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CURVES ON GENERIC HYPERSURFACES

BY HERBERT CLEMENS

1. Introduction

Let

$$V \subseteqq \mathbf{P}^n$$

be a smooth hypersurface of degree $m \geqq 2$ in projective n-space over an algebraically closed field k. By an *immersed curve* on V, we will mean a morphism

$$f: \quad C \rightarrow V$$

which is everywhere of maximal rank from a complete non-singular algebraic curve C. Every such mapping has a normal bundle

$$N_{f,\, V} = f^*(T_V)/T_C.$$

Our purpose in this paper is to prove:

1.1. THEOREM. — *Let* V *be a* generic *hypersurface of degree m in* $\mathbf{P}^n$. *Then* V *does not admit an irreducible family of immersed curves of genus g which cover a variety of codimension* $<$D *where*

$$D = \frac{2-2g}{\deg f} + m - (n+1).$$

Notice that, for example, if $g=0$, Theorem 1.1 says that there are no rational curves on generic V, if $m \geqq 2n-1$.

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2. Normal bundles to curves

Let C be a complete non-singular curve and

$$\varphi: \quad E \to C$$

a vector bundle of finite rank. We will call E *semi-positive* if all quotient bundles of E have non-negative degree.

2.1. LEMMA. — *Let*

$$E_\xi \to C$$

be an algebraic family of vector bundles of rank r over C. *If*

$$E_0 \to C$$

is semi-positive, then $E_\xi \to C$ *is also semi-positive for each generic* ξ *which specializes to* 0.

Proof. — If the lemma is false, there exists a generic point ξ' and a quotient bundle

$$E_{\xi'} \twoheadrightarrow Q_{\xi'}$$

such that

$$0 < s = \text{rank } Q_{\xi'} < r$$

and

$$\deg Q_{\xi'} < 0.$$

Let L be a fixed line bundle on C such that $L \otimes E_\xi$ is generated by global sections for all ξ. So we have a bundle epimorphism

$$C \times k^N \to L \otimes E_\xi,$$

so that $L \otimes E_\xi$ is induced by a map to a Grassmann variety

$$\varphi_\xi: \quad C \to Gr(N-r, N)$$

of a degree equal to

$$\deg E_\xi + r(\deg L).$$

Also $L \otimes Q_{\xi'}$ is induced by a map

$$\psi_{\xi'}: \quad C \to Gr(N-s, N)$$

of degree equal to

$$\deg Q_{\xi'} + s(\deg L). \tag{2.2}$$

Now $\psi_{\xi'}$ specializes to a map

$$\psi_0: \quad C \to \mathrm{Gr}(N-s, N)$$

of degree $\leqq(2.2)$ and so gives a quotient bundle of $L \otimes E_0$ of degree $\leqq(2.2)$. Thus E_0 must have a quotient bundle of negative degree.

2.3. LEMMA. — *If the global sections of* $E \to C$ *span the fibre of the bundle at some point* $p \in C$, *then* E *is semi-positive.*

Proof. — The determinant bundle of any quotient bundle of E has a non-trivial section.

2.4. LEMMA. — *Let*

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

be an exact sequence of bundles over C *such that* E_1 *and* E_3 *are semi-positive. Then* E_2 *is also semi-positive.*

Proof. — Let T be a sub-bundle of E_2 of degree greater than deg E_2. Let S be the minimal sub-bundle of E_2 containing T and E_1. Consider the map

$$\eta: \quad T \oplus E_1 \to S.$$

Then there exists a sub-bundle K of $T \oplus E_1$ such that, for almost all $p \in C$, the mapping η gives an injection

$$((T \oplus E_1)/K)_p \to S_p.$$

Since K is a sub-bundle of E_1, deg $L \leqq$ deg E_1, so that

$$\deg((T \oplus E_1)/K) \geqq \deg T.$$

Therefore deg $S \geqq$ deg T. Thus $\deg(E_2/S) < 0$ contradicting the semi-positivity of E_3.

Let V be a smooth hypersurface of degree m in $\mathbf{P}^n$ and let

$$f: \quad C \to V$$

be an immersion of degree d. Let W be a *generically chosen* hypersurface of degree m in $\mathbf{P}^{n+m}$ such that

$$\mathbf{P}^n . W = V.$$

We wish to prove the following:

2.5. LEMMA. — *The normal bundle* $N_{f, W}$ *to the mapping*

$$f: \quad C \to V \subseteqq W$$

is semi-positive.

