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Representations with cohomology in the discrete spectrum of subgroups of $SO(n, 1)(\mathbb{Z})$ and Lefschetz numbers

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REPRESENTATIONS WITH COHOMOLOGY IN THE DISCRETE SPECTRUM OF SUBGROUPS OF $SO(n, 1)(\mathbb{Z})$ AND LEFSCHETZ NUMBERS

BY JÜRGEN ROHLFS ⁽¹⁾ AND BIRGIT SPEH ⁽²⁾

Introduction

Let G/\mathbb{Q} be a semisimple algebraic group defined over \mathbb{Q} and $\Gamma \subset G(\mathbb{Q})$ an arithmetic subgroup. Then the space $\mathcal{A}(G(\mathbb{R})/\Gamma)$ of automorphic forms in the sense of Langlands [H-Ch] (Chap. I) can be decomposed

$$\mathcal{A}(G(\mathbb{R})/\Gamma) = \mathcal{A}_{\text{cusp}}(G(\mathbb{R})/\Gamma) \oplus \mathcal{A}_{\text{Eis}}(G(\mathbb{R})/\Gamma)$$

where $\mathcal{A}_{\text{cusp}}(G(\mathbb{R})/\Gamma)$, the subspace of cuspidal automorphic forms, is a direct sum

$$\mathcal{A}_{\text{cusp}}(G(\mathbb{R})/\Gamma) = \bigoplus_{\pi \in \hat{G}(\mathbb{R})} H_{\pi}^{m(\pi, \Gamma)}$$

of unitary irreducible representations $\pi \in \hat{G}(\mathbb{R})$ of the real Lie group $G(\mathbb{R})$ with representation space H_{π} and finite multiplicities

$$m(\pi, \Gamma) = \dim \text{Hom}(H_{\pi}, \mathcal{A}_{\text{cusp}}(G(\mathbb{R})/\Gamma)).$$

Here Hom has to be taken in the category of (\mathfrak{g}, K) -modules, where \mathfrak{g} is the Lie algebra of $G(\mathbb{R})$ and K a maximal compact subgroup. The second factor $\mathcal{A}_{\text{Eis}}(G(\mathbb{R})/\Gamma)$ is spanned by Eisenstein series and their residues.

Although it is obvious that for infinitely many $\pi \in \hat{G}(\mathbb{R})$ the multiplicities $m(\pi, \Gamma)$ are non zero, very little is known about the actual dependence of the multiplicity on Γ and π . The simplest example with $G(\mathbb{R}) = \text{SL}_2(\mathbb{R})$ shows that for fixed Γ and general $\pi \in \hat{G}(\mathbb{R})$ which sits in a continuous family of representations only asymptotic results seem to be possible.

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On the other hand, there is a countable set of unitary representations π with the “topological” property that the relative Lie algebra cohomology $H^*(\mathfrak{g}, \mathbf{K}, H_\pi \otimes V) \neq \{0\}$ for some irreducible finite dimensional representation V of $G(\mathbb{R})$. At least some of these π then are forced by topological reasons to appear in $\mathcal{A}(G(\mathbb{R})/\Gamma)$. If $G(\mathbb{R})/\Gamma$ is compact this follows from Matsushima’s formula

$$H^*(X/\Gamma, \tilde{V}) = \bigoplus_{\pi \in \hat{G}(\mathbb{R})} H^*(\mathfrak{g}, \mathbf{K}, H_\pi \otimes V)^{m(\pi, \Gamma)},$$

where X is the space of maximal compact subgroups of $G(\mathbb{R})$ and \tilde{V} the sheaf associated to V on X/Γ .

If now V is sufficiently regular and if $G(\mathbb{R})$ has discrete series representations, there is enough vanishing in Matsushima’s formula and one can compute a certain (small) sum of multiplicities of discrete series representations as a non vanishing topological index like the Gauss-Bonnet-or Riemann-Roch-formula. In fact, in this favourable situation the $m(\pi, \Gamma)$ also can be determined by a trace formula of Selberg type [L1], [A]. Other related information on such multiplicities is given in [D-W] and [C].

If $G(\mathbb{R})$ has no discrete series representations, whether $G(\mathbb{R})/\Gamma$ is compact or not, the above method gives no information on multiplicities whereas the following topological approach is promising.

Assume that $\varepsilon : G(\mathbb{R}) \rightarrow G(\mathbb{R})$ is an automorphism of finite order stabilizing Γ and V . Then the Lefschetz number $L(\varepsilon, \Gamma, V) = \sum_{i=0}^{\infty} (-1)^i \text{tr}(\varepsilon^i | H^i(X/\Gamma, \tilde{V}))$ is defined and methods to compute this number have been developed [R1, R3]. A representation theoretical interpretation of the Lefschetz number contains certain contributions

$$\text{tr}(\varepsilon^* | H^*(\mathfrak{g}, \mathbf{K}, H_\pi \otimes V)) \cdot \text{tr}(\varepsilon | \text{Hom}(H_\pi, \mathcal{A}(G(\mathbb{R})/\Gamma))),$$

if π is equivalent to $\pi \circ \varepsilon$, and some boundary contributions. If ε and V are chosen carefully, this should become manageable and should give the desired non trivial information on multiplicities. One can view this approach as a topological—and that means easier—analogy of the ε -twisted Selberg trace formula. For certain subgroups Γ of $SL_2(\mathbb{C})$ this approach has been carried out in [H3], [R2] and for subgroups of $SL_3(\mathbb{R})$ in [L-S].

Since there are many technical difficulties, we present here only a relatively general example of our approach to compute multiplicities and hope to deal with the general \mathbb{Q} -rank one case later.

We consider in this paper the special orthogonal group $SO(n, 1)/\mathbb{Q}$, $n \geq 2$, associated to the quadratic form $\eta(x_1, \dots, x_{n+1}) = \sum_{i=1}^n x_i^2 - x_{n+1}^2$, $x_i \in \mathbb{Q}$, and denote by ε the standard Cartan involution given by conjugation with the matrix representing η . Let $G = SO(n, 1)(\mathbb{R})^0$ be the connected component of the real Lie group $SO(n, 1)(\mathbb{R})$ and \mathbf{K} be the maximal compact subgroup of G determined by ε . Denote by $\Gamma = \Gamma(m)$ the full congruence subgroup mod m of $SO(n, 1)(\mathbb{Z})$. We assume that 16 divides m . Then

$\Gamma(m) \subset G$ and Γ is torsionfree. Moreover, this assumption is suitable to keep down the amount of not really relevant technical computations.

Before we can state our first results, we have to label certain representations. Assume that $n = 2k + 1$. Then there are no discrete series representations. Corresponding to a certain choice of an ε -stable Cartan subalgebra in $\mathfrak{g} \otimes \mathbb{C}$ and a system of positive roots, there is, for each regular weight λ , exactly one class of unitary representations $A_k(\lambda)$ of $G = \mathrm{SO}(2k + 1, 1)(\mathbb{R})^0$ such that $H^*(\mathfrak{g}, K, A_k(\lambda) \otimes V_\lambda) \neq \{0\}$ where V_λ is irreducible of highest weight λ . Moreover, ε acts as an equivalence on $A_k(\lambda)$ and V_λ , see [V-Z] (Prop. 6.2). We write $U(\mathfrak{g})$ for the enveloping algebra of G . Then we prove:

PROPOSITION. — *Suppose that $G = \mathrm{SO}(2k + 1, 1)(\mathbb{R})^0$, that V_λ (as above) is regular and denote the action induced by ε on $\mathrm{Hom}_{U(\mathfrak{g})}(A_k(\lambda), \mathcal{A}_{\mathrm{cusp}}(G/\Gamma(m)))$ again by ε . Then*

$$\begin{aligned} L(\varepsilon, \Gamma(m), V_\lambda) &= 2(-1)^k \mathrm{tr}(\varepsilon | \mathrm{Hom}_{U(\mathfrak{g})}(A_k(\lambda), \mathcal{A}_{\mathrm{cusp}}(G/\Gamma(m)))) \\ &= 2^{-k} \dim(V_\mu) m^{k(2k+1)} \prod_{i=1}^k \left(\prod_{p|m} (1-p^{-2i}) |\zeta(1-2i)| \right). \end{aligned}$$

Here V_μ is a certain representation of $\mathfrak{so}(n)$, see 3.5, $\prod_{p|m}$ denotes the product over all prime divisors p of m and $|\zeta(1-2i)|$ is the absolute value of the Riemann zeta function at $1-2i$.

We point out that the absence of boundary contributions in the above formula stems from the absence of ε -stable boundary components which results from the assumption $16|m$. The numerical constant in our formula is up to the power 2^{-k} equal to the number of fixpoints $|(X/\Gamma)^\varepsilon|$ of the induced ε -action on X/Γ . If V_λ is not regular we have a similar slightly more complicated formula, see 5.4.

If $n = 2k$ then it is well known that there are up to equivalence exactly two representations $A_k^\pm(\lambda)$ such that for regular λ we have $H^*(\mathfrak{g}, K, A_k^\pm(\lambda) \otimes V_\lambda) \neq \{0\}$ and the $A_k^\pm(\lambda)$ are discrete series representations. Of course, the notation depends again on a system of positive roots.

PROPOSITION. — *Suppose that $G = \mathrm{SO}(2k, 1)(\mathbb{R})^0$ and that λ is regular. Then, denoting the Euler-Poincaré characteristic of $H^*(X/\Gamma(m), \mathbb{C})$ by $\chi(\Gamma(m))$, we have the formula*

$$\begin{aligned} \chi(\Gamma(m)) \cdot \dim(V_\lambda) &= (-1)^k \dim \mathrm{Hom}_{U(\mathfrak{g})}(A_k^+(\lambda) \oplus A_k^-(\lambda), \mathcal{A}_{\mathrm{cusp}}(G/\Gamma(m))) \\ &\quad + h(\Gamma(m)) \cdot \sum_{i=0}^{k-1} (-1)^{2k-1-i} \dim \pi_i(\lambda). \end{aligned}$$

Here $h(\Gamma(m))$ is the number of boundary components and the $\pi_i(\lambda)$ are certain finite dimensional representations of $\mathrm{SO}(n-1)(\mathbb{R})$ determined by the cohomology of the boundary.

Of course $\chi(\Gamma(m))$ is proportional to the volume of $G/\Gamma(m)$. We compute this volume in 3.4.3.

Using an estimate for $h(\Gamma(m))$ we obtain

COROLLARY. — *There is a constant $C > 0$ such that*

$$\dim \operatorname{Hom}_{\mathbb{U}(\mathfrak{g})}(A_k^+(\lambda) \oplus A_k^-(\lambda), \mathcal{A}_{\text{cusp}}(G/\Gamma(m))) \geq \dim V_\lambda \chi(\Gamma(m)) - C m^{k(2k-1)+1}.$$

A similar result was obtained by Barbasch and Moscovici [B-M]. We recall that $\chi(\Gamma(m))$ grows like $m^{k(2k+1)}$.

We now summarize the contents of this paper. In Chapter I a representation theoretical interpretation of $H^*(X/\Gamma, \tilde{V})$ for all V is given. As a starting point we establish the equality

$$H^*(\mathfrak{g}, K, C^\infty(G/\Gamma)) \cong H^*(\mathfrak{g}, K, \mathcal{A}(G/\Gamma))$$

due to W. Casselman (unpublished) for \mathbb{Q} -rank one. In 1.2 cohomology classes given by Eisenstein series at a regular parameter are discussed and in 1.4 we treat classes given by residues of Eisenstein series. We collect in 1.3 the results on (\mathfrak{g}, K) cohomology, see [V-Z], which are needed in this paper. The methods used in Chapter I have been developed in [S1], for generalisations to \mathbb{Q} -rank one groups, see [S2].

In the first three paragraphs of Chapter II we interpret $L(\varepsilon, \Gamma, V)$ in terms of $H^*(\mathfrak{g}, K, \mathcal{A}(G/\Gamma) \otimes V)$ and write the induced ε -action on

$$H^*(\mathfrak{g}, K, H_\pi \otimes V) \otimes \operatorname{Hom}_{\mathbb{U}(\mathfrak{g})}(H_\pi, \mathcal{A}(G/\Gamma))$$

as a product $\varepsilon_1[\pi] \otimes \varepsilon_2[\pi]$ of actions on the factors. Both $\varepsilon_i = \varepsilon_i[\pi]$ depend on a choice of an equivalence $C_\varepsilon : H_{\pi \circ \varepsilon} \rightarrow H_\pi$. Finally, in 2.4 we obtain a formula for the Lefschetz number which has for $G = \operatorname{SO}(n, 1)(\mathbb{R})^0$ and $n = 2k + 1$ the form

$$\begin{aligned} L(\varepsilon, \Gamma, V_\lambda) &= \sum_{i=i(\lambda)}^k (-1)^i 2 \operatorname{tr}(\varepsilon | \operatorname{Hom}_{\mathbb{U}(\mathfrak{g})}(A_i(\lambda), \mathcal{A}_{\text{dis}}(G/\Gamma))) \\ &\quad + \sum_{i=0}^k (-1)^{2k-1-i} \operatorname{tr}(\varepsilon | \operatorname{Hom}_{\mathbb{U}(\mathfrak{g})}(U_i(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma))). \end{aligned}$$

Here $A_i(\lambda)$, $i = i(\lambda), \dots, k$ represent the classes of inequivalent unitary representation π with $H^*(\mathfrak{g}, K, H_\pi \otimes V_\lambda) \neq \{0\}$, each $A_i(\lambda)$ is the Langlands quotient of a principal series representation $U_i(\lambda)$ and $A_i(\lambda)$ sits in $\mathcal{A}_{\text{dis}}(G/\Gamma)$, the space of square integrable functions of $\mathcal{A}(G/\Gamma)$. For the definition of the other principal series representations $U_i(\lambda)$ and the ε -action on the second sum, see 1.3, 2.4. Similar formulas are given for $n = 2k$ and also for other automorphisms than ε .

In Chapter III we show that $(X/\Gamma)^\varepsilon$, the fixpoint set of the ε -action on X/Γ , consists of finitely many points and prove $L(\varepsilon, \Gamma, V) = |(X/\Gamma)^\varepsilon| \operatorname{tr}(\eta | V)$. Here we use our assumption $16 | m$. The fixpoint set $(X/\Gamma)^\varepsilon$ is identified with the first non abelian cohomology $H^1(C, \Gamma)$ of the group $C = \{1, \varepsilon\}$ acting on Γ and the determination of $|H^1(C, \Gamma)|$ is reduced to a volume computation. The trace $\operatorname{tr}(\eta | V_\lambda)$ is given in 3.5 using Weyl's resp. Kostant's [K] character formula.

We give an estimation of the number of $\Gamma(m)$ -conjugacy classes of cuspidal parabolic subgroups in 4.1. Paragraph 4.2 contains a rather long but conceptual proof of the

fact that there are no ε -stable Γ -conjugacy classes of cuspidal parabolic subgroups. This depends again on our assumption $16 \mid m$ and resulting properties of $H^1(C, \Gamma(m))$. As a consequence we show in 4.3 that in the Lefschetz number of the Cartan involution ε the boundary contributions vanish. In Chapter V we finally state our main results.

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I. Automorphic cohomology

Let G be the connected component of $SO(n, 1)(\mathbb{R})$, $n \geq 2$, where $SO(n, 1)(\mathbb{R})$ is the group of all real $(n + 1) \times (n + 1)$ -matrices with determinant 1 which preserve the quadratic

form

$$x_1^2 + x_2^2 + \dots + x_n^2 - x_{n+1}^2.$$

We assume that $\Gamma \subset G$ is a torsionfree subgroup of finite index in $\mathrm{SO}(n, 1)(\mathbb{Z})$. In this section we describe the cohomology of Γ with coefficients in a finite dimensional irreducible representation

$$\delta : G \rightarrow \mathrm{End} V$$

in representation theoretical terms. We analyze the contributions to the cohomology by Eisenstein series in 1.2, the residual spectrum in 1.4 and the cuspidal spectrum in 1.1. In 1.5 we summarize the results. The results of this section are a special case of the situation discussed in [S2]. When it is possible we give direct proofs, but in many instances we only sketch what can be found in detail in [S2].

1.1. THE $(\mathfrak{g}, \mathbb{K})$ -COHOMOLOGY OF THE SPACE OF AUTOMORPHIC FORMS. — Let $\mathbb{K} = \mathrm{SO}(n)(\mathbb{R})$ be the maximal compact subgroup of G , which is embedded by mapping $A \in \mathrm{SO}(n)(\mathbb{R})$ to

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

in obvious notation.

For $\Gamma \subset G$ as above we have by Matsushima's result [B-W]

$$H^*(\Gamma, V) \cong H^*(\mathfrak{g}, \mathbb{K}, C^\infty(G/\Gamma) \otimes V),$$

where V is irreducible for the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} = \mathrm{Lie}(G)$.

We will later prove:

1.1.1. THEOREM. — *Let $G = \mathrm{SO}(n, 1)(\mathbb{R})^0$, $\Gamma \subset G$ as above and $\mathcal{A}(G/\Gamma)$ the space of automorphic forms on G/Γ . Then*

$$H^*(\Gamma, V) \cong H^*(\mathfrak{g}, \mathbb{K}, \mathcal{A}(G/\Gamma) \otimes V).$$

For a definition of automorphic forms see [H-Ch] (1.2). This result is a special case of a theorem announced by W. Casselman in a letter to Borel.

To prove the theorem we first analyze

$$H^*(\mathfrak{g}, \mathbb{K}, \mathcal{A}(G/\Gamma) \otimes V)$$

and then use the results of [S2] and [H4].

1.2. As a $(\mathfrak{g}, \mathbb{K})$ -module

$$\mathcal{A}(G/\Gamma) \cong \mathcal{A}_{\mathrm{cusp}}(G/\Gamma) \oplus \mathcal{A}_{\mathrm{Eis}}(G/\Gamma).$$

Here $\mathcal{A}_{\mathrm{cusp}}(G/\Gamma)$ is the space of cusp forms [H-Ch] (1.4) and $\mathcal{A}_{\mathrm{Eis}}(G/\Gamma)$ is the space spanned by Eisenstein series and their residues (see 1.2 for details). So we have

$$H^*(\mathfrak{g}, K, \mathcal{A}(G/\Gamma) \otimes V) \cong H^*(\mathfrak{g}, K, \mathcal{A}_{\text{cusp}}(G/\Gamma) \otimes V) \oplus H^*(\mathfrak{g}, K, \mathcal{A}_{\text{Eis}}(G/\Gamma) \otimes V).$$

We may replace $\mathcal{A}_{\text{cusp}}(G/\Gamma)$, resp. $\mathcal{A}_{\text{Eis}}(G/\Gamma)$, by $\mathcal{A}_{\text{cusp}}^V(G/\Gamma)$, resp. $\mathcal{A}_{\text{Eis}}^V(G/\Gamma)$, the (\mathfrak{g}, K) -submodule with the same generalized infinitesimal character as V (see [S2], 1.1).

If π_1, \dots, π_r are all the unitary irreducible representations of G up to equivalence so that

$$H^*(\mathfrak{g}, K, \pi_i \otimes V) \neq 0, \quad i=1, \dots, r,$$

then, since $\mathcal{A}_{\text{cusp}}(G/\Gamma)$ is a direct sum of unitary irreducible representations which occur with finite multiplicity [H-Ch] (1. Theorem 1), we have

$$H^*(\mathfrak{g}, K, \mathcal{A}_{\text{cusp}}^V(G/\Gamma) \otimes V) \simeq \bigoplus_{i=1}^r H^*(\mathfrak{g}, K, \pi_i \otimes V)^{m(\pi_i, \Gamma)}$$

with $m(\pi_i, \Gamma) = \text{Hom}_{U(\mathfrak{g})}(\pi_i, \mathcal{A}_{\text{cusp}}(G/\Gamma))$.

So to describe the cohomology of Γ in representation theoretical terms, it suffices to handle the term

$$H^*(\mathfrak{g}, K, \mathcal{A}_{\text{Eis}}^V(G/\Gamma) \otimes V).$$

1.2. EISENSTEIN CLASSES. — Let us start with a description of the submodule structure of $\mathcal{A}_{\text{Eis}}^V(G/\Gamma)$. All indecomposable subrepresentations of $\mathcal{A}_{\text{Eis}}^V(G/\Gamma)$ are in the image of an Eisenstein intertwining operator

$$E: \text{ind}_{\text{MAN}}^G \pi \otimes v^{(l)} \otimes 1. \rightarrow \mathcal{A}(G/\Gamma),$$

where $\text{Re } v \geq 0$, [S 2].

Here $P = \text{MAN}$ is a parabolic subgroup of G , π an irreducible unitary subrepresentation of $L^2(M)$ and $v^{(l)}$ an l -dimensional indecomposable representation of A . Induction means normalized induction and we assume that the induced representation is realized as a subrepresentation of the left regular representation on $C^\infty(G/N)$.

Put $I(P, \pi, v^{(l)}) = \text{ind}_P^G \pi \otimes v^{(l)} \otimes 1$. If we consider a fixed parabolic subgroup P we omit the index P and similarly we omit l if $l=1$.

Remark. — An l -dimensional indecomposable representation $v^{(l)}$ of $A \cong \mathbb{R}_+^*$ has a unique socle v . Furthermore two l -dimensional indecomposable representations with the same socle are isomorphic. This is a consequence of the existence of Jordan-normal forms of complex matrices.

