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The zeta functions of Ruelle and Selberg. I

THE ZETA FUNCTIONS OF RUELLE AND SELBERG   I

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Selberg associated a zeta function to a Fuchsian group Γ with remarkable properties that link the conjugacy classes of Γ to the spectrum of the Laplacian on Γ-invariant functions on the upper halfplane H [S]. As noted by Smale, this zeta function can be interpreted dynamically as a product over the closed orbits of the geodesic flow on $\Gamma\backslash H$, and then generalized to other Axiom A flows [Sm]. Ruelle noted the analogies of such zeta functions with the partition function of a one-dimensional lattice gas and gave further generalizations [R2]. Under strong analyticity assumptions Ruelle used methods from statistical mechanics to show his zeta function has a meromorphic continuation to C, generalizing certain properties of Selberg's zeta function [R1].

In this paper we will develop Ruelle's methods to a suitable generality, providing a large class of meromorphic zeta functions of geometric interest. We will then obtain a formula for the smoothed length spectrum of certain flows in terms of the zeroes and poles of a zeta function, generalizing Randol's results for the geodesic flow on $\Gamma\backslash H$. We will also discuss the analogue of Selberg's functional equation for the suspension flow of an Anosov automorphism of an infranilmanifold, shedding some light on the geometric meaning of these functional equations.

In Part II we will specialize to the geodesic flow on a compact locally symmetric space X of rank one. Generalizing the work of Selberg, Gangolli, Millson, Scott and Hejhal ([S], [G], [M], [Sc], [H]) we will determine the zeroes and poles of the Selberg zeta function associated to a locally homogeneous vector bundle on X and we will prove the functional equation (this has been done independently by Wakayama [W]). We also give a new functional equation for the Ruelle function. Further we find the most precise possible asymptotic estimate for the smoothed length spectrum of X.

In [F5] these zeta functions will be applied to give a connection between the geodesic flow and the Ray-Singer analytic torsion of a closed hyperbolic manifold. This answers a question we posed in [F1] to find an analogue of the Lefschetz formula for counting the closed orbits of this flow. For a cocompact Fuchsian group Γ we have proved the

corresponding formula for the Reidemeister torsion of $\Gamma\backslash\text{PSL}(2, \text{R})$ using Selberg zeta functions and their functional equations [F2].

These methods apply to compute the Ray-Singer holomorphic torsion of a closed complex hyperbolic manifold from closed geodesics [F6], generalizing the Riemann surface case of [RS]. A related computation of the eta invariant on a hyperbolic manifold is due to Millson [M], although he did not need the functional equation.

We will make full use of the symbolic dynamics of Axiom A flows. As we know no fully satisfactory reference, in Section 1 we present an account of this, incorporating certain improvements of Bowen's paper [B1] suggested by his book [B2]. We hope this makes this paper reasonably self contained, although we refer to Bowen for certain proofs.

In Section 2 we study an Axiom A basic set Λ and a lifting ψ of the flow to a vector bundle. We define a Selberg zeta function and a Ruelle zeta function in this situation and deduce their elementary properties and relationships.

In Section 3 we impose certain analyticity conditions on Λ and ψ and a polarization condition on ψ. We then prove that the Selberg and Ruelle functions are meromorphic in $\mathbf{C}$.

In Section 4 we estimate the growth at infinity of the zeroes and poles of these functions, correcting a small error in [R1]. We then compute the length spectrum of Λ from this data.

In Section 5 we give examples of geometric interest, including the suspended Anosov automorphisms mentioned earlier. For a mixing Anosov flow in 3 dimensions we show that the Selberg zeta function is regular at noninteger points.

Section 1. — Basic sets and symbolic dynamics.

