eta(t.w)$ is affine there is a constant $A>0$ such that

$$\|Y_\eta(t.w)\| \leqq A.|t|+\|Y_\eta(0)\|$$

for all t.

SUBLEMMA 1. — *For* $s\in\mathbb{R}$ *and* $z\in\mathbb{R}^k$ *we have*

$$\|Y_\eta(sw+z)\| \leqq A.|s|+\|Y_\eta(z)\|$$

Proof. — Let $c_t:[0,r_t]\to\mathbb{R}^k$ be the unit speed line from z to $t.w$ where $r_t=\|t.w-z\|$. Then $|t-r_t|\leqq\|z\|$. First assume that our claim does not hold for some $s\geqq 0$. Using $\lim_{t\to\infty} c_t(s)=z+sw$ we conclude that there exists $\rho>0$ such that for all t sufficiently large

$$\|Y_\eta(c_t(s))\| \geqq (A+\rho).s+\|Y_\eta(z)\|$$

The convexity of $\|Y_\eta\|$ now implies

$$\|Y_\eta(t.w)\| \geqq (A+\rho).\tau_t+\|Y_\eta(z)\|$$

But this is impossible for large t since $\|Y_\eta(t.w)\|\leqq A.t+\|Y_n(0)\|$ and $|r_t-t|\leqq\|z\|$. The proof for $s\leqq 0$ is similar. □

First we treat the case that $\|Y_\eta^N\|$ is unbounded. We shall show that this case cannot occur since it leads to a contradiction to Lemma 3.4. Since $\|Y_\eta^N\|$ is unbounded and convex there exists a line $t.z_0$ in $\mathbb{R}^k$ such that $\|Y_\eta(tz_0)\|$ grows at least linearly for $t\to\infty$. We choose a sequence $t_i\to\infty$ such that

$$\zeta_i=\|\varphi_*^{t_i z_0}(\eta)\|^{-1}.\varphi_*^{t_i z_0}(\eta)$$

converges to $\zeta\in TV$. Note that

$$Y_{\zeta_i}(z)=\|\varphi_*^{t_i z_0}(\eta)\|^{-1}.Y_\eta(z+t_i z_0)$$

and that $\|\varphi_*^{t_i z_0}(\eta)\|\to\infty$. Since Y_η^T is affine and $\|Y_\eta^N(tz_0)\|$ grows at least linearly we conclude $Y_\zeta^N\neq 0$. On the other hand Sublemma 1 and $\|\varphi_*^{t_i z_0}(\eta)\|\to\infty$ imply that

$$\|Y_\zeta(z+sw)\|=\lim_{i\to\infty}\|Y_{\zeta_i}(z+sw)\|$$

is bounded in $s\in\mathbb{R}$, *i.e.* $\zeta\in PJF_w$. Hence $Y_\zeta^N\neq 0$ and this contradicts Lemma 3.4.

It remains to treat the case that $\|Y_\eta^N\|$ is bounded, *i.e.* Y_η^N is parallel along any line in $\mathbb{R}^k$. For every $v\in V$ we consider the subspace NPJF(v) ("Normally parallel Jacobifields") of $T_v V$ defined by

$$\text{NPJF}(v)=\{\eta\in T_v V \mid Y_\eta^N \text{ is parallel}\}$$

SUBLEMMA 2. — NPJF *is an analytic distribution in* TV.

The proof of Sublemma 2 is similar to the first part of the proof of Lemma 3.3.

Finally we show that Y_η is parallel and $Y_\eta^N=0$ if $\eta \in \text{NPJF}$. This will conclude the proof of Lemma 3.6. Given $\eta \in \text{NPJF}(v)$ choose a curve $v:(-\varepsilon, \varepsilon) \to V$ with $\dot{v}(0)=\eta$ which is everywhere tangent to NPJF. Consider the variation $F(z, t)=F_{v(t)}(z)$ of the flat $F_0=F_{v(0)}$ and the Jacobifields

$$Y_t(z)=Y_{\dot{v}(t)}(z)=\frac{\partial F}{\partial t}(z, t)$$

along F_t. We shall prove that the flats $F_t(\mathbb{R}^k)$ are parallel along $\pi \circ v(t)$. To this end we reparametrize F_t as $\tilde{F}_t=F_t \circ I_t$ where $I_t \in \text{Iso}(\mathbb{R}^k)$ is so chosen that the tangential part of $\tilde{Y}=\partial \tilde{F}/\partial t$ vanishes. According to 1.B we can achieve this by taking I_t to be a solution of

$$\left(\frac{d}{dt} I\right) . I^{-1}=-R$$

where R_t is the infinitesimal isometry of $\mathbb{R}^k$ satisfying

$$F_{t*z}(R_t z)=Y_t^T(z)$$

Then

$$\left\| \frac{\partial \tilde{F}}{\partial t}(z) \right\|=\| Y_t^N(z) \|$$

is independent of z since $Y_t=Y_{\dot{v}(t)}$ and $\dot{v}(t) \in \text{NPJF}$. This implies that all the flats $F_t(\mathbb{R}^k)$ are parallel along $\pi \circ v(t)$. Hence $F_t(\mathbb{R}^k)=F_0(\mathbb{R}^k)$ and, in particular, $Y_\eta^N=Y_0^N=0$. That Y_η is parallel follows as in the proof of Lemma 3.4. □