Proof. — Since we assume throughout that $m \geqq 2$, we can specialize W to a hypersurface X of degree m in $\mathbf{P}^{n+m}$ which contains $\mathbf{P}^n$ and is non-singular at points of $f(C)$. By

Lemma 2.1, it will suffice to prove the assertion of the lemma for

$$f: \quad C \to W$$

where W is generic such that it contains the P^n. From the sequence of normal bundles

$$0 \to N_{f,\, P^n} \to N_{f,\, W} \to f^* N_{P^n,\, W} \to 0$$

and the fact that $N_{f,\, P^n}$ is semi-positive by Lemma 2.3, we need only find some W such that $f^* N_{P^n,\, W}$ is semi-positive. (Use Lemma 2.1 and Lemma 2.4 to see that this is enough.) To this end, consider the sequence

$$(2.6) \qquad 0 \to f^* N_{P^n,\, W} \to f^* N_{P^n,\, P^{n+m}} \xrightarrow{\lambda} f^* N_{W,\, P^{n+m}} \to 0.$$

If we can find some special W for which

$$f^* N_{P^n,\, W} \cong \mathcal{O}_C^{\oplus (m-1)},$$

the proof of Lemma 2.5 will be complete. We do this by direct computation. Suppose $f(C)$ does not intersect the linear space of codimension 2 given by

$$x_0 = x_1 = 0$$

in P^n. *Then let W be the hypersurface given by*

$$x_{n+1} x_0^{m-1} + x_{n+2} x_0^{m-2} x_1 + \ldots + x_{n+m} x_1^{m-1} = 0.$$

In this case, we rewrite the map λ in (2.6) as

$$f^* \mathcal{O}_{P^n}(1)^{\oplus m} \to f^* \mathcal{O}_{P^n}(m)$$

$$(\alpha_j) \to \sum_{j=1}^{m-1} \alpha_j x_0^{m-1-j} x_1^j.$$

It is immediate to see that the kernel of this mapping is generated by

$$(x_1, -x_0, 0, \ldots, 0)$$
$$(0, x_1, -x_0, 0, \ldots, 0)$$

etc.

Since x_0 and x_1 do not vanish simultaneously on $f(C)$

$$f^* N_{P^n,\, W} \cong \mathcal{O}_C^{\oplus (m-1)}.$$

3. Proof of the main theorem

In this final section, we will prove Theorem 1.1. We let V be a generic hypersurface of degree m in P^n and we suppose that there is an irreducible algebraic family g of immersed curves of genus g on V which covers a quasi-projective variety of codimension D in V. For f generic in F, and

$$Y \subseteq P^{n+s}$$

a smooth hypersurface with $Y \cdot P^n = V$, let

$$R \subseteq H^0(N_{f,Y})$$

be any subspace. We denote, for each $p \in C$, the image of the evaluation map

$$R \to (\text{fibre of } N_{f,Y} \text{ at } p)$$
$$\rho \mapsto \rho(p)$$

by R_p. Then there is a unique sub-bundle

$$S \subseteq N_{f,Y}$$

such that $R \subseteq H^0(S)$ and, for almost all $p \in C$, the fibre of S is exactly R_p. Next consider the diagram

$$\begin{array}{c} R \subseteq H^0(N_{f,Y}) \\ \downarrow \nu \\ H^0(N_{V,Y}) \xrightarrow{\mu} H^0(f^* N_{V,Y}). \end{array}$$

Assume now that

(3.1) $$\nu(R) = \mu(H^0(N_{V,Y})).$$

Then the sections of R must generate the fibres of $f^* N_{V,Y}$ at each point. So

$$T = S \cap N_{f,V}$$

is a well-defined sub-bundle of $N_{f,V}$. In fact, we claim that under the above assumptions the sequence

(3.2) $$0 \to N_{f,V}/T \to N_{f,Y}/T \to f^* N_{V,Y} \to 0$$

must be split. To see this, notice that the mapping

$$f^* N_{V,Y} \cong S/T \to N_{f,Y}/T$$

splits the sequence.