Let M be a smooth manifold and φ_t a smooth local flow on M. We consider a compact φ_t-invariant set $\Lambda \subset M$ that is isolated, meaning that for some open neighborhood U of Λ, $\bigcap_{t \in \mathbf{R}} \varphi_t U = \Lambda$. We suppose that Λ is hyperbolic, meaning that the restriction of the tangent bundle to Λ splits continuously as a sum of an unstable, stable, and center bundles $T_\Lambda M = E^u \oplus E^s \oplus E^c$ with these properties:

(1) E^c is one dimensional and spanned by $d\varphi/dt$.

(2) E^u, E^s are invariant under $T\varphi_t$ and the norms $\|T\varphi_t \mid E^s\|$, $\|T\varphi_{-t} \mid E^u\|$ converge exponentially to zero as $t \to \infty$.

Here in (2) we take some Riemannian metric on $T_\Lambda M$ (the choice is irrelevant as Λ is compact) and give the bundle maps the sup norm of the operator norm on the fibers E^s_x, E^u_x. We also suppose the closed orbits of φ are dense in Λ and that Λ is connected. Following Smale's work on Axiom A flows, we call Λ with these properties a *basic set* [Sm]. For instance if X is a negatively curved connected closed manifold then the unit sphere bundle $SX = \Lambda = M$ is a basic set for the geodesic flow [A]. It was, in fact, Anosov's study of these systems that inspired Smale's work on Axiom A. If $\Lambda = M$ we say φ is a *transitive Anosov flow*. If the set U above can be chosen so $\varphi_t U \subset U$ for $t \geq 0$ we say Λ is a *hyperbolic attractor:* if this holds for $t \leq 0$ we call Λ a *hyperbolic repeller.*

Given $x \in \Lambda$ let $W^s(x)$, the *stable manifold* of x, be those points $y \in M$ such that the distance from $\varphi_t x$ to $\varphi_t y$ converges to zero as $t \to \infty$. It is indeed a manifold of dimension $d_s = \dim E^s$, tangent to E_x^s and as smooth as φ. As x varies, $W^s(x)$ varies continuously but *not* smoothly in general, even if φ is real analytic. Near Λ one can choose (topological) coordinate charts so that the sets $W^s(y)$, $y \in \Lambda$, corresponds to unions of parallel subspaces of dimension d_s: one says W^s is a (continuous) *lamination* [HPS]. If M is analytic and if these charts can be chosen analytic we say W^s is an *analytic lamination*. If one replaces φ_t by φ_{-t} (or $t \to \infty$ by $t \to -\infty$) one obtains the *unstable lamination* W^u consisting of the unstable manifolds $W^u(y)$, $y \in \Lambda$, each of dimension $d_u = \dim E^u$. When Λ is a hyperbolic attractor, W^s is a (continuous) foliation of a neighborhood of Λ by smooth leaves. When φ is Anosov, W^s and W^u are called *Anosov foliations*. For the geodesic flow on SX, X a Riemann surface, these are the horocycle foliations by cosets of a nilpotent group [GF].

Let D be a small codimension one disc transverse to φ. We project the laminations W^s, W^u to D along φ, obtaining ("horizontal" and "vertical") laminations $\mathscr{F}^u$, $\mathscr{F}^s$ on D with respective tangent spaces $(E^u \oplus E^c) \cap TD$, $(E^s \oplus E^c) \cap TD$. Thus these are transverse laminations of dimensions d_u, d_s meeting in $\Lambda \cap D$. If $x, y \in D$ then the leaf of $\mathscr{F}^u$ through y meets the leaf of $\mathscr{F}^s$ through x at most once: this point of intersection is denoted $[x, y]$ ("x over, y up"). A compact subset $R \subset D \cap \Lambda$ is called a *rectangle* if $x, y \in R \Rightarrow [x, y] \in R$. Then the 2 equivalence relations induced on R by $\mathscr{F}^u$ and $\mathscr{F}^s$ define quotient spaces ("vertical" and "horizontal") R^V and R^H with a natural homeomorphism $R \to R^H \times R^V$. Let R^0 denote the interior of R in $\Lambda \cap D$: if R^0 is dense in R we say R is *proper*.