4. Closing of flats

In this section we continue to assume that the set $E_{1, k}$ is well structured so that we have a compact submanifold W of SM with the properties stated in Theorem 3.1. In particular W is foliated by the orbits $\psi_w(\mathbb{R}^k)$, $w \in W$, of the $\mathbb{R}^k$-action ψ. Our aim is to find $w \in W$ such that $\psi_w: \mathbb{R}^k \to SM$ is not injective. Then a k-flat $F \subset \tilde{M}$ which is a lift of $F_w(\mathbb{R}^k)=\pi \circ \psi_w(\mathbb{R}^k)$ is mapped to itself by some deck transformation γ which restricts to a nontrivial parallel translation of F. In a first step we prove the existence of a compact subset $G \neq \varnothing$ of W which is ψ-invariant and on which the geodesic flow is normally hyperbolic, *cf.* [HPS]. This means that the normal bundle of the foliation restricted to G splits into two continuous g_*^t-invariant subbundles on which g_*^t expands resp. contracts more sharply than in the directions tangent to the foliation. Then the existence of some $w \in W$ such that ψ_w is not injective can be proved by using the Shadowing Lemma (7A.2) in [HPS].

According to 1.B every Jacobifield $Y \in J^*(v)$, $v \in SM$, can be decomposed into its affine, strong stable and strong unstable components

$$Y = Y^a + Y^{ss} + Y^{su}$$

Correspondingly we shall decompose

$$T_v SM = E_v^a \oplus E_v^{ss} \oplus E_v^{su}$$

and every $\xi \in T_v SM$ into

$$\xi = \xi^a + \xi^{ss} + \xi^{su}$$

Here $\xi \in E_v^a$ if and only if $\xi \in T_v SM$ and $Y_\xi \in J^{*a}(v)$ and so on. Since Jacobifields represent the differential of the geodesic flow g^t this decomposition is invariant under g_*^t, *i.e.* $E_{g^t(v)}^a = g_{*v}^t(E_v^a)$, $E_{g^t(v)}^{ss} = g_{*v}^t(E_v^{ss})$, $E_{g^t(v)}^{su} = g_{*v}^t(E_v^{su})$. Finally we have the space $E_v^p \subset E_v^a$ which consists of the initial conditions of parallel Jacobifields. In this notation property (3.1) (iii) of W says

$$T_w W \cap E_w^a = T_w W \cap E_w^p = \psi_{w*0}(\mathbb{R}^k)$$

Note that E^a, E^{ss} and E^{su} may not be continuous vector bundles over SM. However the following weak continuity properties are trivially true:

Suppose $v_i \in SM$, $\lim v_i = v$. If $\xi_i \in E_{v_i}^p$, resp. $\xi_i \in E_{v_i}^a$ and $\xi = \lim \xi_i$ then $\xi \in E_v^p$, resp. $\xi_i \in E_v^a$. If $\xi_i \in E_{v_i}^{ss}$, resp. $\xi_i \in E_{v_i}^{su}$ and $\xi = \lim \xi_i$ then $\xi \in E_v^p \oplus E_v^{ss}$, resp. $\xi \in E_v^p \oplus E_v^{su}$. Obviously analogous statements hold for the convergence of Jacobifields. The set $G \subset W$ mentioned above will be the set of all $v \in W$ such that we have a splitting

$$T_v W = (E_v^p \cap T_v W) \oplus (E_v^{ss} \cap T_v W) \oplus (E_v^{su} \cap T_v W)$$

To prove that $G \neq \varnothing$ and that G has the properties stated above we need two lemmas.

The first lemma shows that due to property (3.1) (iii) of W all Jacobifields Y_ξ with $\xi \in TW \setminus (E^a \oplus E^{ss})$ grow exponentially.

4.1. Lemma. — *Suppose $w \in W$ and $L \subset T_w W$ is a vectorspace complement to $(E_w^a \oplus E_w^{ss}) \cap T_w W$. Then there exists $t_0 \in \mathbb{R}$ and $\varepsilon > 0$ such that*

$$\| Y_\xi(t+1) \| \geqq (1+\varepsilon) \| Y_\xi(t) \|$$

whenever $\xi \in L$ and $t \geqq t_0$.

