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EXTREMAL RAYS ON SMOOTH THREEFOLDS

BY JÁNOS KOLLÁR

1. Introduction

In [M2] Mori proved a structure theorem for threefolds whose canonical class is not nef. His proof had two parts.

First he investigated the cone of effective curves and proved that part of it is locally polyhedral. This is first proved in positive characteristic and then "lifted back" to characteristic zero by a nontrivial argument.

The second part of his proof is the description of extremal rays on a threefold in characteristic zero. This description relies on very delicate applications of Kodaira's vanishing theorem, and therefore it is not applicable in positive characteristic.

Later development of the theory relied even more on vanishing theorems that are not available (and frequently false) in positive characteristic.

The aim of this article is to develop a method that relies less on vanishing theorems. The emphasis here is on studying the deformation theory of curves in smooth threefolds. This approach is rather independent of the characteristic, and thus leads to the following generalization of [M2], Theorems 3.1-3.

(1.1) MAIN THEOREM. — *Let* X *be a smooth projective threefold over an algebraically closed field* k *of any characteristic. Let* R *be an extremal ray of the closed cone of curves. Then*

(1.1.1) *There is a normal projective variety* Y *and a surjective map* $f: X \to Y$ *such that an irreducible curve* $C \subset X$ *is mapped to a point by* f *iff* $[C] \in R$. *One can always assume that* $f_* \mathcal{O}_X = \mathcal{O}_Y$ *and then* Y *and* f *are unique up to isomorphism.*

The following is a list of all the possibilities for f *and* Y.

(1.1.2) *First case:* f *is birational.*

Let $E \subset X$ *be the exceptional set of* f. *One has the following possibilities for* E, Y *and* f:

(1.1.2.1) *E is a smooth minimal ruled surface with typical fiber* C *and* $C.E = -1$. Y *is smooth and* f *is the inverse of the blowing up of a smooth curve in* Y.

(1.1.2.2) $E \cong \mathbb{P}^2$ *and its normal bundle is* $\mathcal{O}(-1)$. Y *is smooth and* f *is the inverse of the blowing up of a point in* Y.

In the remaining cases Y *has exactly one singular point* P *and* f *is the inverse of the blowing up of* P *in* Y. *Let* $\widehat{\mathcal{O}}_{P, Y}$ *be the completion of the local ring of* $P \in Y$.

(1.1.2.3) $E \cong \mathbb{P}^2$ *and its normal bundle is* $\mathcal{O}(-2)$. $\widehat{\mathcal{O}}_{P, Y} \cong k[[x^2, y^2, z^2, xy, yz, zx]]$.

(1.1.2.4) $E \cong Q$ *where* Q *is a quadric cone in* $\mathbb{P}^3$ *and its normal bundle is* $\mathcal{O}_{\mathbb{P}^3}(-1) | Q$. $\widehat{\mathcal{O}}_{P, Y} \cong k[[x, y, z, t]]/(xy - z^2 - t^3)$.

(1.1.2.5) $E \cong Q$ *where* Q *is a smooth quadric surface in* $\mathbb{P}^3$, *the two families of lines on* Q *are numerically equivalent in* X *and its normal bundle is* $\mathcal{O}_{\mathbb{P}^3}(-1) | Q$. $\widehat{\mathcal{O}}_{P, Y} \cong k[[x, y, z, t]]/(xy - zt)$.

(1.1.3) *Second case: f is not birational.*

Then we have one of the following cases:

(1.1.3.1) $\dim Y = 2$. *Then* Y *is smooth and* f *is a flat conic bundle* (i.e. *every fiber is isomorphic to a conic in* $\mathbb{P}^2$). *If the characteristic is different from two, the general fiber is smooth.*

(1.1.3.2) $\dim Y = 1$. *Then* Y *is a smooth curve and every fiber of* f *is irreducible. Any fiber with reduced scheme structure is a* (*possibly nonnormal*) *Del Pezzo surface.*

(1.1.3.3) $\dim Y = 0$. *Then* X *is a Fano variety* (i.e. $-K_X$ *is ample*).

(1.2) *Remarks.* — (1.2.1) In case (1.1.3.2) I can not prove that all fibers are reduced or that the generic fiber is normal. The situation seems fairly complicated, especially in characteristic two.

(1.2.2) It will be clear that the methods give very little information about Fano varieties. However the result should be very useful in their study. For instance, the results of [N] should imply that if X is a Fano threefold over a field of any characteristic with Picard number at least 6 then X has an extremal face of type (1.1.3.1) or (1.1.3.2).

The original goal I had in mind was to obtain a more direct way of finding extremal rays. The idea is the following. Assume that a threefold X contains a rational curve C_0 such that $C_0 . K_X < 0$. By [M1], Proposition 3, one can deform C_0 keeping it rational. It may degenerate and then we get an algebraic equivalence $C_0 \approx C_1 \cup D_1$ where C_1 is a rational curve such that $C_1 . K_X < 0$. Now continue the procedure with C_1. If X is projective then finally we must get a rational curve C_k which deforms but stays irreducible all the time. Since this C_k is not algebraically equivalent to the sum of other curves in any obvious way, one could hope that it generates an extremal ray. This is false if X itself is uniruled, but is very close to being true otherwise. The following result is proved in Chapter 2:

(1.3) Theorem. — *Let* X *be a smooth projective threefold over an algebraically closed field of any characteristic and let* C_k *be as before. Assume that* $\kappa(X) \geqq 0$. *Then the deformations of* C_k *sweep out a surface* $E \subset X$ *which is one of those listed in* (1.1.2.1-4).

The new method works very efficiently to describe an extremal ray R on a threefold X in two cases:

first, if there is a surface $E \subset X$ such that $R . E < 0$ (this is done in Chapter 2);

second, if the curves in R cover X (this is done in Chapter 4).

The weakness of the method is that at the moment it does not imply that there are no other cases. Fortunalety this follows from a result of Miyaoka-Mori [MM] (*see* Chapter 3).

Some of the above results have analogs for nonprojective threefolds as well. One such example is worked out in Chapter 5. This requires a nonprojective version of a lemma of Ein [E].

It would be very interesting to generalize these results to threefolds with terminal singularities. Mori's original approach to the cone theorem works for threefolds with isolated hypersurface singularities. Almost all of (1.1) can be generalized to threefolds with isolated factorial hypersurface singularities; there are slight problems with (1.1.2.1) and (1.1.3.1). It is interesting to note that rationality of the singularities plays no role; the nonrational singularities will not occur on the exceptional loci.

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Notation

(N.1) For a variety X the $\mathbb{R}$-vectorspace of 1-cycles modulo numerical equivalence will be denoted by $N(X)$. If $F \subset N(X)$ then we will say that a curve C is in F if its class is in F.

(N.2) $NE(X) \subset N(X)$ will denote the convex cone generated by the classes of effective curves. Its closure in the Eucledean topology of $N(X)$ will be denoted by $\overline{NE}(X)$.

(N.3) Assume that K_X is $\mathbb{Q}$-Cartier. A ray $R = \mathbb{R}^+[C] \subset \overline{NE}(X)$ is called an extremal ray if $[C].K_X < 0$ and if $u, v \in \overline{NE}(X)$ and $u+v \in R$ imply that $u, v \in R$.

(N.4) Let E be a Cartier divisor on X and let R be an extremal ray. If $C \subset X$ is a curve such that $[C] \in R$ then the sign of the intersection product $C.E$ depends only on E and R. Thus the notation $R.E < 0$ (resp. $=0$ etc.) makes sense. The ray R is called nef if $R.E \geqq 0$ for every effective divisor E.

(N.5) An extremal ray R is said to cover X if through every point $x \in X$ there is a curve C_x in R. If R covers X then it is obviously nef.

(N.6) NS (X) denotes the Néron-Severi group of X; $NS_{\mathbb{Q}}(X) = \mathbb{Q} \otimes NS(X)$.

(N.7) Algebraic equivalence of cycles will be denoted by $\approx$.

(N.8) We say that a surface X is a (possibly nonnormal) Del Pezzo surface, if X is a reduced, irreducible Gorenstein surface such that ω_X^{-1} is ample.

2. Divisorial Contractions

The main result of this chapter is the following:

(2.1) THEOREM. — *Let* X *be a three dimensional smooth algebraic space resp. a three dimensional complex manifold. Let* S *be a proper smooth minimal ruled surface with typical fiber* F. *Let* $f: S \to X$ *be a morphism. Its image is denoted by* E *and the image of* F *by* C. *Assume that*

(2.1.1.1) $\dim E = 2$;

(2.1.1.2) $C.K_X < 0$;

(2.1.1.3) $C.E < 0$.

Then we have one of the following situations:

(2.1.2.1) *E is a smooth minimal ruled surface with typical fiber* C *and* $C.E = -1$.

(2.1.2.2) $E \cong \mathbb{P}^2$ *and its normal bundle is* $\mathcal{O}(-1)$.

(2.1.2.3) $E \cong \mathbb{P}^2$ *and its normal bundle is* $\mathcal{O}(-2)$.

(2.1.2.4) $E \cong Q$ *where* Q *is a quadric cone in* $\mathbb{P}^3$ *and its normal bundle is* $\mathcal{O}_{\mathbb{P}^3}(-1) \mid Q$.

(2.2) *Remarks.* – (2.2.1) (2.1.1.2) implies (2.1.1.3) if $\kappa(X) \geqq 0$.

