CLAUDE DANTHONY
ARNALDO NOGUEIRA

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MEASURED FOLIATIONS ON NONORIENTABLE SURFACES

BY CLAUDE DANTHONY AND ARNALDO NOGUEIRA (*)

ABSTRACT. — In this paper, we show that, for a nonorientable surface M, almost all measured foliations in $\mathcal{MF}$(M) have a compact leaf which is a one-sided curve. In order to do this, we introduce and study a generalization of interval exchange transformations.

RÉSUMÉ. — Nous montrons que presque tout feuilletage mesuré d'une surface non orientable possède une feuille compacte. Pour cela, nous utilisons les involutions linéaires, qui généralisent les échanges d'intervalles.

Key words: Measured foliation, Nonorientable surface, Interval exchange.

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0. Introduction

Let M be a closed surface of negative Euler characteristics $\chi(M)$. The set of equivalence classes of measured foliations on M is denoted by $\mathcal{MF}(M)$. Recall ([T 1], [T 2], FLP]) that $\mathcal{MF}(M)$ is a PL manifold which is homeomorphic to $\mathbb{R}^{-3\chi(M)} \setminus \{0\}$. Therefore $\mathcal{MF}$ has a Lebesgue measure class.

In the case where M is orientable, we have the following theorem by H. Masur [M]:

THEOREM (Masur). — *If* M *is orientable, then almost all elements of* $\mathcal{MF}$ *are uniquely ergodic*. (The methods we shall introduce allow us to give a new proof of this result.)

When M is nonorientable, it is easy to see that the set of foliations having a compact leaf which is a one-sided curve is open and dense in $\mathcal{MF}$. Therefore the minimal foliations are not dense (recall that, for measured foliations, unique ergodicity implies minimality). However, there exist minimal foliations ([AY] or [G]), and even uncountably many [N 1].

In this paper, we prove the following result:

THEOREM I. — *If* M *is nonorientable, then almost all measured foliations on* M *have a compact leaf which is a one-sided curve.*

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The origin of Masur's theorem is the following:

There is a relation between measured foliations and interval exchange transformations; more precisely, let $\mathscr{F}$ be a measured foliation and $\mathscr{G}$ be its orientation covering foliation, then every first return application on a transversal for $\mathscr{G}$ is an interval exchange transformation. There was a conjecture by M. Keane that almost all interval exchange transformations which preserve orientation are uniquely ergodic. W. Thurston asked the natural equivalent question for measured foliations, which is Masur's theorem.

Keane's conjecture was proved by H. Masur ([M]) and W. Veech ([V 2]).

The second author proved [N 2] that almost all interval exchanges with flips (*i.e.* not orientation preserving) have a periodic point with negative derivative. It was then natural to conjecture the result of Theorem I.

Trying to deduce Theorem I from this result about interval exchange transformations, we first proved that, if E is a linear subset of $\mathscr{M}\mathscr{F}$ containing only orientable foliations, almost all (for the Lebesgue measure of E) elements of E have a compact leaf. This is Theorem II. This result is restricted to orientable foliations because only in this case the first return maps are interval exchanges. Moreover, if we consider the return transformations for the orientation covering foliations, we loose the surjectivity of the application which takes foliations to interval exchanges.

THEOREM II. — *If* M *is nonorientable, and if* E *is a linear subset of* $\mathscr{M}\mathscr{F}$ (M) *given by a train track which contains only orientable foliations, then almost all elements of* E *have a compact leaf.*

In other words, there is a strong relation between *orientable* measured foliations and interval exchange transformations, but there is no relation between measured foliations and interval exchanges good enough to obtain Theorem I.

In Section 1, we introduce the *linear involutions*, a generalization of interval exchanges. The difference between interval exchanges and linear involutions is the same as the difference between orientable foliations and foliations. We use them to prove Theorem I.

Given a foliation $\mathscr{F}$ (even not orientable), and a transverse segment I, we can cut the surface along I, obtaining a new surface M′, two copies of I, namely I_+ and I_-, and a foliation $\mathscr{F}'$ of M′. By Poincaré's recurrence, each regular leaf of $\mathscr{F}'$ which cuts $I_+ \cup I_-$ is a segment. We can then define the involution of $I_+ \cup I_-$ minus finitely many points, which assigns to each point x the other endpoint of the leaf of $\mathscr{F}'$ going through x.

A linear involution is a transformation of the form $\sigma \circ T'$, where T′ is a continuous involution of $I_+ \cup I_-$ minus finitely many points, which preserves Lebesgue measure, and σ is the involution which exchanges I_+ and I_-. The space of linear involutions is the disjoint union of open subsets of linear spaces. Then we prove:

THEOREM III. — *Almost all linear involutions with flip have a periodic point with negative derivative.*

With exactly the same methods, we establish a generalization of the theorem of Veech ([V 2]) to linear involutions:

THEOREM IV. — *Almost all linear involutions without flip are uniquely ergodic.*

The proofs of Theorems III and IV are combinatorial. They use an *induction process* similar to the one introduced by R. Rauzy in [R]. In order to study a linear involution, we look at linear involutions induced on smaller intervals, and construct a sequence of nonnegative matrices which describe how the induced involution depends on the original one. Then we characterize the properties of a linear involution by the properties of the sequence associated to it.

Section 2 is devoted to prove that linear involutions correspond well to measured foliations. First, we recall some definitions about measured foliations and train tracks. Train tracks give the PL structure of $\mathcal{MF}$. We relate this structure with the linear structure of the space of linear involutions. We study carefully the existence, for foliations carried by a given train track, of unstable connections. Then we establish that the applications which take foliations to linear involutions have good properties: they are, for x in a full measure set, linear and onto in a neighborhood of x.

In Section 3, we use the result obtained in Section 2 to deduce Theorem I from Theorem III and Masur's theorem from Theorem IV. In order to illustrate that interval exchanges correspond well to orientable measured foliations, we deduce Theorem II from a result of Nogueira [N 2] on interval exchanges.

In the end of Section 3, we answer some questions, and complete the results by some examples.

These results were announced in [DN].

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1. Linear involutions

1.1. DEFINITIONS, STATEMENT OF THE RESULTS. — Let I be an interval. We denote by $I_+ = I \times \{+1\}$, $I_- = I \times \{-1\}$, and σ the involution of $I_+ \cup I_-$ given by $\sigma((x, \varepsilon)) = (x, -\varepsilon)$.

DEFINITION. — We call *linear involution* an application T of the form $\sigma \circ T'$, where T' is an involution of $I_+ \cup I_-$ without fixed point, continuous except in finitely many points, and which preserves the Lebesgue measure. We denote by Λ the set of linear involutions.

Remarks. — * Let E be an interval exchange transformation on I. We can consider E as a linear involution, setting $T=E$ on I_+ and $T=E^{-1}$ on I_-.

* The set Λ is not a group: the composition of two linear involutions is not a linear involution. But the power of a linear involution is a linear involution: we can iterate the elements of Λ.

A linear involution is characterized by: — the number of intervals exchanged by T, — the way they are exchanged (a permutation), — the derivative (± 1) of T on each interval exchanged, — the lengths of the intervals exchanged. More precisely:

Let $T=\sigma\circ T'$ be a linear involution. We denote by J_k the maximum open intervals where T' is continuous. We denote by p (respectively q) the number of intervals J_k which are contained in I_+ (respectively I_-). The intervals J_k included in I_+ (respectively I_-) are enumerated from 1 to p (respectively, from $p+1$ to $p+q$).

We denote by φ the involution (without fixed point) of $\{1, \ldots, p+q\}$ such that $T'(J_k)=J_{\varphi(k)}$. Let λ_k be the length of the interval J_k. Then $(\lambda_1, \ldots, \lambda_{p+q})$ belongs to the following set:

$$\Lambda_{p,q,\varphi}=\{(\lambda_1, \ldots, \lambda_{p+q})\in(\mathbb{R}_+^*)^{p+q} : \lambda_i=\lambda_{\varphi(i)} \text{ and } \lambda_1+\ldots+\lambda_p=\lambda_{p+1}+\ldots+\lambda_{p+q}\}.$$

In some cases, the last equation is a consequence of the others. This happens exactly when we are considering interval exchanges.

We define $s_\varphi(i)=1$ if J_i and $T'(J_i)$ are both on I_+ or both on I_-, and $s_\varphi(i)=-1$ otherwise ($s_\varphi(i)=1$ if $(p+(1/2)-i)(p+(1/2)-\varphi(i))>0$, and $s_\varphi(i)=-1$ otherwise). Let F be the subset of $\{1, \ldots, p+q\}$ consisting of i such that the derivative of T' on J_i is equal to $s_\varphi(i)$.

We draw some examples in Figure 1. Notice that i belongs to F if, and only if we cannot draw a flow on a band, which stays in the plane of the picture, and which induces T' on J_i as a first return map.

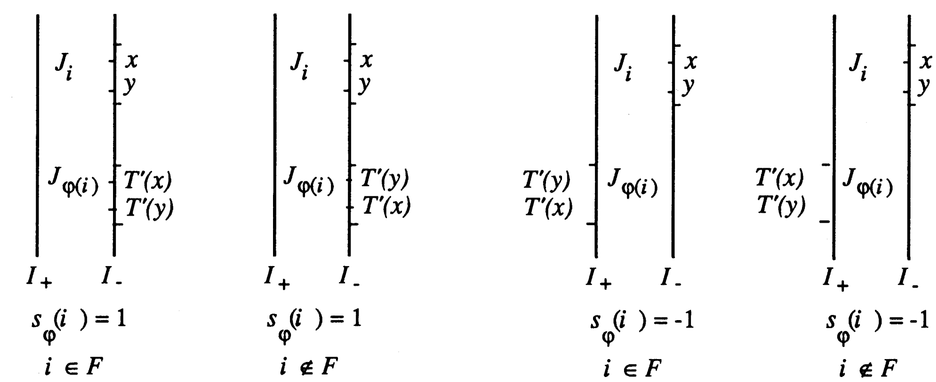


Fig. 1

Definition. — Let T be a linear involution, and $\alpha=(p, q, \varphi, F)$ be as above. We say that T is *of type* α. We denote by Λ_α the set of linear involutions of type α.

Conversely, let p and q be two positive integers, φ be an involution without fixed point of $\{1, \ldots, p+q\}$, and F be a subset of $\{1, \ldots, p+q\}$ which is invariant by φ, and $\alpha=(p, q, \varphi, F)$. The application which assigns to each linear involution the vector $(\lambda_1, \ldots, \lambda_{p+q})$ is a bijection from Λ_α onto $\Lambda_{p, q, \varphi}$. We give to these sets the Lebesgue measure of the second one.