Proof. — Otherwise there exist $\xi_i \in L$, $t_i \to \infty$ and $\varepsilon_i \to 0$ such that

$$\| Y_{\xi_i}(t_i+1) \| < (1+\varepsilon_i) \| Y_{\xi_i}(t_i) \|$$

We define

$$Y_i(t) := \| Y_{\xi_i}(t_i) \|^{-1} Y_{\xi_i}(t_i + t)$$

Then $Y_i \in J^*(g^{t_i} w)$, $\|Y_i(0)\| = 1$ and $\|Y_i(1)\| < 1 + \varepsilon_i$. Hence there exists a subsequence of the Y_i converging to a Jacobifield $Y \in J^*(v)$ for some limit vector $v \in W$ of the $g^{t_i} w$. Since Y_ξ is not in $J^s(w)$ for all $\xi \in L \setminus \{0\}$ a compactness argument shows that there exists $t_p \in \mathbb{R}$ such that $\|Y_\xi(t)\|$ is increasing for $t \geqq t_0$ and for all $\xi \in L \setminus \{0\}$. Hence $\|Y(t)\|$ is non-decreasing and $\|Y(0)\| = 1 = \|Y(1)\|$, *i.e.* $Y \in J^p(v)$ by analyticity. According to (3.1) (iii) the component Y^N of Y normal to the flat F_v vanishes. On the other hand we shall now prove that there exists $s_0 \in \mathbb{R}$ and $\delta > 0$ such that $\|Y_\xi^N(t)\| \geqq \delta \|Y_\xi(t)\|$ whenever $\xi \in L$, $t \geqq s_0$. By continuity this implies $\|Y^N(0)\| \geqq \delta > 0$ and this contradiction will complete our proof.

To proof $\|Y_\xi^N(t)\| \geqq \delta \|Y_\xi(t)\|$ note that $\|Y_\xi^{su}\|'(0) > 0$ for all $\xi \in L \setminus \{0\}$. Hence there exists $\alpha > 0$ such that for all $\xi \in L \setminus \{0\}$:

$$\|Y_\xi^{su}\|'(0) \geqq \alpha \|\xi\|$$

Since $\|Y_\xi^{su}(t)\|$ is convex this implies $\|Y_\xi^{su}(t)\| \geqq \alpha \|\xi\| t$.

Similarly we obtain $A > 0$ such that for all $\xi \in L$, $t \geqq 0$: $\|Y_\xi^{ss}(t)\| \leqq A \|\xi\|$ and $\|Y_\xi^a(t)\| \leqq A \|\xi\| (t+1)$. Since the affine part of Y_ξ^N is orthogonal to $Y_\xi^{su} + Y_\xi^{ss}$ we can estimate for $t \geqq 0$:

$$\|Y_\xi^N(t)\| \geqq \|Y_\xi^{su}(t) + Y_\xi^{ss}(t)\| \geqq \|\xi\| (\alpha t - A)$$

Since the tangential part Y_ξ^T grows at most linearly the proceeding inequality imply the existence of $\delta > 0$, $s_0 \in \mathbb{R}$ such that $\|Y_\xi^N(t)\| \geqq \delta \|Y_\xi(t)\|$ whenever $\xi \in L$ and $t \geqq s_0$. □

We denote by $L_\alpha(W)$, resp. $L_\omega(W)$ the α- resp. ω-limit set of the geodesic flow on the compact g^t-invariant set W.

4.2. LEMMA. — *Suppose* $v \in L_\omega(W)$, *say* $v = \lim g^{t_i} w$ *where* $w \in W$ *and* $\lim t_i = \infty$. *If* $\xi \in T_v W$ *then the strong unstable part* ξ^{su} *of* ξ *lies in* $T_v W$, i.e.

$$T_v W = ((E_v^a \oplus E_v^{ss}) \cap T_v W) \oplus (E_v^{su} \cap T_v W)$$

Proof. — Given $\xi \in T_v W$ choose $\xi_i \in T_w W$ such that $\xi = \lim g_*^{t_i} \xi_i$. Choose $L \subset T_w W$ as in (4.1) and decompose $\xi_i = \xi_i^1 + \xi_i^2$ where $\xi_i^1 \in L$ and $Y_i^2 := Y_{\xi_i^2} \in J^{*a}(w) \oplus J^{ss}(w)$. We set $Y_i^1 := Y_{\xi_i^1}$. Then

$$Y_\xi(t) = \lim_{i \to \infty} (Y_i^1(t+t_i) + Y_i^2(t+t_i))$$

First we want to show that a subsequence of $Y_i^1(t+t_i)$ converges: otherwise we may assume that there exists a sequence $\lambda_i \to 0$ such that $\lambda_i Y_i^1(t+t_i)$ converges to a non-zero $Z \in J^*(v)$. Now (4.1) implies