We will be interested in a finite family of disjoint D_i, $i \in I$, and proper rectangles $R_i \subset D_i$ that fit together well under the flow φ, in the following sense. For certain distinct $i, j \in I$ there are proper rectangles in R_i, R_j of the sort $X \times R_i^V$, $R_j^H \times Y$ and homeomorphisms $X \to R_j^H$, $R_i^V \to Y$ that determine a correspondence C_{ij} from R_i to R_j (indeed, a homeomorphism of certain subrectangles). This correspondence is induced by φ in the sense that there is a continuous return time function $t_{ij} : C_{ij} \to (0, \infty)$ so that $(a, b) \in C_{ij} \Rightarrow b = \varphi_{t_{ij}}(a)$. Here one pictures a family of trajectories going from $X \times R_i^V$ to $R_j^H \times Y$ and C_{ij} as a slice to this family (*Fig. 1*).

The pairs (i, j) for which C_{ij} is defined give a subset of *transitions* $A \subset I \times I$. We let B_{ij} be the box swept out by C_{ij} minus its ends, $B_{ij} = \{ \varphi_t x \mid x \in X \times R_i^V, 0 < t < t_{ij}(x) \}$. We write $B_{ij}^- = X \times R_i^V \subset R_i$ and $B_{ij}^+ = R_j^H \times Y \subset R_j$ for these two ends.

The family $\mathscr{M} = \bigcup_{i \in I} R_i$ is supposed to meet every flowline of $\varphi_t \mid \Lambda$ in time $\ll 1$. The sets B_{ij}^+, i fixed, are supposed to cover R_i but have disjoint interiors. At an interior point $x \in (B_{ij}^+)^0$, $\varphi_{t_{ij}(x)} x = C_{ij} x$ is to be the first point on the forward trajectory from x that lies in $\mathscr{M}$. When all this holds and the R_i are of sufficiently small size we say the R_i are a *Markov family* of local sections on Λ. Such families exist by a theorem of Bowen's, generalizing many earlier constructions [B1, AW, Si, R]. They give use to a useful description of the flow $\varphi \mid \Lambda$ in terms of symbol sequences, as follows.

To A we associate the space Σ_A of all sequences $i_n \in I$, $n \in Z$, with $(i_n, i_{n+1}) \in A$ for all n. Thus Σ_A is the infinite iteration of A as a correspondence from I to I. (Note
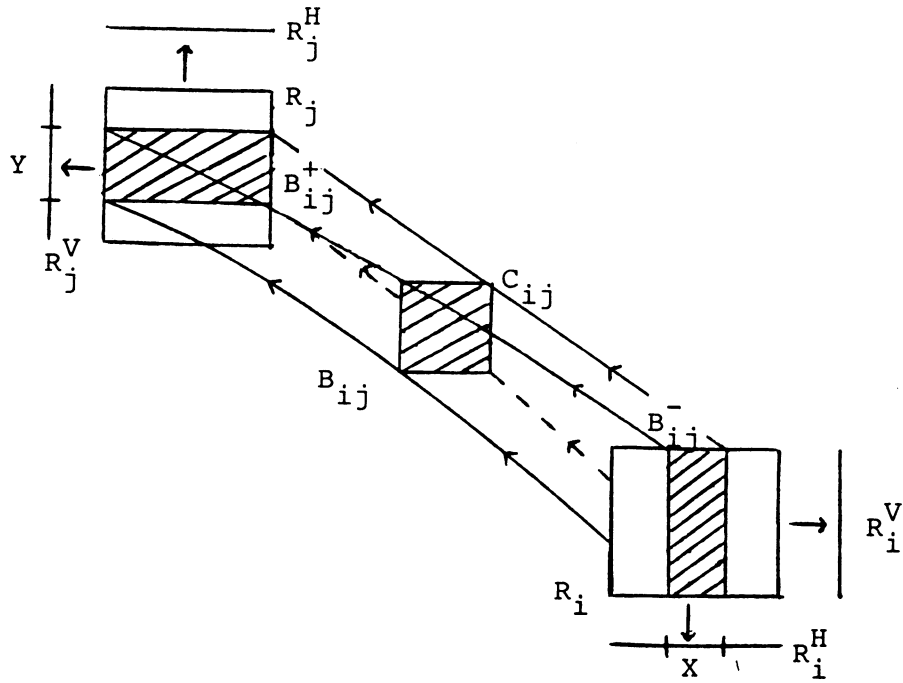
Fig. 1.