Let $0 \neq \Phi_v \in I(\pi, v)$, $\text{Re } v \geq 0$. In [S 1] it was proved that

$$\left(\frac{d}{dv}\right)^l \Phi_v \in I(\pi, v^{(l+1)}) \subset C^\infty(G/N)$$

and

$$\left(\frac{d}{dv}\right)^l \Phi_v \neq 0 \text{ mod } I(\pi, v^{(l)}),$$

The functions

$$\left(\frac{d}{dv}\right)^j \Phi_v, \quad \Phi_v \in I(\pi, v), \quad j < l,$$

span $I(\pi, v^{(l)})$.

If v is very dominant with respect to the choice of N we define following [H-Ch] for $\Phi_v \in I(\pi, v)$ the Eisenstein series

$$E(\pi, \Phi, v, g) = \sum_{\gamma \in \Gamma/\Gamma \cap P} \Phi_v(g\gamma).$$

These Eisenstein series have a meromorphic continuation with a finite number of simple poles for $\text{Real } v \geq 0$. We define the Eisenstein intertwining operator if $\text{Re } v_0 \geq 0$ by

$$E\left(\left(\frac{d}{dv}\right)^j \Phi_{v_0}\right)(g) = \frac{j!}{2\pi i} \int_C \frac{E(\pi, \Phi, v, g)}{(v-v_0)^{j+1}} dv$$

if all the Eisenstein series $E(\pi, \Phi, v)$, $\Phi \in I(\pi, v)$, are non singular at v_0 and otherwise by

$$E\left(\left(\frac{d}{dv}\right)^j \Phi_{v_0}\right)(g) = \frac{j!}{2\pi i} \int_C \frac{(v-v_0) E(\pi, \Phi, v, g)}{(v-v_0)^{j+1}} dv.$$

Here C is a circle around $v_0 \geq 0$, so that the Eisenstein series

$$E(\pi, \Phi, v, g), \quad \Phi_v \in I(\pi, v),$$

have no pole in the disk defined by C except possibly for $v=v_0$, see [H-Ch] (4.7).

1.2.1. THEOREM. — *Suppose that*

$$H^*(\mathfrak{g}, K, I(\pi, v) \otimes V) \neq 0.$$

Then (with a shift in degrees)

$$H^*(\mathfrak{g}, K, I(\pi, v) \otimes V) \cong \Lambda^* \mathbb{C}.$$

Let $v(\pi)$ be the smallest integer so that

$$H^{v(\pi)}(\mathfrak{g}, K, I(\pi, v) \otimes V) \neq 0.$$

Then

$$\lim_{l \rightarrow \infty} H^i(\mathfrak{g}, K, I(\pi, v^{(l)}) \otimes V) = \begin{cases} \mathbb{C} & \text{if } i = v(\pi) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — The first part of the theorem is proved in [B-W] in the case $V = \mathbb{C}$. Using tensor functors [V] we can reduce the case of general coefficients to that case. Since in the case of trivial coefficients the cohomology of π is non trivial only in one degree the second part of the theorem follows from [S 1].

Q.E.D.

1.2.2. COROLLARY. — Suppose that $I(\pi, \nu)$ satisfies the assumptions of 1.2.1 and that in addition ν is non-negative with respect to N and the Eisenstein map

$$E: I(\pi, \nu) \rightarrow \mathcal{A}(G/\Gamma)$$

is injective. Write E^* for the Eisenstein map on cohomology. Then

$$E^i(H^i(\mathfrak{g}, K, I(\pi, \nu) \otimes V)) = \begin{cases} 0 & \text{if } i \neq \nu(\pi) \\ \mathbb{C} & \text{if } i = \nu(\pi). \end{cases}$$

Proof. — By 1.2.1 we have

$$\lim_{i \rightarrow \infty} E^i(H^i(\mathfrak{g}, K, I(\pi, \nu^{(i)}) \otimes V) = 0 \quad \text{if } i \neq \nu(\pi).$$

In [S 1] was proved that in fact the class in

$$E^{\nu(\pi)}(H^{\nu(\pi)}(\mathfrak{g}, K, I(\pi, \nu) \otimes V))$$

has a non-trivial restriction to the boundary of the Borel-Serre compactification. Hence it is non-zero in $H^*(\mathfrak{g}, K, C^\infty(G/\Gamma) \otimes V)$ and so in particular in $H^*(\mathfrak{g}, K, \mathcal{A}(G/\Gamma) \otimes V)$.

Q.E.D.

1.3. REPRESENTATIONS WITH NON-TRIVIAL (\mathfrak{g}, K) COHOMOLOGY. — Before we discuss the non-injective Eisenstein intertwining operator let us recall the classification of the irreducible unitary representations of G with non-trivial (\mathfrak{g}, K) -cohomology. We follow the conventions of [V-Z]. Since the results in this section are a special case of these in [V-Z] we state them without proof and refer to [V-Z] for a proof.

We use small German letters for Lie algebra of a subgroup and denote the complexification by the subscript \mathbb{C} .

Let θ be the Cartan involution of \mathfrak{g} and hence of $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{h}_{\mathbb{C}}$ be a θ stable fundamental Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. If $\mathfrak{g} = \mathfrak{so}(2k, 1)$ then $\mathfrak{h} = \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{g}$ is the Lie algebra of a compact torus. If $\mathfrak{g} \cong \mathfrak{so}(2k+1, 1)$, then $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$, where \mathfrak{t} is the Lie algebra of a maximal torus in K .

We may assume that

$$\mathfrak{a} \cong \mathbb{R}(E_{n+1, n} + E_{n, n+1})$$

where $E_{i, j}$ is the elementary matrix with 1 in place of i, j and 0 otherwise.

Suppose that $n = 2k + 1$ or $n = 2k$. Then all θ -stable parabolic subalgebras $\mathfrak{p}_i = \mathfrak{l}_i \oplus \mathfrak{n}_i$ of \mathfrak{g} have Levi factor isomorphic to

$$\begin{aligned} \mathfrak{l}_{i, \mathbb{C}} &\cong \mathbb{C}^i \oplus \mathfrak{so}(n-2i, 1)_{\mathbb{C}}, & 0 \leq i < k \\ \mathfrak{l}_{k, \mathbb{C}} &\cong \mathbb{C}^k, & i = k. \end{aligned}$$

and unipotent radical \mathfrak{n}_i .

Let $\psi(I_i)$ be a set of positive roots of $(I_i, \mathfrak{h}_{\mathbb{C}})$ such that the sum of those roots, which are also roots of $((I_i \cap \mathfrak{f})_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is dominant with respect to $\psi(I_i)$. Write $\Delta(\mathfrak{n}_i)$ for the roots of $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{n}_{i, \mathbb{C}})$ and put

$$\psi_i = \psi(I_i) \cup \Delta(\mathfrak{n}_i).$$

Let L_i be the normalizer of I_i, \mathbb{C} in G and let

$$\lambda: L_i \rightarrow \mathbb{C}$$

be a character such that

$$(\star) \quad \left\langle \lambda + \frac{1}{2} \sum_{\alpha \in \psi_i} \alpha, \beta \right\rangle \geq 0 \quad \text{for all } \beta \in \psi_i.$$

Here we denote the differential of λ at the identity also by λ .

In [V-Z] it was proved that for each (ψ_i, λ) , $0 \leq i \leq k$ and $n = 2k + 1$, or $0 \leq i < k$ and $n = 2k$ which satisfies (\star) , there exists exactly one irreducible unitary representation $A_i(\lambda)$ so that

$$(1.3.1) \quad H^q(\mathfrak{g}, K, A_i(\lambda) \otimes V_\lambda) = \begin{cases} 0 & q \neq i, \quad n - i \\ \mathbb{C} & q = i, \quad n - i \end{cases}$$

for the finite dimensional irreducible $U(\mathfrak{g})$ -module V_λ with highest weight λ restricted to $\mathfrak{h}_{\mathbb{C}}$.

If $i = k$ and $n = 2k$, then there are two inequivalent choices of ψ_i and so [V-Z], we obtain two representations $A_k^+(\lambda), A_k^-(\lambda)$ with property 1.3.1. Put

$$A_k(\lambda) = A_k^+(\lambda) \oplus A_k^-(\lambda)$$

and

$$A_i(0) = A_i.$$

Since $L_{i_0} \supset L_j$ if $j \geq i_0$, every character $\lambda: L_{i_0} \rightarrow \mathbb{C}$ which satisfies the condition (\star) for ψ_{i_0} , is also a character $\lambda: L_j \rightarrow \mathbb{C}$, $j > i_0$ and it satisfies (\star) for ψ_j , $j > i_0$. We denote by $i(\lambda)$ the smallest integer so that λ is a character of $L_{i(\lambda)}$ and it satisfies (\star) with respect to $\psi_{i(\lambda)}$. Then the representation $A_i(\lambda)$, $i(\lambda) \leq i \leq k$, exists and every irreducible unitary representation with nontrivial cohomology with respect to V_λ is equivalent to a representation $A_i(\lambda)$ if $i \neq k$ or a factor of $A_k(\lambda)$ for some $i \geq i(\lambda)$, [V-Z]. A short computation using (\star) and [V-Z] shows that

(a) if $n = 2k$ and λ is nonsingular then $A_k^+(\lambda), A_k^-(\lambda)$ are the only representations with nontrivial (\mathfrak{g}, K) -cohomology;

(b) if $n = 2k + 1$ and if λ in the notation of Bourbaki equals $(\lambda_1, \dots, \lambda_k)$, $\lambda_i > \lambda_j$, $i > j$ and $\lambda_k \neq 0$, then there are no representations with nontrivial (\mathfrak{g}, K) -cohomology;

(c) if $n = 2k + 1$, and $\lambda = (\lambda_1, \dots, \lambda_k, 0)$, $\lambda_i > \lambda_j > 0$, $i > j$, and λ is nonsingular with respect to ψ_k , then $A_k(\lambda)$ is the only representation with nontrivial (\mathfrak{g}, K) -cohomology.

If $G = \mathrm{SO}(2k, 1)(\mathbb{R})^0$, then $A_k(\lambda)$ is a sum of two discrete series representations, whereas if $G = \mathrm{SO}(2k+1, 1)(\mathbb{R})^0$, then $A_k(\lambda)$ is a tempered unitarily induced principal series representation denoted by $U_k(\lambda)$, [V-Z].

For $\lambda: L_i \rightarrow \mathbb{C}$ which satisfies (\star) there exist *principal series representations* $U_i(\lambda)$, $0 \leq i < k$ so that

$$(1.3.2) \quad H^q(\mathfrak{g}, K, U_i(\lambda) \otimes V_\lambda) = \begin{cases} \mathbb{C} & \text{if } q = n-i \text{ or } n-i-1 \\ 0 & \text{otherwise.} \end{cases}$$

We have for $i(\lambda) \leq i < k$ the exact sequence

$$(1.3.3) \quad 0 \rightarrow A_{i+1}(\lambda) \rightarrow U_i(\lambda) \rightarrow A_i(\lambda) \rightarrow 0.$$

This follows from [B-W] using translation functors [V] (4.5). The formula for the cohomology of an induced representation [B-W] shows that for $0 \leq i < i(\lambda)$ there are also principal series representations $U_i(\lambda)$ with

$$H^q(\mathfrak{g}, K, U_i(\lambda) \otimes V_\lambda) = \begin{cases} \mathbb{C} & \text{if } q = n-i \text{ or } n-i-1 \\ 0 & \text{otherwise} \end{cases}$$

Comparing their composition factors with the unitary dual of G shows however that they don't have unitary quotients, [B-B].

Using [B-W] (III 3.3 and VI 3.1, 3.2, 3.3) and translation functors [V] (4.5) one can show that every principal series representation $I(\pi, \nu)$ with $\mathrm{Re} \nu \geq 0$ and

$$H^*(\mathfrak{g}, K, I(\pi, \nu) \otimes V_\lambda) \neq 0$$

is isomorphic to a representation $U_i(\lambda)$ for some $0 \leq i < n/2$.

Suppose $U_i(\lambda) = I(\pi_i, \nu_i)$ for some $\pi_i \in L^2(M)$, $\nu_i \in \hat{A}$. We define

$$U_i^{(l)}(\lambda) = I(\pi_i, \nu_i^{(l)}).$$

If λ is the trivial character we simply write $U_i^{(l)}$. We have the non splitting short exact sequence

$$(1.3.4) \quad 0 \rightarrow U_i^{(l)}(\lambda) \rightarrow U_i^{(l+1)}(\lambda) \rightarrow U_i(\lambda) \rightarrow 0.$$

In [S1] it is proved that for $l > 1$ and all $q \geq 0$

$$(1.3.5) \quad \dim H^q(\mathfrak{g}, K, U_i^{(l)}(\lambda) \otimes V_\lambda) \cong \dim H^q(\mathfrak{g}, K, U_i(\lambda) \otimes V_\lambda).$$

1.4. RESIDUAL CLASSES. — Since the situation for an arbitrary coefficient system V_λ and for the trivial coefficients \mathbb{C} are very similar, we will in this section sometimes give proofs only for trivial coefficients.

The Eisenstein intertwining operator (explained in 1.2)

$$E: I(\pi, \nu^{(l)}) \rightarrow \mathcal{A}(G/\Gamma) \\ \Phi_\nu \mapsto E(\pi, \Phi, \nu)$$

is injective unless $E(\Phi, \pi, \nu, g)$ has a pole at $\nu = \nu_0$ for some function $\Phi_\nu \in I(\pi, \nu)$. [S2].

1.4.1. LEMMA. — *Suppose that for $\operatorname{Re} \nu > 0$*

$$E: I(\pi, \nu) \rightarrow \mathcal{A}(G/\Gamma)$$

is not injective and suppose that the image J has nontrivial (\mathfrak{g}, K) -cohomology. Then

$$J \cong A_i$$

$$I(\pi, \nu) \cong U_i$$

$$\operatorname{kern} E \cong A_{i+1} \quad \text{for some } 0 \leq i < k.$$

Proof. — We identify the character ν with its differential at the identity. Since the representation in the image of E is contained in the residual spectrum, it is unitary and hence isomorphic to A_i for some i . The Langlands classification [B-W] implies that there is a unique principal series representation so that A_i is a subquotient, but not a subrepresentation, namely, U_i . The last assertion then follows from 1.3.3.

Q.E.D.

Using 1.4.1 and [S2] (4.1.3), we deduce

1.4.2. LEMMA. — *Suppose that*

$$E: U_i^{(l)} \rightarrow \mathcal{A}(G/\Gamma), \quad l \geq 1$$

is an Eisenstein intertwining operator which is not injective. Then

$$\operatorname{kern} E = A_{i+1}.$$

1.4.3. PROPOSITION. — *Suppose that*

$$H^*(\mathfrak{g}, K, U_i^{(l)}/A_{i+1}) \neq 0, \quad l \geq 1.$$

Then

$$\lim_{l \rightarrow \infty} H^j(\mathfrak{g}, K, U_i^{(l)}/A_{i+1}) = \begin{cases} \mathbb{C} & \text{if } j=i \\ 0 & \text{if } j \neq i. \end{cases}$$

Proof. — We have the exact sequence

$$0 \rightarrow A_{i+1} \rightarrow U_i^{(l)} \rightarrow U_i^{(l)}/A_{i+1} \rightarrow 0.$$

Since the direct limit is exact we get for all $j \in \mathbb{N}$ the exact sequence

$$\begin{aligned} \lim_{l \rightarrow \infty} H^{j-1}(\mathfrak{g}, K, U_i^{(l)}/A_{i+1}) &\xrightarrow{\delta} H^j(\mathfrak{g}, K, A_{i+1}) \\ &\rightarrow \lim_{l \rightarrow \infty} H^j(\mathfrak{g}, K, U_i^{(l)}) \rightarrow \lim_{l \rightarrow \infty} H^j(\mathfrak{g}, K, U_i^{(l)}/A_{i+1}) \\ &\xrightarrow{\delta} H^{j+1}(\mathfrak{g}, K, A_{i+1}) \rightarrow \lim_{l \rightarrow \infty} H^{j+1}(\mathfrak{g}, K, U_i^{(l)}). \end{aligned}$$

Using 1.2.1 and 1.3.2 we have

$$\lim_{l \rightarrow \infty} H^j(\mathfrak{g}, K, U_i^{(l)}) = \begin{cases} \mathbb{C} & \text{if } j = n - i - 1 \\ 0 & \text{if } j \neq n - i - 1. \end{cases}$$

For $j \notin \{i, n-i, i+1, n-i-1\}$ we apply 1.3.1 and see that $\lim_{l \rightarrow \infty} H^j(\mathfrak{g}, K, U_i^{(l)}/A_{i+1}) = 0$. If $j = i$ we get $\lim_{l \rightarrow \infty} H^i(\mathfrak{g}, K, U_i^{(l)}/A_{i+1}) \cong \mathbb{C}$ and if $j \in \{i+1, n-i, n-i-1\}$ we easily derive

$$\lim_{l \rightarrow \infty} H^j(\mathfrak{g}, K, U_i^{(l)}/A_{i+1}) = 0$$

and our claim holds.

Q.E.D.

1.4.4. THEOREM. — Suppose that $E : U_i \rightarrow \mathcal{A}(G/\Gamma)$ is an Eisenstein map with $\ker E \cong A_{i+1}$. We write for the map on cohomology

$$E^* : H^*(\mathfrak{g}, K, U_i^{(l)}/A_{i+1}) \rightarrow H^*(\mathfrak{g}, K, \mathcal{A}(G/\Gamma)).$$

Then

$$\lim_{l \rightarrow \infty} E^q(H^q(\mathfrak{g}, K, U_i^{(l)}/A_{i+1})) = \begin{cases} \mathbb{C} & \text{if } q = i \\ 0 & \text{if } q \neq i. \end{cases}$$

Proof. — By 1.4.3

$$\lim_{l \rightarrow \infty} H^q(\mathfrak{g}, K, U_i^{(l)}/A_{i+1}) = 0 \quad \text{if } q \neq i.$$

So it suffices to show that the image of the nontrivial class in degree i is not trivial. The image of this class was constructed by G. Harder in [H2] (Sec. 4) and he proved that it has a nontrivial restriction to a face of the Borel Serre compactification, hence it is nontrivial.

We may also prove its existence as follows. If this form would be a coboundary, then it would be a coboundary in the (\mathfrak{g}, K) -cohomology of an indecomposable submodule of $\mathcal{A}(G/\Gamma)$ which contains $E(U_i) \cong A_i$. But all indecomposable submodules of $\mathcal{A}(G/\Gamma)$ with this property are contained in $E(U_i^{(l)})$ for some $l > 0$.

Q.E.D.

Now after discussing all the submodules of $\mathcal{A}_{\text{Eis}}^V(G/\Gamma)$ which have nontrivial (\mathfrak{g}, K) -cohomology, we finally can prove 1.1.1.

Proof of 1.1.1. — In [S1] it is proved that all the classes in $H^*(\mathfrak{g}, K, \mathcal{A}(G/\Gamma))$ which are constructed in 1.2.2 are, in fact, nontrivial classes in $H^*(\mathfrak{g}, K, C^\infty(G/\Gamma))$. On the other hand, by [S2] (4.2.2) the classes constructed in 1.4.4 are also nontrivial in $H^*(\mathfrak{g}, K, C^\infty(G/\Gamma))$. So using [B2] (Cor. 5.5) for the cuspidal contributions, we see

that the map on cohomology induced by the inclusion of $\mathcal{A}(G/\Gamma)$ into $C^\infty(G/\Gamma)$ is injective. But by [H2] (pp. 158-159) it is also surjective.

Q.E.D.

1.5. A FORMULA FOR THE COHOMOLOGY OF Γ . — Let us summarize our results.

1.5.1. THEOREM. — Let Γ be a torsion free arithmetic subgroup of G . Let $n=2k+1$ or $n=2k$. Then

$$\begin{aligned} \dim H^i(\Gamma, V_\lambda) &= 0 \quad \text{if } i < i(\lambda) \\ &= \dim \operatorname{Hom}_{U(\mathfrak{g})}(A_i(\lambda), \mathcal{A}(G/\Gamma) \cap L^2(G/\Gamma)) \quad \text{if } i(\lambda) \leq i < k \\ &= \dim \operatorname{Hom}_{U(\mathfrak{g})}(A_k(\lambda), \mathcal{A}(G/\Gamma)) \quad \text{if } n=2k+1 \text{ and } i=k \\ &= \dim \operatorname{Hom}_{U(\mathfrak{g})}(A_k, \mathcal{A}_{\text{cusp}}(G/\Gamma)) \\ &\quad + \dim \langle E \in \operatorname{Hom}_{U(\mathfrak{g})}(U_{k-1}(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) / \operatorname{Ker} E = 0 \rangle \\ &\quad \text{if } n=2k \text{ and } i=k \\ &= \dim \operatorname{Hom}_{U(\mathfrak{g})}(A_{n-i}(\lambda), \mathcal{A}_{\text{cusp}}(G/\Gamma)) \\ &\quad + \dim \langle E \in \operatorname{Hom}_{U(\mathfrak{g})}(U_{n-i-1}(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) / \operatorname{Ker} E = 0 \rangle \\ &\quad \text{if } k < i \leq n-i(\lambda) \\ &= \dim \operatorname{Hom}_{U(\mathfrak{g})}(U_{n-i-1}(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) \quad \text{if } n-i(\lambda) < i < n. \end{aligned}$$

Here we write $\langle E \in \operatorname{Hom}_{U(\mathfrak{g})}(U_{n-i-1}(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) / \operatorname{ker} E = 0 \rangle$ for a complementary subspace of

$$\{ E \in \operatorname{Hom}_{U(\mathfrak{g})}(U_{n-i-1}(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) / \operatorname{ker} E \neq 0 \} \text{ in } \operatorname{Hom}_{U(\mathfrak{g})}(U_{n-i-1}(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)).$$

II. Lefschetz numbers of involutions for $H^*(\Gamma, V)$.

A representation theoretical interpretation

Let \underline{G} be an algebraic semisimple group defined over \mathbb{Q} , $G = \underline{G}(\mathbb{R})^0$, K a maximal compact subgroup and $\Gamma \subset G$ an arithmetic torsionfree subgroup. Let $\varepsilon : G \rightarrow G$ be an involutive automorphism so that $\varepsilon(K) \subset K$ and $\varepsilon(\Gamma) \subset \Gamma$. It acts on $C^\infty(G/\Gamma)$ by

$$L(\varepsilon) f : g \rightarrow f(\varepsilon(g))$$

and defines a representation of $G' = G \rtimes (\varepsilon, 1)$ since

$$L(\varepsilon) L(g) = L(\varepsilon(g)) L(\varepsilon).$$

Here $L(g)$ is the left regular representation of G on $C^\infty(G/\Gamma)$. We choose once and for all an ε -stable Borel subgroup of $\underline{G}(\mathbb{C})$.