$$\|Z(t+1)\| \geqq (1+\varepsilon) \|Z(t)\|$$

for all $t \in \mathbb{R}$ since $Y_i^1 = Y_{\xi_i^1}$, $\xi_i^1 \in L$ and $t_i \to \infty$. On the other hand $\lim \lambda_i = 0$ implies

$$Z(t) = -\lim_{i \to \infty} \lambda_i Y_i^2(t+t_i)$$

where $Y_i^2 \in J^{*a}(w) \oplus J^{ss}(w)$. Hence $Z \in J^{*a}(v) \oplus J^{ss}(v)$ and this contradicts the exponential growth of Z. So we may assume that $Y_i^1(t+t_i)$ and $Y_i^2(t+t_i)$ converge individually, say $\lim Y_i^1(t+t_i) = Z^1(t)$, $\lim Y_i^2(t+t_i) = Z^2(t)$ and $Y_\xi(t) = Z^1(t) + Z^2(t)$. As above we have

$$\| Z^1(t+1) \| \geqq (1+\varepsilon) \| Z^1(t) \|$$

for all $t \in \mathbb{R}$, hence $Z^1 \in J^{su}(v)$. Moreover $Z^2 \in J^{*a}(v) \oplus J^{ss}(v)$ so that indeed $Z_1 = (Y_\xi)^{su}$. Since $(Y_\xi)^{su} = Y_{\xi^{su}}$ by definition we obtain $\xi^{su} = \lim g_*^{t_i} \xi_i^1 \in T_v W$. □

As mentioned before we let G denote the set of all $v \in V$ such that we have a splitting

$$T_v W = (E^p \cap T_v W) \oplus (E_v^{ss} \cap T_v W) \oplus (E_v^{su} \cap T_v W)$$

The theorem below proves the properties of G mentioned in the introduction to this section.

4.3. THEOREM:

(i) G *is* ψ*-invariant, compact and* $G \neq \emptyset$, *more specifically* $(\overline{L_\alpha(W)} \cap \overline{L_\omega(W)}) \subset G$. *Over* G *the bundles* $E^p \cap TW$, $E^{ss} \cap TW$, $E^{su} \cap TW$ *are continuous,* g_*^t*-invariant vector bundles.* $E^p \cap TW$ *is the distribution tangent to the leaves* $\psi(\mathbb{R}^k \times \{w\})$, $w \in W$, *and, consequently,* $(E^{ss} \cap TW) \oplus (E^{su} \cap TW)$ *restricted to* G *coincides with the normal distribution of the foliation* $\psi(\mathbb{R}^k \times \{w\})$, $w \in W$.

(ii) *There exists* $t_0 > 0$ *and* $\theta \in (0, 1)$ *such that*

$$\| g_*^t(\xi) \| \leqq \theta \| \xi \| \qquad \textit{if} \quad \xi \in E^{ss} \cap TW \textit{ and } t \geqq t_0,$$

and

$$\| g_*^t(\xi) \| \geqq \theta^{-1} \| \xi \| \qquad \textit{if} \quad \xi \in E^{su} \cap TW \textit{ and } t \geqq t_0,$$

while

$$\| g_*^t(\xi) \| = \| \xi \| \qquad \textit{if} \quad \xi \in E^p \cap TW \textit{ and } t \in \mathbb{R}$$

Note. — 1. We do not exclude the possibility that G has several components and that the fibre dimensions of $E^{ss} \cap TW$ and, consequently, of $E^{su} \cap TW$ are different on different components of G.

2. The statements in (ii) do not only hold over G.

Proof of (ii). — We first show that there exists $\varepsilon > 0$ such that

$$\| Y_\xi(1) \| \geqq (1+\varepsilon) \| Y_\xi(0) \|$$

for all $\xi \in E^{su} \cap TW$. This is similar to the proof of (4.1): otherwise there exists a sequence $\xi_i \in E^{su} \cap TW$ and $\varepsilon_i \to 0$ such that $\| Y_{\xi_i}(0) \| = 1$ and $\| Y_{\xi_i}(1) \| \leqq 1 + \varepsilon_i$. We may assume that the Y_{ξ_i} converge to a Jacobifield Y_ξ with $\xi \in T_w W$. Then