DEFINITION. — We say that T is a *linear involution with flip* if it is of type $\alpha=(p, q, \varphi, F)$, with $F \neq \varnothing$ and $F \neq \{i : s_\varphi(i)=1\}$. This condition is necessary for the existence of a periodic point with negative derivative for T.

In the first section of this paper, we prove the following theorem, which is a generalization of a theorem of Nogueira [N 2].

THEOREM III. — *Almost all linear involutions with flip have a flipped periodic point*, i. e. *there exist* $x_0 \in I_+$ *and* $k \in \mathbb{N}$, *such that* $T^k(x_0)=x_0$ *and the derivative of* T^k *in* x_0 *is equal to* -1.

Remark. — Moreover, it is clear that this property is open in the set of linear involutions.

Using the same methods, one can prove a generalization of a theorem of Masur [M] and Veech [V] and use it to give a proof of Masur's theorem:

THEOREM IV. — *If* $\alpha=(p, q, \varphi, F)$, $F=\varnothing$ *and if* α *is such that the minimal foliations are dense in* Λ_α, *then almost all linear involutions of* Λ_α *are uniquely ergodic.*

1.2. THE INDUCING PROCESS. — Let T be a linear involution of type $\alpha=(p, q, \varphi, F)$, and $(\lambda_1, \ldots, \lambda_{p+q})$ be the lengths of the intervals exchanged by T. There can be three cases:

(i) $\lambda_p=\lambda_{p+q}$;

(ii) $\lambda_p<\lambda_{p+q}$, we set $I'=[0, \lambda_1+\ldots+\lambda_{p-1}]$, $U((x, \varepsilon))=T((x, \varepsilon))$ if $T((x, \varepsilon))$ is in $I' \times \{-1,1\}$, and $U((x, \varepsilon))=T^2((x, \varepsilon))$ otherwise.

(iii) $\lambda_p>\lambda_{p+q}$, we exchange I_+ and I_- and induce on $[0, \lambda_{p+1}+\ldots+\lambda_{p+q-1}]$.

In cases (ii) and (iii), we say that U is the induced of T, and we write: $U=\mathrm{ind}(T)$. In case (i), we cannot induce. This type of induction was first introduced by Rauzy [R].

LEMMA 1.1. — U *is a linear involution which exchanges the same number of intervals as* T, *and which type depends only on* α *and if we are in case* (ii) *or* (iii).

* or $\varphi(p)=\varphi(p+q)$, then all linear involutions of type α are in case (i), and ind(T) can not be defined.

* or $\varphi(p)\neq\varphi(p+q)$, and the equality $\lambda_p=\lambda_{p+q}$ holds only in a hyperplane in Λ_α. ind sends $\Lambda_\alpha \cap \{\lambda_p<\lambda_{p+q}\}$ to $\Lambda_{\alpha'}$ (α' a type), and sends $\Lambda_\alpha \cap \{\lambda_p>\lambda_{p+q}\}$ to $\Lambda_{\alpha''}$ (α'' a type).

Choosing a $\mu_j=\lambda_i=\lambda_{\varphi(i)}$ for each pair $(i, \varphi(i))$, we have an isomorphism:

$\Lambda_\alpha \approx (\mathbb{R}_+^*)^{N+1} \cap S_\alpha$, where $N=(1/2)(p+q)-1$, and S_α is the space of solutions of the equation in μ_i associated to $\lambda_1+\ldots\lambda_p=\lambda_{p+1}+\ldots+\lambda_{p+q}$.

Remark. — If we make an induction, from Λ_α to $\Lambda_{\alpha'}$, this isomorphism determines an isomorphism $\Lambda_{\alpha'} \approx (\mathbb{R}_+^*)^{N+1} \cap S_{\alpha'}$.

LEMMA 1.2. — *Suppose that* ind *sends* $\Lambda_\alpha \cap \{\lambda_p < \lambda_{p+q}\}$ *to* $\Lambda_{\alpha'}$. *The inverse of* ind, *considered from* $S_{\alpha'}$ *to* S_α, *is the restriction of a linear transformation of* Λ_N *given by an elementary* $(N+1)\times(N+1)$ *matrix* E, i. e. *one obtains the lengths of the intervals exchanged by* T *multiplying by* E *the lengths of the intervals exchanged by* U.

Lemmas 1.1 and 1.2 are not difficult. We omit their proofs.

Let T be in Λ_α, and suppose that the lengths of the intervals exchanged by T do not satisfy any rational relation which is not in the definition of $\Lambda_{p,q,\varphi}$. As we are concerned with measure properties, we shall now consider only such T.

We have a sequence ind^k (T), which are in Λ_{α_k}. For each k we have an elementary matrix E_k, which gives the inverse of ind on Λ_{α_k}. We denote by A_k the product $E_1 \ldots E_k$. The matrix A_k is a nonnegative matrix which gives the inverse of ind^k on Λ_{α_k}.

DEFINITION. — The sequence of matrices A_k (or equivalentely the product $E_1 . E_2 \ldots$) is called the *expansion* of T.

As we want to compute measures, it is sometimes easier to use sets of finite measure. For this reason, we projectivize the set of linear involutions. We consider only linear involutions on intervals of total length 1. The projectivized version of a set denoted by Λ will be denoted by Δ:

$$\Delta_{p,q,\varphi} = \{(\lambda_1, \ldots, \lambda_{p+q}) \in (\mathbb{R}_+^*)^{p+q} : \lambda_i = \lambda_{\varphi(i)}, \lambda_1 + \ldots + \lambda_p = \lambda_{p+1} + \ldots + \lambda_{p+q} = 1\}$$

Δ_α is the set of linear involutions such that the lengths of the intervals are in $\Delta_{p,q,\varphi}$.

We denote by Σ_N the standard N-dimensional simplex.

So, we have an application $\underline{\mathrm{ind}}$ which corresponds to ind up to the normalization of the interval I′.

We denote by $\mathscr{L}_A$ the transformation from Σ_N to itself associated to the nonnegative matrix A: $\mathscr{L}_A(x) = A(x)/|A(x)|$. The transformation $\mathscr{L}_{A_k}$ gives the inverse of $\underline{\mathrm{ind}}^k$ on Δ_{α_k}.

W. Veech computed in [V 1] the Jacobian of the transformation $\mathscr{L}_A$ (from Σ_N to Σ_N) for A a nonnegative matrix of determinant ± 1.

$J_A(\nu) = |A\nu|^{-(N+1)}$, for ν in Δ_N. Let $c_j = a_{1,j} + \ldots + a_{N+1,j}$. Then:

$$|A\nu| = \sum_{j=1}^{N+1} c_j \nu_j.$$

We call $\rho(A) = \max\limits_{1 \leq i,\, j \leq N+1} (c_i/c_j)$. Then, for all u, v, $J_A(u)/J_A(v) \geqq \rho(A)^{-(N+1)}$.

1.3. SOME PROPOSITIONS.

DEFINITION. — Let φ be an involution of $\{1, \ldots, p+q\}$ without fixed point. We call φ *reducible* if φ satisfies one of the two conditions below for a positive integer n smaller than p and q.

(i) For all $i \leqq n$, $p < \varphi(i) \leqq p+n$.

(ii) For all $p-n \leqq i \leqq p$, $\varphi(i) \geqq p+q-n$.

Otherwise we say that φ is *irreducible*.

When φ is reducible, the linear involutions of type (p, q, φ, F) can be decomposed into two linear involutions on two distinct intervals.

Let T be a (irrationnal) linear involution of type $\alpha=(p, q, \varphi, F)$. We denote by A_k the expansion of T, and $\alpha_k=(p_k, q_k, \varphi_k, F_k)$ the type of $\operatorname{ind}^k(T)$.

* If the expansion of T is finite, we say $T \in \Delta_\alpha^{\text{fin}}$.

* If the expansion of T is infinite, but from some stage, φ_k is reducible, we write $T \in \Delta_\alpha^{\text{red}}$.

* If the expansion of T is infinite, and there exists a subsequence $k(n)$ such that $\rho(A_{k(n)})$ is bounded, we write $T \in \Delta_\alpha^{\text{bnd}}$.

The key proposition, as in [N 2] and [Ke], is to show that, up to a vanishing set, these are the only three possible cases. We leave the proof of this proposition for the subsection 1.5.

PROPOSITION 1.3. — *The measure of the complement of* $\Delta_\alpha^{\text{fin}} \cup \Delta_\alpha^{\text{red}} \cup \Delta_\alpha^{\text{bnd}}$ *is* 0.

LEMMA 1.4. — *If* T *has a finite expansion*, T *has a flipped periodic point.*

If the expansion is finite, there exists k such that $\varphi_k(p_k)=\varphi_k(p_k+q_k)$. This can be obtained after an induction only if $p_k \in F_k$.

PROPOSITION 1.5. — *Let* $\mathscr{T}$ *be a finite set of types such that, if the type of* T *is in* $\mathscr{T}$, *then the type of* ind T *belongs to* $\mathscr{T}$.

Let X *be a subset of* Δ *which is invariant by* $\underline{\text{ind}}$.

If, for all α *in* $\mathscr{T}$, $\mu(X \cap \Delta_\alpha) \neq 0$, *then, for all* α *in* $\mathscr{T}$, $\mu(X \cap \Delta_\alpha^{\text{bnd}})=\mu(\Delta_\alpha^{\text{bnd}})$.

Proof. – Let C be the minimum, for α in $\mathscr{T}$, of the numbers: $\mu(X \cap \Delta_\alpha)/\mu(\Delta_\alpha)$. Let T be in $\Delta_\alpha^{\text{bnd}}$, A_k be the expansion of T, and $k(n)$ be a subsequence such that $\rho(A_{k(n)}) \leqq R$ for some constant R. We write $V_N=\mathscr{L}_{A_{k(n)}}(\Delta_{\alpha_{k(n)}})$. V_N is a convex neighborhood of T which satisfies:

(*a*) We know the Jacobian of $\mathscr{L}_{A_{k(n)}}$, from Σ_N to Σ_N. Using that: (i) $\mathscr{L}_{A_{k(n)}}$ sends segments to segments, (ii) If $A \subset B \subset H$ a hyperplane in $\mathbb{R}^n$, then the proportion of A in B is equal to the proportion of the pyramid over A (with vertex v) in the pyramid over B (same vertex); we deduce that the proportion of X in the set $\mathscr{L}_{A_{k(n)}}(\Delta_{\alpha_{k(n)}})$ is more than $K=C \times R^{-(N+1)}$, which a positive constant.

(*b*) $\Delta_\alpha=\Sigma_N \cap S_\alpha$. So the "exentricity" of the Δ_α's are bounded. Moreover we have: $\rho(A_{k(n)}) \leqq R$, therefore the "exentricities" of the V_N are bounded.