(2.2.2) In cases (2.1.2.2-4) the curve C can be complicated. Indeed, there are complete families of large degree planar rational curves.

(2.3) THEOREM. — *Let* X *be a smooth projective threefold. Let* R *be an extremal ray. Assume that there is an irreducible surface* $E \subset X$ *such that* $R.E < 0$. *Then the collection of all curves in* R *covers the surface* E *and we have one of the following situations:*

(2.3.1) E *is a smooth minimal ruled surface with typical fiber* C *and* $C.E = -1$. *Only the fibers of* E *are in* R.

(2.3.2) $E \cong \mathbb{P}^2$ *and its normal bundle is* $\mathcal{O}(-1)$.

(2.3.3) $E \cong \mathbb{P}^2$ *and its normal bundle is* $\mathcal{O}(-2)$.

(2.3.4) $E \cong Q$ *where* Q *is a quadric cone in* $\mathbb{P}^3$ *and its normal bundle is* $\mathcal{O}_{\mathbb{P}^3}(-1) \mid Q$.

(2.3.5) $E \cong Q$ *where* Q *is a smooth quadric surface in* $\mathbb{P}^3$ *and its normal bundle is* $\mathcal{O}_{\mathbb{P}^3}(-1) \mid Q$. *Every curve in* E *is in* R.

Proof. – Let us fix an ample divisor H on X. By [M2, 1.4] there is a rational curve $g: \mathbb{P}^1 \to X$ in R. Choose g such that $\deg(g^* K_X) < 0$ is maximal. Since $\deg(g^* K_X) < 0$, by [M1], Proposition 3, there is a nontrivial deformation $G^0: \mathbb{P}^1 \times D^0 \to X$. We can complete this to a map $G: S \to X$ where S is a not necessarily minimal ruled surface over D and G does not contract any (-1)-curve contained in a fiber of S/D.

If S is not minimal then there is a reducible fiber $F = \sum c_k F_k$. Thus $g(\mathbb{P}^1) \approx \sum c_k G(F_k)$. $[G(F_k)] \in R$ since R is extremal; in particular, $G(F_k).K_X < 0$. Since $\sum c_k G(F_k).K_X = g(\mathbb{P}^1).K_X$, this implies that $g(\mathbb{P}^1).K_X < G(F_k).K_X$, which contradicts the choice of g. Thus S is minimal and (2.1) applies. The only missing piece of information is to show that if E is ruled and if every curve in E is in R then we have (2.3.5).

If every curve in E is in R then both $-\mathrm{E}|\mathrm{E}$ and $-\mathrm{K_X}|\mathrm{E}$ are ample, thus $-\mathrm{K_E}=-\mathrm{E}|\mathrm{E}-\mathrm{K_X}|\mathrm{E}$ is also ample. $-\mathrm{K_E}$ is ample only for $\mathrm{E}\cong\mathbb{P}^1\times\mathbb{P}^1$ and for the one point blow-up of $\mathbb{P}^2$. The second case is impossible because there $-\mathrm{K_E}$ is not the sum of two ample divisors.

(2.4) *Proof of* (2.1). – Let us start with the following auxiliary

(2.4.1) LEMMA. — *Let* $g: \mathrm{T}\to\mathrm{B}$ *be a not necessarily minimal smooth proper ruled surface with general fiber* C. *Let* L *be a line bundle on* T *such that*

(2.4.1.1) $\mathrm{L}.\mathrm{C}=1$,

(2.4.1.2) *if* $\mathrm{D}\subset\mathrm{T}$ *is a curve contained in a fiber then* $\mathrm{L}.\mathrm{D}\geqq 0$ *and*

(2.4.1.3) *if* $\mathrm{D}\subset\mathrm{T}$ *is a* (-1)*-curve contained in a fiber then* $\mathrm{L}.\mathrm{D}>0$.

Then T *is a minimal ruled surface over* B.

Proof. – Let $\sum d_i \mathrm{D}_i$ be a singular fiber. Since $\mathrm{L}.\mathrm{C}=\mathrm{L}.\sum d_i\mathrm{D}_i=1$, we conclude that every singular fiber contains at most one (-1)-curve (even counted with multiplicity). It is easy to see that there is no such singular fiber. This shows the result.

(2.4.2) Now let us consider $f: \mathrm{S}\to\mathrm{E}$. We can factor it through the normalization $\bar{\mathrm{E}}$ of E to get $\bar{f}: \mathrm{S}\to\bar{\mathrm{E}}$. Since $\bar{f}(\mathrm{F})$ moves in $\bar{\mathrm{E}}$, $\bar{f}(\mathrm{F}).\bar{f}(\mathrm{F})\geqq 0$ (here we use the intersection theory of [Mu, II (*b*)]. We will consider separately the following alternatives:

(2.4.2.1) $\bar{f}(\mathrm{F}).\bar{f}(\mathrm{F})=0$, *i.e.* for any two fibers F_1 and F_2 of S, either $\bar{f}(\mathrm{F}_1)=\bar{f}(\mathrm{F}_2)$ (as sets) or they are disjoint.

(2.4.2.2) $\bar{f}(\mathrm{F}).\bar{f}(\mathrm{F})>0$.

(2.4.3) Assume that we have the alternative (2.4.2.1). Then $\bar{\mathrm{E}}$ is covered by a family of pairwise disjoint rational curves $\bar{f}(\mathrm{F}_t)$. For generic t the curves $\bar{f}(\mathrm{F}_t)$ have the same Hilbert polynomial, thus there is a one dimensional closed irreducible subset of the Hilbert scheme of $\bar{\mathrm{E}}$ which generically parametrizes the curves $\bar{f}(\mathrm{F}_t)$. Let B be the normalization of this subset and let $\mathrm{U}\to\mathrm{B}$ be the normalization of the universal family over B. The natural morphism $u: \mathrm{U}\to\bar{\mathrm{E}}$ is finite and generically one-to-one. Therefore it is everywhere one-to-one. In characteristic zero this implies that u is an isomorphism and thus we obtain a morphism $\bar{\mathrm{E}}\to\mathrm{B}$. In positive characteristic u factors through a power of the Frobenius, thus again we obtain a morphism $\bar{\mathrm{E}}\to\mathrm{B}$. In both cases the set theoretic fibers are precisely the curves $\bar{f}(\mathrm{F}_t)$.

Let $p: \mathrm{T}\to\bar{\mathrm{E}}\to\mathrm{E}$ be the minimal desingularization of E. $\hat{\mathrm{C}}$ – the proper transform of C – is the general fiber of T/B. Since

$$\mathrm{K_T}\approx p^*\mathrm{K_E}-\text{(effective divisor)},$$

we get that

$$-\mathrm{K_T}.\hat{\mathrm{C}}\geqq -\mathrm{K_E}.\mathrm{C}=-\mathrm{E}.\mathrm{C}-\mathrm{K_X}.\mathrm{C}>0.$$

In particular, T/B is a ruled surface. Therefore $-\mathrm{K_T}.\hat{\mathrm{C}}=2$ and $-\mathrm{K_X}.\mathrm{C}=-\mathrm{E}.\mathrm{C}=1$. Let $\mathrm{L}=-p^*\mathrm{K_X}$. L satisfies (2.4.1.1). If $\mathrm{D}_i\subset\mathrm{T}$ is a component of a fiber then either $p(\mathrm{D}_i)=$(point), or $p(\mathrm{D}_i)\approx\mathrm{C}$. In both cases $\mathrm{L}.\mathrm{D}_i\geqq 0$. Furthermore a (-1)-curve can not be mapped to a point by p since the resolution is minimal. Thus by (2.4.1) T itself is minimal.

Now we can replace S by T and we can thus assume that S is the normalization of E. We can write

$$K_S \approx f^*(E+K_X) - (\text{Conductor of } f).$$

Since $F.K_S=-2$ and $F.f^*E$ and $F.f^*K_X$ are both negative, this implies that $F.f^*E=F.f^*K_X=-1$ and that the conductor is contained in the union of some fibers of S. In particular this implies that $C \cong \mathbb{P}^1$. Let C_t be the (set theoretic) image of any of the fibers F_t. I claim that C_t is also smooth. Since $C_t.K_X<0$, by (5.1) there is a deformation of C_t which does not have $\operatorname{red} C_t$ as its component. Let $\bar{C}$ be a generic deformation of C_t. We may assume that it has no common components with the conductor of f. Since $C_t.E<0$, $\bar{C}$ is contained in E. Let $\tilde{C} \subset S$ be the proper transform of $\bar{C}$. As $\bar{C}$ specializes to C_t, $\tilde{C}$ specializes to a curve $\tilde{C}_t \subset S$. By construction, $f(\tilde{C}_t)=C_t$. Therefore $\tilde{C}_t$ is contained in $F_t \cup$ (conductor of f), hence it is a union of some fibers of S. Consequently, $\tilde{C}$ is a union of some fibers of S, hence $\bar{C} \cong \tilde{C}$. Therefore $\chi(C_t)=\chi(\bar{C}) \geqq 1$. This shows that $C_t \cong \mathbb{P}^1$.

The above argument also shows that if D is a subscheme of X whose support is C_t and D_s is a deformation of D then the one dimensional part of the support of D_s is a union of curves C_{t_i}.

(2.4.4) CLAIM. – *Let* I *be the ideal sheaf of* $C_t \cong \mathbb{P}^1$. *Then* $I/I^2 \cong \mathcal{O} + \mathcal{O}(1)$.