$$\| Y_\xi(0) \| = \| Y_\xi(1) \| = 1$$

and $\|Y_\xi\|$ is monotonic, hence $Y_\xi \in J^p(w)$ by analyticity. By (3.1) (iii) we have $0 \neq \xi \in \psi_{w*0}(\mathbb{R}^k)$. On the other hand $\xi_i \in E^{su}$ implies that $\xi = \lim \xi_i$ is orthogonal to $E^p_w \cap T_w W = \psi_{w*0}(\mathbb{R}^k)$. This contradiction proves the existence of $\varepsilon > 0$ such that $\|Y_\xi(1)\| \geqq (1+\varepsilon)\|Y_\xi(0)\|$ for all $\xi \in E^{su} \cap TW$. To complete the proof of (ii) note that if $-a^2$ is a lower bound for the curvature of M then $\|Y'(t)\| \leqq a \|Y(t)\|$ for all $t \in \mathbb{R}$ and all stable and all unstable Jacobifields, *cf. e.g.* [BBE], sect. 1. Hence on $E^{ss} \oplus E^{su}$ the norms $\|\xi\|$ and $\|\pi_*(\xi)\|$ are equivalent,

$$\|\xi\| \geqq \|\pi_*(\xi)\| \geqq (1+a^2)^{-1/2} \|\xi\|$$

Together with our first estimate this implies (ii) for $E^{su} \cap TW$. One can take $\theta^{-1} = 1+\varepsilon$ and $t_0 = 2 + \ln(1+a^2) \ln((1+\varepsilon))^{-1}$. The proof for $E^{ss} \cap TW$ is analogous. The last claim in (ii) is obvious.

Proof of (i). — Lemma (4.2) states that

$$((E^a_w \oplus E^{ss}_w) \cap T_w W) \oplus (E^{su}_w \cap T_w W) = T_w W$$

provided $w \in L_\omega(W)$. We shall use (ii) to show that this decomposition holds also if $w \in \overline{L_\omega(W)}$. If $w_i \in L_\omega(W)$ converge to w we may assume that $(E^a_{w_i} \oplus E^{ss}_{w_i}) \cap T_w W$ and $(E^{su}_{w_i} \cap T_{w_i} W)$ converge to subspaces L_1 and L_2 of $T_w W$. Then $L_1 \subset E^a_w \oplus E^{ss}_w$ and the uniform estimate (ii) implies that $L_2 \subset E^{su}_w$. In particular $L_1 \cap L_2 = \{0\}$ and hence $L_1 \oplus L_2 = T_w W$. This implies the above decomposition also for $w \in \overline{L_\omega(W)}$. In complete analogy we obtain for all $w \in \overline{L_\alpha(W)}$

$$((E^a_w \oplus E^{ss}_w) \cap T_w W) \oplus (E^{su}_w \cap T_w W) = T_w W$$

Recalling that $E^a \cap TW = E^p \cap TW$ we see that the preceeding statements imply $(\overline{L_\alpha(W)} \cap \overline{L_\omega(W)} \subset G$. Since $L_\alpha \cap L_\omega$ contains every minimal set of the geodesic flow restricted to the compact manifold W we see that $G \neq \varnothing$. Moreover the preceding arguments also show that G is closed and that $E^{ss} \cap TW$ and $E^{su} \cap TW$ are continuous vector bundles over G.

Finally we prove the ψ-invariance of G, the remaining statements in (i) being trivial. Suppose $v \in G$ and $\bar{v} = \psi(z, v) = \psi^z(v)$ for some $z \in \mathbb{R}^k$. We want to prove that $\bar{v} \in G$. It suffices to show that $\psi^z_*(E^p \cap TW) \subset E^p$, $\psi^z_*(E^{ss} \cap TW) \subset E^{ss}$ and $\psi^z_*(E^{su} \cap TW) \subset E^{su}$. If $\xi \in E^p$ then $\|g^t_* \xi\| = \|Y_\xi(t)\|$ is constant. Since ψ^z_* is uniformly bounded also $\|\psi^z_*(g^t_* \xi)\| = \|g^t_*(\psi^z_* \xi)\|$ is bounded in t. In particular $\|Y_{\psi^z_* \xi}(t)\|$ is bounded, hence $\psi^z_* \xi \in E^p$. If $\xi \in E^{ss} \cap TW$ then $\lim_{t \to \infty} \|g^t_* \xi\| = 0$ by (ii). The same argument as above shows that $\lim_{t \to \infty} \|Y_{\psi^z_* \xi}(t)\| = 0$, hence $\psi^z_* \in E^{ss}$. The proof for $\xi \in E^{su} \cap TW$ is analogous. □

We now turn to the problem to apply the results from [HPS] to our situation. We adjust our notation to the symbols used in [HPS].