Let (δ, V_δ) be a finite dimensional irreducible representation of G so that $\delta \cong \delta \circ \varepsilon$. We define a representation of G' on V_δ —and hence on $C^\infty(G/\Gamma) \otimes V_\delta$ —by the requirement that ε acts identically on the highest weight vector of V_δ .

The involution ε acts on the Lie algebra as well. Therefore we have an action $L^*(\varepsilon)$ on

$$H^*(\mathfrak{g}, K, C^\infty(G/\Gamma) \otimes V_\delta)$$

We defined the Lefschetz number of ε by

$$L(\varepsilon, \Gamma, V_\delta) = \sum_i (-1)^i \text{tr } L^i(\varepsilon).$$

If $V_\delta = \mathbb{C}$ we write $L(\varepsilon, \Gamma)$. We first show that this definition of the Lefschetz number of ε coincides with the usual geometric definition (2.1). Then we define the Lefschetz number of ε on the (\mathfrak{g}, K) -cohomology of a unitary representation and compute it in 2.2 for unitary representations of $G = SO(n, 1)(\mathbb{R})^0$. In 2.3 we show that $L^*(\varepsilon)$ can be written as a product of a ‘‘Hecke operator’’ and of an ‘‘operator on the cohomology of the representation at infinity’’ and in 2.4 prove a formula which is a representation theoretical interpretation of the Lefschetz fixpoint theorem.

2.1. THE LEFSCHETZ NUMBER OF AN INVOLUTION ON THE (\mathfrak{g}, K) -COHOMOLOGY OF AUTOMORPHIC FORMS. — To simplify the notation we prove the theorems in this part only in the special case $V_\delta = \mathbb{C}$.

The involution ε operates on the forms on $K \backslash G/\Gamma$. So we have an action ε^* of ε on $H_{\text{de Rham}}^*(K \backslash G/\Gamma, V_\delta)$ and define the geometric Lefschetz number

$$\mathcal{L}(\varepsilon, \Gamma, V_\delta) = \sum_i (-1)^i \text{tr } \varepsilon^i$$

2.1.1. PROPOSITION. — *Under the above assumptions*

$$\mathcal{L}(\varepsilon, \Gamma, V_\delta) = L(\varepsilon, \Gamma, V_\delta)$$

Proof. — Assume that $V_\delta = \mathbb{C}$. We have to show that the following diagram is commutative.

$$\begin{array}{ccc} H_{\text{de Rham}}^*(K \backslash G/\Gamma, \mathbb{C}) & \xrightarrow{\varepsilon^*} & H_{\text{de Rham}}^*(K \backslash G/\Gamma, \mathbb{C}) \\ \parallel & & \parallel \\ H^*(\mathfrak{g}, K, C^\infty(G/\Gamma)) & \xrightarrow{L^*(\varepsilon)} & H^*(\mathfrak{g}, K, C^\infty(G/\Gamma)) \end{array}$$

We recall the definition of the complexes giving the different cohomology groups and the induced action of ε on these complexes.

We have an obvious left G -action on $C^\infty(G/\Gamma)$ and hence by differentiation a left \mathfrak{g} -action which induces by restriction a left $U(\mathfrak{k})$ action of the enveloping algebra of \mathfrak{k} on $C^\infty(G/\Gamma)$. We denote the Cartan decomposition of \mathfrak{g} with respect to K by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. — Then \mathfrak{p} is a left \mathfrak{k} -module, the dual \mathfrak{p}^* of \mathfrak{p} carries the contragredient \mathfrak{k} -action and induces

a left $U(\mathfrak{f})$ -module structure on $\Lambda^* \mathfrak{p}^*$. By definition. see [B-W](I), the relative Lie algebra cohomology $H^*(\mathfrak{g}, K, C^\infty(G/\Gamma))$ is computed from the complex

$$\Lambda^* \mathfrak{p}^* \otimes_{U(\mathfrak{f})} C^\infty(G/\Gamma)$$

and $L^*(\varepsilon)$ is induced by the map $\varepsilon \otimes L(\varepsilon)$ with $\alpha \otimes f \mapsto \varepsilon \alpha \otimes f \circ \varepsilon$ for $\alpha \in \Lambda^* \mathfrak{p}^*$, $f \in C^\infty(G/\Gamma)$. The de Rham cohomology is computed from the complex $A^*(K \backslash G/\Gamma)$ of C^∞ -differential forms on $K \backslash G/\Gamma$ with the obvious action of ε given by

$$\omega \mapsto \varepsilon^* \omega \quad \text{for } \omega \in A^*(K \backslash G/\Gamma).$$

The vertical isomorphisms of our last diagram are induced by the map

$$\Phi_1 : \Lambda^* \mathfrak{p}^* \otimes_{U(\mathfrak{f})} C^\infty(G/\Gamma) \rightarrow A^*(K \backslash G)^\Gamma$$

where $A^*(K \backslash G)^\Gamma$ are the right- Γ -invariant differential forms on $K \backslash G$, combined with the identification

$$\pi^{*-1} A^*(K \backslash G)^\Gamma \xrightarrow{\sim} A^*(K \backslash G/\Gamma)$$

where $\pi : K \backslash G \rightarrow K \backslash G/\Gamma$ is the covering map. For $\omega \in \Lambda^* \mathfrak{p}^*$, $f \in C^\infty(G/\Gamma)$ and $g \in G$ representing $q \in K \backslash G$ we have by definition $\Phi_1(\omega \otimes f)(q) = f(g) \cdot R(g^{-1})^* \omega$. Here $R(g)$ denotes right translation with g on $K \backslash G$ and \mathfrak{p}^* is identified with the cotangent space of $K \backslash G$ at the point given by K . Let $\Phi = \pi^{*-1} \circ \Phi_1$. Then it suffices to show:

2.1.2. LEMMA. — *The following diagram is commutative*

$$\begin{array}{ccc} \Lambda^* \mathfrak{p}^* \otimes_{U(\mathfrak{f})} C^\infty(G/\Gamma) & \xrightarrow{\varepsilon \otimes L(\varepsilon)} & \Lambda^* \mathfrak{p}^* \otimes_{U(\mathfrak{f})} C^\infty(G/\Gamma) \\ \downarrow \Phi & & \downarrow \Phi \\ A^*(K \backslash G/\Gamma) & \xrightarrow{\varepsilon^*} & A^*(K \backslash G/\Gamma) \end{array}$$

Proof. — If $\alpha \in \Lambda^* \mathfrak{p}^*$, $f \in C^\infty(G/\Gamma)$ then

$$(\Phi_1 \circ (\varepsilon \otimes L(\varepsilon))(\alpha \otimes f))(q) = \Phi_1(\varepsilon(\alpha) \otimes f \circ \varepsilon)(q) = f(\varepsilon g) R(g^{-1})^* \varepsilon(\alpha).$$

On the other hand

$$\varepsilon^*(\Phi_1(\alpha \otimes f))(q) = \Phi_1(\alpha \otimes f)(\varepsilon(g)) \circ \varepsilon_* = f(\varepsilon g) (R(\varepsilon g^{-1})^* \alpha) \circ \varepsilon_*.$$

But

$$(R(\varepsilon g^{-1})^* \alpha) \circ \varepsilon_* = \alpha \circ R(\varepsilon g)^{-1} \circ \varepsilon_* = \alpha \circ \varepsilon_* \circ R(g^{-1})_* = R(g^{-1})^* \varepsilon(\alpha).$$

Here $q \in K \setminus G$ is represented by $g \in G$ and the lower $*$ indicates induced maps on exterior powers of the tangent bundle. Since ε^* and π^* commute the chain follows.

Q.E.D.

2.1.3. *Remark.* — If $G = \mathrm{SO}(n, 1)(\mathbb{R})^0$ then by 1.1.1 the Lefschetz number of ε on $H^*(\mathfrak{g}, K, C^\infty(G/\Gamma) \otimes V_\delta)$ also is equal to the geometric Lefschetz number $\mathcal{L}(\varepsilon, \Gamma, V_\delta)$.

2.2. THE LEFSCHETZ NUMBER OF AN INVOLUTION ON THE (\mathfrak{g}, K) -COHOMOLOGY OF A REPRESENTATION. — Before we can use the results of Section I to find a representation theoretical interpretation of the operation of ε^* , let us define the Lefschetz number of an indecomposable representation π and compute it for special ε and special π .

If $\pi \cong \pi \circ \varepsilon$ we put $L(\varepsilon, \pi) = 0$.

If $\pi \not\cong \pi \circ \varepsilon$ then there exists an intertwining operator

$$\begin{aligned} C_\varepsilon : H_\pi &\rightarrow H_\pi, \\ C_\varepsilon \pi(g) &= \pi(\varepsilon g) \cdot C_\varepsilon \end{aligned}$$

and

$$C_\varepsilon^2 = \mathrm{Id}.$$

C_ε is unique only up to a sign. Define

$$\varepsilon_1 : \mathrm{Hom}_K(\Lambda^* \mathfrak{p}, H_\pi) \rightarrow \mathrm{Hom}_K(\Lambda^* \mathfrak{p}, H_\pi)$$

by

$$\omega \rightarrow C_\varepsilon^{-1} \varepsilon^* \omega.$$

Then ε_1 induces a map

$$H^*(\mathfrak{g}, K, \pi) \rightarrow H^*(\mathfrak{g}, K, \pi).$$

and we define

$$L(\varepsilon_1, \pi) = \sum (-1)^i \mathrm{tr} \varepsilon_1^i.$$

Remarks. — (a) The Lefschetz number of an indecomposable representation is defined only up to a sign, since it depends on the particular choice of C_ε .

(b) If ε is the identity and $C_\varepsilon = 1$ we obtain the Euler-Poincaré characteristic $\chi(\pi)$.

(c) For a finite dimensional representation δ with $\delta \circ \varepsilon \simeq \delta$ we define analogously

$$L(\varepsilon, \pi, V_\delta) = L(\varepsilon, \pi \otimes V_\delta).$$

If ε is an inner automorphism, then the following lemma helps to compute the Lefschetz numbers.

2.2.1. LEMMA. — *Suppose that ε is an inner automorphism of G , π a (\mathfrak{g}, K) -module. Then*

$$|L(\varepsilon_1, \pi)| = |\chi(\pi)|.$$

Proof. — Suppose $\varepsilon(g) = \alpha g \alpha^{-1}$.

Since ε has order 2, α has finite order and thus $\alpha \in K$.

Let $\omega \in \text{Hom}_K(\Lambda^* \mathfrak{p}, H_n)$, $X \in \Lambda^* \mathfrak{p}$. Then

$$(\varepsilon_1 \omega)(X) = C_\varepsilon^{-1} \varepsilon^* \omega(X) = C_\varepsilon^{-1} \omega(\text{ad } \alpha(X)) = C_\varepsilon^{-1} \pi(\alpha) \omega(X).$$

The lemma follows, since we can choose $C_\varepsilon = \pi(\alpha)$.

Q.E.D.

Similarly we can prove

2.2.2. LEMMA. — Suppose that $\varepsilon = \varepsilon' \varepsilon''$ where ε' is an outer automorphism, ε'' an inner automorphism. For a (\mathfrak{g}, K) -module π we have

$$|L(\varepsilon_1, \pi)| = |L(\varepsilon'_1, \pi)|.$$

If $G = \text{SO}(n, 1)(\mathbb{R})^0$, then $|\text{Aut } G / \text{Inner Aut } G| \leq 2$. So to compute the Lefschetz numbers of a representation with respect to all the involutive automorphisms of G by 2.2.1, 2.2.2 we have to consider at most two involutions.

2.2.3. LEMMA. — Suppose $G = \text{SO}(2k+1, 1)(\mathbb{R})^0$. Let $A_i(\lambda), V_\lambda$ be as in 1.3. Then

$$\begin{aligned} \chi(A_i(\lambda) \otimes V_\lambda) &= 0, & i = i(\lambda), \dots, k \\ |L(\varepsilon_1, A_i(\lambda) \otimes V_\lambda)| &= 2, & i = i(\lambda), \dots, k, \end{aligned}$$

where ε is the Cartan involution.

Proof. — The first assertion follows from 1.3.

To prove the second assertion we construct representatives of the cohomology classes. Let $\omega_i \in \Lambda^r(\hat{\mathfrak{u}}_i \cap (\mathfrak{p} \otimes \mathbb{C}))$ where $r = \dim \hat{\mathfrak{u}}_i \cap (\mathfrak{p} \otimes \mathbb{C})$, $\omega_i \neq 0$. We pick a basis Y_j , $2i < j < n$, as follows:

$$Y_j = E_{j, n+1} + E_{n+1, j}$$

Put

$$v_i = Y_{2i+1} \wedge Y_{2i+2} \wedge \dots \wedge Y_n$$

Let $\tilde{\omega}_i, \tilde{v}_i$ be the dual forms to ω_i, v_i , and f the highest weight vector in the lowest K -type of $A_i(\lambda) \otimes V_\lambda$. Then

$$\tilde{\omega}_i \otimes f, \quad \tilde{\omega}_i \wedge \tilde{v}_i \otimes f$$

are representatives of the cohomology classes of $A_i(\lambda) \otimes V_\lambda[V-Z]$. Hence to compute the Lefschetz number it suffices to compute the operation of ε^* on \tilde{v}_i . A quick calculation shows that $\varepsilon^*(\tilde{v}_i) = -\tilde{v}_i$.

Q.E.D.

2.2.4. LEMMA. — Suppose $G = \text{SO}(2k, 1)(\mathbb{R})^0$. Let $A_i(\lambda)$ be as in 1.3. Then

$$|\chi(A_i(\lambda) \otimes V_\lambda)| = 2, \quad i = i(\lambda), \dots, k-1$$

and

$$\chi(A_k^\pm(\lambda) \otimes V_\lambda) = (-1)^k.$$

The proof is analogous to 2.2.3.

Remark. — If $G = \mathrm{SO}(2k, 1)(\mathbb{R})^0$ and $k > 1$, then G has no outer automorphisms.

2.3. A PRODUCT FORMULA. — Let (π, H_π) be an indecomposable reposition of G , $\pi \circ \varepsilon \simeq \pi$ and (δ, V) an irreducible finite dimensional representation of G , $\delta \circ \varepsilon \cong \delta$. We choose $C_\varepsilon : H_\pi \rightarrow H_\pi$ and ε_1 as in 2.2 and fix the ε -action on V as in the introduction of this chapter.

We define a map

$$\varepsilon_2 : \mathrm{Hom}_{U(\mathfrak{g})}(H_\pi, \mathcal{A}(G/\Gamma)) \rightarrow \mathrm{Hom}_{U(\mathfrak{g})}(H_\pi, \mathcal{A}(G/\Gamma))$$

by

$$\varepsilon_2(T)(w)(g) = T(C_\varepsilon(w))(\varepsilon g)$$

for $w \in H_\pi$, $g \in G/\Gamma$ and $T \in \mathrm{Hom}_{U(\mathfrak{g})}(H_\pi, \mathcal{A}(G/\Gamma))$.

The evaluation map

$$H_\pi \otimes V \otimes \mathrm{Hom}_{U(\mathfrak{g})}(H_\pi, \mathcal{A}(G/\Gamma)) \rightarrow \mathcal{A}(G/\Gamma) \otimes V$$

given by $w \otimes v \otimes T \rightarrow T(w) \otimes v$ for $w \in H_\pi$, $v \in V$, $T \in \mathrm{Hom}_{U(\mathfrak{g})}(H_\pi, \mathcal{A}(G/\Gamma))$ induces a map

$$\tau : H^*(\mathfrak{g}, K, H_\pi \otimes V) \otimes \mathrm{Hom}_{U(\mathfrak{g})}(H_\pi, \mathcal{A}(G/\Gamma)) \rightarrow H^*(\mathfrak{g}, K, \mathcal{A}(G/\Gamma) \otimes V).$$

2.3.1. LEMMA. — *With the notation introduced above the following diagram is commutative*

$$\begin{array}{ccc} H^*(\mathfrak{g}, K, H_\pi \otimes V) \otimes \mathrm{Hom}_{U(\mathfrak{g})}(H_\pi, \mathcal{A}(G/\Gamma)) & \xrightarrow{\tau} & H^*(\mathfrak{g}, K, \mathcal{A}(G/\Gamma) \otimes V) \\ \downarrow \varepsilon_1 \otimes \varepsilon_2 & & \downarrow L^*(\varepsilon) \end{array}$$

$$H^*(\mathfrak{g}, K, H_\pi \otimes V) \otimes \mathrm{Hom}_{U(\mathfrak{g})}(H_\pi, \mathcal{A}(G/\Gamma)) \xrightarrow{\tau} H^*(\mathfrak{g}, K, \mathcal{A}(G/\Gamma) \otimes V).$$

Proof. — If $\alpha \in \Lambda^* \mathfrak{p}^*$, $w \in H_\pi$, $v \in V$, $T \in \mathrm{Hom}_{U(\mathfrak{g})}(H_\pi, \mathcal{A}(G/\Gamma))$ and if $\alpha \otimes w \otimes v$ represents $\omega \in H^*(\mathfrak{g}, K, H_\pi \otimes V)$ then $L^*(\varepsilon) \circ \tau(\omega \otimes T)$ is represented by $\varepsilon^* \alpha \otimes T(\omega) \circ \varepsilon \otimes \varepsilon(v)$.

On the other hand $\tau \circ \varepsilon_1 \otimes \varepsilon_2(\omega \otimes T)$ is represented by

$$\varepsilon^*(\alpha) \otimes \varepsilon_2(T)(C_\varepsilon^{-1}(w)) \otimes \varepsilon(v) = \varepsilon^*(\alpha) \otimes T(C_\varepsilon^{-1} \circ C_\varepsilon w) \circ \varepsilon \otimes \varepsilon(v),$$

whence the result holds.

Q.E.D

So we can write $L^*(\varepsilon)$ as a product of a ‘‘Hecke operator’’ ε_2 and an involution ε_1 on the cohomology of $\pi \otimes V$. Since the choice of ε_1 and ε_2 depends on the choice of C_ε this product decomposition is not unique. We are free to choose $\varepsilon_1, \varepsilon_2$ for each π

separately. If we want to emphasize the dependence of the choice on π we will write $\varepsilon_1[\pi], \varepsilon_2[\pi]$.

2.4. A FORMULA FOR THE LEFSCHETZ NUMBER. — Assume now that $G = \text{SO}(n, 1)(\mathbb{R})^0$, that $\Gamma \subset \text{SO}(n, 1)(\mathbb{Z})$ is torsion free of finite index, $\varepsilon(\Gamma) = \Gamma$ and $\lambda \circ \varepsilon = \lambda$.

We now fix $\varepsilon_i[\pi], i = 1, 2$ by requiring that $\varepsilon_1[A_i(\lambda)]$ acts trivial on

$$H^i(\mathfrak{g}, K, A_i(\lambda) \otimes V_\lambda) \quad \text{if } i(\lambda) \leq i < k$$

and by asking that $\varepsilon_1[U_i(\lambda)]$ acts identically on $H^{n-i-1}(\mathfrak{g}, K, U_i(\lambda) \otimes V_\lambda)$ for $0 \leq i < k$. If $n = 2k + 1$ and $U_k(\lambda) = A_k(\lambda)$ or if $n = 2k$ and $A_k(\lambda) = A_k^+(\lambda) \oplus A_k^-(\lambda)$ we take the obvious extension of the above definition. We now write simply ε instead of $\varepsilon_i[\]$. Our convention implies that if $L(\varepsilon, A_i(\lambda) \otimes V_\lambda) \neq 0$ then

$$L(\varepsilon, A_i(\lambda) \otimes V_\lambda) = (-1)^i 2 \quad \text{for } i(\lambda) \leq i \leq k.$$

Moreover, if ε is inner, ε acts identically on $H^*(\mathfrak{g}, K, A_i(\lambda) \otimes V_\lambda)$ and $H^*(\mathfrak{g}, K, U_i(\lambda) \otimes V_\lambda)$. We recall that $L^2(G/\Gamma) \cap \mathcal{A}(G/\Gamma) = \mathcal{A}_{\text{dis}}(G/\Gamma)$.