Proof. – Since C_t is smooth, rational and $C_t.K_X=-1$ we known that $I/I^2 \cong \mathcal{O}(-a)+\mathcal{O}(a+1)$ for some $a \geqq 0$.

Let J be the ideal sheaf generated by I^2 and by the $\mathcal{O}(a+1)$ factor of the above decomposition. Then $D=\operatorname{Spec} \mathcal{O}_X/J$ has support on C_t and satisfies $D.K_X=-2$ and $\chi(D)=2-a$. By (5.1), D has a two dimensional family of deformations. Let D_s be any deformation of D. If D_s is disconnected then $D_s=C_{t_1} \cup C_{t_2}$, hence $\chi(D)=\chi(D_s)=2$. Thus $a=0$. If every deformation D_s of D is connected then $\operatorname{supp} D_s=C_{t_s}$ for some t_s. D has a two parameter family of deformations and C_t is a one parameter family, therefore D has a nontrivial deformation which leaves the support unchanged. Hence there is a one parameter family of maps

$$I/I^2 \cong \mathcal{O}(-a)+\mathcal{O}(a+1) \to \mathcal{O}(a+1).$$

This is impossible, hence the claim.

Thus we see that E itself is a smooth minimal surface. This completes the first case.

(2.4.5) Assume that we have the alternative (2.4.2.2). Then for any fiber F_t of S we have that $\bar{f}(F_1) \cap \bar{f}(F_t) \neq 0$. Therefore $\bar{f}^{-1}\bar{f}(F_1)) \subset S$ has an irreducible component Z′ which is a (possibly multiple) section of S. Therefore by (4.4) we see that $N(\bar{E})$ is generated by the classes of $\bar{f}(Z')=\bar{f}(F_1)$ and by $\bar{f}(F_t)$. Thus $\dim N(\bar{E})=1$. Let $p: T \to E$ be the minimal desingularization of E and let $\hat{C}$ be the proper transform of C in T. As in (2.4.3) we get that $K_T . \hat{C} < 0$, thus T is birationally ruled. Therefore T is either $\mathbb{P}^2$ or it is ruled over a curve B. We ignore the first case for the moment.

Let $L = -p^* K_X$. L is nef, it intersects all (-1)-curves positively and $L . \hat{C} = 1$ since

$$-2 = K_T . \hat{C} = K_X . C + E . C - (\text{effective divisor}) . \hat{C}.$$

Thus by (2.4.1) we see that T is minimal. T can not be the normalization of E since it has $\dim N(\bar{E}) = 1$. Thus p must contract some curve.

(2.4.6) LEMMA. — *Let* T *be a smooth minimal ruled surface and let* $D \subset T$ *be a contractible curve. Then* D *is irreducible.*

Proof. — The components of D are linearly independent in N(T), hence D is irreducible since $\dim N(T) = 2$.

(2.4.7) As we saw, T is either $\mathbb{P}^2$ or it is minimal ruled over a curve B. In the latter case p contracts an irreducible curve $D \subset T$. As before we have the adjunction formula

$$(2.4.7.1) \qquad -K_T = -p^*(E) - p^*(K_X) + (\text{effective curve}).$$

$\hat{C}$ is either a line in $\mathbb{P}^2$ or a fiber of T/B. Taking intersection numbers with $\hat{C}$ in the above formula we obtain

$$a + b + c = \begin{cases} 3 & \text{if } T \cong \mathbb{P}^2, \\ 2 & \text{if } T \text{ is minimal ruled,} \end{cases}$$

where $a = -p^*(E) . \hat{C}$, $b = -p^*(K_X) . \hat{C}$ are positive integers and $c = (\text{effective curve}) . \hat{C}$ is a nonnegative integer.

If $T \cong \mathbb{P}^2$ then there are three numerical possibilities:

(2.4.8.1) $a = 1$, $b = 2$ and $c = 0$. In this case $E \cong \mathbb{P}^2$ with normal bundle $\mathcal{O}(-1)$.

(2.4.8.2) $a = 2$, $b = 1$ and $c = 0$. In this case $E \cong \mathbb{P}^2$ with normal bundle $\mathcal{O}(-2)$.

(2.4.8.3) $a = 1$, $b = 1$ and $c = 1$. We will exclude this case. From Riemann-Roch we get that

$$\chi(\mathcal{O}_E(E \mid E)) = \frac{E . (E - K_E)}{2} + \chi(\mathcal{O}_E) = \frac{3}{2} + \chi(\mathcal{O}_E),$$

which is impossible.

If T is is a minimal ruled surface then $a = b = 1$ and $c = 0$. Thus

$$K_T = p^*(E + K_X) - (\text{some fibers of } T/B).$$

Therefore

$$2 p_a(D) - 2 = D . (D + K_T) = D^2 - D . (\text{some fibers of } T/B).$$

Since D is contractible, $D^2 < 0$. This implies that $p_a(D) = 0$. If $D^2 = -1$ then $p : T \to E$ factors through $\mathbb{P}^2$, which impossible since p is the minimal resolution. Thus $D^2 \leq -2$

hence $D^2=-2$ and we obtain the last case:

(2.4.8.4) E is the quadric cone.

This completes the proof of (2.1).

(2.5) COMPLEMENT. — *Assumptions and notation as in* (2.3). *There is a contraction map* $f: X \to Y$ *onto a three dimensional normal projective variety* Y *such that an irreducible curve* $D \subset X$ *is contracted to a point iff* D *is in* R. f *can be described case by case as follows:*

(2.5.1) Y *is smooth and* f *is the inverse of the blowing up of a smooth curve in* Y.

(2.5.2) Y *is smooth and* f *is the inverse of the blowing up of a point in* Y.

In the remaining cases Y *has exactly one singular point* P *and* f *is the inverse of the blowing up of* P *in* Y. *The singularity at* P *is formally equivalent to the following singularities:*

(2.5.3) $\operatorname{Spec} k[[x^2, y^2, z^2, xy, yz, zx]]$.

(2.5.4) $\operatorname{Spec} k[[x, y, z, t]]/(xy-z^2-t^3)$.

(2.5.5) $\operatorname{Spec} k[[x, y, z, t]]/(xy-zt)$.

Proof. — First we prove that the contraction map $f: X \to Y$ exists. This will be done using the method of Castelnuovo (*cf.* [H, V.5.7]).

Let F be the fiber of E in case (2.3.1) and any line on E in the other cases. Let M be a very ample line bundle on X. If necessary we replace M with a suitable multiple and then we can consider the following new line bundle

$$\bar{M} = M \otimes \mathcal{O}\left(-\frac{M.F}{E.F}E\right).$$

We claim that $\bar{M}$ is generated by global sections and that the Stein factorization of the resulting morphism is exactly f. As in [H, V.5.7], this follows once we know that

$$(2.5.6) \qquad H^1(E, M\otimes\mathcal{O}(-cE)\,|\,E)=0 \qquad \text{for} \quad 0 \leqq c < -\frac{M.F}{E.F},$$

and $\bar{M}\,|\,E$ is generated by global sections. The second condition is clear in the cases (2.3.2-5) and (2.5.6) follows from

$$H^1(E, \mathcal{O}(n))=0 \qquad \text{for} \quad n>0.$$

We have to be more careful in case (2.3.1). The Néron-Severi group of E is generated by F and a section S. One can easily see that there is a constant k such that

$$(2.5.7) \qquad H^1(E, \mathcal{O}(mS+nF))=0 \qquad \text{for} \quad n>k(m+1); \quad m \geqq 0.$$

Now let $E\,|\,E \cong \mathcal{O}(-S+dF)$ and $M\,|\,E \cong \mathcal{O}(aS+bF)$. (2.5.7) implies (2.5.6) if

$$(2.5.8) \qquad \frac{b}{a} \geqq |d|+2k,$$

which is of course not necessarily satisfied. Here we have to use that F generates an extremal ray. Choose a line bundle H which is a supporting function of [F] [*cf.* (3.1)]. Then $H|E \cong \mathcal{O}(eF)$ for some $e>0$. Thus if we replace M be $M \otimes H^s$ for s sufficiently large then (2.5.8) becomes true and $\bar{M}|E$ is generated by global sections. This proves the existence of $f: X \to Y$.

The description of f is well known for (2.3.1-2) thus we concentrate on the remaining cases. First one verifies by explicit computation that by blowing-up the singularities given in (2.5.3-5) we get a smooth threefold with the required exceptional divisor. Now we need to see that a formal neighbourhood of E is isomorphic to the one we obtained by blowing-up. This is done using [M2, (3.33)] (*see* also [HR], Lemma 9). The required computations are very similar to those done at the end of [M2], section 7. Only the case (2.5.4) needs extra care. Here there can be two different infinitesimal extensions of $\mathcal{O}_E$ by $\mathcal{O}_E(1)$ because of the singular point of E. However one of them has embedding dimension 4 at the singular point. Therefore we have the other extension on both threefolds.

3. Division into cases

(3.1) Definition. – Let R be an extremal ray on a variety X. A divisor H (or more precisely, its class $[H] \in NS_{\mathbb{Q}}(X)$) is called a supporting function of R if $z.H \geqq 0$ for every $z \in \overline{NE}$ and $z.H=0$ iff $z \in R$.