that in composing correspondences one "remembers" the intermediate points.)   We can also iterate the correspondence $C = \bigcup_{(i, j) \in A} C_{ij}$ from $\mathcal{M}$ to $\mathcal{M}$ to obtain the sequence space $\Sigma_C \subset \mathcal{M}^{\mathbf{Z}}$.   In the product space topology Σ_A and Σ_C are compact and metrizable.   There is a natural map $p: \Sigma_C \to \Sigma_A$ sending the sequence (c_n), $c_n \in C_{i_n i_{n+1}}$, to the sequence (i_n).   There is also a natural map $q: \Sigma_C \to \mathcal{M}$ mapping a sequence to its 0th term.   The hyperbolicity of φ implies p is 1-1.   The Markov property of $\mathcal{M}$ shows that p is onto.   The composite $\pi = q \circ p^{-1}: \Sigma_A \to \mathcal{M}$ is the discrete version of *symbolic dynamics*: the sequence $(i_n) \in \Sigma_A$ maps to a point $x = \pi(i_n) \in R_{i_0}$ whose φ-trajectory runs through $\ldots R_{i_{-1}}$, R_{i_0}, $R_{i_1} \ldots$ in that order.   (Even though this trajectory may also hit other rectangles, this is enough to specify x).

To better describe φ, we produce a space Γ, a flow f_t on Γ with Σ_A as cross-section, and a surjective map $\pi: \Gamma \to M$ extending $\pi: \Sigma_A \to \mathcal{M}$ that is a semiconjugacy (i. e. $\varphi_t \pi = \pi f_t$ for all $t \in \mathbf{R}$).   To do this, define $t: \Sigma_A \to (0, \infty)$ by the rule $t(i_n) = t_{i_0 i_1} \pi(i_n)$.   Let $\sigma_A: \Sigma_A \to \Sigma_A$ be the shift homeomorphism $\sigma_A(i_n) = (i_{n+1})$: it is called the subshift of finite type determined by A.   Then following the usual suspension flow construction, we let Γ be the compact quotient space of $\Sigma_A \times \mathbf{R}$ by the discontinuous group action generated by the homeomorphism $(\alpha . \tau) \to (\sigma_A \alpha, \tau - t(\alpha))$.   The translation flow $(\alpha, \tau) \to (\alpha, \tau + t)$ induces a flow $f_t: \Gamma \to \Gamma$ called the *suspension flow* of σ_A with *return time* $t(\alpha)$.   One identifies Σ_A with the image of $\Sigma_A \times 0$ in Γ.   The semiconjugacy π is induced from the map $\Sigma_A \times \mathbf{R} \to M$ sending (α, t) to $\varphi_t(\pi\alpha)$ (note this is compatible

with our discontinuous group and so determines a map $\Gamma \to M$). All told π gives description of the trajectories of φ in terms of transitions and return times.

If $x \in M$ satisfies $\varphi_t x = x$ for some $t > 0$ then we say the trajectory γ_0 through x is a *prime periodic orbit* for φ. If one considers a connected finite cover γ of γ_0 with m sheets, one says γ is a *periodic orbit* of *multiplicity* $m = \mu(\gamma)$ and one writes $\gamma = \gamma_0^m$. Given that $\varphi_t x = x$, $t > 0$, the covering space $\mathbf{R}/t\,\mathbf{Z} \to \gamma_0$ induced by φ is such a periodic orbit. We then write $t = l(\gamma)$ for its *length* (or period). These quantities play a vital role in zeta functions.