2.4.1. THEOREM. — Suppose $G = \text{SO}(2k, 1)(\mathbb{R})^0, k \geq 1$. Let $\varepsilon: G \rightarrow G$ be an involutive inner automorphism and V_λ an irreducible finite dimensional representation of G with $\lambda \circ \varepsilon = \lambda$. Then

$$\begin{aligned} L(\varepsilon, \Gamma, V_\lambda) &= \sum_{i=i(\lambda)}^{k-1} (-1)^i 2 \text{tr } \varepsilon | \text{Hom}_{U(\mathfrak{g})}(A_i(\lambda), \mathcal{A}_{\text{dis}}(G/\Gamma)) \\ &\quad + (-1)^k \text{tr } \varepsilon | \text{Hom}_{U(\mathfrak{g})}(A_k(\lambda), \mathcal{A}_{\text{dis}}(G/\Gamma)) \\ &\quad + \sum_{i=0}^{k-1} (-1)^{n-i-1} \text{tr } \varepsilon | \text{Hom}_{U(\mathfrak{g})}(U_i(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)). \end{aligned}$$

Proof. — Using 1.5.1 and the above conventions for C_ε we have

$$\begin{aligned} \text{tr } \varepsilon^i | H^i(\Gamma, V_\lambda) &= 0 \quad \text{if } i < i(\lambda) \\ &= \text{tr } \varepsilon | \text{Hom}_{U(\mathfrak{g})}(A_i(\lambda), \mathcal{A}_{\text{cusp}}(G/\Gamma)) \\ &\quad + \text{tr } \varepsilon | \{T \in \text{Hom}_{U(\mathfrak{g})}(U_i(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) / \ker T \neq 0\} \quad \text{if } i(\lambda) \leq i \leq k-1 \\ &= \text{tr } \varepsilon | \text{Hom}_{U(\mathfrak{g})}(A_k(\lambda), \mathcal{A}_{\text{cusp}}(G/\Gamma)) \\ &\quad + \text{tr } \varepsilon | \langle T \in \text{Hom}_{U(\mathfrak{g})}(U_{k-1}(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) / \ker T = 0 \rangle \quad \text{if } i = k. \\ &= \text{tr } \varepsilon | \text{Hom}_{U(\mathfrak{g})}(A_{n-i}(\lambda), \mathcal{A}_{\text{cusp}}(G/\Gamma)) \\ &\quad + \text{tr } \varepsilon | \langle T \in \text{Hom}_{U(\mathfrak{g})}(U_{n-i-1}(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) / \ker T = 0 \rangle \quad \text{if } k+1 \leq i \leq n-i(\lambda) \\ &= \text{tr } \varepsilon | \text{Hom}_{U(\mathfrak{g})}(U_{n-i-1}(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) \quad \text{if } n-i(\lambda)+1 \leq i \leq n-1. \end{aligned}$$

Hence

$$L(\varepsilon, \Gamma, V_\lambda) = \sum_{i=i(\lambda)}^{k-1} 2(-1)^i \text{tr } \varepsilon | \text{Hom}_{U(\mathfrak{g})}(A_i(\lambda), \mathcal{A}_{\text{cusp}}(G/\Gamma))$$

$$\begin{aligned}
 & + \sum_{i=i(\lambda)}^{k-1} (-1)^i \operatorname{tr} \varepsilon \left| \{T \in \operatorname{Hom}_{U(\mathfrak{g})}(U_i(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) / \ker T \neq 0\} \right. \\
 & + (-1)^k \operatorname{tr} \varepsilon \left| \operatorname{Hom}_{U(\mathfrak{g})}(A_k(\lambda), \mathcal{A}_{\text{dis}}(G/\Gamma)) \right. \\
 & + \sum_{i=k}^{n-i(\lambda)} (-1)^i \operatorname{tr} \varepsilon \left| \langle T \in \operatorname{Hom}_{U(\mathfrak{g})}(U_{n-i-1}(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) / \ker T = 0 \rangle \right. \\
 & + \sum_{i=n-i(\lambda)+1}^{n-1} (-1)^i \operatorname{tr} \varepsilon \left| \operatorname{Hom}_{U(\mathfrak{g})}(U_{n-i-1}(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) \right. .
 \end{aligned}$$

The first two sums can be rewritten as

$$\begin{aligned}
 & \sum_{i=i(\lambda)}^{k-1} 2(-1)^i \operatorname{tr} \varepsilon \left| \operatorname{Hom}_{U(\mathfrak{g})}(A_i(\lambda), \mathcal{A}_{\text{dis}}(G/\Gamma)) \right. \\
 & + \sum_{i=i(\lambda)}^{k-1} (-1)^{i-1} \operatorname{tr} \varepsilon \left| \{T \in \operatorname{Hom}_{U(\mathfrak{g})}(U_i(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) / \ker T \neq 0\} \right. .
 \end{aligned}$$

Taking $j = n - i - 1$ as a new variable for the summation in the last two sums contributing to $L(\varepsilon, \Gamma, V_\lambda)$ the result follows.

Q.E.D.

Remark. — If $k > 1$ then $\operatorname{SO}(2k, 1)$ has only inner automorphisms.

With the same technic we prove:

2.4.2. THEOREM. — Suppose $G = \operatorname{SO}(2k+1, 1)(\mathbb{R})^0$, $k \geq 1$. Let $\varepsilon: G \rightarrow G$ be an involutive automorphism and V_λ an irreducible finite dimensional representation of G with highest weight λ on which ε acts.

(a) Suppose that ε is inner. Then

$$L(\varepsilon, \Gamma, V_\lambda) = \sum_{i=0}^k (-1)^{n-i-1} \operatorname{tr} \varepsilon \left| \operatorname{Hom}_{U(\mathfrak{g})}(U_i(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) \right.$$

(b) Suppose that ε is an outer automorphism. Then

$$\begin{aligned}
 L(\varepsilon, \Gamma, V_\lambda) = \sum_{i=i(\lambda)}^k (-1)^i 2 \operatorname{tr} \varepsilon \left| \operatorname{Hom}_{U(\mathfrak{g})}(A_i(\lambda), \mathcal{A}_{\text{dis}}(G/\Gamma)) \right. \\
 + \sum_{i=0}^k (-1)^{n-i-1} \operatorname{tr} \varepsilon \left| \operatorname{Hom}_{U(\mathfrak{g})}(U_i(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) \right. .
 \end{aligned}$$

III. Computation of Euler-Poincaré characteristics and Lefschetz numbers

Let Γ be the full congruence subgroup mod $m \in \mathbb{N}$ of $\operatorname{SO}(n, 1)(\mathbb{Z})$. In order to avoid unpleasant computations over the 2-adic numbers we assume that 8 divides m . Using

Harder's Gauss-Bonnet theorem [H1] we derive an explicit formula for the Euler-Poincaré characteristic $\chi(\Gamma)$ of Γ . Let V be a complex valued finite dimensional representation of $O(n, 1)(\mathbb{R})$. Then the standard Cartan involution $\varepsilon: A \mapsto {}^t A^{-1}$ acts on V and on Γ and the Lefschetz number $L(\varepsilon, \Gamma, V)$ is defined. In 3.1 we explain our method to compute this number, in 3.4 we give its value. The cohomological computations needed for this are contained in 3.3. In 3.5 we compute the trace of the ε -action on V .

3.1. FIXPOINTS AND LEFSCHETZ NUMBER. — We recall the methods which have been developed to compute Lefschetz numbers in [R1, R3].

3.1.1. Assume that a finite group C acts from the left on an arithmetic group Γ . If $\sigma \in C$ and $\gamma \in \Gamma$ we write ${}^\sigma \gamma$ for the action of σ on γ . The notion of the first nonabelian cohomology $H^1(C, \Gamma)$ is explained in [Se1]. Let $\gamma = \{\gamma_\sigma\}$, $\sigma \in C$, be a cocycle for $H^1(C, \Gamma)$. Then ${}^\gamma \Gamma$ is the group Γ endowed with a new left C -action given by $\sigma \xi = \gamma_\sigma {}^\sigma \xi \gamma_\sigma^{-1}$, $\xi \in \Gamma$, $\sigma \in C$. Let $\Gamma(\gamma)$ be the subgroup of Γ consisting of all elements which are fixed by the new γ -twisted action of C . If V is a vector space and a Γ - and C -module in a compatible way, i.e. ${}^\sigma(\xi v) = {}^\sigma \xi {}^\sigma v$ for $v \in V$, $\sigma \in C$, $\xi \in \Gamma$, then we denote by ${}^\gamma V$ the vector space V with the twisted C action given by ${}^\sigma v = \gamma_\sigma {}^\sigma v$, $\sigma \in C$, $v \in V$. We observe that C and ${}^\gamma \Gamma$ act twisted in a compatible way on ${}^\gamma V$.

It is known [B-S] that $H^1(C, \Gamma)$ is a finite set. If $\gamma = \{\gamma_\sigma\}$, $\sigma \in C$, is a cocycle for $H^1(C, \Gamma)$ then obviously the Euler-Poincaré characteristic $\chi(\Gamma(\gamma))$ of $\Gamma(\gamma)$ depends only on the cohomology class represented by γ . Let $\varepsilon \in C$ then $\text{tr}(\varepsilon|{}^\gamma V)$, the trace of ε acting γ -twisted on V , depends only on the cohomology class represented by γ .

THEOREM 3.1.2 (Lefschetz fixpoint formula). — *Assume that the finite cyclic group $C = \langle \varepsilon \rangle$ generated by $\varepsilon \in C$ acts on a torsion free arithmetic group Γ . Let V be a finite dimensional C -vector space on which ε and Γ act in a compatible way. Then*

$$L(\varepsilon, \Gamma, V) = \sum_{\bar{\gamma} \in H^1(C, \Gamma)} \chi(\Gamma(\gamma)) \text{tr}(\varepsilon|{}^\gamma V),$$

where in the summation for each $\bar{\gamma}$ some representing cocycle γ has been chosen.

Proof. — We only sketch the argument which will appear in [R 3]. Let Γ be a discrete subgroup of the real semisimple Lie group G and denote by X the space of maximal compact subgroups of G . Then ε extends to an automorphism of X/Γ . Denote by V also the local system on X/Γ given by the Γ -action on V and abbreviate $F = (X/\Gamma)^\varepsilon$ for the fixpoints of ε on X/Γ . Using Poincaré-duality and the methods developed in [R1] we get

$$\sum_{\bar{\gamma} \in H^1(C, \Gamma)} \chi(\Gamma(\gamma)) \text{tr}(\varepsilon|{}^\gamma V) = L_c(\varepsilon, F, V|F)$$

where the Lefschetz number $L_c(\varepsilon, F, V|F)$ is computed from the ε -action on cohomology with compact supports $H_c^*(F, V|F)$. Using the long cohomology sequence with compact

supports of the pair $(X/\Gamma, F)$ then the theorem follows since by the classical Lefschetz theorem $0 = L_c(\varepsilon, X/\Gamma \setminus F, V | X/\Gamma \setminus F)$.

Q.E.D.

We observe that the number $\text{tr}(\varepsilon | \gamma V)$ in fact depends only on the image of the cohomology class represented by γ in $H^1(C, GL(V))$. Hence we have

COROLLARY 3.1.3. — *Assume that the natural map*

$$H^1(C, \Gamma) \rightarrow H^1(C, GL(V))$$

is trivial. Then

$$L(\varepsilon, \Gamma, V) = \text{tr}(\varepsilon | V) \sum_{\bar{\gamma} \in H^1(C, \Gamma)} \chi(\Gamma(\gamma)).$$

3.1.4. We explain now how we evaluate the right side of the Lefschetz fixpoint formula 3.1.2 in our special situation.

Let Γ be the full congruence subgroup mod $m = \prod p^{n_p} \in \mathbb{N}$, $n_2 \geq 3$, of $SO(n, 1)(\mathbb{Z})$. According to [M] this group is torsion free. Let $C = \langle \varepsilon \rangle$ be the group generated by the standard Cartan involution $A \rightarrow {}^t A^{-1}$ and suppose that the Γ - and C -module V is given by a representation of $O(n, 1)(\mathbb{R})$. We write $\Gamma_\infty = SO(n, 1)(\mathbb{R})^0$, where the upper index zero denotes the connected component, and Γ_p for the closure of Γ in $SO(n, 1)(\mathbb{Q}_p)$ with respect to the p -adic topology. The inclusions $\Gamma \rightarrow \Gamma_p$ then induce a map

$$h : H^1(C, \Gamma) \rightarrow \prod_v H^1(C, \Gamma_v)$$

and we will show in 3.3.3 that the components of $\text{im}(h)$ are nontrivial only at the place $2=v$. We can identify a cocycle for $H^1(C, \Gamma)$ with a $b \in \Gamma$ such that $b = {}^t b$. For such a b then there is an $a \in \Gamma_\infty$ such that $b = a^{-1} {}^t a^{-1}$. We consider $O(n)(\mathbb{R})$ as a subgroup of $SO(n, 1)(\mathbb{R})$ by mapping $A \in O(n)(\mathbb{R})$ to $\begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix}$. Then $\Gamma(b) = \Gamma \cap a O(n)(\mathbb{R}) a^{-1} = \{1\}$, since Γ is torsion free and $O(n)(\mathbb{R})$ is compact. The image of h in $H^1(C, \Gamma_\infty)$ is trivial. Hence we can apply 3.1.3 and get

$$L(\varepsilon, \Gamma, V) = \text{tr}(\varepsilon | V) |H^1(C, \Gamma)|.$$

In 3.3.4 we reduce the computation of the number of classes in $H^1(C, \Gamma)$ to the computation of a class-number which is determined in 3.4.2.

3.2. ON THE SPINOR NORM. — Later on we will have to apply at several instances the strong approximation property. In order to do so we have to work with the image of the simply connected covering in $SO(n, 1)$. We now collect some technical preparations for this.

3.2.1. Consider a local ring or field k with $2 \in k^*$, a quadratic form $q: k^n \rightarrow k$ and assume that the bilinear form $b_q(x, y) = q(x+y) - q(x) - q(y)$, $x, y \in k^n$, is nondegenerate. Let $O(q)(k)$ be the set of k -rational points of the orthogonal group corresponding to q . The spinor norm is a homomorphism $N: O(q)(k) \rightarrow k^*/k^{*2}$ with the following

property: If $a \in k^n$, $q(a) \in k^*$, and if τ_a is the reflection given by $x \mapsto x - ab_q(x, a)/q(a)$, $x \in k^n$, then $N(\tau_a) \equiv q(a) \pmod{k^{*2}}$. We recall that according to a classical result of Cartan-Dieudonné [L] every element of $O(q)(k)$ is a product of such reflections.

3.2.2. Consider the quadratic form $h: k^{n+1} \rightarrow k$ given by

$$h(x_1, \dots, x_n, x_0) = q(x_1, \dots, x_n) - x_0^2; \quad x_1, \dots, x_n, x_0 \in k^{n+1}.$$

We have an injective homomorphism $i: O(q)(k) \rightarrow SO(h)(k)$ given in matrix notation by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix}$. If $A \in O(q)(k)$, we have for the spinor norms the equation $N(iA) = N(A) \det A \pmod{k^{*2}}$. Using the Cartan-Dieudonné result we have to check this only for reflections. If $v = (a, 0) \in k^{n+1}$, $q(a) \neq 0$ and $e = (0, \dots, 0, 1) \in k^{n+1}$ then $i(\tau_a) = \tau_v \tau_e$ and the formula follows.

If $g: k^n \rightarrow k$ is given by $q(x_1, \dots, x_n) = \sum_{i=1}^n x_i^2$ we write $O(n)$ instead of $O(q)$. We call a quadratic form $q: \mathbb{Q}_p^n \rightarrow \mathbb{Q}_p$ integral if $q(x) = q_b(x) = {}^t x b x$; $x \in \mathbb{Q}_p^n$ is considered as a column vector, and $b \in M_n(\mathbb{Z}_p)$ is a symmetric $n \times n$ matrix with coefficients in the ring \mathbb{Z}_p .

LEMMA 3.2.3. — *Let $q: \mathbb{Q}_p^n \rightarrow \mathbb{Q}_p$ be an integral quadratic form and denote by $U(j)$, $1 \leq j \in \mathbb{N}$, the full congruence subgroup of $O(q)(\mathbb{Z}_p) \pmod{p^j}$. Then the spinor-norm of $U(j)$ is trivial if $p=2$ and $j \geq 3$ or if $p \neq 2$ and $j \geq 1$.*

Proof. — We have a bijection between $U(j)$, $j \geq 2$ if $p=2$, and

$$T := \{A \in M_n(\mathbb{Z}_p) \mid -bA = {}^t(bA)\}$$

given by $A \mapsto \exp(p^j A)$, where $\exp(\)$ is the usual exponential of a matrix and $b \in M_n(\mathbb{Z}_p)$, $b = {}^t b$, is the integral matrix which determines q . Hence $U(j)$ consists of squares if $p=2$ and $j \geq 3$ or if $p \neq 2$ and $j \geq 1$ and the claim holds.

Q.E.D.

LEMMA 3.2.4.. — (i) *The spinor norm N induces a surjection*

$$N: SO(n, 1)(\mathbb{Z}) \rightarrow \mathbb{Z}[1/2]^*/\mathbb{Z}[1/2]^{*2}.$$

(ii) *If $j \geq 3$ and if $\Gamma = \Gamma(2^j)$ is the full congruence subgroup of $SO(n, 1)(\mathbb{Z}) \pmod{2^j}$ then $N(\Gamma(2^j)) \equiv 1$.*

Proof. — If $\gamma \in SO(n, 1)(\mathbb{Z})$ then $N(\gamma) \in \mathbb{Q}^*/\mathbb{Q}^{*2}$ can be represented by a square free number m . On the other hand $\gamma \in SO(n, 1)(\mathbb{Z}_p)$ and $N(\gamma) \in \mathbb{Z}_p^*/\mathbb{Z}_p^{*2} \subset \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$ if $p \neq 2$. Hence $p \neq 2$ does not divide m and $N(\gamma) \in \mathbb{Z}[1/2]^*/\mathbb{Z}[1/2]^{*2} \cong \{\pm 1, \pm 2\}$. Using 3.2.2 and the fact that $O(n)(\mathbb{Z})$ has elements of spinor norm 1 and 2 the first claim follows. The natural map $\mathbb{Z}[1/2]^*/\mathbb{Z}[1/2]^{*2} \rightarrow \mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$ is injective and by 3.2.3 the group $U(j)$, $j \geq 3$, has trivial spinor norm. Now $\Gamma(2^j) \subset U(j)$ and the second claim holds.

Q.E.D.

If $j \in \mathbb{N}$ let $\Gamma_p(j)$ be the full congruence subgroup of $SO(n, 1)(\mathbb{Z}_p)$. We have $\Gamma_p(0) = SO(n, 1)(\mathbb{Z}_p)$. If $\Gamma = \Gamma(m)$, $m = \prod p^{n_p}$, then obviously $\Gamma_p \subset \Gamma_p(n_p)$ where Γ_p is

the closure of Γ in $\Gamma_p(0)$. We denote by $\Gamma_p^1(j)$ the subgroup of $\Gamma_p(j)$ of elements of spinor norm 1. To have a smooth notation we introduce $n_\infty = 0$ and $\Gamma_\infty(0) = \text{SO}(n, 1)(\mathbb{R})$ and $\Gamma_\infty = \{x \in \Gamma_\infty(0) \mid N(x) \equiv 1\}$.

LEMMA 3.2.5. — *If $m = \Pi p^{n_p}$, $n_2 \geq 3$, and if Γ is the full congruence subgroup mod m of $\text{SO}(n, 1)(\mathbb{Z})$ then $\Gamma_p = \Gamma_p^1(n_p)$ for all primes p .*

Proof. — Strong approximation [E].

Q.E.D.

3.3. ON THE NONABELIAN COHOMOLOGY. — Let $\Gamma = \Gamma(m)$ be the full congruence subgroup mod m of $\text{SO}(n, 1)(\mathbb{Z})$ and assume throughout that 8 divides m . This paragraph contains the necessary local and global cohomological computations. The methods used here have been developed in [R1]. In order to be able to sum up the fixpoint contribution to the Lefschetz numbers of ε explicitly we use strong approximation and work on the kernel of the spinor norm. We use the notation $\text{SO}(\eta)^1(\mathbb{R}) = \ker(\text{SO}(\eta)(\mathbb{R}) \xrightarrow{N} \mathbb{R}^*/\mathbb{R}^{*2})$ where \mathbb{R} is any ring with $2 \in \mathbb{R}^*$ and N denotes the spinor norm. Here $\eta = q_\eta$ is the quadratic form given by

$$\eta(x_1, \dots, x_n, x_0) = \sum_{i=1}^n x_i^2 - x_0^2, \quad x_i \in \mathbb{Z}.$$

LEMMA 3.3.1. — (i) *If $p = 2$ and $j \geq 3$ or if $p \neq 2$ and $j \geq 1$, then the map $H^1(\mathbb{C}, \Gamma_p(j)) \rightarrow H^1(\mathbb{C}, \Gamma_p(0))$ induced by the inclusion $\Gamma_p(j) \subset \Gamma_p(0)$ is trivial.*

(ii) *If $p \neq 2$ and $j \geq 1$, then $H^1(\mathbb{C}, \Gamma_p(j)) = \{1\}$.*

(iii) *If $p = 2$ and $j \geq 3$, then $H^1(\mathbb{C}, \Gamma_2(j)) \cong (\mathbb{Z}/2\mathbb{Z})^n$.*

(iv) *If $p \neq 2$, then the natural map*

$$H^1(\mathbb{C}, \Gamma_p(0)) \rightarrow H^1(\mathbb{C}, \text{SO}(\eta)(\mathbb{F}_p))$$

induced by the projection $\text{SO}(\eta)(\mathbb{Z}_p) \rightarrow \text{SO}(\eta)(\mathbb{Z}_p/p\mathbb{Z}_p)$ is injective.