(3.2) Proposition ([M2, 3.15]). — *Let* X *be a projective variety of dimension n with* $\mathbb{Q}$*-factorial singularities and let* R *be an extremal ray with supporting function* H. *The following two statements are equivalent:*

(3.2.1) $H^n>0$;

(3.2.2) *there is an irreducible divisor* $D \subset X$ *such that* $D.R<0$.

Proof. – One only has to note that in Mori's original proof the use of Kodaira's vanishing was not essential. If $H^n>0$ then some multiple of H defines a birational map (*cf.* [F, 6.5]). Thus if M is ample on X then there is an effective divisor $D' \sim kH-M$. Since $D'.R<0$, some irreducible component D of D′ will serve.

If $D.R<0$ then $kH-D$ is ample for $k \gg 0$ by [M2, 3.7]. Thus $h^0(m(kH-D)) \leqq h^0(m(kH))$ grows as m^n, hence $H^n>0$.

The following is a slightly different version of the main result of [MM].

(3.3) Proposition ([MM]). — *Let* X *be a smooth projective variety of dimension n. Let* D *be a smooth proper curve and let* $f: D \to X$ *be a nonconstant map. Let* M *be an ample divisor on* X. *Assume that* $-K_X.D>0$. *Then through every point* $x \in f(D)$ *there is a rational curve* $L \subset X$ *such that*

$$M.L \leqq 2(n+1)\frac{M.D}{-K_X.D}.$$

Proof. — This is proved in [MM] if $f(\mathrm{D})$ is not rational. (In [MM] the existence of such a curve is claimed only for general points of $f(\mathrm{D})$. Once L exists for general points, we can specialize to any point of $f(\mathrm{D})$.) Let us assume then that $f(\mathrm{D})$ is rational. Let $k=\deg(\mathrm{D}/f(\mathrm{D}))$. If

$$k \geqq \frac{-\mathrm{K_X}.\mathrm{D}}{2(n+1)}$$

then let $\mathrm{L}=f(\mathrm{D})$ with reduced scheme structure. Thus

$$\mathrm{M}.\mathrm{L} \leqq \frac{\mathrm{M}.\mathrm{D}}{k} \leqq 2(n+1)\frac{\mathrm{M}.\mathrm{D}}{-\mathrm{K_X}.\mathrm{D}}.$$

Otherwise

$$k \leqq \frac{-\mathrm{K_X}.\mathrm{D}}{2(n+1)}.$$

Assume furthermore that the ground field has positive characteristic p. Let $f_s : \mathrm{D} \to \mathrm{X}$ be the composition of f with the *sth* power of the Frobenius map. As in [MM], Theorem 5, the deformation space of the map f_s fixing the images of b different points of D has dimension at least

$$p^s(-\mathrm{K_X}.\mathrm{D})+n(1-g(\mathrm{D}))-nb.$$

The space of deformations whose image is $f(\mathrm{D})$ has dimension at most $2\deg f_s = 2p^s k$. Therefore, if

$$p^s(-\mathrm{K_X}.\mathrm{D})+n(1-g(\mathrm{D}))-nb-2p^s\frac{-\mathrm{K_X}.\mathrm{D}}{2(n+1)}>0,$$

then we can deform f_s in a one parameter family in such a way that the image of the family is a surface in X. The assumption that $f(\mathrm{D})$ is nonrational is used in [MM], Theorem 4, only to ensure the validity of the last claim. Thus we can use [MM], Theorem 4, to find a rational curve L which satisfies

$$\mathrm{M}.\mathrm{L} \leqq 2\frac{p^s \mathrm{M}.\mathrm{D}}{b}.$$

As in [MM], choosing b as large as possible and letting s go to infinity gives the result.

The characteristic zero case can be reduced to the positive characteristic case as in [MM].

The following result should be viewed as a weak version of the Contraction Theorem.

(3.4) THEOREM. — *Let* X *be a smooth projective variety of dimension n. Let* R *be an extremal ray of* X. *Then exactly one of the following conditions is satisfied:*

(3.4.1) *There is an irreducible divisor* $\mathrm{E} \subset \mathrm{X}$ *such that* $\mathrm{R}.\mathrm{E}<0$ (*hence* R *covers at most* E).

(3.4.2) R *covers* X.

Proof. — Pick a curve C which is in R and any point $x \in X - C$. Blow up x and consider the new smooth variety $p: Y = B_x X \to X$. Let $F \subset Y$ be the exceptional divisor. If $p^* H - \varepsilon F$ is nef, where H is a supporting function of R, then

$$0 \leqq (p^* H - \varepsilon F)^n = H^n - \varepsilon^n.$$

Thus $H^n > 0$ and by (3.2) there is a divisor $E \subset X$ such that $E . C < 0$.

If $p^* H - \varepsilon F$ is not nef then there is a curve $\bar{D} \subset Y$ such that $p^* H . \bar{D} \leqq \varepsilon F . \bar{D}$. Let $D = p(\bar{D})$ and let M be a very ample divisor on X. The previous inequality now gives

$$H . D \leqq \varepsilon . \operatorname{mult}_x D \leqq \varepsilon M . D.$$

The function $\max\{-K_X . Z, H . Z\}$ is strictly positive on $\overline{NE}(X)$, thus there is a $\delta > 0$ such that

$$\max\{-K_X . Z, H . Z\} \geqq \delta M . Z.$$

If $\varepsilon < \delta$ then these two inequalities imply that $-K_X . D \geqq (\delta/\varepsilon) H . D$. Thus by choosing ε small enough, we can assume that

$$-K_X . D \geqq 4(n+1) H . D.$$

Apply (3.3) with the ample divisor $M + kH$. Thus we find a rational curve L through $x \in D \subset X$ such that

$$M . L + kH . L \leqq 2(n+1) \frac{M . D + kH . D}{-K_X . D} = 2(n+1) \frac{M . D}{-K_X . D} + \frac{k}{2}.$$

For k large this implies that $H . L = 0$, *i.e.* L is R. Since x was arbitrary, this completes the proof.

4. Covering Case

In this section we describe those extremal rays that cover X. It is natural to look at the more general situation of having a covering family of curves, not necessarily one that comes from an extremal ray. Much of the theory will be valid with few assumptions.

(4.1) Basic set-up. — (4.1.1) We will consider diagrams as follows:

$$\begin{array}{ccc} U & \xrightarrow{F} & X \\ {\scriptstyle p}\downarrow & & \\ Z & & \end{array}$$

(4.1.2) Here X is our threefold which for the moment we only assume to be normal and projective. Z is an irreducible normal surface, $p: U \to Z$ is a proper (not necessarily flat) morphism with one dimensional fibers whose generic fiber is an irreducible and reduced curve. Finally we assume that F is surjective, *i.e.* U is a covering family of curves.

In typical examples Z is a closed subset of a Hilbert scheme or a Chow variety and U is the corresponding universal family.

(4.1.3) C_z will denote the fiber of p over $z \in Z$ with reduced scheme structure and D_z will be the set theoretic image of C_z. For $z \in Z$ sufficiently general we will also write C_{gen} resp. D_{gen}.

(4.1.4) We will also assume that the covering is generically minimal in the weak sense that:

(4.1.4.1) for sufficiently general z, $C_z \to D_z$ is birational, and

(4.1.4.2) for sufficiently general z, $D_z = D_{z'}$ implies that $z = z'$.

The notation and assumptions of (4.1) remain in force for the rest of the section.

(4.2) Lemma. — *Notation as in* (4.1). *Then either:*

(4.2.1) D_{gen} *intersects infinitely many other curves* D_z;

or:

(4.2.2) D_{gen} *does not intersect any other curve* D_z.

Proof. — If F is generically $k:1$ and $k>1$ then through a general point there are at least k different D_z, thus we have the first case. If F is $1:1$ then for every $x \in X$ the fiber $F^{-1}(x)$ is connected. Thus if D_z and $D_{z'}$ pass through a point x then there are infinitely many other such.

(4.3) Further assumptions. — Assume also that

(4.3.1) C_{gen} is rational;

(4.3.2) For every z, every component of D_z is numerically equivalent to a multiple of D_{gen};

(4.3.3) X is $\mathbb{Q}$-factorial.

Before describing the cases satisfying (4.2.1) we need a simple lemma:

(4.4) Lemma. — *Let* $f: W \to Z$ *be a proper surjective morphism between algebraic varieties. Assume that every fiber is one dimensional and that the generic fiber is a rational curve. Let* $i: Z' \subsetneq W$ *be a closed subvariety such that* $f: Z' \to Z$ *is surjective. Then*

$$N(W) = \langle \textit{components of fibers}, \ i_* N(Z') \rangle.$$

Proof. — Let $C \subset W$ be any curve. Then $f^{-1}(f(C))$ is a possibly reducible surface whose general fiber over $f(C)$ is a connected curve with rational components. It is clear that on this surface any (possibly multiple) section [e.g. $Z' \cap f^{-1}(f(C))$] and the components of fibers generate the set of curves modulo algebraic equivalence.