Choose some $v \in G$ and let $m = \dim(E^{ss}_v \cap T_v W)$. We consider the set

$$\Lambda = \{ w \in G \mid \dim(E^{ss}_w \cap T_w W) = m \}$$

This set is non-empty, compact and ψ-invariant. The last property follows easily from $\psi_*^z(E^{ss} \cap TW) \subset E^{ss}$, *cf.* the end of the proof of 4.3. Let $\mathscr{L}$ denote the restriction of our foliation to the ψ-invariant compact subset Λ of W. Then $\mathscr{L}$ is a C^∞-smoothable lamination of Λ in the sense of [HPS], p. 123. Throughout the rest of this section we shall use the following important fact which is a consequence of Theorem 4.3:

For every $t \geqq t_0$ and every $r \in \mathbb{N}$ the diffeomorphism $f = g^t : W \to W$ is r-normally hyperbolic to $\mathscr{L}$. The notion "r-normally hyperbolic to $\mathscr{L}$" is defined in [HPS], p. 116.

For the convenience of the reader we recall some of the terminology and some of the results from [HPS] that we need. For $\varepsilon > 0$ smaller that the injectivity radius of M let $B_\varepsilon \subset \mathbb{R}^k$ be the closed ball of radius ε and for $w \in W$ set $P_w^\varepsilon = \psi_w(B_\varepsilon)$. We call P_w^ε the ε-plaque with center w. The family $\{ \psi_w | B_\varepsilon \,|\, w \in W \}$ is a plaquation of the foliated manifold W in the sense of [HPS], p. 72. Now we fix some $t \geqq t_0$ and consider the diffeomorphism $f = g^t : W \to W$. A δ-pseudo orbit of f is a sequence w_n, $n \in \mathbb{Z}$, in W such that $d(f(w_n), w_{n+1}) < \delta$ for all $n \in \mathbb{Z}$. We say that a δ-pseudo orbit w_n of f respects the ε-plaquation if $f(w_n) \in P_{w_{n+1}}^\varepsilon$. The first fact that we need is:

4.4. LEMMA. — *There exists $\varepsilon > 0$ such that f is ε-plaque expansive on Λ. This means: if $(v_n)_{n \in \mathbb{Z}}$, $(w_n)_{n \in \mathbb{Z}}$ are ε-pseudo orbits in Λ which respect the ε-plaquation and if $d(v_n, w_n) < \varepsilon$ for all $n \in \mathbb{Z}$ then $v_n \in P_{w_n}^\varepsilon$.*

Proof. — This follows from the normal hyperbolicity of f and the smoothness of $\mathscr{L}$, *cf.* [HPS], Theorem (7.4). In our situation one can also give a simple geometric argument to prove (4.4) even for ε-pseudo orbits in W. □

Next we want to show that the Shadowing Lemma (7A.2) from [HPS] holds in our situation. Recall that a sequence v_n in W ε-shadows a sequence $w_n \in W$ if $d(v_n, w_n) < \varepsilon$ for all n.

4.5. LEMMA. — *For every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo orbit of f in Λ can be ε-shadowed by an ε-pseudo orbit for f in Λ which respects the ε-plaquation of Λ.*

Proof. — We have to verify that the following hypotheses of [HPS], (7A.2) are satisfied in our situation:

(*a*) If L and L′ are leaves of $\mathscr{L}$ then $W^s(L) \cap L'$ and $W^u(L) \cap L'$ are open in L′.

(*b*) There exists $\varepsilon > 0$ such that $W_\varepsilon^s(\Lambda) \cap W_\varepsilon^u(\Lambda) = \Lambda$.

Before we can prove this we have to remember the meaning of $W^s(L)$, $W_\varepsilon^s(L)$, etc.

For $v \in \Lambda$ set

$$W^{ss}(v) = \{ w \in W \,|\, \lim_{t \to \infty} d(g^t w, g^t v) = 0 \}$$

If $\varepsilon > 0$ is smaller than the injectivity radius set

$$W_\varepsilon^{ss}(v) = \{ w \in W^{ss}(v) \,|\, d(g^t w, g^t v) \leqq \varepsilon \text{ for all } t \geqq 0 \}$$

Finally $W^s(L) = \bigcup_{v \in L} W^{ss}(v)$ and $W_\varepsilon^s(\Lambda) = \bigcup_{v \in \Lambda} W_\varepsilon^{ss}(v)$. The sets $W^{su}(v)$, $W_\varepsilon^{su}(v)$, $W^u(L)$ and $W_\varepsilon^u(\Lambda)$ are defined analogously. From [HPS], Theorem (6.1), we know that $W^{ss}(v)$,

$W^{su}(v)$, $W^s(L)$ and $W^u(L)$ are projections to SM of submanifolds of $S\tilde{M}$ and that the tangent spaces of $W^{ss}(v)$, resp. $W^{su}(v)$ at v are $E_v^{ss} \cap T_v W$, resp. $E_v^{su} \cap T_v W$. Moreover the tangent spaces to $W^{ss}(v)$, resp. $W^{su}(v)$ vary continuously with their footpoints in $\bigcup_{v \in \Lambda} W^{ss}(v)$, resp. $\bigcup_{v \in \Lambda} W^{su}(v)$, *cf.* [HPS], Theorem (6.1) (*e*).