(*c*) The diameter of V_N goes to 0.

We can apply the Lebesgue density theorem, and we see that almost surely T belongs to X. □

1.4. PROOF OF THEOREM III. – Let $\alpha=(p, q, \varphi, F)$ be a type such that: $F \neq \varnothing$, $F \neq \{i : s_\varphi(i)=1\}$, and let T be a linear involution of type α.

* If T has a finite expansion, T has a flipped periodic point.

* If $T \in \Delta_\alpha^{red}$, we look at linear involutions which exchange less intervals.

Therefore, in order to prove Theorem III, we must prove that almost all $T \in \Delta_\alpha^{bnd}$ has a flipped periodic point. We shall do this by the use of Proposition 1.5.

For a given N, we define:

$\mathcal{T} = \{\alpha = (p, q, \varphi, F) : p+q = 2N+2, F \neq \varnothing, F \neq \{i : s_\varphi(i) = 1\}\}$. $\mathcal{T}$ is a finite set of types, which is stable by induction. Let X be the set of linear involutions whose types are in $\mathcal{T}$, and which have a flipped periodic point. We claim that for all α in $\mathcal{T}$, we have: $\mu(X \cap \Delta_\alpha) \neq 0$.

We can apply Proposition 1.5 to $\mathcal{T}$ and X, and we see that X is of full measure in Δ_α^{bnd}. This proves Theorem III. □

Proof of the claim. — If there exists i in F such that $s_\varphi(i) = -1$, it is very easy, because every linear involution in Δ_α such that $\lambda_i > 1/2$ has a flipped fixed point.

Otherwise, there exists i in F, $1 \leqq i \leqq p$, with $s_\varphi(i) = 1$, and there exists j not in F, $p < j \leqq p+q$, with $s_\varphi(j) = 1$. We have the picture in Figure 2.

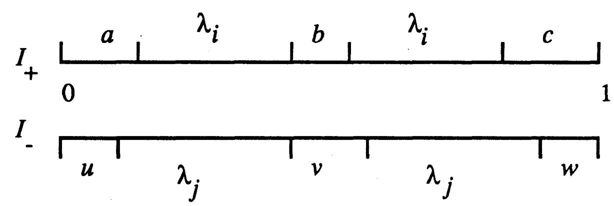


Fig. 2

We can choose *both* λ_i and λ_j arbitrarily near 1/2. We choose T in the open set $\{a, b, c, u, v, w < 1/100\}$. Then $1/2 - 1/50 < \lambda_i, \lambda_j < 1/2$.

Let $J =]a, a+\lambda_i[\times \{+1\}$, we have length $(J) > 1/2 - 1/50$.

Let $K = T(J) \cap [1-w-\lambda_j, 1-w[\times \{-1\}$. We have length $(K) > 1/2 - 2/50$.

Then $T(K) \cap T^{-1}(K) \neq \varnothing$.

But T^2 has derivative -1 on $T^{-1}(K)$. Therefore T^2 has a flipped fixed point.

1.5. Proof of Proposition 1.3. — Similar results are proved in [Ke] (*see* Corollary 1.8) and [N.2] (*see* Lemma 3.4). The latter deals with interval exchanges with flips. We recall that the difference is that linear involutions are not defined by a whole Euclidean space (or simplex), as interval exchanges do, but a subspace in it. Therefore we loose the explicit Jacobian, which is computed for an application from a simplex to itself. Here we follow the reasonning developed in [N 2] to prove Lemma 3.4, whose proof follows from Propositions 3.5, 3.6 and 3.7.

The Jacobian plays a role in the proof of Proposition 3.5 in [N 2], therefore one needs to prove a version of this result for linear involutions, that is, in a subspace setting:

Lemma 1.6. — *Let A_k be the expansion of* T. *Then at any stage k_0 where the matrix A_{k_0} is positive, the probability of the sum of a column increasing by a factor of* K *before being added to another column is less than* $N^2/(K-1)$.

In [N 2], the proof of Proposition 3.6 is a combinatorial one which works the same in our context. We obtain:

LEMMA 1.7. — *Assume that there exist positive integers k and m such that at the k-th and the $(k+m)$-th stage of the inducing process for* T *the induced involutions are of the same type. Then between the two stages of the expansion, each column has been added to some other one or has had some other one added to it.*

Using Lemmas 1.6 and 1.7, we obtain as in [N 2]:

LEMMA 1.8. — *Let* A_m *the m-th matrix in the expansion of* T, *be positive. Let* $C>0$ *be such that for a collection of columns of* A_m, $v_1, \ldots, v_l$, $|v_i|/|v_j| \leqq C$, *for all* $1 \leqq i, j \leqq l$. *Then, with probability* $\mu>0$, *one of the* v_i *will be added to a column outside the collection before the maximum of the* $|v_i|$ *increase by a factor of* K. *The probability* μ *is independent of* A_m *depending only on* C *and* K.

The proof of Proposition 1.3 follows, as in [N 2] from these 3 Lemmas.

Proof of Lemma 1.6. – Let $S=\Delta_\alpha=\Sigma_N \cap S_\alpha$. Let u be is S and $A=A_{k_0}(u)$ be positive. Let S′ be the hyperplane in Σ_N such that u is in $\mathscr{L}_A(S')$. Let c_i be the sum of the i-th column of A. Let $B=A_{k_0+r}(u)$ be the (k_0+r)th element of the expansion of u. We assume that there exists a positive constant K, and that B has been obtained from A in such a manner that any column was added only to the first column and the sum of the first column of B equals $K c_i$.

Let S″ be the hyperplane in Σ_N such that $\mathscr{L}_B(S'')$ contains u. Let P be the probability asked in the lemma. We claim that:

$$P=\frac{\mu_{N-1}(\mathscr{L}_B(S''))}{\mu_{N-1}(\mathscr{L}_A(S'))} \leqq \frac{N^2}{K-1}.$$

There exist positive integers m and n, with $m+n \leqq N+1$, such that, after a suitable reordering of the coordinates, we can write:

$$S'=\{u \in \Sigma_N : u_1+\ldots+u_m=u_{m+1}+\ldots+u_{m+n}\}.$$

For any $2 \leqq i \leqq N+1$, let $\tilde{K}=(1/N)(c_1/c_i)(K-1)$, and set $S_i'=\{u \in S' : u_i \geqq \tilde{K} u_1\}$. Then we have:

$$\mathscr{L}_B(S'')=\bigcup_{i=2}^{N+1} \mathscr{L}_A(S_i') \quad \text{and} \quad P \leqq \sum_{i=2}^{N+1} \frac{\mu_{N-1}(\mathscr{L}_A(S_i'))}{\mu_{N-1}(\mathscr{L}_A(S'))}.$$

In order to deal with the rates on the right hand side, we introduce rates of N dimensional volumes in Σ_N which are also images of $\mathscr{L}_A$. This will allow us to use the computation of the Jacobian in 1.2.

We note that $\mathscr{L}_A \Sigma_N$ is convex. Now we take v in Σ_N which is not in S′. Therefore $\mathscr{L}_A(v)$ does not belong to S. Joining $\mathscr{L}_A(v)$ to each point in $\mathscr{L}_A(S')$, we obtain a pyramid of base S′ and high ε, which we call $P_{\mathscr{L}_A S'}(\varepsilon)$, contained in $\mathscr{L}_A \Sigma_N$. Moreover

$P_{\mathscr{L}_A S'}(\varepsilon)$ is the image by $\mathscr{L}_A$ of a pyramid of base S′ and high γ, namely $P_{S'}(\gamma)$, obtained joining v to each point in S′, that is, $P_{\mathscr{L}_A S'}(\varepsilon)=\mathscr{L}_A(P_{S'}(\gamma))$.

Similarly, we obtain pyramids $P_{S'_i}(\gamma)$ and $P_{\mathscr{L}_A S'_i}(\varepsilon)$ such that $P_{\mathscr{L}_A S'_i}(\varepsilon)=\mathscr{L}_A(P_{S'_i}(\gamma))$, for any $2 \leqq i \leqq N+1$. We have, for any $2 \leqq i \leqq N+1$,

$$\frac{\mu_{N-1}(\mathscr{L}_A(S'_i))}{\mu_{N-1}(\mathscr{L}_A(S'))}=\frac{\mu_{N-1}(P_{\mathscr{L}_A S'_i}(\varepsilon))}{\mu_{N-1}(P_{\mathscr{L}_A S'}(\varepsilon))}=\frac{\mu_{N-1}(\mathscr{L}_A(P_{S'_i}(\gamma)))}{\mu_{N-1}(\mathscr{L}_A(P_{S'}(\gamma)))}=\frac{\displaystyle\int_{P_{S'_i}(\gamma)} J_A}{\displaystyle\int_{P_{S'}(\gamma)} J_A}$$

This implies the relation (*) which follows:

$$\frac{\mu_{N-1}(\mathscr{L}_A(S'_i))}{\mu_{N-1}(\mathscr{L}_A(S'))}=\frac{\displaystyle\int_{S'_i} J_A}{\displaystyle\int_{S'} J_A}$$

Let $k=N+1-(m+n)$, we have:

$$\int_{S'} J_A=\int_{\Gamma}(c_1 z_1+.+c_k z_k+.+c_{k+m} x_m+c_{k+m+1} y_1+.+c_{k+m+n} y_n)^{-(N+1)} \times dz_1 . dz_k\, dx_1 . dx_{m-1}\, dy_1 . dy_{n-1}$$

where

$$x_m=\frac{1}{2}(1-z_1-\ldots-z_k)-x_1-\ldots-x_{m-1},$$

and

$$y_n=\frac{1}{2}(1-z_1-\ldots-z_k)-y_1-\ldots-y_{n-1},$$

and the domain of integration Γ is given by:

$$0 \leqq z_1 \leqq 1, \qquad 0 \leqq z_2 \leqq 1-z_1,$$

$$0 \leqq z_k \leqq 1-z_1-\ldots-z_{k-1}, \qquad 0 \leqq x_1 \leqq \frac{1}{2}(1-z_1-\ldots-z_k),$$

$$0 \leqq x_{m-1} \leqq \frac{1}{2}(1-z_1-\ldots-z_k)-x_1-\ldots-x_{m-2},$$

$$0 \leqq y_1 \leqq \frac{1}{2}(1-z_1-\ldots-z_k), \qquad 0 \leqq y_{n-1} \leqq \frac{1}{2}(1-z_1-\ldots-z_k)-y_1-\ldots-y_{n-2}.$$

In order to evaluate the rates on the right hand side of (*), we have to consider the above integral restricted to five subsets.