(4.5) Theorem. — *Notation and assumptions as in* (4.1) *and* (4.3). *Assume that* $F: U \to X$ *satisfies* (4.2.1). *Then* *either:*

(4.5.1) $\dim N(X)=1$; *in particular* X *is Fano if* $D_{gen} . K_X<0$;

or:

(4.5.2) $\dim N(X)=2$ *and there is a map* $q: X \rightarrow E$ *onto a smooth curve* E *such that every* D_z *is contained in a fiber of* q. *The fibers of* q *are all irreducible. If* X *has isolated singularities and* $D_{gen} . K_X<0$ *then the general fiber is a (possibly nonnormal) Del Pezzo surface.*

Proof. — By (4.2.1) there is an irreducible curve $A \subset F^{-1}(D_{gen})$ such that $p(A)$ is one dimensional. Let $S=F(p^{-1}(p(A))) \subset X$. Apply (4.4) with $W=S$, $Z'=A$, $Z=p(A)$ and (4.3.2) to conclude that every curve in S is numerically equivalent to a multiple of D_{gen}.

Now assume that the intersection number $D_{gen} . S$ is positive. Then S intersects every curve D_x. Again apply (4.4) setting

$$W=U \times_Z F^{-1}(S),\ Z=Z$$

and

$$Z'=\{\text{a suitable irreducible component of } F^{-1}(S)\}$$

to conclude that $\dim N(X)=1$.

Now assume that $D_{gen} . S=0$. Thus for any D_z either $D_z \subset S$ or they are disjoint. Letting D_{gen} vary we get a family of disjoint surfaces that cover X. As in (2.4.3) this gives rise to the required map $q: X \rightarrow E$. By construction every D_z is contained in a fiber of q. If there is a reducible fiber $g^{-1}(e)=S_1 \cup S_2$ then we can find a component D_1 of some D_z such that $D_1 \subset S_1$ and D_1 intersects S_2. In particular, $D_1 . S_2>0$. On the other hand, D_{gen} is disjoint from S_2, thus $D_{gen} . S_2=0$. This contradicts (4.3.2).

(4.6) THEOREM. — *Notation and assumptions as in* (4.1) *and* (4.3). *Assume that* $F: U \rightarrow X$ *satisfies* (4.2.2). Then there is a morphism $g: X \rightarrow Y$ *onto a normal projective surface* Y *such that* $g_* \mathcal{O}_X=\mathcal{O}_Y$ *and the fibers of* g *are precisely the curves* D_z *(at least set theoretically).*

If $D_{gen} . K_X<0$ *and the characteristic is different from two then* g *is generically a* $\mathbb{P}^1$ *bundle (in the étale topology). In characteristic two the generic fiber can also be a planar double line.*

The proof will rest on the following:

(4.7) LEMMA. — *Notation and assumptions as in* (4.1) *and* (4.3). *Assume that* $F: U \rightarrow X$ *satisfies* (4.2.2). *Then*

(4.7.1) *If* $A \subset X$ *is an irreducible curve which is numerically equivalent to a multiple of* D_{gen} *then* A *is a component of some* D_z.

(4.7.2) *If* D_z *and* $D_{z'}$ *intersect then they are equal.*

Proof. — In the first case, if A is not a component of some D_z, the curves D_z that intersect A sweep out a surface $S \subset X$. In the second case let $x \in D_z \cap D_{z'}$. Since $F^{-1}(x)$ is connected, the curves $D_{z''}$ passing through x sweep out a surface $S \subset X$. Applying (4.4) to $W=p^{-1}(p(F^{-1}(x)))$, $Z'=F^{-1}(x)$ we see that in both cases every curve in S is numerically equivalent to a multiple of D_{gen}.

Pick a point $s\in S$ and a general curve $s\in G\subset X$. The curves D_g that intersect G sweep out a surface $T\subset X$. We can take a curve $s\in B\subset S$ which is not contained in T. Therefore $B.T>0$. On the other hand, $D_{gen}.T=0$ since they are disjoint. This contradicts the assertion that B is numerically equivalent to a multiple of D_{gen}.

(4.8) CONSTRUCTION OF g. — Consider the Chow variety of curves of X [HP, X.8]. Let Z'_{gen} be the point corresponding to D_{gen} and let Z' be the closure of Z'_{gen}. By (4.7) the variety Z' parametrizes certain 1-cycles whose support is always a curve of the form D_z. Let $p':U'\to Z'$ be the universal family and let $F':U'\to X$ be the natural map. All the above considerations apply for the new covering family.

I claim that F' is $1:1$ on closed points. Any curve C_z maps injectively by the definition of the Chow variety. By (4.7.2), if D_z and $D_{z'}$ intersect then they are equal. Therefore the cycles corresponding to z and z' have the same support, but the multiplicities of the components may be different. However, as we observed above, this implies that there is a one parameter family $z_t\in Z$ such that the cycles parametrised by z_t all have the same support. The multiplicity is a discrete invariant, so this leads to a contradiction. Thus F' is $1:1$ on closed points.

If the characteristic is zero, this implies that F' is an isomorphism. Thus we can take $Y=Z'$.

In positive characteristic the map F' is purely inseparable hence it factors through a power of the Frobenius. Thus we still get a map $X\to Z'$. Take Stein factorization to get $g:X\to Y$.

(4.9) THE GENERIC STRUCTURE OF g. — Consider any irreducible curve $B\subset Y$ and let $F\subset g^{-1}(B)$ be a reduced surface. Assume that F is locally principal except possibly at finitely many points. For $b\in B$ let $C_b\subset F$ be an irreducible component of $g^{-1}(b)$. Let $n:\bar{F}\to F$ be the normalization of F. Let $\bar{C}_b\subset\bar{F}$ be the proper transform of C_b. For sufficiently general b, the surface $\bar{F}$ is smooth along $\bar{C}_b$ and we have the following adjunction formula:

$$-K_{\bar{F}}.\bar{C}_b=-K_X.C_b-F.C_b+(\text{conductor of } n).\bar{C}_b.$$

Note that $-K_X.C_b>0$, $-F.C_b\geqq 0$ and $(\text{conductor of } n).\bar{C}_b\geqq 0$. Therefore $-K_{\bar{F}}.\bar{C}_b>0$, hence $\bar{F}$ is a ruled surface with typical fiber $\bar{C}_b$. In particular $-K_{\bar{F}}.\bar{C}_b=2$. Thus we have to consider three cases:

(4.9.1) *General case*. — $K_X.C_b=2$. Then the conductor does not intersect C_b, hence n is an isomorphism near $\bar{C}_b$. In particular $C_b\cong\mathbb{P}^1$. By the adjunction formula its normal bundle has degree zero, hence is of the form $\mathcal{O}(a)+\mathcal{O}(-a)$. On the other hand its normal bundle inside F is $\mathcal{O}$, thus $\mathcal{O}$ is a subbundle of $\mathcal{O}(a)+\mathcal{O}(-a)$. This implies that $a=0$. Therefore C_b is the whole fiber and we conclude that the general fiber of g is $\mathbb{P}^1$ and it has trivial normal bundle. Thus g is generically a $\mathbb{P}^1$-bundle.

(4.9.2) *Special case.* — $K_X . C_b = 1$, $-F . C_b = 0$. Then the conductor intersects $\bar{C}_b$ transversally at a single point. Therefore $n \mid \bar{C}_b$ is an isomorphism. Thus $C_b \cong \mathbb{P}^1$.

Assume now in addition that B is a general member of a very ample linear system; in particular $F = g^{-1}(B)$ is itself reduced and the general fiber of $F \to B$ is irreducible. Let I be the ideal sheaf of C_b in $\mathcal{O}_F$. Let $I^{(k)}$ be the k-th symbolic power of I and let $\mathrm{gr}^k I = I^{(k)}/I^{(k+1)}$. $\mathrm{gr}^k I$ is a rank one torsion free sheaf on C_b, hence it is locally free and $\mathrm{gr}^k I \cong \mathcal{O}(-e_k)$ for some e_k. Let $\bar{I}$ be the ideal sheaf of $\bar{C}_b$ in $\mathcal{O}_{\bar{F}}$. This is locally free. Let $\bar{I}^k$ be the k-th power of $\bar{I}$ and let $\mathrm{gr}^k \bar{I} = \bar{I}^k/\bar{I}^{k+1}$. Note that $\mathrm{gr}^k \bar{I} \cong \mathcal{O}$. We have a natural map $\mathrm{gr}^k I \to n_* \mathrm{gr}^k \bar{I} \cong \mathrm{gr}^k \bar{I} \cong \mathcal{O}$ which is generically an isomorphism since n is generically an isomorphism along C_b. Thus the numbers e_k are nonnegative.

Let $E \subset F$ be the fiber of g over b. Then $\mathcal{O}_E = \mathcal{O}_F/I^{(t)}$ for some t and $I^{(t)}$ is generated by one section. Thus $\mathrm{gr}^t I$ has a section. Since it has nonpositive degree, we obtain that $\mathrm{gr}^t I \cong \mathcal{O}$. Also, $I^{(mt)}/I^{(mt+t)} \cong \mathcal{O}_F/I^{(t)}$, thus the sequence e_i is periodic with period t.

Sine E is a fiber, its normal bundle is trivial, hence $\omega_E \cong \omega_X \mid E$. Therefore

$$\chi(\omega_E) = \chi(\mathcal{O}_E) + c_1(\omega_X)[E] = \chi(\mathcal{O}_E) + t . c_1(\omega_X)[\mathrm{red}\, E] = \chi(\mathcal{O}_E) - t.$$

On the other hand, $\chi(\omega_E) = -\chi(\mathcal{O}_E)$. Therefore we obtain that

$$\chi(\mathcal{O}_E) = \frac{t}{2} = \sum_{i=0}^{t-1} (1 - e_i). \tag{4.9.2.1}$$

Let $P = \{ i \in \mathbb{N} \mid e_i = 0 \}$. The existence of the multiplication map $\mathrm{gr}^i I \otimes \mathrm{gr}^j I \to \mathrm{gr}^{i+j} I$ implies that P is closed under addition. By (4.9.3) P has density 1/2, thus P consists of all even numbers. Hence $e_i = 0$ if i is even and $e_i = 1$ if i is odd.