To prove (*a*) we show more specifically: if $v \in \Lambda$, $z \in \mathbb{R}^k$ then $W^{ss}(\psi^z(v)) = \psi^z(W^{ss}(v))$. This implies $W^s(L) \cap L' = L'$ if $W^s(L) \cap L' \neq \varnothing$. Now suppose $w \in W^{ss}(v)$, *i.e.* $w \in W$ and $\lim_{t \to \infty} d(g^t w, g^t v) = 0$. Since ψ^z commutes with the geodesic flow g^t we obtain $\psi^z(w) \in W^{ss}(\psi^z(v))$ and thus $\psi^z(W^{ss}(v)) \subset W^{ss}(\psi^z(v))$. This implies our claim. Finally we show that (*b*) is true independently of ε. We obviously have $\Lambda \subset (W_\varepsilon^s(\Lambda) \cap W_\varepsilon^u(\Lambda))$. Conversely suppose $v \in W^{ss}(v_+) \cap W^{su}(v_-)$ with $v_+, v_- \in \Lambda$. Let T denote the tangent space of $W^{ss}(v_+)$ at v. Since Λ is compact we can find $v_0 \in \Lambda$ and a sequence $t_i \to \infty$ such that $\lim_{i \to \infty} g^{t_i} v = v_0$. The above mentioned continuity of the tangent spaces to $W^{ss}(w)$, $w \in \Lambda$, implies that

$$\lim_{i \to \infty} g_*^{t_i}(T) = T_{v_0}(W^{ss}(v_0)) = E_{v_0}^{ss} \cap T_{v_0} W$$

It is easy to see that this implies $T \subset E_v^{ss}$. Similarly we see that the tangent space S of $W^{su}(v_-)$ at v is contained in E_v^{su}. By the definition of Λ we have $\dim(E_w^{ss} \cap T_w W) = m$ and $k + m + \dim(E_w^{su} \cap T_w W) = \dim W$. Since

$$\dim W^{ss}(v_+) = \dim(E_{v_+}^{ss} \cap T_{v_+} W), \qquad \dim W^{su}(v_-) = \dim(E_{v_-}^{su} \cap T_{v_-} W)$$

and v_+, $v_- \in \Lambda$ we obtain $k + \dim T + \dim S = \dim W$ and $\dim T = m$. Since $T \subset E_v^{ss}$, $S \subset E_v^{su}$ this implies $v \in \Lambda$. □

Using 4.4 and 4.5 we are now able to prove

4.6. Theorem. — *There exists $w \in W$ such that $\psi_w : \mathbb{R}^k \to W$ is not injective.*

Proof. — As before we fix $t \geqq t_0$ and consider $f = g^t : W \to W$. Since Λ is compact and f-invariant we can find a minimal set $K \subset \Lambda$ of the action of f on Λ. Choose $\varepsilon > 0$ according to 4.4 and $\delta > 0$ according to 4.5 and for $\varepsilon/2$. Choose $v \in K$. Then there exists $j > 0$ such that $d(f^j(v), v) < \delta$. Let $(v_n)_{n \in \mathbb{Z}}$ denote the j-periodic sequence defined by $v_i = f^i(v)$ for $0 \leqq i < j$. Then $(v_n)_{n \in \mathbb{Z}}$ is a δ-pseudo orbit for f in Λ. By 4.5 there exists an ε-pseudo orbit $(w_n)_{n \in \mathbb{Z}}$ for f in Λ which $\varepsilon/2$-shadows (v_n) and which respects the ε-plaquation. Since $v_{n+j} = v_n$ we obtain $d(w_n, w_{n+j}) < \varepsilon$ for all $n \in \mathbb{Z}$. Hence we can apply 4.4 to $(w_n)_{n \in \mathbb{Z}}$ and $(w_{n+j})_{n \in \mathbb{Z}}$ and conclude that $w_{n+j} \in P_{w_n}^\varepsilon$. In particular we can find $z \in \mathbb{R}^k$ with $\|z\| < \varepsilon$ such that $w_j = \psi_{w_0}(z)$. On the other hand (w_n) is an ε-pseudo orbit for $f = g^t$ which respects the ε-plaquation. Hence we can find a sequence $z_n \in \mathbb{R}^k$ with $\|z_n\| \geqq |n|(t - \varepsilon)$ such that $\psi_{w_0}(z_n) = w_n$, in particular $\|z_j\| \geqq j(t - \varepsilon)$, and $\psi_{w_0}(z_j) = w_j = \psi_{w_0}(z)$. Since we may assume that $t - \varepsilon > \varepsilon$ we see that $z_j \neq z$, *i.e.* ψ_{w_0} is not injective. □