Case 1: $x_2 \geqq \tilde{K} z_1$.

Case 2: $x_1 \geqq \tilde{K} z_1$.

Case 3: $z_1 \geqq \tilde{K} x_1$.

Case 4: $y_1 \geqq \tilde{K} x_1$.

Case 5: $x_2 \geqq \tilde{K} x_1$.

For short, we will assume that S'_i satisfies case 1, and prove that the rates in (*) are smaller than $N/(K-1)$. Any other case will be the same. We have:

$$\int_{S'_i} J_A = \int_{\Gamma_1} (c_1 z_1 + c_2 z_2 + \dots)^{-(N+1)} dz_1 \dots dz_k \, dx_1 \dots dx_{m-1} \, dy_1 \dots dy_{n-1},$$

with $\Gamma_1 = \{0 \leqq z_1 \leqq 1/(1+\tilde{K}),\ \tilde{K} z_1 \leqq z_2 \leqq 1 - z_1, \dots\}$.

Therefore we have:

$$\int_{S'_i} J_A = \frac{1}{1+\tilde{K}} \int_{\Gamma} (c_1 z_1 + c_2 z_2 + \dots)^{-(N+1)} dz_1 \dots dz_k \, dx_1 \dots$$

Let $L = (1 + ((K-1)/N))/(1 + (c_1/c_i)((K-1)/N))$. In order to evaluate the right hand side of (*), we consider two cases:

(i) $L \leqq 1$:

$$\int_{S'_i} J_A \leqq \frac{N}{K-1} \int_{\Gamma} (c_1 z_1 + c_2 z_2 + \dots)^{-(N+1)} dz_1 \dots dz_k \, dx_1 \dots \leqq \frac{N}{K-1} \int_{S'} J_A$$

(ii) $L > 1$:

$$\int_{S'_i} J_A = \frac{1}{1+\tilde{K} L^{N+1}} \int_{\Gamma} \left(c_1 z_1 + c_2 \frac{z_2}{L} + c_3 \frac{z_3}{L} \dots \right)^{-(N+1)} dz_1 \dots dz_k \, dx_1 \dots$$

$$= \frac{1}{1+\tilde{K} L^3} \int_{\Gamma_2} (c_1 z_1 + c_2 z_2 + \dots)^{-(N+1)} dz_1 \dots dz_k \, dx_1 \dots$$

where

$$\Gamma_2 = \left\{ 0 \leqq z_1 \leqq 1,\ 0 \leqq z_2 \leqq \frac{1}{L}(1 - z_1),\ 0 \leqq z_k \leqq \frac{1}{L}(1 - z_1 - \dots - z_{k-1}), \dots \right\} \subset \Gamma.$$

Therefore:

$$\int_{S'_i} J_A \leqq \frac{N}{K-1} \int_{\Gamma} (c_1 z_1 + c_2 z_2 + \dots)^{-(N+1)} dz_1 \dots dz_k \, dx_1 \dots \leqq \frac{N}{K-1} \int_{S'} J_A.$$

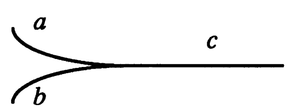
This implies that $P \leqq N^2/(K-1)$, and this proves the claim. □

2. The relation linear involutions – measured foliations

In this section we shall show that there exist applications from measured foliations to linear involutions, which are, in a sense to be defined, locally linear and onto.

2.1. MEASURED FOLIATIONS AND TRAIN TRACKS. – We consider (as in the whole paper) a closed surface of negative Euler characteristics. A *measured foliation* $(\mathcal{F}, \lambda)$ of M is a foliation $\mathcal{F}$ with isolated singularities which are saddles with at least three prongs. This foliation $\mathcal{F}$ has a transverse measure λ, regular with respect to the Lebesgue measure, invariant by holonomy and with support M. This type of foliations, which appears naturally in quadratic differentials, was first considered by W. Thurston in [T 1]. *See* [FLP] for more details. In our notations we will oftenly omit to mention the measure λ. One can introduce an equivalence relation between two measured foliations. Two foliations are equivalent if one goes from one to the other via isotopies and Whitehead operations (create or collapse a connection). W. Thurston showed that the space of equivalence classes of measured foliations, endowed with a good topology, is homeomorphic to $(\mathbb{R}^{-3\chi(\mathrm{M})}) - \{0\}$.

A *train track* τ in M is a closed graph embedded in M such that each edge is smooth, and each vertex is the extremity of three edges which are tangent to a line, and, besides, one in one direction of the line, and the other two in the opposite direction (as a switch of a rail-way, see *Fig.* 3). Since the train track τ has a tangent space, we say that τ is orientable if its tangent space is orientable. Throughout the paper, we consider only train tracks τ such that each component of $\mathrm{M}-\tau$ is homeomorphic to a disk whose boundary has at least three cusps.



$\mu(a)+\mu(b)=\mu(c)$

Fig. 3

Let τ be a train track, and E be the number of edges of τ. An application μ which associates to each edge e a positive number $\mu(e)$ is a *weight system* for τ, if at each vertex it satisfies the equation of compatibility (see *Fig.* 3). We call $\mathrm{E}(\tau)$ the space of weight systems for τ. If we consider in $\mathbb{R}^{\mathrm{E}}$ the vector subspace S of solutions of the compatibility equations, $\mathrm{E}(\tau)$ is equal to $(\mathbb{R}_+^*)^{\mathrm{E}} \cap \mathrm{S}$. So, if $\mathrm{E}(\tau)$ is nonempty, $\mathrm{E}(\tau)$ is an open cell in S. Therefore there is in $\mathrm{E}(\tau)$ a Lebesgue measure.

Now we choose a weight system μ on τ. Replace each edge e of τ by a foliated rectangle with transverse Lebesgue measure of total length $\mu(e)$, and we glue all these rectangles by their sides, using the transverse measure (as in Figure 4). We obtain a foliation $\mathcal{F}_0$ of a neighborhood V of τ, with a transverse measure. It is clear that V does not depend on the weight system. For each edge e of τ, we chose a vertical segment I_e in the middle of the rectangle as in Figure 4.

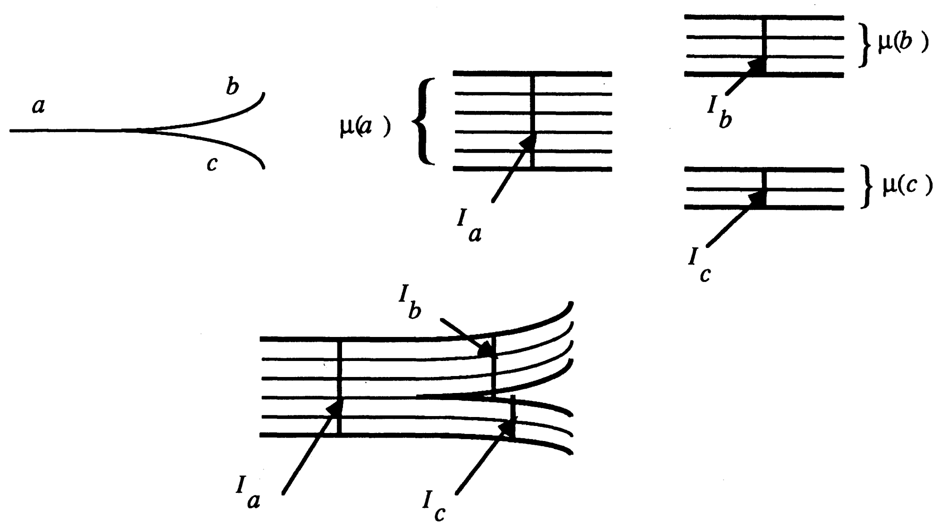


Fig. 4

Now, since the components of $M-\tau$ (and hence $M-V$) are homeomorphic to disks with at least three horns, we can collapse each of these components, creating one (and only one) saddle, as in Figure 5. We obtain a measured foliation $\mathscr{F}$ of M, and say that $\mathscr{F}$ is carried by τ. We also call $E(\tau)$ the space of foliations carried by τ. Note that every measured foliation is carried by a train track. All these definitions were introduced by W. Thurston in [T 2], chap. 9.

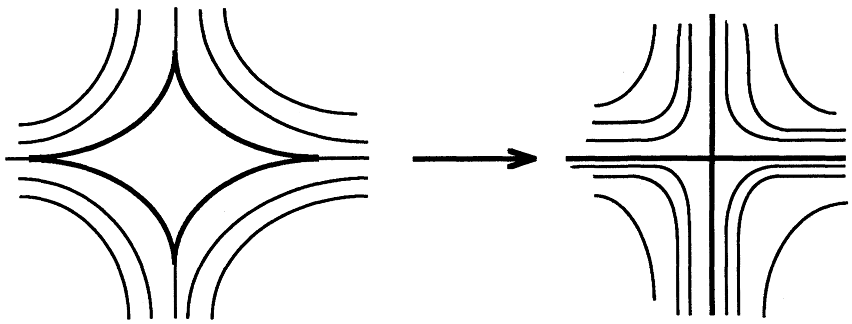


Fig. 5

This defines an application from $E(\tau)$ into $\mathscr{MF}(M)$. One can show that this application is continuous and injective. Moreover, $\mathscr{MF}(M)$ has a PL structure whose charts are given by *complete* train tracks, *i.e.* the train tracks τ such that each component of $M-\tau$ is a triangle.

2.2. First return on the transversals. – Let $\mathscr{F}$ be a measured foliation, and I be an interval transverse to $\mathscr{F}$. As said in the introduction, we can cut the surface M along I, creating a new surface M′, and a foliation $\mathscr{F}'$ of M′. The boundary of M′ is the union of two copies of I, namely I_+ and I_-. The Poincaré's recurrence theorem says that each leaf of $\mathscr{F}'$ coming from $\partial M'$ either goes to a singularity, or cuts $\partial M'$ another time. Therefore, except for finitely many points, each leaf of $\mathscr{F}'$ which cuts $I_+ \cup I_-$ is a segment. This defines an involution T′ of $I_+ \cup I_-$ but finitely many points, which assigns to each point x the other endpoint of the leaf of $\mathscr{F}'$ going through x. Since $\mathscr{F}$ is a measured foliation, T′ preserves the Lebesgue measure of I. Therefore $T=\sigma\circ T'$ is a linear involution.

The linear involution we obtain is of type $\alpha=(p, q, \varphi, F)$. If is now easy to give an interpretation of the set F which appears in the definition of linear involutions. Note that i belongs to F if and only if, for x in J_i, the closed curve which is the union of the leaf of $\mathscr{F}'$ going through x and a piece of I, is a one-sided curve.