To use symbolic dynamics to study zeta functions, one must understand the relationship between the periodic orbits of φ_t and those of f_t. If γ is a prime periodic orbit of φ and $\gamma \cap R_i \subset R_i^0$ for all i then there is only one symbol sequence in Σ_A corresponding to a given $x \in \gamma \cap M$. This gives a unique periodic orbit δ for f that lifts γ and π gives a homeomorphism from δ to γ. But if γ meets $R_i - R_i^0$ for some i there may be a finite number of periodic orbits $\delta_1, \ldots, \delta_n$ over γ, where each δ_i covers γ with some multiplicity.

Manning showed how to account for this in the Axiom A diffeomorphism case by using auxiliary subshifts [Ma1], then Bowen improved this calculation by using certain signed transition matrices and, separately, giving the argument for flows ([B1], [B2]). We will put these improvements together here.

For a fixed $i \in I$, we let $B_i = \bigcup_{j \in I} (B_{ij} \cup B_{ij}^-)$. Given a nonnegative integer $m \geqq 0$ we let the mth symbol set I_m be all pairs (i, P) where $i \in I$ and P is an $m+1$ element subset of I containing i such that R_i meets $\cap B_j$, where j runs over $P - \{i\}$. The m-th transition set $A_m \subset I_m \times I_m$ consists of the pairs $((i, P), (j, Q))$ with $Q - P = \{j\}$ and $(j', j) \in A$ where $\{j'\} = P - Q$. Clearly $I_0 = I$, $A_0 = A$. We will see that by using all the subshifts Σ_{A_m} we can count periodic orbits in φ precisely.

Note there is a natural correspondence $P \cong Q$ determined by the above transition in A_m, where j' corresponds to j and $P - \{j'\}$ maps identically to $Q - \{j\}$. Given a sequence $(i_n, P_n) \in \Sigma_{A_m}$, let $j_0 = i_0$ and construct the sequence $\ldots j_{-1}, j_0, j_1, \ldots$ with $j_n \in P_n$ corresponding to $j_{n+1} \in P_{n+1}$ in this way. Dropping any repeated indices gives a collapsed sequence $(\ldots k_{-1}, k_0, k_1, \ldots) \in \Sigma_A$ with $k_0 = i_0$. This defines a continuous map $\gamma_m : \Sigma_{A_m} \to \Sigma_A$. We let $\pi_m : \Sigma_{A_m} \to M$ by $\pi_m = \pi \circ \gamma_m$. Let $t_m : \Sigma_{A_m} \to (0, \infty)$ give the time it takes $\pi_m(i_n, P_n)$ to go from R_{i_0} to R_{i_1}. Let $f_{mt} : \Gamma_m \to \Gamma_m$ be the suspension flow of the shift σ_{A_m} with return time t_m. We extend π_m to $\pi_m : \Gamma_m \to M$ by $\pi_m f_{mt} = \varphi_t \pi_m x$, $x \in \Sigma_{A_m}$. This is well defined and gives a semiconjugacy of flows from f_m to φ.

Now we order I in a fixed way. This gives an order preserving correspondence of each $m+1$ element subset of I with $\{0, \ldots, m\}$. If $((i, P), (j, Q)) \in A_m$ we obtain a permutation of $\{0, \ldots, m\}$ from our correspondence of P and Q. Taking its sign gives $\varepsilon_m : A_m \to \{\pm 1\}$. Clearly $\varepsilon_0 \equiv 1$.