Proof. — Let $T = \{A \in M_{n+1}(\mathbb{Z}_p) \mid \eta A \eta = -{}^t A\}$. Then the usual exponential map of matrices induces a bijection

$$T \xrightarrow{\sim} \Gamma_p(j) \quad \text{by } A \mapsto \exp(p^j A)$$

if $p = 2$ and $j \geq 2$ or if $p \neq 2$ and $j \geq 1$. If $B \in \Gamma_p(j)$ represents a cocycle, i. e. $B = {}^t B$ and $B = \exp(p^j A)$, we put $D = \exp(p^{j-1} A/2)$ and get ${}^t D D = D^2 = \exp(p^j A) = B$. Therefore (ii) holds and the natural map $H^1(\mathbb{C}, \Gamma_2(j+1)) \rightarrow H^1(\mathbb{C}, \Gamma_2(j))$ is trivial. Hence (i) holds.

To prove (iii) we observe that \exp induces an isomorphism $T/2T \rightarrow \Gamma_2(j)/\Gamma_2(j+1)$ of commutative groups with ε -action. Here on the left the \mathbb{C} -action is induced by

$A \mapsto {}^tA, A \in T$, i. e. we have a trivial ε -action on both sides and $H^1(C, T/2T) \cong T/2T$. Using a twisted version of (i) we get an injection

$$f: H^1(C, \Gamma_2(j)) \hookrightarrow H^1(C, \Gamma_2(j)/\Gamma_2(j+1)) \simeq T/2T$$

If $A = {}^tA \in \Gamma_2(j)$ represents a cohomology class, then $A = \exp(2^j B), B \in T, B = {}^tB$ and there is a column vector $b \in \mathbb{Z}_2^n$ such that $B = \begin{pmatrix} 0 & b \\ {}^t b & 0 \end{pmatrix}$. Hence the image of f is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$ and (iii) holds.

By Hensel's Lemma the natural map $SO(\eta)(\mathbb{Z}_p) \rightarrow SO(\eta)(\mathbb{F}_p)$ is surjective. Therefore we get an exact sequence

$$H^1(C, \Gamma_p(1)) \rightarrow H^1(C, \Gamma_p(0)) \xrightarrow{f} H^1(C, SO(\eta)(\mathbb{F}_p)).$$

Now by (ii) we have $H^1(C, \Gamma_p(1)) = \{1\}$. Since this argument holds as well in a twisted situation f is injective and (iv) holds.

Q.E.D.

LEMMA 3.3.2. — *The inclusions $\Gamma \hookrightarrow SO(\eta)(\mathbb{Q})$ resp. $\Gamma \hookrightarrow \prod_{v \neq 2} \Gamma_v(n_v)$ induce trivial maps*

$$H^1(C, \Gamma) \rightarrow H^1(C, SO(\eta)(\mathbb{Q}))$$

resp.

$$H^1(C, \Gamma) \rightarrow \prod_{v \neq 2} H^1(C, \Gamma_v(n_v)).$$

Proof. — If $b \in \Gamma$ represents a class in $H^1(C, \Gamma)$ then $b = {}^t b$ and $\eta b \eta b = 1$. For $v \in \mathbb{Z}^{n+1} =: L$ we have $(\eta b + \text{Id})v/2 \in \mathbb{Z}^{n+1}$ and $(\eta b - \text{Id})v/2 \in \mathbb{Z}^{n+1}$ since in particular $b \equiv 1 \pmod 2$ by assumption. Hence if $L_{\pm} := \{v \in L/\eta bv = \pm v\}$, then $L_+ \oplus L_- = L$. Now, for $u \in L_+, w \in L$ we have ${}^t w b u = {}^t w \eta u$ and for $v \in L_-$ we have ${}^t w b v = -{}^t w \eta v$. Moreover

$${}^t u b v = -{}^t u \eta v = -{}^t v \eta u = -{}^t v b u = -{}^t u b v = 0.$$

Therefore L_+ and L_- are orthogonal with respect to the quadratic forms q_b and q_η induced by b and η . Since $\det b = 1$, we have $\det(q_b|_{L_+}) = \sigma \in \{\pm 1\}$. According to 3.3.1(i) the natural map $H^1(C, \Gamma) \rightarrow H^1(C, \Gamma_2(0))$ is trivial. Thus there is an $a \in SO(\eta)(\mathbb{Z}_2)$ such that ${}^t a b a = 1$ or equivalently $a^{-1} \eta b a = \eta$. Therefore $\dim L_+ = n$ and $\dim L_- = 1$.

We have $\det(q_\eta|_{L_-}) = -\sigma$. If $\sigma = -1$ then $q_b|_{L_-}$ is negative definite and since q_η has signature $(n, 1)$ we have $(n-1, 2)$ as signature of q_b . Let $c_v(b)$ be the Hasse-Witt invariant of q_b over \mathbb{Q}_v , [Se3]. Then $\prod_v c_v(b) = 1$ and $c_p(b) = 1$ for all p . Hence

$c_\infty(b) = 1$. But if b has signature $(n-1, 2)$ then $c_\infty(b) = -1$. Therefore $\sigma = -1$ is impossible and $\det(q_b|_{V_+}) = 1$.

Let F be one of the fields $\mathbb{Q}, \mathbb{R}, \mathbb{Z}/p\mathbb{Z}, p \neq 2$. By Witt's cancellation theorem we get $q_\eta | L_+ \otimes F \cong g |_{F^n}$ where g is the standard quadratic form with $g(x_1, \dots, x_n) = \sum_{i=1}^n x_i^2$.

Therefore there exists a basis e_1, \dots, e_n of $L_+ \otimes F$ such that $'e_i \eta e_j = \delta_{ij}$. If we choose an $0 \neq e_0 \in L_-$ such that $q_\eta(e_0) = -1$ and choose the order of e_1, \dots, e_n correctly we find a $B \in \text{SO}(\eta)(F)$ with $'B \eta B = \eta$ and $'B b B = \text{Id}$. Now for $F = \mathbb{Q}$ the first claim results and using 3.3.1 (iv) if $p \neq 2$, $\mathbb{F}_p = F$ and $F = \mathbb{R}$ if $v = \infty$ the second claim is established.

Q.E.D

LEMMA 3.3.3. — *The inclusions $\Gamma \hookrightarrow \text{SO}(\eta)^1(\mathbb{Q})$ resp. $\Gamma \hookrightarrow \prod_{v \neq 2} \Gamma_v$ induce trivial maps*

$$H^1(\mathbb{C}, \Gamma) \rightarrow H^1(\mathbb{C}, \text{SO}(\eta)^1(\mathbb{Q}))$$

resp.

$$H^1(\mathbb{C}, \Gamma) \rightarrow \prod_{v \neq 2} H^1(\mathbb{C}, \Gamma_v).$$

Proof. — We have $\Gamma = \Gamma(m)$, $m = \prod p^{n_p}$ and $\Gamma_p^1 = \Gamma_p(n_p)$ if $n_p \neq 0$. Using the last Lemma we see that we have to work only with $v = \infty$ or $v = p, p \nmid m$, in order to prove the last claim. In this situation $\Gamma_p = \text{SO}(\eta)^1(\mathbb{Z}_p)$ and $\Gamma_\infty = \text{SO}(\eta)^1(\mathbb{R}) = \text{SO}(\eta)(\mathbb{R})^0$. From the exact sequence

$$1 \rightarrow \Gamma_p \rightarrow \Gamma_p(0) \rightarrow \mathbb{Z}_p^*/\mathbb{Z}_p^{*2} \rightarrow 1$$

we get the long exact cohomology sequence

$$\text{O}(n)(\mathbb{Z}_p) \xrightarrow{\tilde{N}} \mathbb{Z}_p/\mathbb{Z}_p^{*2} \xrightarrow{\partial} H^1(\mathbb{C}, \Gamma_p) \rightarrow H^1(\mathbb{C}, \Gamma_p(0))$$

here $\tilde{N}(x) \equiv N(ix) \equiv N(x) \det x$, see 3.2.2. If $a \in \mathbb{Z}_p^*$ is a nonsquare, then the equation $x^2 + y^2 = -a$ is solvable for $x, y \in \mathbb{Z}_p$. If $u = (x, y, 0, \dots, 0) \in \mathbb{Z}_p^n$, then $\tilde{N}(u) = a$, see 3.2.2. Hence the map $H^1(\mathbb{C}, \Gamma_p) \rightarrow H^1(\mathbb{C}, \Gamma_p(0))$ has trivial fibre through the distinguished element.

If $v = \infty$, we get as above an exact sequence

$$\text{O}(n)(\mathbb{R}) \xrightarrow{\tilde{N}} \mathbb{R}^*/\mathbb{R}^{*2} \xrightarrow{\partial} H^1(\mathbb{C}, \text{SO}(\eta)^1(\mathbb{R})) \rightarrow H^1(\mathbb{C}, \text{SO}(\eta)(\mathbb{R}))$$

and the map \tilde{N} is surjective. Therefore using Lemma 3.3.2 we see that the second claim of the Lemma holds.

To prove the first claim we consider the exact sequence

$$\text{O}(n)(\mathbb{Q}) \xrightarrow{\tilde{N}} \mathbb{Q}^*/\mathbb{Q}^{*2} \xrightarrow{\partial} H^1(\mathbb{C}, \text{SO}(\eta)^1(\mathbb{Q})) \rightarrow H^1(\mathbb{C}, \text{SO}(\eta)(\mathbb{Q}))$$

Hence using Lemma 3.3.2 we see that if b is the image of $\gamma \in H^1(\mathbb{C}, \Gamma)$ in $H^1(\mathbb{C}, \text{SO}(\eta)^1(\mathbb{Q}))$ then there is a class $\bar{a} \in \mathbb{Q}^*/\mathbb{Q}^{*2}$ represented by $a \in \mathbb{Q}$ such that $\partial(\bar{a}) = b$. From the first part of our proof we know that the equation $x_1^2 + x_n^2 = -a$ is

solvable over all \mathbb{Q}_v . Hence by the theorem of Hasse-Minkowski [Se 3] this equation is solvable over \mathbb{Q} and there is an $x \in O(n)(\mathbb{Q})$ with $\tilde{N}(x) = \bar{a}$. Thus the first claim holds.

Q. E. D.

Let \mathbb{A} be the ring of adèles over \mathbb{Q} and $\mathbb{A}^* = J$ be the ideles. Then the local spinor norms induce a map $N : SO(\eta)(\mathbb{A}) \rightarrow J/J^2$. We denote by $SO(\eta)^1(\mathbb{A})$ the kernel of this map.

PROPOSITION 3.3.4. — *Let $\Gamma = \Gamma(m)$ be the full congruence subgroup of $SO(\eta)(\mathbb{Z})$ mod m and assume that 8 divides m . Then we have an exact sequence of pointed sets*

$$1 \rightarrow SO(\eta)^1(\mathbb{Q})^\varepsilon \backslash SO(\eta)^1(\mathbb{A})^\varepsilon / (\prod \Gamma_v)^\varepsilon \xrightarrow{\partial} H^1(C, \Gamma) \xrightarrow{h} H^1(C, \Gamma_2) \rightarrow 1.$$

The map ∂ is injective.

Proof. — We recall from 3.2.5 that Γ_v is the closure of Γ in $SO(\eta)(\mathbb{Z}_p)$ if $v = p$ and $\Gamma_\infty = SO(\eta)^1(\mathbb{R})$ and that $\Gamma_p = \Gamma_p^1(n_p)$ if $m = \prod p^{n_p}$.

The map h is induced by the inclusion $\Gamma \hookrightarrow \Gamma_2$. We define ∂ as follows. If $b \in SO(\eta)^1(\mathbb{A})^\varepsilon$ represents a double class \bar{b} then by strong approximation in $SO(\eta)^1(\mathbb{A})$, [E], there is a $g \in SO(\eta)^1(\mathbb{Q})$ and a $\gamma \in \prod \Gamma_v$ such that $b = g\gamma$. Since $b = {}^\varepsilon b$ we get

$$g^{-1} {}^\varepsilon g = \gamma^\varepsilon \gamma^{-1} = \gamma_\varepsilon \in SO(\eta)^1(\mathbb{Q}) \cap \prod \Gamma_v = \Gamma.$$

We define $\partial(\bar{b})$ to be the class in $H^1(C, \Gamma)$ represented by the cocycle γ_ε . Obviously ∂ is well defined and $h\partial$ is trivial.

If $A = {}^t A \in \Gamma$ represents a class in the kernel of h then using 3.3.3 there is a $g \in SO(\eta)^1(\mathbb{Q})$ and a $\gamma \in \prod \Gamma_v$ such that $A = g^{-1} {}^\varepsilon g = \gamma^{-1} {}^\varepsilon \gamma$. Then

$b := g\gamma^{-1} \in SO(\eta)^1(\mathbb{A})^\varepsilon$. We get a map which is defined on the kernel of h which is inverse to ∂ . Hence ∂ is injective and the sequence is exact at the middle.

We show that h is surjective. If $\gamma_\varepsilon \in \Gamma_2$ is a cocycle then there is an $a \in SO(\eta)^1(\mathbb{Z}_2)$ such that $a^{-1} {}^\varepsilon a = \gamma_\varepsilon$. Using strong approximation we find a $b \in SO(\eta)(\mathbb{Z})$ which is so close to a in $SO(\eta)(\mathbb{Z}_2)$ such that $b^{-1} {}^\varepsilon b$ and γ_ε determine the same class in $H^1(C, \Gamma_2)$.

Q. E. D.

Remark 3.3.5. — The other fibres of h are determined by twisting [Se 1]. If $\gamma \in \Gamma$ represents the class $\bar{\gamma} \in H^1(C, \Gamma)$ then the fibre $h^{-1}(h(\bar{\gamma}))$ of h through $\bar{\gamma}$ is

$${}^\gamma SO(\eta)^1(\mathbb{Q})^\varepsilon \backslash {}^\gamma SO(\eta)^1(\mathbb{A})^\varepsilon / {}^\gamma (\prod \Gamma_v)^\varepsilon.$$

We know from 3.3.3 that there is an $a \in \prod_{v \neq 2} \Gamma_v \times \Gamma_2(0)$ and a $b \in SO(\eta)^1(\mathbb{Q})$ such that

$\gamma = a^{-1} {}^\varepsilon a = b^{-1} {}^\varepsilon b$. We get a bijection induced by $c \mapsto bcb^{-1}$ of $h^{-1}(h(\bar{\gamma}))$ with $SO(\eta)^1(\mathbb{Q})^\varepsilon \backslash SO(\eta)^1(\mathbb{A})^\varepsilon / ba^{-1} (\prod \Gamma_v^\varepsilon) ab^{-1}$.

3.4. EXPLICITE FORMULAS. — In this paragraph we use the notations established up to now. We apply the method explained in 3.1.4 to obtain explicite formulas for the Lefschetz numbers and the Euler-Poincaré characteristic.

3.4.1. To shorten our notation in the main results of this chapter we introduce some abbreviations. Let

$$(d_1, \dots, d_l) = \begin{cases} (2, 4, \dots, 2l) & \text{if } n=2l+1 \\ (2, 4, \dots, 2l-2, l) & \text{if } n=2l. \end{cases}$$

Write $\chi : \mathbb{Z} \rightarrow \{\pm 1, 0\}$ for the number theoretical character given by $\chi(n)=0$ if n is even and $\chi(n)=(-1)^{(n-1)/2}$ if n is odd. We define

$$c(n, m) = \begin{cases} m^{n(n-1)/2} \prod_{p|m} \prod_{i=1}^l (1-p^{-d_i}) & \text{if } n \not\equiv 2 \pmod{4} \\ m^{n(n-1)/2} \prod_{p|m} (1-\chi(p)p^{-d_i}) \prod_{i=1}^{l-1} (1-p^{-d_i}) & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

We denote by $\zeta(s)$, $s \in \mathbb{C}$, the Riemann zeta function and by $L(\chi, s)$, $s \in \mathbb{C}$, the L-function associated to the character χ .

PROPOSITION 3.4.2. — *Let Γ be the full congruence subgroup mod $m \in \mathbb{N}$ of $\mathrm{SO}(\eta)(\mathbb{Z})$ and assume that 8 divides m . Let ε with ${}^\varepsilon A = {}^t A^{-1}$, $A \in \mathrm{SO}(\eta)(\mathbb{R})$ be the standard Cartan involution and let X be the space of maximal compact subgroups of $\mathrm{SO}(\eta)(\mathbb{R})$. Then ε induces an involution $\varepsilon : X/\Gamma \rightarrow X/\Gamma$; the fixpoint set $(X/\Gamma)^\varepsilon$ of this involution consists of finitely many points, their number is equal to $|\mathrm{H}^1(\mathbb{C}, \Gamma)|$ and*

$$|\mathrm{H}^1(\mathbb{C}, \Gamma)| = \begin{cases} (-1)^{l(l+1)/2} c(n, m) \prod_{i=1}^l \zeta(1-d_i) & \text{if } n=2l+1 \\ 2^{l-1} c(n, m) \prod_{i=1}^l \zeta(1-d_i) & \text{if } n=2l \text{ and } l \text{ is even} \\ 2^{-l} c(n, m) L(\chi, 1-d_l) \prod_{i=1}^{l-1} \zeta(1-d_i) & \text{if } n=2l \text{ and } l \text{ is odd.} \end{cases}$$

Proof. — We recall that the set $\mathrm{H}^1(\mathbb{C}, \Gamma)$ is finite [B-S]. If $b \in \Gamma$, $b = {}^t b$, represents a class $\gamma \in \mathrm{H}^1(\mathbb{C}, \Gamma)$ then the corresponding component $F(\gamma)$ of $(X/\Gamma)^\varepsilon$ is the image of ${}^b \mathrm{SO}(\eta)(\mathbb{R})^\varepsilon$ in X/Γ [Se 1]. But $b = a^{-1} {}^\varepsilon a$ for some $a \in \mathrm{SO}(\eta)(\mathbb{R})$, see 3.3.2. Therefore ${}^b \mathrm{SO}(\eta)(\mathbb{R})^\varepsilon \cong \mathrm{O}(n)(\mathbb{R})$, which is compact represents a point in X and the first part of our claim holds.

From 3.3.1 (iii) and 3.3.4 together with the remark 3.3.5 we have

$$|(X/\Gamma)^\varepsilon| = 2^n |\mathrm{SO}(\eta)^1(\mathbb{Q})^\varepsilon \setminus \mathrm{SO}(\eta)^1(\mathbb{A})^\varepsilon / (\prod \Gamma_v)^\varepsilon|.$$

We have a map $f = \prod f_v : \mathrm{SO}(\eta)^1(\mathbb{A})^\varepsilon \rightarrow \prod \{\pm 1\}$ given by

$$\left(\dots, \begin{pmatrix} \mathbf{B}_v & 0 \\ 0 & \det \mathbf{B}_v \end{pmatrix}, \dots \right) \mapsto (\dots, \det \mathbf{B}_v, \dots).$$

Here for the spinor norm $N(B_v) = \det(B_v)$. If $v = p$ and $p \nmid m$ or $v = \infty$ then there is a $\begin{pmatrix} B_v & 0 \\ 0 & \det B_v \end{pmatrix} \in \Gamma_v$ such that $\det(B_v) = -1$ and if $p \mid m$ then $\Gamma_p = \Gamma_p(n_p)$, see 3.2.5, and $\det B_v = 1$. Therefore we have an induced surjection, again denoted by f

$$f: \mathrm{SO}(\eta)^1(\mathbb{Q})^\varepsilon \backslash \mathrm{SO}(\eta)^1(\mathbb{A})^\varepsilon / \prod_v \Gamma_v^\varepsilon \rightarrow \{\pm 1\} \backslash \prod_{p \mid m} \{\pm 1\},$$

where $\{\pm 1\}$ is considered as diagonally imbedded in $\prod_{p \mid m} \{\pm 1\}$. If $\sigma = \prod_{p \mid m} \sigma_p \in \prod_{p \mid m} \{\pm 1\}$, then the fibre of f over the class $\bar{\sigma}$ represented by σ in $\{\pm 1\} \backslash \prod_{p \mid m} \{\pm 1\}$ is

$$f^{-1}(\bar{\sigma}) = \mathrm{SO}(\eta)^1(\mathbb{Q})^\varepsilon \backslash \mathrm{SO}(\eta)^1(\mathbb{A})^\varepsilon(\sigma) / \prod_v \Gamma_v^\varepsilon$$

where

$$\mathrm{SO}(\eta)^1(\mathbb{A})^\varepsilon(\sigma) = \{ B \in \mathrm{SO}(\eta)^1(\mathbb{A})^\varepsilon / \prod_{p \mid m} f_p(B_p) = \pm \sigma \}$$

and $\mathrm{SO}(\eta)^1(\mathbb{Q})^\varepsilon$ acts on this set. Consider the inclusion $L: \mathrm{SO}(n)(\) \subset \mathrm{SO}(n, 1)(\)$ given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. In $\mathrm{SO}(n)$ again by the upper index 1 we denote elements of spinor norm 1. Let $U_v = L^{-1}(\Gamma_v)$. Then we have a bijection

$$\mathrm{SO}(n)^1(\mathbb{Q}) \backslash \mathrm{SO}(n)^1(\mathbb{A}) / \prod_v U_v \cong f^{-1}(\bar{\sigma})$$

as follows:

If $B \in \mathrm{SO}(\eta)^1(\mathbb{A})^\varepsilon$ and $f(B) = \sigma$ we map a double class represented by $c \in \mathrm{SO}(n)^1(\mathbb{A})$ to the double class represented by cB . Obviously we get all classes represented by $\mathrm{SO}(\eta)^1(\mathbb{A})^\varepsilon(\sigma)$ and the map is surjective. If $c_1, c_2 \in \mathrm{SO}(n)^1(\mathbb{A})$ and if there is a $g \in \mathrm{SO}(\eta)^1(\mathbb{Q})$, $g = \begin{pmatrix} h & 0 \\ 0 & \det h \end{pmatrix}$, $h \in \mathrm{O}(n)(\mathbb{Q})$, and a $\gamma \in \prod \Gamma_v^\varepsilon$, $\gamma = \prod \gamma_v$, $\gamma_v = \begin{pmatrix} u_v & 0 \\ 0 & \det u_v \end{pmatrix}$ such that $gc_1 B \gamma = c_2 B$ then $\det(h) \cdot \det(u_v) = 1$ for all v . But $N(\gamma_v) = N(u_v) \det(u_v) = 1$, see 3.2.2. The spinor norm of the congruence subgroup mod 8 of $\mathrm{O}(n)(\mathbb{Z}_2)$ is trivial. Hence $N(u_2) = 1 = \det(u_2) = \det(h)$. Therefore $h \in \mathrm{SO}^1(\eta)(\mathbb{Q})$, $\gamma_v = \begin{pmatrix} u_v & 0 \\ 0 & 1 \end{pmatrix}$, $u_v \in U_v$ and the above map is injective. We get for $s = \#\{p/p \mid m\}$:

$$|(\mathbb{X}/\Gamma)^\varepsilon| = 2^n 2^{s-1} |\mathrm{SO}(n)^1(\mathbb{Q}) \backslash \mathrm{SO}(n)^1(\mathbb{A}) / \prod_v U_v|.$$

Choose a Tamagawa measure $\omega = \prod_v \omega_v$ on $\mathrm{SO}(n)(\mathbb{A})$ coming from a rational invariant differential form ω . Let $p : \mathrm{Spin}(n) \rightarrow \mathrm{SO}(n)$ be the natural projection from the universal covering $\mathrm{Spin}(n)$ to $\mathrm{SO}(n)$. Then $p^* \omega$ determines a Tamagawa measure $\tau = \prod_v \tau_v$ on $\mathrm{Spin}(n)(\mathbb{A})$ and

$$\mathrm{vol}_\tau(\mathrm{Spin}(n)(\mathbb{Q}) \backslash \mathrm{Spin}(n)(\mathbb{A})) = 1, \quad \text{see [O].}$$

For all places v of \mathbb{Q} we have the exact sequence

$$1 \rightarrow \{ \pm 1 \} \rightarrow \mathrm{Spin}(n)(\mathbb{Q}_v) \xrightarrow{p} \mathrm{SO}(n)(\mathbb{Q}_v) \xrightarrow{N} \mathbb{Q}_v^* / \mathbb{Q}_v^{*2}.$$

Let $\tilde{U}_v := p^{-1}(U_v)$. Then p induces a bijection

$$\mathrm{Spin}(n)(\mathbb{Q}) \backslash \mathrm{Spin}(n)(\mathbb{A}) / \prod_v \tilde{U}_v \xrightarrow{\sim} \mathrm{SO}(n)^1(\mathbb{Q}) \backslash \mathrm{SO}(n)^1(\mathbb{A}) / \prod_v U_v.$$

Write $\mathrm{Spin}(n)(\mathbb{A}) = \bigcup_{i=1}^h \mathrm{Spin}(n)(\mathbb{Q}) x_i \prod_v \tilde{U}_v$ as a disjoint union of double classes. If $\tilde{g} \in \mathrm{Spin}(n)(\mathbb{Q})$, $\tilde{u} \in \prod_v \tilde{U}_v$ and $x_i = \tilde{g} x_i \tilde{u}$ write $g = p(\tilde{g})$, $u = p(\tilde{u})$ and $p(x_i) = g_i \gamma_i$ with $g_i \in \mathrm{SO}(n)^1(\mathbb{Q})$ and $\gamma_i \in \prod_v \Gamma_v$. Here we consider $\mathrm{SO}(n)^1(\mathbb{A})$ as a subgroup of $\mathrm{SO}(n)^1(\mathbb{A})$ and use strong approximation. We get

$$g_i^{-1} g g_i = \gamma_i u^{-1} \gamma_i^{-1} \in \Gamma \cap \gamma_i \left(\prod_v U_v \right) \gamma_i^{-1}.$$

Since Γ is torsion free and $\prod_v U_v$ is compact, we find $g = u = 1$ and $\tilde{g} = \tilde{u}$ is in the center $\{ \pm 1 \} = \mu_2$ of $\mathrm{Spin}(n)(\mathbb{Q})$. Therefore we have an injection

$$x_i \prod_v \tilde{U}_v / \mu_2 \hookrightarrow \mathrm{Spin}(n)(\mathbb{Q}) \backslash \mathrm{Spin}(n)(\mathbb{A}).$$

Using the invariance of the Tamagawa measure we obtain

$$h = 2 \prod_v \mathrm{vol}_{\tau_v}(\tilde{U}_v)^{-1}.$$

We now compute the local volumes. For $v = \infty$ or $v = p$ and $p \mid m$, $m = \prod p^{n_p}$ we have $\mathrm{vol}_{\tau_v}(\tilde{U}_v) = 2 \mathrm{vol}_{\omega_v}(U_v)$ and get therefore

$$\mathrm{vol}_{\tau_\infty}(\tilde{U}_\infty) = 2 \mathrm{vol}_{\omega_\infty}(\mathrm{SO}(n)(\mathbb{R})), \quad \mathrm{vol}_{\tau_p}(\tilde{U}_p) = 2 \mathrm{vol}_{\omega_p}(U_p(n_p)),$$

where $U_p(n_p)$ is the full congruence subgroup mod p^{n_p} of $\mathrm{SO}(n)(\mathbb{Z}_p)$. If p is prime to m we have an exact sequence

$$1 \rightarrow \{ \pm 1 \} \rightarrow U_p \rightarrow \mathrm{SO}(n)(\mathbb{Z}_p) \rightarrow \mathbb{Z}_p^* / \mathbb{Z}_p^{*2} \rightarrow 1,$$

and consequently $\text{vol}_{\tau_p}(\tilde{U}_p) = \text{vol}_{\omega_p}(\text{SO}(n)(\mathbb{Z}_p))$. Summarizing we get

$$|(X/\Gamma)^\varepsilon| = 2^{n-1} \text{vol}_{\omega_\infty}(\text{SO}(n)(\mathbb{R}))^{-1} \prod_p \text{vol}_{\omega_p}(U_p(n_p))^{-1}.$$

Using the formulas for local volumes given in [R1] (4.6, 4.7) and the functional equation of the $\zeta(\)$ - and $L(\chi, \)$ -function, we arrive at the formula stated in the proposition.

Q. E. D.

PROPOSITION 3.4.3. — *Let Γ be the full congruence subgroup mod $m \in \mathbb{N}$ of $\text{SO}(n, 1)(\mathbb{Z})$ and assume that 8 divides m . Then we have the following formula for the Euler-Poincaré characteristic $\chi(\Gamma)$ of Γ :*

$$\chi(\Gamma) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^l (-1)^{l(l+1)/2} 2^{1-n-s} c(n+1, m) \prod_{i=1}^l \zeta(1-d_i) & \text{if } n=2l, \end{cases}$$

where $s = \#\{p \text{ a prime}/p | m\}$.

Proof. — Using Harder’s Gauss-Bonnet theorem [H1] we have $\chi(\Gamma) = 0$ if n is odd. We assume now that $n = 2l$ is even. Then $-\eta \in \text{SL}_{n+1}(\mathbb{Z})$, $\text{SO}(-\eta)(\) = \text{SO}(\eta)(\)$, and the methods of [R1] (§4) apply. Let ω be an invariant rational differential form of highest degree on $\text{SO}(\eta)/\mathbb{Q}$ and $\omega = \prod_v \omega_v$ be the Tamagawa measure determined by ω . Then we have an Euler-Poincaré measure

$$e_\chi = (-1)^l \text{vol}_{\omega_\infty}(\text{SO}(n+1)(\mathbb{R}))^{-1} \omega_\infty \text{ on } \text{SO}(\eta)(\mathbb{R}), \quad [\text{R1}](4.6).$$

Let G/\mathbb{Q} be the universal covering of $\text{SO}(\eta)/\mathbb{Q}$ and $p : G \rightarrow \text{SO}(\eta)$ be the natural projection. Then the differential form $p^*\omega$ induces a Tamagawa measure $\tau = \prod_v \tau_v$ on $G(\mathbb{A})$. Since the spinor norm of Γ is trivial, see 3.2.4, Γ is contained in $\text{SO}(\eta)(\mathbb{R})^0$, the connected component of $\text{SO}(\eta)(\mathbb{R})$, and moreover $p(G(\mathbb{R})) = \text{SO}(\eta)(\mathbb{R})^0$. Therefore

$$\int_{\text{SO}(\eta)(\mathbb{R})/\Gamma} \omega_\infty = 2 \int_{\text{SO}(\eta)(\mathbb{R})^0/\Gamma} \omega_\infty = 2 \int_{G(\mathbb{R})/\tilde{\Gamma}} \tau_\infty,$$

where $\tilde{\Gamma} = p^{-1}(\Gamma)$. Let $\tilde{\Gamma}_p$ be the closure of $\tilde{\Gamma}$ in $G(\mathbb{Q}_p)$ and $\tilde{\Gamma}_\infty = G(\mathbb{R})$. Using the exact sequences

$$1 \rightarrow \{\pm 1\} \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow 1$$

and

$$1 \rightarrow \{\pm 1\} \rightarrow \tilde{\Gamma}_p \rightarrow \text{SO}(\eta)(\mathbb{Z}_p) \rightarrow \mathbb{Z}_p^*/\mathbb{Z}_p^{*2} \rightarrow 1$$

resp.

$$\mathfrak{h} = \left\{ \left(\begin{array}{cccc} & & & 1 \\ & & & -1 \\ & & \ddots & \\ & & & & & & & 1 \\ & & & & & & & -1 \\ & & & & & & & & & & & & & & & & & & & 1 \end{array} \right) t_i \mid t_i \in \mathbb{C} \right\} \text{ if } n+1=2l.$$

Writing $\sum_{i=1}^l e_i t_i = h \in \mathfrak{h}$ and denoting $\varepsilon_i, i=1, \dots, l$, the dual basis of $e_i, i=1, \dots, l$, we choose a system of positive roots as in [Bo]. We consider $SO(n, \mathbb{C})$ as a subgroup of $SO(n+1)(\mathbb{C})$ given in obvious notation by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ and take $\mathfrak{h}_n := \mathfrak{so}(n)(\mathbb{C}) \cap \mathfrak{h}$ as a Cartan subalgebra of $\mathfrak{so}(n)(\mathbb{C})$. We choose positive root spaces in $\mathfrak{so}(n)(\mathbb{C})$ and $\mathfrak{so}(n+1)(\mathbb{C})$ in a compatible way. Denote by δ resp. δ_n half of the sum of the positive roots in $\mathfrak{so}(n+1)(\mathbb{C})$, resp. $\mathfrak{so}(n)(\mathbb{C})$.

3.5.3. We observe that an irreducible representation V_λ of $SO(n+1)(\mathbb{C})$ with highest weight λ admits an η -action if and only if $\lambda \circ \varepsilon = \lambda$ where ε is of course the action induced on the dual of \mathfrak{h} by conjugation with η on \mathfrak{h} . We fix from now on the action of η on such a V_λ by requiring that η acts trivial on highest weight vectors of V_λ . If $n=2l$ then $\lambda \circ \varepsilon = \lambda$ for all weights. If $n=2l-1$ then $\lambda \circ \varepsilon = \lambda$ if and only if $\lambda = (\lambda_1, \dots, \lambda_{l-1}, 0)$ with respect to the basis $\varepsilon_1, \dots, \varepsilon_l$.

3.5.4. PROPOSITION. — *Let V_λ be an irreducible representation of $SO(n+1)(\mathbb{C})$ with highest weight λ such that η acts on V_λ . In the notation introduced above let $\mu = \lambda |_{\mathfrak{h}_n} + \delta |_{\mathfrak{h}_n} - \delta_n$. Then μ is an integral dominant weight for $\mathfrak{so}(n)(\mathbb{C})$ and if V_μ denotes the corresponding irreducible (spin-) representation of $\mathfrak{so}(n)(\mathbb{C})$ we have*

$$\text{tr}(\eta | V_\lambda) = 2^{l-1} \dim(V_\mu).$$

Proof. — (i) Assume $n=2l$. Then $-\eta \in SO(n+1)(\mathbb{C})$ and $-\eta = \exp(2\pi i h)$ with $h = 1/2 \sum_{i=1}^l e_i$. But $-\eta$ acts on a highest weight vector $0 \neq v \in V_\lambda$ by $(-\eta)v = \exp 2\pi i \langle \lambda, h \rangle v$ where $\langle \cdot, \cdot \rangle$ denotes the evaluation of linear forms on elements. Hence

$$\text{tr}(\eta | V_\lambda) = \exp 2\pi i \langle \lambda, h \rangle \text{tr}(-\eta | V_\lambda).$$

By Weyl's character formula

$$\text{tr}(-\eta | V_\lambda) = \lim_{t \rightarrow h} \frac{\sum_{w \in W} \det(w) \exp 2\pi i \langle w(\lambda + \delta), t \rangle}{\sum_{w \in W} \det(w) \exp 2\pi i \langle w\delta, t \rangle}$$

where W is the Weyl group corresponding to \mathfrak{h} . The denominator of this formula is equal to

$$\begin{aligned} \exp 2\pi i \langle \delta, t \rangle \prod_{\alpha > 0} (1 - \exp -2\pi i \langle \alpha, t \rangle) \\ = \exp 2\pi i \langle \delta, t \rangle \prod_{j=1}^l (1 - \exp -2\pi i \langle \varepsilon_j, t \rangle) \prod_{\beta > 0} (1 - \exp -2\pi i \langle \beta, t \rangle) \end{aligned}$$

where β runs over the positive roots of $\mathfrak{so}(n)(\mathbb{C})$. We put $t = h + \tau$, write $W = W_0 \cup w_0 W_0$ where W_0 is the Weyl group for $\mathfrak{so}(n)(\mathbb{C})$ corresponding to \mathfrak{h}_n and w_0 acts on \mathbb{R}^l by multiplying the last coordinate with -1 . Then

$$\begin{aligned} \prod_{\beta > 0} (1 - \exp -2\pi i \langle \beta, t \rangle) &= \exp(-2\pi i \langle \delta_n, t \rangle) \sum_{w \in W_0} \det(w) \exp 2\pi i \langle w \delta_n, t \rangle \\ &= \exp(-2\pi i \langle \delta_n, \tau \rangle) \sum_{w \in W_0} \det(w) \exp 2\pi i \langle w \delta_n, \tau \rangle. \end{aligned}$$

We consider the nominator of the character formula. For $w \in W_0$ we have

$$\begin{aligned} \langle w_0 w(\lambda + \delta), t \rangle &= \langle w(\lambda + \delta), w_0 \tau \rangle + \langle w(\lambda + \delta), w_0 h - h \rangle \\ &\quad + \langle \lambda + \delta, w^{-1} h - h \rangle + \langle \gamma + \delta, h \rangle. \end{aligned}$$

Since $\delta \equiv h \pmod{\mathbb{Z}}$, $w^{-1} h - h \equiv 0 \pmod{2\mathbb{Z}}$, we get

$$\exp 2\pi i \langle w_0 w(\lambda + \delta), t \rangle = -\exp 2\pi i \langle \lambda + \delta, h \rangle \exp 2\pi i \langle \lambda + \delta, w_0 \tau \rangle.$$

Therefore

$$\mathrm{tr}(-\eta | V_\lambda) = \exp(2\pi i \langle \lambda, h \rangle) 2^{1-l} \lim_{\tau \rightarrow 0} \frac{\sum_{w \in W_0} \det(w) \exp 2\pi i \langle w(\lambda + \delta), \tau \rangle}{\sum_{w \in W_0} \det(w) \exp 2\pi i \langle w \delta_n, \tau \rangle}.$$

Since $\lambda + \delta = \lambda + \delta - \delta_n + \delta_n = \mu + \delta_n$ the claim holds.

(ii) Assume that $n+1 = 2l$. We now use Kostant's character formula [K] (Thm. 7.5) for the disconnected group $O(n+1)(\mathbb{C})$. If $T \subset SO(n+1)(\mathbb{C})$ is the Cartan subgroup corresponding to \mathfrak{h} then its normalizer H in $O(n+1)(\mathbb{C})$ has two connected components.

For $a \in H$ close to η we write $a = \eta \exp 2\pi i \left(\sum_{j=1}^l e_j t_j \right)$. We use Kostant's notation. His group W_a can be identified with W_0 , the Weyl group of \mathfrak{h}_n in $\mathfrak{so}(n)(\mathbb{C})$. Using that λ and δ have as a last component zero with respect to the basis $\varepsilon_1, \dots, \varepsilon_l$ Kostant's

formula reads

$$\text{tr}(\eta|V_\lambda) = \lim_{t \rightarrow 0} \frac{\sum_{w \in W_0} \det(w) \exp 2\pi i \langle w(\lambda + \delta) - \delta, t \rangle}{\sum_{w \in W_0} \det(w) \exp 2\pi i \langle w\delta - \delta, t \rangle}.$$

and the result follows easily.

Q.E.D.

IV. Boundary contributions

In order to be able to apply the formulas for the Euler-Poincaré characteristic in 1.5 and for the Lefschetz number in 2.4 for estimates of multiplicities of representations with (\mathfrak{g}, K) -cohomology we have to understand the numbers $h(\Gamma)$ resp. $h_\varepsilon(\Gamma)$ of Γ -conjugacy classes resp. ε -stable Γ -conjugacy classes of proper parabolic subgroups of $SO(n, 1)/\mathbb{Q}$. In 4.1 we give an estimate of $h(\Gamma)$, in 4.2 we compute $h_\varepsilon(\Gamma)$ and in 4.3 we evaluate the boundary contributions occurring in 2.4.

4.1. AN ESTIMATE OF THE NUMBER OF Γ -CONJUGACY CLASSES OF PARABOLIC SUBGROUPS. — For the application we have in mind an estimate $h(\Gamma) \leq Cm^\alpha$ with some constants $0 < C \in \mathbb{R}$ and $0 < \alpha < n(n+1)/2$ will be sufficient. Here $\Gamma = \Gamma(m)$. After some preparations in 4.1.1 such an estimate is given in 4.1.2.

4.1.1. We recall the description of parabolic subgroups of $SO(n, 1)/\mathbb{Q}$. Let $H = \mathbb{Q}(e_n + e_{n+1}) \subset \mathbb{Q}^{n+1} = \mathbb{Q}e_1 + \dots + \mathbb{Q}e_{n+1}$ be a totally isotropic subspace of \mathbb{Q}^{n+1} with respect to the quadratic form $\eta(x_1, \dots, x_{n+1}) = \sum_{i=1}^n x_i^2 - x_{n+1}^2$. Then $SO(n, 1)(\mathbb{Q})$ acts on \mathbb{Q}^{n+1} and the stabilizer $P(\mathbb{Q})$ of H in $SO(n, 1)(\mathbb{Q})$ is called a parabolic subgroup. All proper parabolic subgroups defined over \mathbb{Q} are stabilizers of 1-dim isotropic subspaces and we have a bijection.

$$SO(n, 1)(\mathbb{Q})/P(\mathbb{Q}) \rightarrow \mathcal{P}$$

given by $g \mapsto gPg^{-1}$ where \mathcal{P} denotes the set of proper parabolic subgroups defined over \mathbb{Q} . Of course Γ acts by conjugation on \mathcal{P} and we get an induced bijection

$$\Gamma \backslash SO(n, 1)(\mathbb{Q})/P(\mathbb{Q}) \simeq \Gamma \backslash \mathcal{P}$$

and we have to compute $h(\Gamma) = |\Gamma \backslash SO(n, 1)(\mathbb{Q})/P(\mathbb{Q})|$. It is well known that $h(\Gamma) < \infty$, see [B-S].