This implies that

$$h^0(\mathcal{O}_E) = \chi(\mathcal{O}_E) = \frac{t}{2}.$$

Since E is the general fiber, $h^0(\mathcal{O}_E) = 1$ and we get $t = 2$. Thus $\mathcal{O}_E$ is an extension

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_E \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$

Such extensions are classified by $H^1(\mathbb{P}^1, T_{\mathbb{P}^1}(-1)) = 0$ (*cf.* [H], III.4.10 Exercise). Therefore g is a conic bundle whose general fiber is a planar double line. The double line is regular only in characteristic two, therefore the special case can occur only in characteristic two.

(4.9.3) *Reducible case.* — $K_X . C_b = 1$, $-F . C_b = 1$. Then the conductor does not intersect $\bar{C}_b$. Therefore n is an isomorphism near C_b. Thus $C_b \cong \mathbb{P}^1$ and its normal bundle is $\mathcal{O} + \mathcal{O}(-1)$. Since $F . C_b = -1$, C_b can not be the support of a whole fiber. This can not occur over a general point of Y.

(4.10) THEOREM. — *Let* X *be a normal threefold with* $\mathbb{Q}$*-factorial singularities. Let* $R \subset \overline{NE}(X)$ *be an extremal ray and assume that the rational curves in* R *cover* X. *Then* *we have one of the following cases:*

(4.10.1) $\dim N(X)=1$; *in particular* X *is Fano.*

(4.10.2) $\dim N(X)=2$ *and there is a map* $g:X\to E$ *onto a smooth curve* E *such that if* $C\subset X$ *is an irreducible curve then* $[C]\in R$ *iff* $g(C)=$*point. The fibers of g are all irreducible.*

(4.10.3) *There is a morphism* $g:X\to Y$ *onto a normal projective surface* Y *such that if* $C\subset X$ *is an irreducible curve then* $[C]\in R$ *iff* $g(C)=$*point. If the characteristic is different from two then g is generically a* $\mathbb{P}^1$ *bundle (in the étale topology). In characteristic two the generic fiber can also be a planar double line.*

Proof. — There are only countably many maximal families of rational curves on X. By assumption there is one family whose members are in R and cover X. Let $C_{\mathrm{gen}}\subset X$ be the generic curve of this family. Let $z_{\mathrm{gen}}\in \mathrm{Hilb}\,X$ be the corresponding point of the Hilbert scheme. Let $Z'\subset \mathrm{Hilb}\,X$ be the closure of z_{gen}. In general Z′ has dimension larger than two, so let $Z\subset Z'$ be a sufficiently general two dimensional closed subvariety. Let $p:U\to Z$ be the universal family over Z and let $F:U\to X$ be the natural map.

F is surjective since R covers X and Z is sufficiently general. (4.1.4) is satisfied since Z is a subset of the Hilbert scheme. (4.3.2) is satisfied since R is an extremal ray. The other assumptions are clear from the construction. Thus (4.5) and (4.6) imply (4.10).

(4.11) COMPLEMENT. — *Let* $g:X\to Y$ *be as in* (4.10.3). *Then*

(4.11.1) *If* X *has only finitely many singular points then there is a finite set* $S\subset Y$ *such that*

$$g:\quad X-g^{-1}(S)\to Y-S$$

is a flat conic bundle.

(4.11.2) *If* X *is smooth then* Y *is smooth and g is a flat conic bundle.*

Proof. — Let

$$S=\{y:\text{some component of } g^{-1}(y) \text{ is not smooth}\}\cup\{\mathrm{Sing}\,Y\}\cup g\{\mathrm{Sing}\,X\}.$$

This set is finite by (4.9). Since $X-g^{-1}(S)$ and $Y-S$ are smooth, g is flat over $Y-S$. For $y\in Y-S$ the fiber $g^{-1}(y)$ has no embedded points, $g^{-1}(y).K_X=-2$ and $\chi(g^{-1}(y))=1$. Therefore we have the following possibilities for $g^{-1}(y)$:

(4.11.3.1) $g^{-1}(y)$ is reduced and irreducible. Then $g^{-1}(y)\cong\mathbb{P}^1$.

(4.11.3.2) $g^{-1}(y)$ is reduced and reducible. Then $g^{-1}(y)$ has two components intersecting transversally: two lines intersecting in the plane.

(4.11.3.3) $g^{-1}(y)$ is nonreduced and irreducible. Then $g^{-1}(y)$ is an infinitesimal extension of $\mathbb{P}^1$ with a line bundle of degree -1. This is a planar double line.

This proves (4.11.1).

Now assume that X is smooth. If C_y is a component of $g^{-1}(y)$ then by (5.1) C_y deforms in a flat family. A general deformation is a component of a fiber over $Y-S$ hence a smooth rational curve. Therefore $C_y\cong\mathbb{P}^1$.

Next I want to prove that g is flat. Let H be the normalization of the component of the Hilbert scheme of X which parametrises the fibers of g over $Y-S$ and their specializations. Let $f: U \to H$ be the universal family. We have the following commutative diagram:

(4.11.4)
$$\begin{array}{ccc} U & \xrightarrow{p} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ H & \xrightarrow{q} & Y \end{array}$$

Both p and q are isomorphisms over $Y-S$. U is reduced and irreducible over $Y-S$ and f is flat, thus U is reduced and irreducible. In particular p is an isomorphism if it is one-to-one. Therefore, if q is an isomorphism then so is p.

Let D_y be the reduced fiber of g over $y \in Y$. We want to show that there is exactly one 1-cycle E_y in H whose support is in D_y. We have two cases:

(4.11.5.1) $D_y . K_X = -2$. Then $D_y = E_y$ is the only possibility.

(4.11.5.2) $D_y . K_X = -1$. Then $D_y \cong \mathbb{P}^1$.

Let I be the ideal sheaf of D_y. Since $D_y \cong \mathbb{P}^1$ we can decompose $I/I^2 \cong \mathcal{O}_C(-a) + \mathcal{O}_C(a+1)$. We will prove that $a=1$.

Let C_{gen} be the generic fiber of g. By (4.10.3) $\chi(C_{gen}) = 1$ and $C_{gen} . K_X = -2$ (C_{gen} may not be reduced). We can specialize this curve to a one dimensional connected subscheme $C' \subset X$ such that $\operatorname{supp} C' = D_y$. In general C' may have some embedded points; let C'' be the scheme obtained by removing them. $C'' . K_X = -2$, thus the ideal sheaf J of C'' satisfies $I^2 \subset J \subset I$ and I/J has rank one. Also, $1 = \chi(C') \geqq \chi(C'') = 1 + \chi(I/J)$. This implies that $\deg(I/J) \leqq -1$. Therefore $a \geqq 1$.

If $a > 1$ then let J' be the ideal sheaf of $\mathcal{O}_X$ generated by I^2 and by the $\mathcal{O}_C(a+1)$ summand of I/I^2. Let $D = \operatorname{Spec} \mathcal{O}_X/J'$. Note that $\chi(D) = 2-a$. $D . K_X = -2$ hence by (5.1) D moves in an at least two dimensional family. If D_s is a deformation of D then every component of D_s is a component of a fiber of g, thus it is a smooth rational curve. D_s can not have two connected components, since this would give $\chi(D) = \chi(D_s) = 2$. D_s can not be contained in the open set where X is a $\mathbb{P}^1$-bundle (or a double line bundle in characteristic two) since there every curve D' whose support is a single fiber and which satisfies $D' . K_X = -2$ always has $\chi(D') \geqq 1$. Therefore $\operatorname{supp} D_s$ moves in a one dimensional family only. Thus there has to be a one dimensional family of deformations keeping the support fixed. This is impossible since the $\mathcal{O}_C(a+1)$ summand is unique. This proves that we must have $a=1$ and $E = \operatorname{Spec} \mathcal{O}_X/J$ is the only possibility for E.

This proves that $g=f$ is flat. Now Y is smooth by [Ma], 21.D, and as before we see that g is a conic bundle.

(4.12) *Example.* — In characteristic two it is possible that every fiber is a double line. For example, in $\mathbb{P}^2 \times \mathbb{P}^2$ with homogeneous coordinates $(x:y:z, u:v:w)$ consider the smooth hypersurface $X = (xu^2 + yv^2 + zw^2 = 0)$. Projection to the first factor makes it into a conic bundle and every fiber is a double line.

We can also get an example of a threefold Z in characteristic two which has Kodaira dimension $-\infty$ and even satisfies condition (NC) of [MM] but Z is not separably uniruled. The example is a special case of a recent construction of E. Sato [S].

Using X one can easily construct a threefold Z which is a conic bundle over a nonuniruled smooth surface S such that the general fiber is a double line. Clearly $\kappa(Z)=-\infty$.

Let T be the component of the Hilbert scheme of Z parametrising the reduced fibers of Z. It is easy to see that the natural map $T \to S$ is purely inseparable of degree 4. (In the example of X the function field of T is $k(\sqrt{x/z}, \sqrt{y/z})$.) Let $U \to T$ be the universal family. The natural map $U \to Z$ is also purely inseparable of degree 4.