Continuing with the same assumptions, we wish to show that

$$L \otimes T$$

is semi-positive, where L, as above, is line bundle

$$f^* \mathcal{O}_{P^n}(1).$$

To see this, let $p \in C$ be a point such that the sections in the vector space R given above generate the fibre of S at p. Let

$$t_p \in (\text{fibre of T at } p).$$

By Lemma 2.3, to prove the semi-positivity of $L \otimes T$, it suffices to find a meromorphic section τ of T such that:

(i) $\tau(p) = t_p$,

(ii) the polar locus of τ is either 0 or is a hyperplane section of $f(C)$.

To accomplish this, choose a section of $\rho \in R$ such that

$$\rho(p) = t_p.$$

If $\rho \in H^0(N_{f,V})$, set $\tau = \rho$. If $\rho \notin H^0(N_{f,V})$ then by (3.1), ρ determines a non-trivial section of $f^* N_{V,Y}$ which is the restriction of a section $\bar{\rho}$ of $N_{V,Y}$. Now let

$$N_{V,Y} \to \mathcal{O}_V(1)$$

be a projection such that $\bar{\rho}$ maps to a non-trivial section of $\mathcal{O}_V(1)$.

Choose a base-point free pencil on $f^* H^0(\mathcal{O}_V(1))$ which comes from a two-dimensional subspace

$$R_0 \subseteq R$$

such that $\rho \in R_0$. Let R_1 be an affine line in R_0 which passes through ρ but does not contain the origin of R_0. We define our section τ of T by the rule

$$\tau(q) = \rho'(q)$$

where ρ' is the unique section in R_1 whose image in $H^0(f^* \mathcal{O}_V(1))$ vanishes at q.

We are now ready to complete the proof of Theorem 1.1. Since V is generic, we can find an irreducible family $\mathcal{F}$ of curves of genus g in

$$W \subseteq \mathbf{P}^{n+m}$$

such that:

(i) if $f \in \mathcal{F}$, then (image f) spans a linear space of dimension $\leq n$;

(ii) for generically chosen $f \in \mathcal{F}$, the tangent space to $\mathcal{F}$ at f maps isomorphically to a subspace

$$R \subseteq H^0(N_{f,W})$$

satisfying (3.1) for $Y = W$,

(iii) $$f \in \mathfrak{g} \subseteq \mathcal{F},$$

where $\mathfrak{g}$ is the family of curves on V postulated at the beginning of paragraph 3.

(We simply use the deformations of f into curves on $K \cdot W$ where K is a linear space of dimension n in $\mathbf{P}^{n+m}$.)

So we are in the situation considered earlier in paragraph 3. Thus we have associated to R the sub-bundles

$$S \subseteqq N_{f,\,W}$$

and

$$T = S \cap N_{f,\,V}$$

giving a split sequence

$$0 \to N_{f,\,V}/T \to N_{f,\,W}/T \to L^{\oplus m} \to 0 \tag{3.3}$$

Also $L \otimes T$ is semi-positive.

By Lemma 2.5, $N_{f,\,W}$ is semi-positive, and so therefore is

$$N_{f,\,V}/T$$

since it is a quotient of $N_{f,\,W}$. In particular

$$\deg N_{f,\,V}/T \geqq 0.$$

On the other hand there is a unique sub-bundle

$$T_V \subseteqq T$$

such that the sections of the tangent space to $\mathfrak{g}$ at f, considered as a subspace of $H^0(N_{f,\,V})$, lie in T_V and generate almost all fibres of T_V. Referring to the first part of paragraph 3,

$$\text{rank } T_V = (n-2) - D$$

so that

$$\text{rank}(T/T_V) \leqq D.$$

Now by the adjunction formula

$$\deg N_{f,\,V} = (n+1-m)(\deg L) - (2-2g).$$

On the other hand

$$\deg N_{f,\,V} = \deg(T/T_V) + \deg T_V + \deg(N_{f,\,V}/T)$$

$$\geqq \deg(T/T_V).$$

Since $L \otimes T$ is semi-positive

$$\deg(L \otimes T/L \otimes T_V) \geqq 0$$

so

$$\deg(T/T_V) \geqq -rk(T/T_V)(\deg L).$$

Putting everything together

$$(n+1-m)(\deg L)-(2-2g) \geqq -(rk\, T/T_V)(\deg L).$$

Let

$$\alpha = \frac{2-2g}{\deg L}$$

Then

$$rk\,(T/T_V) \geqq \alpha + m - (n+1)$$

so that

$$D \geqq \alpha + m - (n+1).$$

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