4.1.2. PROPOSITION. — *There is a constant $0 < C \in \mathbb{R}$ such that for all $m \in \mathbb{N}$ we have an estimate*

$$h(\Gamma(m)) \leq Cm^{(1/2)n(n-1)+1}.$$

Proof. — Consider the natural projection

$$\Gamma(m) \backslash \mathrm{SO}(n, 1)(\mathbb{Q}) / \mathrm{P}(\mathbb{Q}) \rightarrow \mathrm{SO}(n, 1)(\mathbb{Z}) \backslash \mathrm{SO}(n, 1)(\mathbb{Q}) / \mathrm{P}(\mathbb{Q}).$$

If $g \in \mathrm{SO}(n, 1)(\mathbb{Q})$ represents a class on the right the fibre E_g of this map over g is isomorphic to $\Gamma(m) \backslash \mathrm{SO}(n, 1)(\mathbb{Z}) / \mathrm{SO}(n, 1)(\mathbb{Z}) \cap g \mathrm{P}(\mathbb{Q}) g^{-1}$. Let U^g be the unipotent radical of $\mathrm{P}^g = g \mathrm{P} g^{-1}$ and abbreviate

$$U^g(0) = \mathrm{SO}(n, 1)(\mathbb{Z}) \cap U^g(\mathbb{Q}), \quad U^g(m) = \Gamma(m) \cap U^g(\mathbb{Q}).$$

Then $|E_g| \leq [\mathrm{SO}(n, 1)(\mathbb{Z}) : \Gamma(m)] [U^g(0) : U^g(m)]^{-1}$. We have an inclusion

$$\mathrm{SO}(n, 1)(\mathbb{Z}) / \Gamma(m) \hookrightarrow \prod_{p|m} \mathrm{SO}(n, 1)(\mathbb{Z}_p) / \Gamma_p(m_p) \quad \text{with} \quad m = \prod_p p^{n_p}.$$

Choose invariant local measures ω_p such that $\mathrm{vol}_{\omega_p}(\Gamma_p(s)) = p^{-s(n+1)n/2}$ if $s \geq 1$. Then, see [R 1] (§4), for all $p \neq 2$ we get $\mathrm{vol}_{\omega_p}(\mathrm{SO}(n, 1)(\mathbb{Z}_p)) \leq 1$. Hence there is a constant A independent of m such that $|\mathrm{SO}(n, 1)(\mathbb{Z}) / \Gamma(m)| \leq A m^{n(n+1)/2}$. Let $U_p^g(0)$ resp. $U_p^g(m_p)$ be the closure of $U^g(0)$ resp. $U^g(m)$ in $U^g(\mathbb{Q}_p)$. Since U^g has strong approximation

$$U^g(0) / U^g(m) = \prod_{p|m} U_p^g(0) / U_p^g(m_p).$$

Choose invariant measures ω_p on $U^g(\mathbb{Q}_p)$ such that $\mathrm{vol}_{\omega_p}(U_p^g(s)) = p^{-(n-1)s}$ if s is big enough. Here $n-1 = \dim U_g$. Then $\mathrm{vol}_{\omega_p}(U_p^g(0)) = 1$ for almost all p .

Hence there is a constant $0 < C_g \in \mathbb{R}$ such that $[U_g(0) : U_g(m)] \geq C_g^{-1} \cdot m^{n-1}$ for all m . Since $\mathrm{SO}(n, 1)(\mathbb{Z}) \backslash \mathrm{SO}(n, 1)(\mathbb{Q}) / \mathrm{P}(\mathbb{Q})$ is finite, we can choose a $C \in \mathbb{R}$ bigger than all AC_g and our claim holds.

Q.E.D.

4.1.3. *Remark.* — Using adelic methods it is possible to compute the precise value of $h(\Gamma(m))$, see the proof of 4.2.4.

4.2. ON THE NUMBER OF ε -STABLE Γ -CONJUGACY CLASSES OF PARABOLIC SUBGROUPS. — For applications to estimates of multiplicities we need an estimate $h_\varepsilon(\Gamma(m)) \leq C m^\alpha$ with some constants $0 < C \in \mathbb{R}$ and $0 < \alpha < n(n-1)/2$. Such an estimate however, would give a wrong geometric picture since we will show that $h_\varepsilon(\Gamma(m)) = 0$ if 16 divides m . This means that no point of $(X/\Gamma(m))^\varepsilon$ is close to the boundary of $X/\Gamma(m)$. We think that our method to obtain this is of interest in its own.

4.2.1. On the set \mathcal{P} of proper parabolic subgroups defined over \mathbb{Q} of $\mathrm{SO}(n, 1)/\mathbb{Q}$ conjugation by η induces an involution ε . Since $\Gamma = \Gamma(m)$ is ε -stable we have an induced involution, again called ε on $\Gamma \backslash \mathcal{P}$. We have a bijection

$$\Gamma \backslash \mathrm{SO}(n, 1)(\mathbb{Q}) / \mathrm{P}(\mathbb{Q}) \rightarrow \Gamma \backslash \mathcal{P}$$

and by transport of structure an involution—for psychological reasons now called σ —on the left side. We describe σ as follows:

Choose

$$h = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix} \in \mathrm{SO}(n, 1)(\mathbb{Z}).$$

Then $\eta h = h \eta$, $h^2 = 1$ and $h P h = \eta P \eta = \varepsilon(P)$. We have an involution $g \mapsto {}^\sigma g := {}^\varepsilon g h$ on $\mathrm{SO}(n, 1)(\mathbb{Q})$. If $\gamma \in \Gamma$, $g \in \mathrm{SO}(n, 1)(\mathbb{Q})$ and $q \in P(\mathbb{Q})$ then ${}^\sigma(\gamma g q) = {}^\varepsilon \gamma {}^\sigma g {}^\tau q$ where ${}^\tau q = h \eta q \eta h$. Hence σ -induces an involution—again denoted by σ —on $\Gamma \backslash \mathrm{SO}(n, 1)(\mathbb{Q})/P(\mathbb{Q})$ and the bijection $\Gamma \backslash \mathcal{P} \rightarrow \Gamma \backslash \mathrm{SO}(n, 1)(\mathbb{Q})/P(\mathbb{Q})$ is equivariant with respect to the σ -resp. ε -action. We point out, that σ is not compatible with the group action on $\mathrm{SO}(n, 1)(\mathbb{Q})$ and that σ should be viewed as a Hecke operator.

The computation of $(\Gamma \backslash \mathrm{SO}(n, 1)(\mathbb{Q})/P(\mathbb{Q}))^\sigma$ can be divided in a local and a global part. For the local part we need the following explicit Iwasawa decomposition.

4.2.3. LEMMA. — *We have for all primes p an Iwasawa-decomposition*

$$\mathrm{SO}(n, 1)(\mathbb{Z}_p) P(\mathbb{Q}_p) = \mathrm{SO}(n, 1)(\mathbb{Q}_p).$$

Proof. — See [T] (3.3.2). Of course here an ad hoc proof is possible, which is almost trivial if $p \neq 2$ and uses for $p = 2$ the classification of quadratic forms over \mathbb{Z}_2 , see [J] (p. 91).

Q.E.D.

Since $(X/\Gamma)^\varepsilon$ consists of finitely many isolated points the boundary $\partial(\bar{X}/\Gamma)$ of the Borel-Serre compactification of X/Γ contains no fixpoints of ε . This corresponds to the fact that there are non ε -stable proper parabolic subgroups defined over \mathbb{Q} . In principle it could happen that ε acts freely on a component of $\partial(\bar{X}/\Gamma)$. We now show that this is impossible if $\Gamma = \Gamma(m)$ and m is bigg enough. The argument depends on the fact that the \mathbb{R} -rank of $\mathrm{SO}(n, 1)$ is one.

In the proof of the following result we abbreviate $H^1(g, A) = H^1(C, A)$ if g generates the group C acting on A .

4.2.4. PROPOSITION. — *Let $\Gamma = \Gamma(m)$ be the full congruence subgroup mod m of $\mathrm{SO}(n, 1)(\mathbb{Z})$ and assume that 16 divides m . Then the set of ε -stable Γ -conjugacy classes of proper parabolic subgroups defined over \mathbb{Q} of $\mathrm{SO}(n, 1)/\mathbb{Q}$ is empty.*

Proof. — At first we explain the idea of the proof. If $g \in \mathrm{SO}(n, 1)(\mathbb{Q})$ represents a class in $(\Gamma \backslash \mathrm{SO}(n, 1)(\mathbb{Q})/P(\mathbb{Q}))^\sigma$ then

$${}^\sigma g = r_\varepsilon^{-1} g q_\tau \quad \text{and} \quad {}^{\sigma\sigma} g = g = {}^\varepsilon r_\varepsilon^{-1} r_\varepsilon^{-1} g q_\tau {}^\tau q_\tau$$

with $r_\varepsilon \in \Gamma$, $q_\tau \in P(\mathbb{Q})$. Assume that it is possible to choose g , r_ε , q_τ such that $r_\varepsilon {}^\varepsilon r_\varepsilon = 1$. Then r_ε gives a class in $H^1(\varepsilon, \Gamma)$ which bounds in $H^1(\varepsilon, \mathrm{SO}(n, 1)(\mathbb{Q}))$, see

3.3.3, i. e. there is a $y \in \mathrm{SO}(n, 1)(\mathbb{Q})$ such that $q_\tau = y^{-1} h^\tau y$. But then

$$y P y^{-1} = h^\tau y P^\tau y^{-1} h = \varepsilon (y P y^{-1})$$

would be a ε -stable parabolic which does not exist, as we will see in a moment.

We have a bijection

$$\mathrm{SO}(n, 1)(\mathbb{Q})/P(\mathbb{Q}) \xrightarrow{\sim} F(\mathbb{Q}) := \left\{ (x_1, \dots, x_{n+1}) \in \mathbb{P}_n(\mathbb{Q}) / \sum_{i=1}^n x_i^2 - x_{n+1}^2 = 0 \right\}.$$

Here $\mathbb{P}_n(\mathbb{Q})$ is the rational projective space.

The induced σ -action on $F(\mathbb{Q})$ is given by

$$(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n, -x_{n+1}).$$

Hence

$$\mathcal{P}^\varepsilon \cong (\mathrm{SO}(n, 1)(\mathbb{Q})/P(\mathbb{Q}))^\sigma = \mathbb{P}_{n-1}(\mathbb{Q}) \cap F(\mathbb{Q})$$

where $\mathbb{P}_{n-1}(\mathbb{Q}) \subset \mathbb{P}_n(\mathbb{Q})$ is considered as the subspace of $\mathbb{P}_n(\mathbb{Q})$ defined by $x_{n+1} = 0$. Since $\sum_{i=1}^n x_i^2 \neq 0$ unless $x_1 = \dots = x_n = 0$ we have $\mathcal{P}^\varepsilon = \emptyset$.

Of course F is defined over \mathbb{Z} , $F(\mathbb{Q}_p) = F(\mathbb{Z}_p)$, $F(\mathbb{Z}/p^{m_p}\mathbb{Z})$ makes sense and σ acts on these sets. Here $m = \prod p^{n_p}$ as usual. We write $u := m/2 = \prod p^{m_p}$, i. e. $m_p = n_p$ if $p \neq 2$ and $n_2 - 1 = m_2 \geq 3$, and abbreviate $G_p := F(\mathbb{Z}/p^{m_p}\mathbb{Z})^\sigma$ if $p \neq 2$ and

$$G_2 := \mathrm{im}(F(\mathbb{Z}/2^{m_2+1}\mathbb{Z})^\sigma \rightarrow F(\mathbb{Z}/2^{m_2}\mathbb{Z})^\sigma).$$

We get a natural map.

$$\mathrm{Im}((\Gamma(m) \backslash \mathcal{P})^\varepsilon \rightarrow (\Gamma(m/2) \backslash \mathcal{P})^\varepsilon) \rightarrow \prod_{p|m} G_p$$

induced by the obvious maps

$$\Gamma(m) \backslash \mathrm{SO}(n, 1)(\mathbb{Q})/P(\mathbb{Q}) \rightarrow \prod_p \Gamma_p(m_p) \backslash \mathrm{SO}(n, 1)(\mathbb{A}_f)/P(\mathbb{A}_f)$$

and

$$\Gamma_p(m_p) \backslash \mathrm{SO}(n, 1)(\mathbb{Q}_p)/P(\mathbb{Q}_p) \rightarrow \Gamma_p(m_p) \backslash F(\mathbb{Q}_p) \rightarrow F(\mathbb{Z}/p^{m_p}\mathbb{Z}),$$

where \mathbb{A}_f is the ring of finite adeles.

We assume from now on that $(\Gamma(m) \backslash \mathcal{P})^\varepsilon \neq \emptyset$ and show that then necessarily $n \geq 5$ and that the natural maps $F(\mathbb{Q}_p)^\sigma \rightarrow G_p$ are surjective. To see this, let $(a_1, \dots, a_{n+1}) \in \mathbb{Z}_p^{n+1}$ represent a point $\bar{a} \in G_p$. Then $a_{n+1} \equiv 0 \pmod{p^{m_p}}$ and if $p = 2$ then $\sum_{i=1}^n a_i^2 \equiv 0 \pmod{2^{m_2+1}}$. By Hensel's lemma there is a $b = (b_1, \dots, b_n, 0) \in F(\mathbb{Z}_p)^\sigma$ with image

\bar{a} . Since $m_2 \neq 0$ and $\sum_{i=1}^n x_i^2 = 0$ is solvable over \mathbb{Q}_p , $p \mid m$, if and only if $n \geq 5$ our claim holds.

Now we rewrite $\Gamma \backslash \mathcal{P}$ using adèles. Since we have an equality of spinor norms $N(\mathbb{P}(\mathbb{Q})) = N(\mathrm{SO}(n, 1)(\mathbb{Q}))$ we have a σ -equivariant bijection

$$\Gamma \backslash \mathrm{SO}(n, 1)(\mathbb{Q}) / \mathbb{P}(\mathbb{Q}) \xrightarrow{\sim} \Gamma \backslash \mathrm{SO}(n, 1)^1(\mathbb{Q}) / \mathbb{P}^1(\mathbb{Q}).$$

where the index 1 indicates always subgroups of spinor norm 1. By 4.2.3 we have $\mathrm{SO}(n, 1)(\mathbb{Z}_p) \mathbb{P}(\mathbb{Q}_p) = \mathrm{SO}(n, 1)(\mathbb{Q}_p)$ for all p . Since $(\Gamma \backslash \mathcal{P})^\varepsilon \neq \emptyset$ by assumption we assume $n \geq 5$ and get $\mathrm{SO}(n, 1)^1(\mathbb{Z}_p) \mathbb{P}^1(\mathbb{Q}_p) = \mathrm{SO}(n, 1)^1(\mathbb{Q}_p)$. Here we use $N(\mathrm{SO}(n, 1)(\mathbb{Z}_p)) = N(\mathbb{P}(\mathbb{Q}_p) \cap \mathrm{SO}(n, 1)(\mathbb{Z}_p))$, see 3.1. Abbreviate

$$\hat{K} := \prod_p \mathrm{SO}(n, 1)^1(\mathbb{Z}_p) \quad \text{and} \quad \hat{K}(m) = \prod_p \Gamma_p(m_p).$$

Using strong approximation we get a bijection

$$\Gamma(m) \backslash \mathrm{SO}(n, 1)(\mathbb{Q}) / \mathbb{P}(\mathbb{Q}) \xrightarrow{\sim} \hat{K}(m) \backslash \hat{K} \mathbb{P}^1(\mathbb{A}_f) / \mathbb{P}^1(\mathbb{Q}).$$

the induced σ -action on the right is given by ${}^\sigma(\gamma g q) = {}^\varepsilon \gamma {}^\sigma g {}^\tau q$ for $\gamma \in \hat{K}(m)$, $g \in \hat{K} \cdot \mathbb{P}^1(\mathbb{A}_f)$, $q \in \mathbb{P}^1(\mathbb{Q})$ and ${}^\sigma g = {}^\varepsilon g h$.

Consider the natural σ -equivariant projection

$$\pi: \hat{K}(u) \backslash \hat{K} \mathbb{P}^1(\mathbb{A}_f) / \mathbb{P}^1(\mathbb{Q}) \rightarrow \hat{K}(u) \backslash \hat{K} \mathbb{P}^1(\mathbb{A}_f) / \mathbb{P}^1(\mathbb{A}_f)$$

where $u = m/2$. Since

$$(\hat{K}(u) \backslash \hat{K} \mathbb{P}^1(\mathbb{A}_f) / \mathbb{P}^1(\mathbb{A}_f))^\sigma = (\hat{K}(u) \backslash \hat{K} / \hat{K} \cap \mathbb{P}^1(\mathbb{A}_f))^\sigma$$

the first part of our proof shows that in the image of $(\hat{K}(u) \backslash \hat{K} \mathbb{P}^1(\mathbb{A}_f) / \mathbb{P}^1(\mathbb{Q}))^\sigma$ there is a class which can be represented by a $\gamma \in \hat{K}$ such that

$$\gamma^{-1} {}^\sigma \gamma = \gamma^{-1} h {}^\tau \gamma =: c_\tau \in \mathbb{P}^1(\mathbb{A}_f).$$

Here we use $F(\mathbb{Q}_p) = \mathrm{SO}(n, 1)^1(\mathbb{Z}_p) \mathbb{P}^1(\mathbb{Q}_p) / \mathbb{P}^1(\mathbb{Q}_p)$. Let $\bar{\gamma}$ be the class in $\hat{K}(u) \backslash \hat{K} \mathbb{P}^1(\mathbb{A}_f) / \mathbb{P}^1(\mathbb{Q})$ given by γ and denote by $E_{\bar{\gamma}} = \pi^{-1}(\{\pi(\bar{\gamma})\})$ the fibre of π through $\bar{\gamma}$. Then $E_{\bar{\gamma}}$ is σ -stable, $(E_{\bar{\gamma}})^\sigma \neq \emptyset$ and we have a bijection

$$E_{\bar{\gamma}} \rightarrow \hat{K}(u) \backslash \hat{K}(u) \mathbb{P}^1(\mathbb{A}_f) / \mathbb{P}^1(\mathbb{Q})$$

given by $e \mapsto \gamma^{-1} e$. The induced involution on the right is given by $y \mapsto c_\tau {}^\tau y$ for $y \in \hat{K}(u) \mathbb{P}^1(\mathbb{A}_f)$ with $c_\tau = \gamma^{-1} h {}^\tau \gamma \in \hat{K} \cap \mathbb{P}^1(\mathbb{A}_f)$ and we have a σ -equivariant bijection of the right side with $\hat{K}(u) \cap \mathbb{P}^1(\mathbb{A}_f) \backslash \mathbb{P}^1(\mathbb{A}_f) / \mathbb{P}^1(\mathbb{Q})$. Consider the natural projection $\mathbb{P}^1(\mathbb{A}_f) \rightarrow \mathbb{P}^1(\mathbb{A}_f) / U(\mathbb{A}_f) =: M(\mathbb{A}_f)$ and denote the image of $\hat{K}(u) \cap \mathbb{P}^1(\mathbb{A}_f)$ in $M(\mathbb{A}_f)$

by $\hat{K}_M(u)$. Let $M(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Q})/U(\mathbb{Q})$. Since U has strong approximation we get a σ -equivariant bijection

$$\hat{K}(u) \cap \mathbb{P}^1(\mathbb{A}_f) \backslash \mathbb{P}^1(\mathbb{A}_f)/\mathbb{P}^1(\mathbb{Q}) \xrightarrow{\sim} \hat{K}_M(u) \backslash M(\mathbb{A}_f)/M(\mathbb{Q}).$$

There is a natural character $\alpha: P \rightarrow G_m$ given by

$$q(0, \dots, 0, 1, 1) = \alpha(q)(0, \dots, 0, 1, 1), \quad q \in P,$$

which is trivial on U . Hence α induces a homomorphism $|\alpha|: M(\mathbb{A}_f) \rightarrow \mathbb{R}^*$ given by $|\alpha|(\dots, m_p, \dots) = \prod_p \|\alpha(m_p)\|_p$ where $\|\cdot\|_p$ is the standard absolute value

on \mathbb{Q}_p^* and $(\dots, m_p, \dots) \in M(\mathbb{A}_f)$. Let $M(\mathbb{A}_f)(1) = \{m \in M(\mathbb{A}_f) \mid |\alpha|(m) = 1\}$ and $M(\mathbb{Q})(1) = M(\mathbb{A}_f)(1) \cap M(\mathbb{Q})$. Since $|\alpha|(\hat{K}_M(u)) = 1$ and $|\alpha|(M(\mathbb{A}_f)) = |\alpha|(M(\mathbb{Q}))$ we get a σ -equivariant bijection

$$\hat{K}_M(u) \backslash M(\mathbb{A}_f)(1)/M(\mathbb{Q})(1) \xrightarrow{\sim} \hat{K}_M(u) \backslash M(\mathbb{A}_f)/M(\mathbb{Q}).$$

We observe that $M(\mathbb{Q})(1) \subset M(\mathbb{R})(1) = \{m \in \mathbb{P}^1(\mathbb{R})/U(\mathbb{R}) \mid \|\alpha(m)\|_\infty = 1\}$ is compact, where $\|\cdot\|_\infty$ is the standard absolute value on \mathbb{R} .

If $y \in M(\mathbb{A}_f)(1)$ then we have a bijection $\hat{K}_M(u) \times M(\mathbb{Q})(1) \rightarrow \hat{K}_M(u)yM(\mathbb{Q})(1)$ sending (r, m) to rym . To see this let S be a split torus contained in the radical RP of P . Then one has an isomorphism of algebraic groups $Z(S) \xrightarrow{\sim} P/U$ over \mathbb{Q} where $Z(S)$ is the centralizer of S and we can view $M(\mathbb{A}_f)$ as $Z(S)(\mathbb{A}_f)$. A look at the corresponding Lie algebras shows that $\hat{K}_M(u)$ and $\hat{K}(u) \cap Z(S)(\mathbb{A}_f)$ coincide at all primes $p \mid m$ and that these groups are commensurable. Hence there is a lattice Λ in \mathbb{Q}^{n+1} which differs only at some primes, not dividing m from \mathbb{Z}^{n+1} such that $\hat{K}_M(u)$ stabilizes Λ . Consider now $y \in M(\mathbb{A}_f) \subset Z(S)(\mathbb{A}_f) \subset \mathrm{SL}_{n+1}(\mathbb{A}_f)$. By strong approximation $y = vg$ with $g \in \mathrm{SL}_{n+1}(\mathbb{Q})$ and $v \in \mathrm{SL}_{n+1}(\mathbb{A}_f)$ stabilizes $\Lambda \otimes \prod_{\substack{\mathbb{Z} \\ p}} \mathbb{Z}_p$. If $\Gamma_\Lambda \subset \mathrm{SL}_{n+1}(\mathbb{Q})$ is

the subgroup stabilizing Λ and if $\Gamma_\Lambda(u)$ is its full congruence subgroup mod u , then by an old result of Minkowski [M] the group $\Gamma_\Lambda(u)$ is torsion free since $u \geq 4$. Considering $rym = y = rvgm = vg$ we get $gm^{-1}g = v^{-1}rv \in \Gamma_\Lambda(u) \cap gM(\mathbb{Q})(1)g^{-1} = \{1\}$ since $M(\mathbb{R})(1)$ is compact. Hence $r = m = 1$ and our claim about the bijection holds.