Assume that Z is separably uniruled. Then there is a surface F and a separable dominant rational map

$$F \times \mathbb{P}^1 \dashrightarrow Z.$$

By the assumption on S and the definition of $U \to T$ we get the following diagram

(4.12.1)
$$\begin{array}{ccccc} F \times \mathbb{P}^1 & \to & U & \to & Z \\ \downarrow & & \downarrow & & \downarrow \\ F & \to & T & \to & S \end{array}$$

This contradicts the assumption that $F \times \mathbb{P}^1 \dashrightarrow Z$ is separable.

(4.13) *Remark*. — (4.13.1) Let $g: X \to Y$ be as in (4.11.1). The surface Y can have only very special singularities. Indeed, for $y \in Y$ let $x \in g^{-1}(y)$ be a smooth point of X. Let $x \in H \subset X$ be a general smooth surface germ. Then $g: (x, H) \to (y, Y)$ is finite and surjective. In characteristic zero this implies that (y, Y) is a quotient singularity [B, 2.8]. In positive characteristic the situation is less clear.

(4.13.2) If X has only hypersurface singularities and the characteristic is different from two then one can prove that Y is smooth and g is flat. I don't know how to prove this in characteristic two.

5. The nonprojective case

(5.1) Ein's Deformation Lemma. — *Let* X *be a three dimensional smooth algebraic space resp. a three dimensional complex manifold. Let* $C \subset X$ *be a one dimensional proper subscheme without embedded points. Then the dimension of any component of the Hilbert (resp. Douady) scheme containing* [C] *is at least* $-C.K_X$.

Remark. — This is proved in [E], Lemma 5, for $X=\mathbb{P}^3$, for $X=\mathbb{P}^3$. The same proof works for any projective X, but needs some changes in general.

Proof. — Before we get into the general case, let us review the case when X is projective.

Let I be the ideal sheaf of C. I has a length two resolution by locally free sheaves

$$(5.1.1) \qquad 0 \to \mathrm{F} \to \mathrm{E} \to \mathrm{I} \to 0.$$

First Hom it into $\mathcal{O}_X$ to get

$$(5.1.2) \qquad 0 \to \mathrm{Hom}(\mathrm{I}, \mathcal{O}_X) \to \mathrm{E}^* \to \mathrm{F}^* \to \mathrm{Ext}^1(\mathrm{I}, \mathcal{O}_X) \to 0,$$

and note that

$$(5.1.3) \qquad \mathrm{Ext}^1(\mathrm{I}, \mathcal{O}_X) \cong \mathrm{Ext}^2(\mathcal{O}_C, \mathcal{O}_X) \cong \omega_C \otimes \omega_X^{-1}.$$

Next Hom (5.1.1) into $\mathcal{O}_C$ to get

$$(5.1.4) \qquad 0 \to \mathrm{Hom}(\mathrm{I}, \mathcal{O}_C) \to \mathrm{E}^* \,|\, \mathrm{C} \to \mathrm{F}^* \,|\, \mathrm{C} \to \mathcal{Q} \to 0.$$

From (5.1.2) we see that $\mathcal{Q} \cong \omega_C \otimes \omega_X^{-1}$. Also note that $\mathrm{Hom}(\mathrm{I}, \mathcal{O}_C) \cong \mathrm{Hom}(\mathrm{I}/\mathrm{I}^2, \mathcal{O}_C)$. Thus we have the following exact sequence:

$$(5.1.5) \qquad 0 \to \mathrm{Hom}(\mathrm{I}/\mathrm{I}^2, \mathcal{O}_C) \to \mathrm{E}^* \,|\, \mathrm{C} \to \mathrm{F}^* \,|\, \mathrm{C} \to \omega_C \otimes \omega_X^{-1} \to 0.$$

Since I is the ideal sheaf of a curve on a threefold, its first Chern class is zero. Therefore $c_1(\mathrm{E}) = c_1(\mathrm{F})$. Now we can calculate $\chi(\mathrm{Hom}(\mathrm{I}/\mathrm{I}^2, \mathcal{O}_C))$ from the sequence (5.1.5) and we obtain that

$$\chi(\mathrm{Hom}(\mathrm{I}/\mathrm{I}^2, \mathcal{O}_C)) = -\mathrm{C}.\mathrm{K}_X.$$

A curve in a smooth threefold has no local deformation obstructions, thus the above χ is a lower bound for the dimension of the Hilbert scheme at [C] (*see* [G], VI.5).

If X is not quasi-projective then the resolution (5.1.1) need not exist. However we will be able to mimick the above proof.

First choose n sufficiently large such that for any local section $f \in \Gamma(\mathrm{U}, \mathcal{O}_X)$

$$(5.1.6) \qquad f.\mathrm{I} \subset \mathrm{I}^{(n)} \quad \Rightarrow \quad f \in \mathrm{I}.$$

Now let $n\mathrm{C} = \mathrm{Spec}\, \mathcal{O}_X/\mathrm{I}^{(n)}$; the n-th order thickening of C. This is one dimensional, hence projective. We have the following partial analog of the free resolution (5.1.1):

$$(5.1.7) \qquad 0 \to \mathrm{F}_n \to \mathrm{E}_n \to \mathrm{I} \otimes \mathcal{O}_{n\mathrm{C}} \to 0.$$

Here we can choose E_n to be locally free but F_n will not be locally free. Locally in the étale or Eucledean topology on X we can lift the above complex to a free resolution as follows. Let $\mathrm{U} \subset \mathrm{X}$ be a sufficiently small open set. Then we can have

$$(5.1.8) \qquad 0 \to \mathrm{F}_U \to \mathrm{E}_U \to \mathrm{I}_U \to 0$$

such that $\mathrm{E}_U \otimes \mathcal{O}_{n\mathrm{C}} \cong \mathrm{E}_n$. Tensoring (5.1.8) by $\mathcal{O}_{n\mathrm{C}}$ we obtain

$$(5.1.9) \qquad 0 \to \mathrm{Tor}_U^1(\mathrm{I}, \mathcal{O}_{n\mathrm{C}}) \to \mathrm{F}_U \otimes \mathcal{O}_{n\mathrm{C}} \to \mathrm{E}_n \,|\, \mathrm{U} \to \mathrm{I}_U \otimes \mathcal{O}_{n\mathrm{C}} \to 0.$$

Note that $\mathrm{Tor}^1(\mathrm{I}, \mathcal{O}_{n\mathrm{C}}) \cong \mathrm{Tor}^2(\mathcal{O}_\mathrm{C}, \mathcal{O}_{n\mathrm{C}})$, in particular it is killed by I. By (5.1.6) this implies that

$$\text{(5.1.10)} \qquad \mathrm{im}\,[\mathrm{Tor}^1_\mathrm{U}(\mathrm{I}, \mathcal{O}_{n\mathrm{C}})] \subset \mathrm{I}\,.\,(\mathrm{F}_\mathrm{U} \otimes \mathcal{O}_{n\mathrm{C}}).$$

Therefore

$$\text{(5.1.11)} \qquad \mathrm{Hom}_\mathrm{U}(\mathrm{F}_n, \mathcal{O}_\mathrm{C}) \cong \mathrm{Hom}_\mathrm{U}(\mathrm{F}_\mathrm{U} \otimes \mathcal{O}_{n\mathrm{C}}, \mathcal{O}_\mathrm{C}).$$

Next Hom (5.1.7) into $\mathcal{O}_\mathrm{C}$ to get

$$\text{(5.1.12)} \quad 0 \to \mathrm{Hom}_{n\mathrm{C}}(\mathrm{I}/\mathrm{I}^2, \mathcal{O}_\mathrm{C}) \to \mathrm{Hom}_{n\mathrm{C}}(\mathrm{E}_n, \mathcal{O}_\mathrm{C}) \to \mathrm{Hom}_{n\mathrm{C}}(\mathrm{F}_n, \mathcal{O}_\mathrm{C}) \to \mathrm{Ext}^1_{n\mathrm{C}}(\mathrm{I} \otimes \mathcal{O}_{n\mathrm{C}}, \mathcal{O}_\mathrm{C}) \to 0.$$

This is the sequence we will use as a replacement for (5.1.5). By the above remarks locally on U this sequence is isomorphic to

$$\text{(5.1.13)} \quad 0 \to \mathrm{Hom}_\mathrm{U}(\mathrm{I}/\mathrm{I}^2, \mathcal{O}_\mathrm{C}) \to \mathrm{Hom}_\mathrm{U}(\mathrm{E}_\mathrm{U}, \mathcal{O}_\mathrm{C}) \to \mathrm{Hom}_\mathrm{U}(\mathrm{F}_\mathrm{U}, \mathcal{O}_\mathrm{C}) \to \mathrm{Ext}^1_\mathrm{U}(I, \mathcal{O}_\mathrm{C}) \to 0.$$

This gives canonical local isomorphisms

$$\text{(5.1.14)} \qquad \mathrm{Ext}^1_{n\mathrm{C}}(\mathrm{I} \otimes \mathcal{O}_{n\mathrm{C}}, \mathcal{O}_\mathrm{C}) \,|\, \mathrm{U} \cong \mathrm{Ext}^1_\mathrm{U}(\mathrm{I}, \mathcal{O}_\mathrm{C}) \cong \omega_\mathrm{C} \otimes \omega_\mathrm{X}^{-1} \,|\, \mathrm{U}.$$

The two sides of (5.1.14) are defined on $n\mathrm{C}$, thus they are globally isomorphic over $n\mathrm{C}$.