Let τ be a train track on M. We associated to each edge e of τ a segment I_e which is transverse to all foliations carried by τ. Moreover, the family of segments I_e cuts all leaves of the foliations carried by τ. For each edge of τ, the operation above described gives an application from $E(\tau)$ into the set Λ of all linear involutions. Then we have E (the number of edges of τ) applications Φ_i from $E(\tau)$ into Λ. The remaining of Section 2 is devoted to the proof of the following proposition, which gives the naturality of the applications Φ_i with respect to the linear structures of Λ and $E(\tau)$.

Remark. – We want to find a closed leaf in the foliation, when the surface is nonorientable. So we need to look at all I_e, because the closed leaf cuts only some of the intervals I_e.

PROPOSITION 2.1. — *There exists a subset $\mathscr{A}$ of $E(\tau)$ such that:*

1. *The measure of $E(\tau)-\mathscr{A}$ is* 0.
2. *For all $\mathscr{F}_0$ in $\mathscr{A}$, there exists a neighborhood V of $\mathscr{F}_0$ in $E(\tau)$, such that:*

(i) *For all i, there exists a type α_i such that $\Phi_i(V)\subset\Lambda_{\alpha_i}$*
(i.e. *the type of the linear involution $\Phi_i(\mathscr{F})$ is constant in* V)

(ii) *$\Phi_i|_V$ is the restriction of an onto linear map.*

2.3. UNSTABLE CONNECTIONS. – This subsection is devoted to the definition of the set $\mathscr{A}$ of Proposition 2.1.

In Proposition 2.1, we want to assure the local stability of the type of the linear involutions. If $\mathscr{F}_0$ has a connection and the foliations near $\mathscr{F}_0$ do not, then the type of the linear involutions will not be the same for $\mathscr{F}_0$ and the foliations near $\mathscr{F}_0$. For this reason, we began studying the "stability" of the connections.

We call *connection path* any smooth path γ in τ satifying the two following properties:

(i) γ goes from one vertex to another, leaving the origin (and coming at the end) by the edge of maximal weight at this vertex.

(ii) γ can be approximated in M by an injective path.

This definition is justified by the following fact. If $\mathscr{F}$ is carried by τ and has a connection, there corresponds to this connection a connection path in τ. (A connection of a foliation carried by τ always "follows" a connection path in τ.)

We shall speak often of the left and the right of smooth paths in τ. This means that we choose arbitrarily a right (and left) side at the beginning of the path, and transport this throughout the path. One consequence of this is that, if the path γ has a double point, the notion of right and left at this point could not be the same for the two values of the time parameter, say $t_1 \neq t_2$, such that $\gamma(t_1)=\gamma(t_2)$. The choice of right and left at the origin will have no importance, the derived conditions will be symmetric.

Let γ be a connection path of τ. We draw the picture (in fact in the universal covering of τ) of γ (*Fig.* 6).

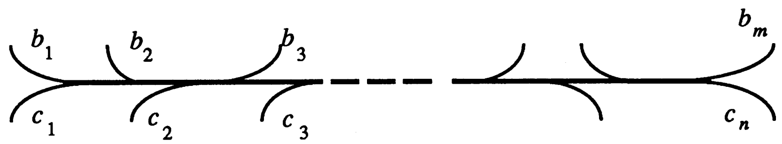


Fig. 6

We denote by $b_1, \ldots, b_m$ the edges coming on γ by the left and by $c_1, \ldots, c_n$ the edges coming on γ by the right. We define $\varepsilon_j=1$ if b_j comes on γ with a positive direction, and $\varepsilon_j=-1$ if b_j comes on γ with a negative direction. We define $\varepsilon'_j=1$ if c_j comes on γ with a positive direction, and $\varepsilon'_j=-1$ if c_j comes on γ with a negative direction (e. g.: in Figure 6, $\varepsilon_1=\varepsilon_2=\varepsilon'_1=\varepsilon'_2=\varepsilon'_3=1$ and $\varepsilon_3=\varepsilon_m=\varepsilon'_n=-1$.)

Let μ be a weight system on τ. If there is, in the foliation associated to μ, a connection following γ, the following equality is clearly satisfied:

$$\varepsilon_1 \mu(b_1)+\ldots+\varepsilon_m \mu(b_m)=0.$$

This equation is equivalent, since μ is a weight system, to the following one:

$$\varepsilon'_1 \mu(c_1)+\ldots+\varepsilon'_n \mu(c_n)=0.$$

We call the first equation E_γ.

If we consider this equation in $\mathbb{R}^E$ (E = number of edges of τ), we have a space of solutions H_γ which is, in general, a hyperplane. Recall that, if S is the space of solutions of the equations of compatibility at each vertex, $E(\tau)=S \cap (\mathbb{R}_+^*)^E$. So there can be two cases:

(i) S is not contained in H_γ. We say that γ is an *unstable connection path*. $H_\gamma \cap E(\tau)$ is a hyperplane of $E(\tau)$. The foliations which are not in this hyperplane have no connection following γ.

(ii) $S \subset H_\gamma$ (the equation E_γ is a linear combination of the equations of compatibility of τ). We say that γ is a *stable connection path*.

There exist countably many connection paths.

Definition. — We define $\mathscr{A}$ in the following way: $\mathscr{A}=E(\tau)-\bigcup_{\gamma\text{ unstable}} H_\gamma$.

The set $\mathscr{A}$ equals $E(\tau)$, but countably many hyperplanes. It follows that the complement of $\mathscr{A}$ has measure 0.

2.4. Local stability. – Let τ be a train track on M. We have the applications $\Phi_i : E(\tau) \to \Lambda$, defined in 2.2. Let $\mathscr{F}_0$ be a foliation in $\mathscr{A}$ associated to a weight system μ_0. We shall show that the type of the linear involutions is constant in a neighborhood of $\mathscr{F}_0$.

Lemma 2.2. — *There exists a neighborhood*, W, *of* μ_0 *in* $E(\tau)$, *and types* α_i *such that for all i*, $\Phi_i(V) \subset \Lambda_{\alpha_i}$.

This result is a direct consequence of the following lemma:

Lemma 2.3. — *For all i, there exists a neighborhood*, V_i, *of* μ_0 *in* $E(\tau)$, *and a type* α_i *such that* $\Phi_i(V_i) \subset \Lambda_{\alpha_i}$.

We will consider the linear involutions induced by the foliations on I_i, the transverse interval which corresponds to an edge e of τ.

Definitions. – We define $\tilde{\Gamma}$ as the set of all smooth paths α (a sequence of edges) of τ such that the first edge is e and α goes through e at most twice, if twice, the last edge of α is also e. If $\alpha \in \tilde{\Gamma}$, we call length of α, and write $lg(\alpha)$, the number of edges of τ in α.

We denote by Γ the set of all paths of $\tilde{\Gamma}$ which are closed (the two extremities are e).

Let α be in $\tilde{\Gamma}$, we choose the left and the right side of α, and draw the picture (*Fig.* 7). In fact, the picture is in the universal covering of τ.

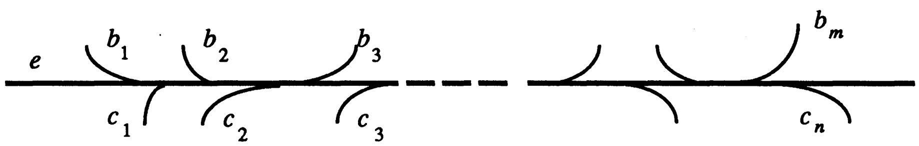


Fig. 7

We denote by $b_1, \ldots, b_m$ the edges coming on α by the left, and by $c_1, \ldots, c_n$ the edges coming on α by the right. We define $\varepsilon_j = 1$ if b_j comes on α with a positive direction, and $\varepsilon_j = -1$ if b_j comes on α with a negative direction. We define $\varepsilon'_j = 1$ if c_j comes on α with a positive direction, and $\varepsilon'_j = -1$ if c_j comes on α with a negative direction (e.g.; in Figure 7, $\varepsilon_1 = \varepsilon_2 = \varepsilon'_1 = \varepsilon'_2 = \varepsilon'_3 = 1$ and $\varepsilon_3 = \varepsilon_m = \varepsilon'_n = -1$.)

Let μ be in $E(\tau)$, we define:

$$L_\alpha(\mu) = \min_{1 \leq j \leq m} \left(\sum_{i=1}^{j} \varepsilon_i \mu(b_i) \right) \quad \text{and} \quad R_\alpha(\mu) = \min_{1 \leq j \leq n} \left(\sum_{i=1}^{j} \varepsilon'_i \mu(c_i) \right)$$

and set $l_\alpha(\mu) = \mu(e) + L_\alpha(\mu) + R_\alpha(\mu)$.

Remarks. – (i) The foliation $\mathscr{F}_\mu$ defined by the weight system μ has leaves going along α if, and only if $l_\alpha(\mu) > 0$. Moreover, the transverse measure of the band of leaves of $\mathscr{F}$ going along α is exactly $l_\alpha(\mu)$.

(ii) For a given α, the map $\mu \to l_\alpha(\mu)$ is a continuous function, since it is an infimum of linear maps.

(iii) Let $\Gamma_+(\mu)=\{\gamma\in\Gamma : l_\gamma(\mu)>0\}$. To each $\gamma\in\Gamma_+(\mu)$ corresponds a band of leaves which goes from I_i to I_i, *i.e.* a small interval of the linear involution $\Phi_i(\mathscr{F}_\mu)$. Hence $\Gamma_+(\mu)$ is finite. The type of the linear involution $\Phi_i(\mathscr{F}_\mu)$ is entirely determined by $\Gamma_+(\mu)$. For example, the flip set F corresponds to the elements of $\Gamma_+(\mu)$ which are one-sided curves. The numbers $l_\gamma(\mu)$, for γ in $\Gamma_+(\mu)$, are the lengths of the intervals exchanged by $\Phi_i(\mathscr{F}_\mu)$.

In order to prove Lemma 2.3, we have to show that, if $\mu_0\in\mathscr{A}$, there exists V_i, a neighborhood of μ_0, such that for all μ in V_i, $\Gamma_+(\mu)=\Gamma_+(\mu_0)$.

We denote by N the maximum of the $lg(\gamma)$ for γ in $\Gamma_+(\mu_0)$ (which is finite). We set $\tilde\Gamma_1=\{\alpha\in\tilde\Gamma : lg(\alpha)\leqq N+1\}$ and $\tilde\Gamma_2=\{\alpha\in\tilde\Gamma : lg(\alpha)=N+1\}$. These two sets are finite.

Scholium 2.4. — *If* α *is in* $\tilde\Gamma_2$, *then* $l_\alpha(\mu_0)\leqq 0$.