4.7. Corollary. — *There exists a k-flat $F \subset \tilde{M}$ and a deck transformation γ such that $\gamma F = F$ and $\gamma_{|F}$ is a non-trivial translation.*

Proof. — From 4.6 we obtain $w \in W$ and $z_1 \neq z_2 \in \mathbb{R}^k$ such that $\psi(z_1, w) = \psi(z_2, w)$. Let $z = z_1 - z_2 \neq 0$. Since ψ is an $\mathbb{R}^k$-action we obtain $\psi^w(x+z) = \psi_w(x)$ for all $x \in \mathbb{R}^k$. Let $F : \mathbb{R}^k \to \tilde{M}$ be a lift to $\tilde{M}$ of $F_w = \pi \circ \psi_w : \mathbb{R}^k \to M$. Since $p \circ F(x+z) = p \circ F(x)$ for all $x \in \mathbb{R}^k$ there exists a deck transformation γ such that $F(x+z) = \gamma(F(x))$ for all $x \in \mathbb{R}^k$. Hence γ maps the flat $F(\mathbb{R}^k) \subset \tilde{M}$ to itself and $\gamma_{|F(\mathbb{R}^k)}$ corresponds to the translation by $z \neq 0$. □

5. Proof of the Theorem

We collect the results of sections 2, 3, 4 to prove the theorem stated in the introduction.

Let $n = \dim(M)$. We assume inductively that the theorem is true for all manifolds M' with $\dim(M') < n$. Exactly as in [S1] section 2, this implies the theorem for manifolds of dimension $\leqq n$ whose universal cover has a (non-trivial) euclidean de Rham factor.

First we assume that for some $s \geqq 2$ the set $E_{s,k}(M)$ as defined in section 2 is well structured, while $E_{s-1,k}(M)$ is not well structured. Then, by Theorem 2.5, for all flags $(\sigma_s, \ldots, \sigma_k) \in E_{s,k}(M)$, the set P_{σ_s} is closed. Fix $(\sigma_s, \ldots, \sigma_k)$ and consider a lift $(\tau_s, \ldots, \tau_k) \in E_{s,k}(\tilde{M})$. Let $M = \tilde{M}/\Gamma$.

Recall that $\tilde{H} = P_{\tau_s}$ is a complete totally geodesic submanifold of $\tilde{M}$. Since P_{σ_s} is closed there exists a subgroup Δ of Γ which operates on $\tilde{H}$ with compact quotient. Then $H = \tilde{H}/\Delta$ with the metric induced from $\tilde{H} \subset \tilde{M}$ is a compact analytic Riemannian manifold of nonpositive curvature whose universal cover $\tilde{H}$ has a euclidean de Rham factor of dimension $s \geqq 2$. Since moreover $\dim H \leqq \dim M = n$ the theorem holds for H. Since H contains a k-flat we obtain a closed k-flat in H and hence also in M.

Hence we are left with the case that $E_{1,k}(M)$ is well structured and we can apply the results of sections 3 and 4. According to Corollary 4.7 there exists a k-flat $F \subset \tilde{M}$ which is invariant under a deck transformation γ which restricts to a non-trivial parallel translation of F. We denote by $\tilde{H} \subset \tilde{M}$ the union of all geodesics which are translated by γ. Then $F \subset \tilde{H}$ and $p(\tilde{H})$ is a compact subset of M since it consists of all closed geodesics in the free homotopy class determined by γ. If $v \in S\tilde{M}$ is an initial vector of a geodesic translated by γ then $\tilde{H}$ is contained in the parallel space P_v. The isometry γ splits on $P_v = \mathbb{R} \times Q$ as $\gamma(t, g) = (t + L, \gamma_1 q)$ where γ_1 is an isometry of the totally geodesic submanifold $Q \subset \tilde{M}$. Hence $\tilde{H}$ is the totally geodesic submanifold $\mathbb{R} \times \mathrm{Fix}(\gamma_1)$ of $\tilde{M}$. In particular $\tilde{H}$ has a non-trivial euclidean de Rham factor. Since $p(\tilde{H})$ is compact and contains the k-flat $p(F)$ we can argue as above and conclude that $p(\tilde{H})$ and hence also M contains a closed k-flat. □

Remark. — A closer look at the proofs of (4.6) and (4.7) shows that we can achieve that γ translates F into a regular direction. Then $\tilde{H} = F$ so that F itself is projected onto an isometrically immersed flat k-torus in M, *cf.* also [BBS] (4.7).

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