Next we need that $c_1(\mathrm{Hom}(\mathrm{E}_n, \mathcal{O}_\mathrm{C})) = c_1(\mathrm{Hom}(\mathrm{F}_n, \mathcal{O}_\mathrm{C}))$. This follows from local considerations again. If we restrict (5.1.8) to $\mathrm{U}-\mathrm{C}$ then we get

$$\text{(5.1.15)} \qquad 0 \to \mathrm{F}_{\mathrm{U}-\mathrm{C}} \to \mathrm{E}_{\mathrm{U}-\mathrm{C}} \to \mathrm{I}_{\mathrm{U}-\mathrm{C}} \to 0.$$

Since $\mathrm{I}_{\mathrm{U}-\mathrm{C}} \cong \mathcal{O}_{\mathrm{U}-\mathrm{C}}$ we have a canonical isomorphism

$$\det \mathrm{E}_{\mathrm{U}-\mathrm{C}} \otimes \det^{-1} \mathrm{F}_{\mathrm{U}-\mathrm{C}} \cong \mathcal{O}_{\mathrm{U}-\mathrm{C}}.$$

Since C has codimension two, this extends to an isomorphism

$$\det \mathrm{E}_\mathrm{U} \otimes \det^{-1} \mathrm{F}_\mathrm{U} \cong \mathcal{O}_\mathrm{U}.$$

This will give us a canonical isomorphism

$$\text{(5.1.16)} \quad \det(\mathrm{Hom}(\mathrm{E}_n, \mathcal{O}_\mathrm{C})) \otimes \det^{-1}(\mathrm{Hom}(\mathrm{F}_n, \mathcal{O}_\mathrm{C})) \cong \det \mathrm{F}_\mathrm{U} \otimes \det^{-1} \mathrm{E}_\mathrm{U} \,|\, \mathrm{C} \cong \mathcal{O}_\mathrm{C}.$$

The two sides again are defined globally over $n\mathrm{C}$ hence they are isomorphic. Now we get the same formula for $\chi(\mathrm{Hom}(\mathrm{I}/\mathrm{I}^2, \mathcal{O}_\mathrm{C}))$ as before.

(5.2) Corollary. — *Let* $\mathrm{C} \subset \mathrm{X}$ *be as in* (5.1). *Let* $x_1, \ldots, x_n \in \mathrm{C}$. *Let* $\mathrm{Def}(\mathrm{C})_{x_1, \ldots, x_n}$ *denote those deformations of* C *that pass through all the points* $x_1, \ldots, x_n$. *Then*

$$\dim \mathrm{Def}(\mathrm{C})_{x_1, \ldots, x_n} \geqq -\mathrm{C}\,.\,\mathrm{K}_\mathrm{X} - 2n.$$

If there is a surface $C \subset S \subset X$ *such that every deformation of* C *is contained in* S *then*

$$\dim \operatorname{Def}(C)_{x_1, \ldots, x_n} \geqq -C.K_X - n.$$

Proof. — In the first case, it is two conditions to pass through a given point x_i, in the second case only one condition.

As an application of the nonprojective case of (5.1) we prove a variant of (2.3) for nonprojective threefolds.

(5.3) THEOREM. — *Let* X *be a three dimensional smooth proper algebraic space resp. a three dimensional compact complex manifold. Assume that* X *contains a rational curve which has negative intersection with the canonical class. Assume furthermore that* $\kappa(X) \geqq 0$. *Then one of the following holds:*

(5.3.1) *either* X *contains rational curves* B_i *such that* $\sum B_i$ *is algebraically equivalent to zero;*

(5.3.2) *or* X *contains a surface* E *which is one of those listed in* (2.1.2.1-4).

Remark. — Algebraic equivalence is usually not defined for nonalgebraic manifolds. It will become clear from the proof that the usual notion makes sense in our case.

Proof. — Since $\kappa(X) \geqq 0$, there is a pluricanonical divisor D. Let $f_1 : \mathbb{P}^1 \to C_1$ be the required rational curve. Since $C_1 . K_X < 0$, $C_1 \subset D$. There is a nontrivial deformation $f_{1,t}$ of f_1 and all the images $f_{1,t}(\mathbb{P}^1)$ are contained in D. Let D_1 be the irreducible component of D that contains all these curves.

We claim that D_1 is an algebraic space. Indeed, its desingularization contains infinitely many rational curves. Therefore it can not have algebraic dimension zero [BPV], p. 129, and it also can not be elliptic without sections.

Thus, as in (2.3) we get a morphism from a not necessarily minimal ruled surface Z to X which does not contract components of fibers. If Z is minimal then (2.1) applies and we get the second alternative.

Otherwise there is a reducible fiber and so C_1 is algebraically equivalent to $C_2 + B_2$. Here $C_2 . K_X < 0$ and B_2 is nonempty (possibly $C_2 = B_2$). Continuing in this manner we get a series of curves and algebraic equivalences

$$C_i \approx C_{i+1} + B_{i+1}.$$

If we ever stop then we get the second possibility. Otherwise we have an infinite sequence of curves as above. I claim that there must be repetitions in the sequence C_i. If $C_i = C_j$ and $j > i$ then

$$\sum_{k=i+1}^{j} B_k \approx 0.$$

This is the first possibility.

The singular locus of D contains only finitely many irreducible curves. Thus we have a repetition if infinitely many of the C_i are contained in the singular locus of D.

Therefore it is sufficient to treat the case when none of the C_i are contained in the singular locus of D. If C_1 is contained in the (unique) irreducible component D_1, then the above deformation must happen in D_1. The same argument applies for C_2, thus we get that every deformation happens inside D_1. Moreover, if $\bar{D}_1$ is the normalization of D_1 then we can lift the curves C_i and all the deformations to $\bar{D}_1$.

Thus we have the following situation:

$\bar{D}_1$ is a normal, proper two dimensional algebraic space. It contains two infinite sequences of nonzero effective curves $\bar{C}_i$ and $\bar{B}_i$ such that

$$\bar{C}_i \approx \bar{C}_{i+1} + \bar{B}_{i+1}.$$

Let $g: \tilde{D}_1 \to \bar{D}_1$ be a desingularization and let H be an ample divisor on $\tilde{D}_1$. Let $\bar{H} = g_*(H)$. This $\bar{H}$ is of course not necessarily Cartier, but one can define intersection numbers of effective curves and of $\bar{H}$. These are always positive rational numbers since H is ample. The possible denominators depend only on the singularities of the surface, hence they form a bounded set. Therefore the equality

$$\bar{H}.\bar{C}_1 = \bar{H}.\bar{C}_j + \sum_{i=2}^{j} \bar{H}.\bar{B}_i$$

leads to a contradiction for large j. This proves the theorem.

REFERENCES

[BPV] W. BARTH, C. PETERS and A. VAN DE VEN, *Compact Complex Surfaces*, Springer, 1984.

[B] E. BRIESKORN, *Rationale Singularitäten komplexer Flächen* (*Invt. Math.*, Vol. 4, 1968, pp. 336-358).

[C] S. D. CUTKOSKY, *Elementary Contractions of Gorenstein Threefolds* (*Math. Ann.*, Vol. 280, 1988, pp. 521-525).

[E] L. EIN, *Hilbert Scheme of Smooth Space Curves* (*Ann. scient. Éc. Norm. Sup.*, Vol. 19, 1986, pp. 469-478).

[F] T. FUJITA, *Semipositive line bundles* (*J. Fac. Sci. Univ. Tokyo*, Vol. 30, 1983, pp. 353-378).

[G] A. GROTHENDIECK, *Fondements de la Géométrie Algébrique*, Secrétariat Math., Paris, 1962.

[H] R. HARTSHORNE, *Algebraic Geometry*, Springer, 1977.

[HP] W. HODGE and D. PEDOE, *Methods of Algebraic Geometry*, Cambridge Univ. Press, 1952.

[HR] H. HIRONAKA and H. ROSSI, *On the Equivalence of Embeddings of Exceptional Complex Spaces* (*Math. Ann.*, Vol. 156, 1964, pp. 313-333).

[Ma] H. MATSUMURA, *Commutative Algebra*, second edition, Benjamin/Cummings, 1980.

[MM] Y. MIYAOKA and S. MORI, *A Numerical Criterion for Uniruledness* (*Ann. of Math.*, Vol. 124, 1986, pp. 65-69).

[M 1] S. MORI, *Projective Manifolds with Ample Tangent Bundles* (*Ann. of Math.*, Vol. 110, 1979, pp. 593-606).

[M 2] S. MORI, *Threefolds whose Canonical Bundles are not Numerically Effective* (*Ann. of Math.*, Vol. 116, 1982, pp. 133-176).

[Mu] D. MUMFORD, *The Topology of Normal Singularities of an Algebraic Surface and a Criterion for Simplicity* (*Publ. Math. I.H.E.S.*, Vol. 9, 1961, pp. 5-22).

[N] V. V. NIKULIN, *Algebraic Threefolds and Diagram Method* (*Math. USSR Izv.*, to appear).

[S] E. SATO, *On a Criterion of Uniruledness in Positive Characteristic*, preprint, 1989.

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