Proof. — If $l_\alpha(\mu_0)>0$, there exists a band of leaves of $\mathscr{F}_0$ along α. These leaves must cut I_i another time (Poincaré's recurrence) but this will be done after a path of length superior than N, which is impossible by definition of N. □

Scholium 2.5. — Let $\alpha\in\tilde\Gamma_1$, *there exists a neighborhood* V_α *of* μ_0 *in* $E(\tau)$ *such that:*

(i) *if* $l_\alpha(\mu_0)>0$, *then* $l_\alpha(\mu)>0$ *for all* $\mu\in V_\alpha$;

(ii) *if* $l_\alpha(\mu_0)\leqq 0$, *then* $l_\alpha(\mu)\leqq 0$ *for all* $\mu\in V_\alpha$.

Proof. — We consider 3 cases.

Case 1. — If $l_\alpha(\mu_0)>0$, then there exists V_α, a neighborhood of μ_0, such that $l_\alpha(\mu)>0$ for μ in V_α, by continuity of l_α.

Case 2. — If $l_\alpha(\mu_0)<0$, then there exists V_α, a neighborhood of μ_0, such that $l_\alpha(\mu)<0$ for μ in V_α, by continuity of l_α.

Case 3. — $l_\alpha(\mu_0)=0$. We choose $1\leqq j_1\leqq m$ and $1\leqq j_2\leqq n$ such that:

$$L_\alpha(\mu_0)=\sum_{i=1}^{j_1}\varepsilon_i\,\mu_0(b_i)\qquad\text{and}\qquad R_\alpha(\mu_0)=\sum_{i=1}^{j_2}\varepsilon'_i\,\mu_0(c_i)$$

We draw the picture in Figure 8.

By the choice of j_1, $\sum_{i=1}^{j_1}\varepsilon_i\,\mu_0(b_i)\leqq\sum_{i=1}^{j_1+1}\varepsilon_i\,\mu_0(b_i)$, hence $\varepsilon_{j_1+1}=1$.

We call e' the edge of α after the point y. If j_3 is the index of the last edge arriving on α by the left before y, we have:

$$\mu_0(e')=\mu_0(e)+\sum_{i=1}^{j_2}\varepsilon'_i\,\mu_0(c_i)+\sum_{i=1}^{j_3}\varepsilon_i\,\mu_0(b_i)$$
$$=\mu_0(e)+R_\alpha(\mu_0)+L_\alpha(\mu_0)+\sum_{j_1<i\leqq j_3}\varepsilon_i\,\mu_0(b_i)=\sum_{j_1<i\leqq j_3}\varepsilon_i\,\mu_0(b_i).$$

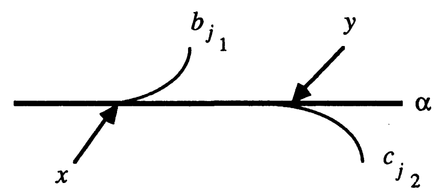


Fig. 8

Since $\mu_0(e')>0$, we must have $j_3>j_1$, that is, the edge b_{j_1+1} arrives on α between the points x and y (see *Fig.* 9).

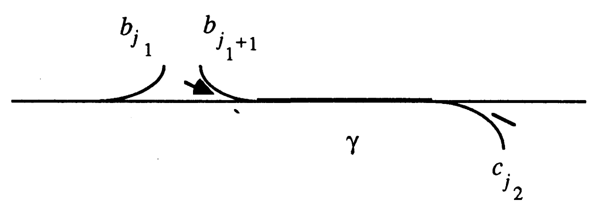


Fig. 9

We look at the connection path γ, which is the part of α between the endpoints of b_{j_1+1} and c_{j_2} indicated by the two arrows in Figure 9. If we sum the weights of μ_0 arriving and leaving γ by the right, we find $\mu_0(e)+R_\alpha(\mu_0)+L_\alpha(\mu_0)$ which is 0. This says that μ_0 satisfies the equation E_γ for all μ in $E(\tau)$, $\mu(e)+\sum_{i=1}^{j_2}\varepsilon'_i\,\mu(c_i)+\sum_{i=1}^{j_1}\varepsilon_i\,\mu(b_i)=0$.

This implies that $l_\alpha(\mu)=\mu(e)+R_\alpha(\mu)+L_\alpha(\mu)\leqq 0$. $\square$

We now conclude the proof of Lemma 2.3.

We denote by V_i the neighborhood of $\mathscr{F}_0$ which is the intersection of the V_α, for α in $\tilde{\Gamma}_1$ (which is a finite set). We will show that $\Gamma_+(\mu)=\Gamma_+(\mu_0)$, for all μ in V_i.

First inclusion: $\Gamma_+(\mu_0)\subset\Gamma_+(\mu)$. Let $\gamma\in\Gamma_+(\mu_0)$, then $\gamma\in\Gamma_1$ and $l_\gamma(\mu_0)>0$. By Scholium 2.5, $l_\gamma(\mu)>0$ for all μ in V_i. Therefore $\gamma\in\Gamma_+(\mu)$ for all μ in V_i.

Second inclusion: $\Gamma_+(\mu)\subset\Gamma_+(\mu_0)$. If $\gamma\notin\Gamma_+(\mu_0)$, then $l_\gamma(\mu_0)\leqq 0$. If $\gamma\in\tilde{\Gamma}_1$, by Scholium 2.5, $l_\gamma(\mu)\leqq 0$, for all μ in V_i. Therefore $\gamma\notin\Gamma_+(\mu)$.

If $\gamma\notin\tilde{\Gamma}_1$, $lg(\gamma)>N+1$. The $N+1$ first edges along γ give a path α in $\tilde{\Gamma}_2$. By Scholium 2.4, $l_\alpha(\mu_0)\leqq 0$. By Scholium 2.5, $l_\alpha(\mu)\leqq 0$ for all μ in V_i. Since α is a part of γ, $L_\gamma(\mu)\leqq L_\alpha(\mu)$ and $R_\gamma(\mu)\leqq R_\alpha(\mu)$, so $l_\gamma(\mu)\leqq l_\alpha(\mu)\leqq 0$. Then $\gamma\notin\Gamma_+(\mu)$. $\square$

2.5. LINEARITY AND SURJECTIVITY.

LEMMA 2.6. — *For* μ_0 *in* $\mathscr{A}$, *let* W *be given by lemma* 2.2. *There exists a neighborhood* V *of* μ_0 *in* W, *such that* $\Phi_{i\,|\,V}$ *is the restriction of an onto linear map.*

Proof. — *Linearity*. — The lengths of the intervals exchanged by $\Phi_i(\mu)$ are the numbers $l_\gamma(\mu)$ for γ in $\Gamma_+(\mu_0)$. We have $l_\gamma(\mu)=\mu(e)+R_\gamma(\mu)+L_\gamma(\mu)$, where $\mu(e)$ is linear in μ, and L_γ and R_γ are minimum of linear maps in μ.

If this minimum is reached for μ_0 at only one index, we can find a neighborhood of μ_0 such that the minimum is reached by the same index, and therefore the formula is linear. Assume there exist $r<s$ such that:

$$L_\gamma(\mu_0)=\sum_{i=1}^{r}\varepsilon_i\,\mu_0(b_i)=\sum_{i=1}^{s}\varepsilon_i\,\mu_0(b_i)$$

By definition, $L_\gamma(\mu_0)\leqq\sum_{i=1}^{r+1}\varepsilon_i\,\mu_0(b_i)$, then $\varepsilon_{r+1}=1$.

A similar reasonning shows that $\varepsilon_r=\varepsilon_s=-1$. Then we have the picture in Figure 10. Let δ be the connection path which is the part of γ between the endpoints of b_{r+1} and b_s, indicated by arrows in Figure 10.

We have $\sum_{i=r+1}^{s}\varepsilon_i\,\mu_0(b_i)=0$, then μ_0 satisfies the equation E_δ associated to δ as in 2.3. Since μ_0 is in $\mathscr{A}$, this equation is always satisfied, that is, for all μ, $\sum_{i=r+1}^{s}\varepsilon_i\,\mu(b_i)=0$, therefore $\sum_{i=1}^{r}\varepsilon_i\,\mu(b_i)=\sum_{i=1}^{s}\varepsilon_i\,\mu(b_i)$. Therefore, the maps corresponding to indices r and s are equal. Therefore the formula defining $L_\gamma(\mu)$ is linear in a neighborhood V of μ_0.

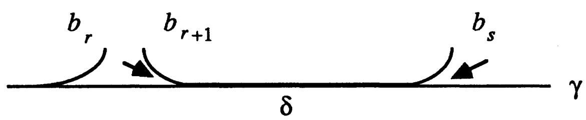


Fig. 10

Surjectivity. — Let $\mathscr{F}_0$, defined by the weight system μ_0, be fixed, and let $\lambda_1, \ldots, \lambda_{p_i+q_i}$ be the lengths of the intervals exchanged by $\Phi_i(\mathscr{F}_0)$, which is the linear involution induced on an interval I_i. Let $\delta_1, \ldots, \delta_{p_i+q_i}$ be nonnegative real numbers such that $\delta_j=\delta_{\varphi_i(j)}$ and $\delta_1+\ldots+\delta_{p_i}=\delta_{p_i+1}+\ldots+\delta_{p_i+q_i}$. For each pair $(j, \varphi(j))$, we choose a segment L_j of a leaf of $\mathscr{F}_0$ whose endpoints lie on I_i, and which goes from the j-th interval to the $\varphi(j)$-th interval. We cut the neighborhood of τ along I_i and all L_j. We glue in place of L_j a foliated rectangle of width δ_j, and glue back the two copies of I_i (see *Fig.* 11). We obtain a new foliation $\mathscr{F}$ which is carried by τ, and such that the lengths of the intervals exchanged by $\Phi_i(\mathscr{F})$ are $\lambda_j+\delta_j$. This proves the surjectivity. □

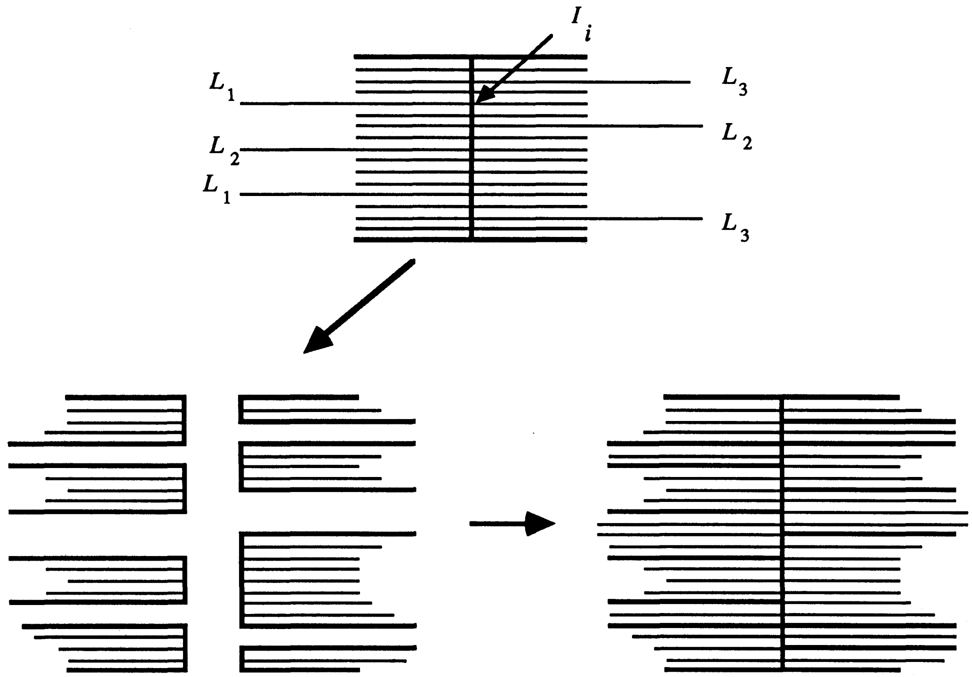


Fig. 11

3. Applications to measured foliations

In this section, we shall use the results of Sections 1 and 2 to prove some results about measured foliations.