Now we are finally able to see cocycles. If $y \in M(\mathbb{A}_f)(1)$ represents an element in $(\hat{K}_M(u) \backslash M(\mathbb{A}_f)(1)/M(\mathbb{Q})(1))^\sigma$ then ${}^\sigma y = d_\tau {}^\tau y = r_\tau^{-1} y m_\tau$ with uniquely determined $r_\tau \in \hat{K}_M(u)$, $m_\tau \in M(\mathbb{Q})(1)$, where d_τ is the image of c_τ in $M(\mathbb{A}_f)$. Since ${}^{\sigma\sigma} y = 1$ we have $y = d_\tau {}^\tau r_\tau^{-1} d_\tau^{-1} r_\tau^{-1} y m_\tau {}^\tau m_\tau$. Hence r_τ determines a cocycle for $H^1(\tau, {}^d \hat{K}_M(u))$ and m_τ a cocycle for $H^1(\tau, M(\mathbb{Q})(1))$, where the upper index d indicates twisting with the cocycle d_τ . We observe that the cohomology classes obtained depend only on the fixpoint represented by y . We read ${}^\sigma y = d_\tau {}^\tau y = r_\tau^{-1} y m_\tau$ as $r_\tau d_\tau = y m_\tau {}^\tau y^{-1}$ and remark that $r_\tau d_\tau$ represents a class in $H^1(\tau, \hat{K}_M)$, where \hat{K}_M is the image of $\hat{K} \cap P(\mathbb{A}_f)$ in $M(\mathbb{A}_f)$. Hence we have natural maps

$$H^1(\tau, {}^d \hat{K}_M(u)) \rightarrow H^1(\tau, {}^d \hat{K}_M) \xrightarrow{\sim} H^1(\tau, \hat{K}_M) \rightarrow H^1(\tau, M(\mathbb{A}_{gf}))$$

and $H^1(\tau, M(\mathbb{Q})(1)) \rightarrow H^1(\tau, M(\mathbb{A}_f))$. The equation $r_\tau d_\tau = y m_\tau^\tau y^{-1}$ then means that we have a map

$$\partial: (\mathbb{K}_M(u) \backslash M(\mathbb{A}_f)/M(\mathbb{Q}))^\sigma \rightarrow H^1(\tau, {}^d\hat{\mathbb{K}}_M(u)) \prod_{H^1(\tau, M(\mathbb{A}_f))} H^1(\tau, M(\mathbb{Q}))$$

where the fibre product on the right is constructed with the above two maps.

In the next part we show that the natural map

$$H^1(\tau, {}^c\hat{\mathbb{K}}(u) \cap P(\mathbb{A}_f)) \prod_{H^1(\tau, P^1(\mathbb{A}_f))} H^1(\tau, P^1(\mathbb{Q})) \rightarrow H^1(\tau, {}^d\hat{\mathbb{K}}_M(u)) \prod_{H^1(\tau, M(\mathbb{A}_f))} H^1(\tau, M(\mathbb{Q}))$$

is surjective.

Since $U(\mathbb{A}_f)$ and $U(\mathbb{Q})$ have normal series with abelian 2-divisible quotients we easily get

$$H^1(\tau, M(\mathbb{A}_f)) \overset{\sim}{\leftarrow} H^1(\tau, P^1(\mathbb{A}_f)) \quad \text{and} \quad H^1(\tau, M(\mathbb{Q})) \overset{\sim}{\leftarrow} H^1(\tau, P^1(\mathbb{Q})).$$

Here we use [Se 1] (I-70) and divide successively out the groups of the normal series. Write $\hat{\mathbb{K}}_M(u) = \Pi W_p$. Then the same argument gives locally at primes $p \neq 2$ a bijection $H^1(\tau, {}^cV_p) \overset{\sim}{\rightarrow} H^1(\tau, {}^dW_p)$ with $V_p = \Gamma_p(m_p) \cap P^1(\mathbb{Q}_p)$. But if $p=2$ then W_2 is a direct factor of V_2 and we get a surjection

$$H^1(\tau, {}^c\hat{\mathbb{K}}(u) \cap P(\mathbb{A}_f)) \rightarrow H^1(\tau, {}^d\hat{\mathbb{K}}_M(u))$$

and our claim holds.

Since by assumption $(\hat{\mathbb{K}}_M(u) \backslash M(\mathbb{A}_f)/M(\mathbb{Q}))^\sigma \neq \emptyset$ we thus have an equation $r_\tau c_\tau = q p_\tau^\tau q^{-1}$ with $r_\tau \in \hat{\mathbb{K}}(u) \cap P(\mathbb{A}_f)$, $q \in P^1(\mathbb{A}_f)$ and $p_\tau \in P^1(\mathbb{Q})$. By strong approximation we can write $q = v g^{-1}$ with $g \in \text{SO}(n, 1)^1(\mathbb{Q})$ and $v \in \hat{\mathbb{K}}(u)$ and get

$$(\gamma v)^{-1} \gamma r_\tau \gamma^{-1} h^\tau(\gamma v) = g p_\tau^\tau g^{-1} =: \gamma_\tau \in \text{SO}(n, 1)(\mathbb{Q}) \cap \hat{\mathbb{K}}.$$

But then $r_\varepsilon := \gamma_\tau h$ is a cocycle for

$$H^1(\tau, {}^h\Gamma(u)) = H^1(\varepsilon, \Gamma(u)), \quad {}^\sigma g = r_\varepsilon^{-1} g p_\tau,$$

and as explained in the beginning of the proof we are done.

Q.E.D.

4.3. CONTRIBUTIONS OF THE BOUNDARY TO LEFSCHETZ NUMBERS. — In the representation theoretical interpretation of Lefschetz numbers given in 2.4 we have boundary contributions of the form.

$$\text{tr}(\varepsilon | \text{Hom}_{U(\mathfrak{g})}(U_i(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)))$$

where $U_i(\lambda) = I(\pi_i(\lambda), v_i(\lambda))$ is a certain principal series representation, see 1.3, and the definition of ε involves an equivalence $C_\varepsilon: U_i(\lambda) \circ \varepsilon \rightarrow U_i(\lambda)$. We denote here by ε the standard Cartan involution. In this chapter we compute these contributions.

PROPOSITION 4.3.1. — Assume that $\Gamma = \Gamma(m)$ and that 16 divides m . Then for $0 \leq i < k$ we have

$$\dim \text{Hom}_{U(\mathfrak{g})}(U_i(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) = h(\Gamma) \dim \pi_i(\lambda)$$

and

$$\text{tr}(\varepsilon | \text{Hom}_{U(\mathfrak{g})}(U_i(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma))) = 0.$$

Proof. — Choose a vector $0 \neq v \in U_i(\lambda)$ which is of dominant weight in the minimal K-type π of $U_i(\lambda)$, see [V]. Then v generates $U_i(\lambda)$ as $U(\mathfrak{g})$ -module and we have an injection

$$\text{Hom}_{U(\mathfrak{g})}(U_i(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma)) \rightarrow \mathcal{A}_{\text{Eis}}(G/\Gamma)$$

mapping T to $T(v)$. We call the image of this map F . Then

$$T(v) = \sum_{s=1}^{h(\Gamma)} E(\varphi_s, \mu) + \text{res}_{\mu} E(\psi_s, \mu).$$

Here $\mu \in \mathbb{C}$ with $\text{real}(\mu) > 0$ is determined by λ and $\psi_s, \varphi_s \in \text{Hom}_{\mathbb{M}}(\pi_i(\lambda), L^2(\mathbb{M}))$ are considered as elements $\psi_s, \varphi_s \in C^\infty(G/A_s N_s)$ with K-type π prescribed by v , $E(\psi_s, \mu) := E(\psi_s, -\mu)$ in the notation of [H—CH] (IV §5) and $P_s = M_s A_s N_s$ is the Langlands decomposition of the parabolic P_s corresponding to a boundary component. It is understood that the φ_s are such functions that their Eisenstein series is holomorphic

at μ . Let W be the linear subspace of functions φ in $\text{Hom}_{\mathbb{K}}(\pi, \bigoplus_{s=1}^{h(\Gamma)} C^\infty(G/A_s N_s))$ such

that $E(\varphi, \mu)$ is holomorphic at μ in obvious notation and choose a complementary subspace V . Then we write

$$T(v) = E(\varphi, \mu) + \text{res}_{\mu} E(\psi, \mu)$$

and any choice of $\varphi \in W, \psi \in V$ determines a $T \in \text{Hom}_{U(\mathfrak{g})}(U_i(\lambda), \mathcal{A}_{\text{Eis}}(G/\Gamma))$. To prove our first result we have to show that $T(v) = 0$ implies $\varphi = 0 = \psi$.

Assume that $T(v) = 0$. If $E(\varphi, \mu) \neq 0$ then $E(\varphi, \mu) \notin L^2(G/\Gamma)$ and we have always $\text{res}_{\mu} E(\psi, \mu) \in L^2(G/\Gamma)$. Hence we get $0 = E(\varphi, \mu) = \text{res}_{\mu} E(\psi, \mu)$. Since $\text{res}_{\mu} E(\psi, \mu) = 0$ the automorphic form $E(\psi, \mu)$ has no poles in a neighborhood of μ . Hence by definition

$\psi \in W \cap V = \{0\}$. We have $E(\varphi, \mu) = 0$. We write as in the beginning $\varphi = \sum_{s=1}^{h(\Gamma)} \varphi_s$ and

$$E(\varphi, \mu) = \sum_{s=1}^{h(\Gamma)} E(\varphi_s, \mu) \text{ and have to show that all } \varphi_s = 0.$$

We recall [H—CH] that there are maps $c(s, r, \mu) : L^2(M_r) \rightarrow L^2(M_s)$ depending analytically on μ for real $\mu > 0$ such that the constant Fourier coefficient of $E(\varphi_r, \mu)$ with respect to P_s is of the form

$$\delta_{sr} \varphi_r a^\mu + c(s, r, \mu) \varphi_r a^{-\mu}$$

where for $g = k_s a_s n_s$ (decomposition with respect to $P_s = M_s \hat{A}_s N_s$, $k_s \in K$) we have $a(g)^\mu = a_s^\mu$ where A_s is identified with \mathbb{R}_+^* , the positive real numbers.

Now $\sum_s E(\varphi_s, \mu) = 0$ if and only if all its constant Fourier coefficients

$$\sum_{s=1}^{h(\Gamma)} \delta_{rs} \varphi_s a^\mu + c(r, s, \mu) \varphi_s a^{-\mu} = 0, \quad [H - Ch](I, \S 5).$$

Writing $a^\mu = \exp t\mu$, $t \in \mathbb{R}$, and applying the differential operator $(\partial/\partial t) + \mu$ we get

$$2\mu \delta_{rs} \varphi_s a^\mu = 0.$$

Since $\mu \neq 0$ we have $\varphi_r = 0$ and our first claim holds.

We have $C_\varepsilon(v) = \pm v$. The induced ε -action on $\sum_s E(\varphi_s, \mu) =: E(\varphi, \mu)$ is given by ${}^\varepsilon E(\varphi, \mu) = \pm E({}^\varepsilon \varphi, \mu)$. But since $16 \nmid m$ the P_s are permuted without fixpoints, see 4.2, and the second claim holds.

Q.E.D.

If $G = \mathrm{SO}(2k+1, 1)(\mathbb{R})^0$ then $A_k(\lambda) = U_k(\lambda) = I(\pi_k(\lambda), \nu_k(\lambda))$ is an irreducible unitarily induced representation with parameter real $(\nu_k(\lambda)) = 0$. The proof given for 4.3.1 does not work in this situation and we have to present another argument.

PROPOSITION 4.3.2. — Assume $G = \mathrm{SO}(2k+1, 1)(\mathbb{R})^0$ and $\Gamma = \Gamma(m)$ where $16 \nmid m$. Then

$$\dim \mathrm{Hom}_{U(\mathfrak{g})}(U_k(\lambda), \mathcal{A}_{\mathrm{Eis}}(G/\Gamma)) = h(\Gamma) \dim(\pi_k(\lambda))$$

and

$$\mathrm{tr}(\varepsilon | \mathrm{Hom}_{U(\mathfrak{g})}(U_k(\lambda), \mathcal{A}_{\mathrm{Eis}}(G/\Gamma))) = 0.$$

Proof. — We have $H^k(\mathfrak{g}, K, U_k(\lambda) \otimes V_\lambda) \cong \mathbb{C}$ and an ε -equivariant injection

$$H^k(\mathfrak{g}, K, U_k(\lambda) \otimes V_\lambda) \otimes \mathrm{Hom}_{U(\mathfrak{g})}(U_k(\lambda), \mathcal{A}_{\mathrm{Eis}}(G/\Gamma)) \rightarrow H^k(\Gamma, V_\lambda)$$

whose image we denote by F . The elements of F can be viewed as cohomology classes given by regular Eisenstein series at the parameter zero. Hence [H2] (§4) the restriction map to the boundary $\partial(\bar{X}/\Gamma)$ of the Borel-Serre compactification \bar{X}/Γ of X/Γ identifies F with $\mathrm{im}(r^k: H^k(\Gamma, V_\lambda) \rightarrow H^k(\partial(\bar{X}/\Gamma), V_\lambda))$. Since $\lambda \circ \varepsilon = \lambda$ the representation V_λ is isomorphic to its contragredient representation. Hence we have a nondegenerate intersection pairing $\langle \cdot, \cdot \rangle$ on $H^k(\partial(\bar{X}/\Gamma), V_\lambda) =: E$ and using cohomology with coefficients we see as in [Se2] (Lemma 11) that F is its own orthogonal F^\perp in E with respect to

$\langle \cdot, \cdot \rangle$. Hence in particular $\dim F = (1/2) \dim E$. We have $\partial(\bar{X}/\Gamma) = \bigcup_{i=1}^{h(\Gamma)} V_i$ as a disjoint

sum, where $V_i \cong N_i/N_i \cap \Gamma$ and $P_i = M_i A_i N_i$ is the Langlands decomposition of the parabolic P_i representing the boundary component V_i and $H^k(V_i, V_\lambda) = H^k(\mathfrak{n}_i, V_\lambda)$, where \mathfrak{n}_i is the Lie-algebra of N_i . The involution ε acts on $\partial(\bar{X}/\Gamma)$ orientation preserving and

leaves $\langle \cdot, \cdot \rangle$ invariant. Hence by [Se 2] (Lemma 12) F^ε is its own orthogonal in E^ε with respect to $\langle \cdot, \cdot \rangle$. Now $\text{tr}(\varepsilon|E) = 0$ since ε permutes the V_i 's without fixed points, see 4.3. Hence $\dim F^\varepsilon = (1/2) \dim E^\varepsilon = (1/4) \dim E$ and the second claim holds. Using Kostant's formula for the cohomology $H^k(\mathfrak{n}_i, V_\lambda)$, [K], we easily get the first result we claimed.

Q.E.D.

V. Lefschetz numbers and multiplicities of representations

We now state consequences of the results obtained in the last two chapters. We use the formulas given in 2.4 and the notation established in the first two chapters without further comment.

5.1. PROPOSITION. — *Suppose that $G = \text{SO}(2k, 1)(\mathbb{R})^0$ and that $\Gamma = \Gamma(m)$, where 16 divides m . Denote by ε the standard Cartan involution and by V_λ an irreducible representation of G . Then*

$$\begin{aligned} \text{(i)} \quad \chi(\Gamma, V_\lambda) &= \chi(\Gamma) \cdot \dim(V_\lambda) \\ &= \sum_{i=i(\lambda)}^{k-1} (-1)^i 2 \dim \text{Hom}_{U(\mathfrak{g})}(A_i(\lambda), \mathcal{A}_{\text{dis}}(G/\Gamma)) \\ &\quad + (-1)^k \dim \text{Hom}_{U(\mathfrak{g})}(A_k(\lambda), \mathcal{A}_{\text{dis}}(G/\Gamma)) + h(\Gamma) \sum_{i=0}^{k-1} (-1)^{2k-i-1} \dim \pi_i(\lambda) \\ \text{(ii)} \quad L(\varepsilon, \Gamma, V_\lambda) &:= L(\varepsilon, \Gamma, \mathbb{C}) \text{tr}(\eta|V_\lambda) \\ &= \sum_{i=i(\lambda)}^{k-1} (-1)^i 2 \text{tr}(\varepsilon| \text{Hom}_{U(\mathfrak{g})}(A_i(\lambda), \mathcal{A}_{\text{dis}}(G/\Gamma)) \\ &\quad + (-1)^k \text{tr}(\varepsilon| \text{Hom}_{U(\mathfrak{g})}(A_k(\lambda), \mathcal{A}_{\text{dis}}(G/\Gamma)). \end{aligned}$$

As an easy consequence, see 1.3., we have

COROLLARY 5.2. — *Assume in addition to the above conditions that V_λ is regular. Then $A_k(\lambda)$ occurs only in the space of cuspidal automorphic forms and*

$$\begin{aligned} \text{(i)} \quad \chi(\Gamma, V_\lambda) &= (-1)^k \dim \text{Hom}_{U(\mathfrak{g})}(A_k(\lambda), \mathcal{A}_{\text{cusp}}(G/\Gamma)) + h(\Gamma) \sum_{i=0}^{k-1} (-1)^{2k-i-1} \text{tr} \pi_i(\lambda). \\ \text{(ii)} \quad L(\varepsilon, \Gamma, V_\lambda) &= (-1)^k \text{tr}(\varepsilon| \text{Hom}_{U(\mathfrak{g})}(A_k(\lambda), \mathcal{A}_{\text{cusp}}(G/\Gamma)) \end{aligned}$$

The precise values of $\chi(\Gamma)$ and $L(\Gamma, \varepsilon, \mathbb{C})$ and $\text{tr}(\eta|V_\lambda)$ have been computed in Chapter III. Using the estimate for $h(\Gamma)$ given in 4.1.2 we get:

5.3. COROLLARY. — *Under the assumptions made in 5.2 there exists a constant $C > 0$ such that for all m with $16|m$ we have*

$$\dim \text{Hom}_{U(\mathfrak{g})}(A_k(\lambda), \mathcal{A}_{\text{cusp}}(G/\Gamma(m))) \geq |\chi(\Gamma(m))| \dim V_\lambda - C m^{k(2k-1)+1}.$$

Remarks. — (i) From 3.4.3 we know that $\chi(\Gamma(m))$ grows with respect to m like $m^{k(2k+1)}$. Hence we obtain that the discrete series representation $A_k(\lambda)$ occurs in the space of cuspidal automorphic forms for m big enough.

(ii) Barbasch and Moscovici [B-M] have a method to attack the individual contributions of $A_k^\pm(\lambda)$.

(iii) We did not try to interpret $\sum_{i=0}^{k-1} \dim \pi_i(\lambda) (-1)^{2k-i-1}$. At least for low dimensional examples this contribution is non zero.

(iv) It is possible to interpret $(-1)^k \dim \text{Hom}_{U(\mathfrak{g})}(A_k(\lambda), \mathcal{A}_{\text{cusp}}(G/\Gamma))$ as the Euler characteristic of Γ in L^2 -cohomology, [S 2].

Next we state our results for $G = \text{SO}(2k+1, 1)(\mathbb{R})^0$. We emphasize that there are no discrete series representations.

5.4. PROPOSITION. — *Suppose that $G = \text{SO}(2k+1, 1)(\mathbb{R})^0$ and that $\Gamma = \Gamma(m)$, where 16 divides m . Denote by ε the standard Cartan involution and by V_λ an irreducible representation of G .*

$$(i) \quad \chi(\Gamma(m), V_\lambda) = 0 = h(\Gamma) \sum_{i=0}^k (-1)^{2k-i-1} \dim \pi_i(\lambda).$$

(ii) *If ε acts on V_λ then*

$$L(\varepsilon, \Gamma, V_\lambda) = L(\varepsilon, \Gamma, \mathbb{C}) \text{tr}(\eta | V_\lambda) = \sum_{i=i(\lambda)}^k (-1)^i 2 \text{tr}(\varepsilon | \text{Hom}_{U(\mathfrak{g})}(A_i(\lambda), \mathcal{A}_{\text{dis}}(G/\Gamma))).$$

5.5. COROLLARY. — *Assume in addition that V_λ is regular. Then $A_k(\lambda)$ only occurs in the space of cuspidal automorphic forms and*

$$L(\varepsilon, \Gamma, V_\lambda) = (-1)^k 2 \text{tr}(\varepsilon | \text{Hom}_{U(\mathfrak{g})}(A_k(\lambda), \mathcal{A}_{\text{cusp}}(G/\Gamma))).$$

Remark. — Using the computation of $L(\varepsilon, \Gamma, V_\lambda)$ made in 3.4.2 we see that the multiplicity of $A_k(\lambda)$ in $\mathcal{A}_{\text{cusp}}(G/\Gamma)$ grows with respect to m at least as $m^{n(n-1)/2}$, $n = 2k+1$.

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