3.1. Nonorientable surfaces. — Let M be a nonorientable surface. We call $\mathscr{C}$ the subset of $\mathscr{MF}$ of foliations which have a compact regular leaf which is a one-sided curve. (This property is not modified by Whitehead operations.) Note that $\mathscr{C}$ is an open set in $\mathscr{MF}$.

Proposition 3.1. — *$\mathscr{C}$ is dense in $\mathscr{MF}$* (M).

Before we prove this proposition, we give some definitions. We denote by $\mathscr{S}$ the set of isotopy classes of closed simple curves in M which do not bound a disk or a Möebius band. Let $\gamma_1, \ldots, \gamma_n$ be n elements of $\mathscr{S}$ such that, if $i \neq j$, $\gamma_i \cap \gamma_j = \varnothing$, and γ_i and γ_j do not bound an annulus. Let $\lambda_1, \ldots, \lambda_n$ be n positive real numbers. We call such a $(\gamma_1, \ldots, \gamma_n, \lambda_1, \ldots, \lambda_n)$ a *multicurve*, and denote by $\mathscr{MS}$ the set of all multicurves.

To each multicurve, we can associate a measured foliation ([FLP], exp. 5, § III). So $\mathscr{MS}$ is a subset of $\mathscr{MF}$. The elements of $\mathscr{MF}$ are the foliations whose regular leaves are compact. The following result is well known.

Lemma 3.2. — *$\mathscr{MS}$ is dense in $\mathscr{MF}$.*

Proof. — Let $\mathcal{F}_0$ in $\mathcal{MF}$ be carried by a train track τ, and let μ_0 be the weight system on τ which gives $\mathcal{F}_0$. In $E(\tau)$, the weight systems with rational weights are dense, since the compatibility equations at each vertex have integer coefficients. Therefore, we can approximate $\mathcal{F}_0$ by a foliation $\mathcal{F}$ given by a rational weight system μ. It is easy to see that all regular leaves of $\mathcal{F}$ must be compact. □

The next lemma is a generalization of Proposition 1.2 of [S].

LEMMA 3.3. — *Each element* $(\gamma_1, \ldots, \gamma_n, \lambda_1, \ldots, \lambda_n)$ *of* $\mathcal{MS}$ *can be approximated by a multicurve* $(\delta_1, \ldots, \delta_p, \mu_1, \ldots, \mu_p)$ *where at least one* δ_i *is a one-sided curve. In particular,* $\mathcal{MS}$ *is in the closure of* $\mathcal{C}$.

Proof. — If one γ_i is a one-sided curve, we are done. If not, we first approximate $(\gamma_1, \ldots, \gamma_n, \lambda_1, \ldots, \lambda_n)$ by $(\gamma_1, \ldots, \gamma_n, \lambda'_1, \ldots, \lambda'_n)$, where each λ'_i is a rational number. We call k the least common multiple of the denominators of the λ'_i. Let C be a one-sided curve in M which cuts at least one γ_i, and such that the number of times C intersects γ_i, namely l_i, is minimal in the isotopy class of C. Let l be the least common multiple of the l_i. We construct a train track, and for each p a weight system as in Figure 12. We obtain foliations $\mathcal{F}_p$ wich are multicurves (rational weights) and in $\mathcal{C}$. Moreover, it is clear that $\lim(\mathcal{F}_p)=(\gamma_1, \ldots, \gamma_n, \lambda'_1, \ldots, \lambda'_n)$. This proves Lemma 3.3 □

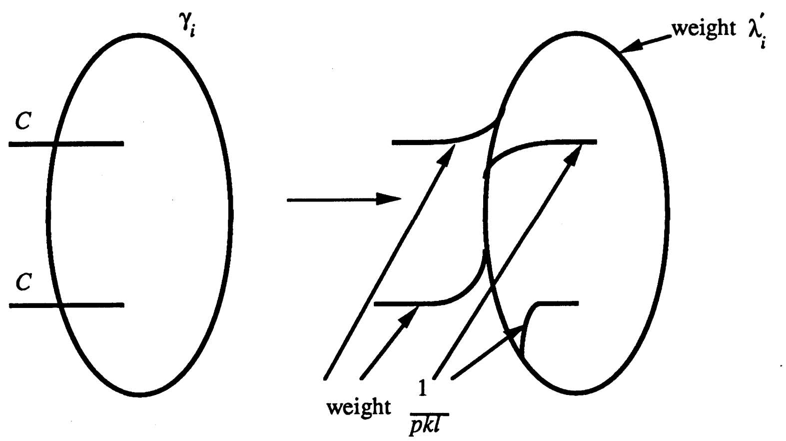


Fig. 12

Proposition 3.1 is a direct consequence of Lemmas 3.2 and 3.3 A corollary of 3.1 is:

COROLLARY 3.4. — *Let* τ *be a complete train track, and* $\mathcal{A}$, Φ_i, $\alpha_i=(p_i, q_i, \varphi_i, F_i)$ *be as in proposition* 2.1. *Let* $\mathcal{F}_0 \in \mathcal{A}$. *Then there exists i such that* $\varnothing \neq F_i \neq \{j : s_{\varphi_i}(j) > 0\}$.

Proof. — Let $\mathcal{F}_0$ be in $\mathcal{A}$. By Proposition 2.1, there exist a neighborhood V of $\mathcal{F}_0$ in $E(\tau)$, and $\alpha_i=(p_i, q_i, \varphi_i, F_i)$ such that $\Phi_i(V) \subset \Lambda_{\alpha_i}$. Since τ is complete, V is also a neighborhood of $\mathcal{F}_0$ in $\mathcal{MF}$. Therefore, by Proposition 3.1, there is an element of $\mathcal{C}$ in V. This implies that there exists i such that $\varnothing \neq F_i \neq \{j : s_{\varphi_i}(j) > 0\}$. □

Proof of Theorem I. — Let M be a nonorientable surface, and τ be a complete train track on M. If $\mathscr{F}_0$ is in $\mathscr{A}$, by Propositions 2.1 and 3.4, there exist a neighborhood $V_{\mathscr{F}_0}$ of $\mathscr{F}_0$ in $E(\tau)$, and i, $\alpha_i=(p_i, q_i, \varphi_i, F_i)$ be such that the restriction of Φ_i to $V_{\mathscr{F}_0}$ is linear and onto, and that $\varnothing \neq F_i \neq \{j : s_{\varphi_i}(j) > 0\}$. By Theorem III, almost all elements of $V_{\mathscr{F}_0}$ belong to $\mathscr{C}$ Since $E(\tau)$ is separable, there exists J countable in $\mathscr{A}$ such that $\mathscr{A} \subset \bigcup_{x \in J} V_x$. Then we have: $\mathscr{A} - \mathscr{C} \subset \bigcup_{x \in J} (V_x - \mathscr{C})$. Therefore the set $\mathscr{A} - \mathscr{C}$ has measure 0. As $E(\tau) - \mathscr{A}$ is also of measure 0, $E(\tau) - \mathscr{C}$ has measure 0. Since this is true for all complete train track, Theorem I is proved. □

3.2. Orientable surfaces. — Here, M is an orientable surface. The following proposition is well known. One can prove it using the fact that the mapping class group acts minimally on $\mathscr{PMF}$ (M) when the surface is orientable.

Proposition 3.5. — *If* M *is orientable, the minimal foliations are dense in* $\mathscr{MF}$ (M).

We call $\mathscr{E}$ the set of the foliations of $\mathscr{MF}$ (M) which are uniquely ergodic. Let τ be a complete train track on M. If $\mathscr{F}_0$ is in the set $\mathscr{A}$ of Proposition 2.1, there exist a neighborhood $V_{\mathscr{F}_0}$ of $\mathscr{F}_0$ in $E(\tau)$, and $\alpha_i=(p_i, q_i, \varphi_i, F_i)$ such that the restriction of Φ_i to $V_{\mathscr{F}_0}$ is linear and onto. Since M is orientable, $F_i=\varnothing$. Since τ is complete, $V_{\mathscr{F}_0}$ is a neighborhood of $\mathscr{F}_0$ in $\mathscr{MF}$ (M), and therefore, by Proposition 3.5, minimal foliations are dense in $V_{\mathscr{F}_0}$, hence it is sufficient to look at one interval. We can use Theorem IV, which says that almost all elements of Λ_{α_i} are uniquely ergodic, and we deduce by Proposition 3.5 that the measure of $V_{\mathscr{F}_0} - \mathscr{E}$ is 0. Using the same reasonning as in 3.1, we establish the following theorem, due to Masur.

Theorem (Masur [M]). — *If* M *is orientable, almost all elements of* $\mathscr{MF}$ (M) *are uniquely ergodic.*

3.3. More on nonorientable surfaces. — In fact, when we began working on the subject, our goal was to was to prove Theorem I using the result of A. Nogueira [N 2] about interval exchange transformations with flips.

The programm was: – first to establish the properties of the applications which take foliations to interval exchanges, (as in Section 2 of this work) – second to show that on nonorientable surface, almost all measured foliations induce interval exchange with flips on transverse segments.

But this programm works only when one is considering orientable foliations. Why?

Given a measured foliation $\mathscr{F}$ and a transverse segment I, we can look at the first return application on $\tilde{I}$ for the orientation covering foliation. This defines an interval exchange transformation.

So, given a train track τ on M, it is possible to define, as in Section 2, a family of applications from $E(\tau)$ into the set of interval exchange transformations. We can *almost* establish an equivalent to Proposition 2.1. The type of the interval exchange will be locally constant, and the applications locally linear, but not onto, if τ is not orientable.

In other terms, the interval exchanges correspond well to orientable foliations. It is why we were led to introduce the linear involutions, which generalize interval exchanges as foliations generalize orientable foliations.

Before introducing linear involutions, we proved the following theorem, which asserts that the set of foliations with a compact leaf which is a one-sided curve is of full measure not only in $\mathcal{MF}$ (M), but also in some linear subsets of $\mathcal{MF}$ (M), the ones which contain only orientable foliations.

THEOREM II. — *Let* M *be a nonorientable surface, and* τ *be an orientable train track on* M. *Almost all foliations of* $E(\tau)$ *have a compact leaf which is a one-sided curve.*

This is not a consequence of Theorem I, because $E(\tau)$ is a null set in $\mathcal{MF}$ (M). The proof of Theorem II is similar to the proof of Theorem I. We only need a Proposition similar to Corollary 3.4.

PROPOSITION 3.6. — *Let* τ *be an orientable train track on a nonorientable surface. Let* $\mathcal{F}_0$ *be in the set* $\mathcal{A}$ *given by proposition* 2.1. *There exists an* i *such that the interval exchange* $\Phi_i(\mathcal{F}_0)$ *has a flip.*

Let τ, $\mathcal{A}$, $\mathcal{F}_0$ be as in Proposition 3.6. We look at the first return maps $\Phi_i(\mathcal{F}_0)$ on the intervals I_i, for $i=1, \ldots, E$ (number of edges of τ). Let S be a subset of $\{1, \ldots, E\}$. We denote by A_S the union of all leaves of $\mathcal{F}_0$ which are not connections, and cut I_i if, and only if $i \in S$.

LEMMA 3.7. — A_S *is open.*

Proof. — Let i be in S and x be in I_i. The point x is not in a connection, therefore one of the two half-leaves of $\mathcal{F}_0$ beginning at x, namely L_x, is regular. There exists a neighborhood I' of x in I_i such that, if y is in I', the leaf containing y is not a connection. If L_x is compact, all leaves nearby are compact, hence they cut the same intervals, then they are in A_S. Otherwise, let J be the interior of the closure of $I' \cap L_x$. By ([FLP], exposé 9), x belongs to J.

Assume that there exists y in J such that the leaf containing y cuts a I_j. There exists a neighborhood J' of y in J such that all leaves cutting J' also cut I_j. But $L_x \cap J$ is dense in J, hence L_x cuts J', therefore it cuts I_j. As x is in A_S, $j \in S$. This proves Lemma 3.7. □

Let $S \neq S'$. By definition, $A_S \cap A_{S'} = \varnothing$, and by Lemma 3.7, $A_{S'}$ is open. Then we have: $(\bar{A}_S - A_S) \cap A_{S'} = \varnothing$. Therefore $(\bar{A}_S - A_S)$ is a union of connections of $\mathcal{F}_0$. As $M = \bigcup \bar{A}_S$, M is the union of two-manifolds with boundary, which are glued along the boundary (union of connections of $\mathcal{F}_0$).

We want to show that one of the interval exchanges has a flip. We assume the contrary.

Hypothesis. — None of the induced interval exchanges has a flip.

Let $i \in S$. We denote by N_i the closure of the union of leaves which cut I_i. N_i contains $\bar{A}_S$. It is clear that N_i is a two-manifold with boundary.

LEMMA 3.8. — *Since* $\mathcal{F}_0 \in \mathcal{A}$, N_i *is orientable.*

Proof. — N_i is the union of the closure of all bands of leaves which correspond to first return. Let B be such a band and γ be a piece of leaf in B with endpoints on I_i. We complete γ to obtain a closed path α using the segment of I_i between the endpoints of γ. We call α the elementary path associated to B. Since B does not correspond to a flip, α is a two-sided curve.

We look at B as the rectangle $(a, b)\times(0, 1)$, with $(a, b)\subset I_i$ and $\{x\}\times(0, 1)$ on a leaf. We set $\partial B=((a, b)\times\{0, 1\})\cup(\{a, b\}\times(0, 1))$. The interval (a, b) is in the interior of N_i. Since B is a band of first return, the leaf leaving I_i by the point $(a, 0)$ goes back to I_i. Therefore there exists x_a, x_b in $[0, 1]$ such that the point (a, x_a) and (b, x_b) are in ∂N_i. Let y_a and z_a (respectively y_b and z_b) be the infimum and supremum of such x_a (respectively x_b). We claim that: $\{a\}\times[y_a, z_a]\subset\partial N_i$ and $\{b\}\times[y_b, z_b]\subset\partial N_i$.

If not, there exists t_a in $[y_a, z_a]$ such that the point (a, t_a) is in $\operatorname{int} N_i$. We call K the connected component in $\operatorname{int} N_i\cap\{a\}\times(y_a, z_a)$ which contains the point (a, t_a). It is clear that K is a connection of $\mathscr{F}_0$ which separates B from another band B′ (see *Fig.* 13).

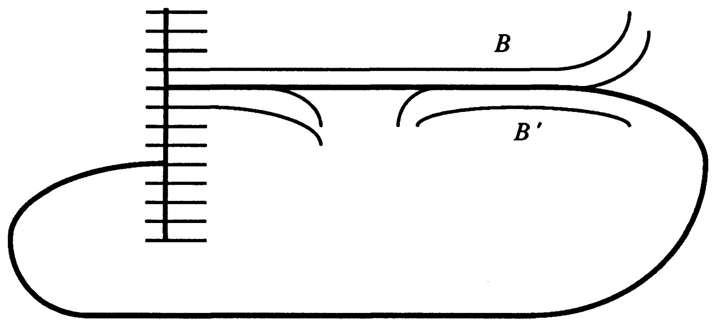


Fig. 13

We cut along I_i and the bold segment of Figure 13, add a band of width ε, and reglue. We obtain a foliation carried by τ which has no connection corresponding to K. This contradicts the fact that the connections in $\mathscr{A}$ are stable.

Then B is topologically a disk whose boundary is splitted into two segments in ∂N_i and two open intervals in $\operatorname{int} N_i$.

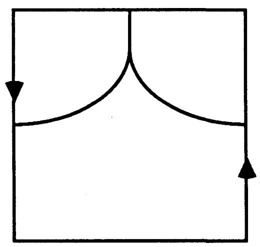
Now we look at a path α in N_i. Each segment of α in B which is not homotopic to 0 relatively to ∂B, is homotopic to a leaf in B. This says that each segment in B is homotopic to the elementary path associated to B. Therefore the elementary paths generate $\pi_1(N_i)$, hence N_i is orientable. $\square$

$\bar{A}_S$ is a submanifold of N_i, so it is orientable. Therefore M is a union of orientable pieces glued together along connections of $\mathscr{F}_0$. Let γ be a such connection corresponding to a connection path δ in τ. Since μ_0 is in $\mathscr{A}$, δ is a stable connection path. We parametrize δ.

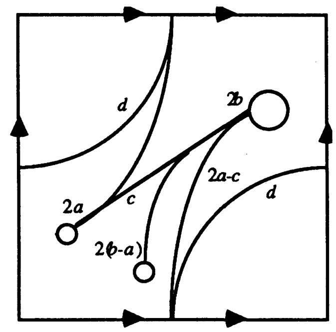
LEMMA 3.9. — *If $\delta(t_1)=\delta(t_2)$, the closed curve $\delta([t_1, t_2])$ is a two-sided curve.*

Proof. — Suppose $\delta([t_1, t_2])$ is a one-sided curve. This means that γ cuts twice the same I_i, and the first return application has a flip, which contradicts our hypothesis. $\square$

Therefore δ is a stable connection path which satisfies Lemma 3.9. Then we can apply Lemma 3.4 in [D] to conclude that δ is injective. Using Lemma 3.5 in [D], we



14 a



14 b

Fig. 14

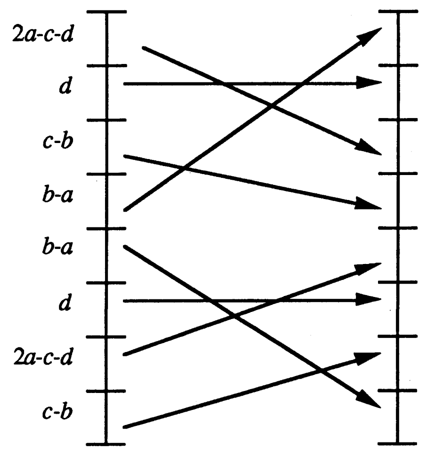


Fig. 15

see that γ is a closed connection. This connection must be a two-sided curve. Then using Lemma 3.6 of [D], we see that γ separates M. Therefore, M is a union of orientable pieces, glued along closed curves which separate M. This can happen only if M is orientable. This proves Proposition 3.6. □

3.4. Remarks. – Looking at Theorems I and II, one can ask some some questions.

First, if τ is an orientable train track on an orientable surface, are almost every foliations carried by τ uniquely ergodic? The answer is no. There exist ([D]) orientable train tracks on orientable surfaces which carry no minimal foliations, hence no uniquely ergodic ones.

The second question is the following: if τ is a train track on a nonorientable surface, have almost all foliations carried by τ a compact leaf? The answer is no. Here is a counterexample.

We construct a train track on the connected sum M of a torus and three projective planes as follows. Each part on a P^2 minus a disk is as in Figure 14*a*. The part of the train track on T^2 minus 3 disks is as in Figure 14*b*.

M-τ is an octogon. We choose generators for the weight systems, and write the numbers on the Figure 14*b*. We shall look at the first return map for the orientation covering foliation on the edge of weight $2a$. If we look at the open set in $E(\tau)$ given by $a<b<c<2a$ and $d<2a-c$, we obtain interval exchanges without flips described in Figure 15.

If a, b, c and d are rationally independent, it follows from a result of M. Keane [Ka] that the foliation given by the weight system is minimal, hence it has no compact regular leaf. This implies that in $E(\tau)$ a set of non zero measure contains foliations with no compact regular leaf.

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C. DANTHONY,
Laboratoire de mathématiques,
École Normale Supérieure de Lyon,
46, allée d'Italie, 69364 Lyon Cedex 7
et U.R.A. au C.N.R.S. n° 746.
A. NOGUEIRA,
Instituto de matemática,
U.F.R.J., Caixa Postal 68530,
21944 Rio de Janeiro R. J.,
Brésil.