If $x \in \Sigma_{A_m}$ and $\sigma_{A_m}^p x = x$, $p > 0$, we let $\delta = \delta(x, p)$ be the closed orbit of f_m corresponding to the cycle $x, \ldots, \sigma_m^{p-1} x$ and set $\varepsilon(\delta) = \prod_{j=0}^{p-1} \varepsilon_m(x_j, x_{j+1})$. Thus $\varepsilon(\delta)$ is the sign of the permutation of P_0 induced by our correspondence $P_0 \to P_1 \to \ldots \to P_p = P_0$, where $x_j = (i_j, P_j)$. We now give the key combinatorial fact that makes the Manning-Bowen counting argument work.

PROPOSITION 1. — *For γ a closed orbit of φ and $m \geq 0$ let δ run over the closed orbits of f_m with length $l(\delta) = l(\gamma)$ that cover γ under π_m. Then*

$$\frac{1}{\mu(\gamma)} = \sum_m (-1)^m \sum_\delta \frac{\varepsilon(\delta)}{\mu(\delta)}.$$

Proof. — Let γ_0 be the prime closed orbit underlying γ. Say $R_{i_0}, R_{i_1}, \ldots, R_{i_n} = R_{i_0}$ are the successive rectangles γ_0 meets at points $x_0, x_1, \ldots, x_n = x_0$ respectively. Set $Q_j = \{i \mid x_j \in B_i\}$.

Bowen showed there is a bijection $\theta_j \colon Q_j \to Q_{j+1}$ that fixes $Q_j \cap Q_{j+1} = Q_{j+1} - \{R_{i_j}\}$. Let $\theta = \theta_{n-1} \circ \ldots \circ \theta_0$ be the permutation of Q_0 obtained by tracing γ_0.

Now in any $y = (i_n, P_n) \in \Sigma_{A_m}$ with $\pi_m y = x_0$, the sequence P_n determines the sequence i_n as $P_n - P_{n+1} = \{i_n\}$. Also $P_0 \subset Q_0$. Moreover P_0 determines the P_n, since $P_1 = \theta_0 P_0$, etc. Conversely any $m + 1$ element subset $P_0 \subset Q_0$ generates a sequence P_n in this way and in turn a sequence $(i_n, P_n) \in \Sigma_{A_m}$. Thus $\pi_m^{-1} y_0$ corresponds naturally to the set of $m + 1$ element subsets of Q_0 [B1].

Thus a prime closed orbit δ of f_m over γ_0 determines an $m + 1$ element subset $P_\delta \subset Q_0$. If δ covers γ_0 with multiplicity $o(\delta)$ then $\theta^{o(\delta)}$ preserves P_δ and permutes it with sign $\varepsilon(\delta)$. Moreover $\theta^k P_\delta \neq P_\delta$ if $0 < k < o(\delta)$. This is a bijective correspondence pairing δ and an orbit of $m + 1$ element subsets of Q_0 under θ.

We now geometrize this setup. Let $\Delta \subset \mathbf{R}^{P_0}$ be the standard simplex with vertices P_0. θ induces an affine map $\beta \colon L \to L$ of the affine space L of points with coordinate sum 1. We can choose a Morse function h on L with critical points at the barycenters of the faces of Δ so that the barycenter of an m-dimensional face has Morse index m. This can be done so h is invariant under coordinate permutations. Let $\alpha \colon L \to L$ be the composition of B with the time one flow of the vector field -grad h. Then α is a Morse-Smale diffeomorphism with periodic points at the barycenters and a "source at ∞". We choose a smooth regular neighborhood N of Δ that's preserved by α.

Now we apply the Lefschetz formula to the iterates of $\alpha \mid N$. Each δ gives a periodic orbit of period $o(\delta)$ and Lefschetz index $(-1)^m$ whose unstable orientation is preserved by $\alpha^{o(\delta)}$ if $\varepsilon(\delta) = +1$, otherwise reversed. Thus the Lefschetz zeta function of α is

$$\zeta(x) = \prod_{m \geq 0} (1 - \varepsilon(\delta) x^{o(\delta)})^{(-1)^{m+1}}.$$

But clearly α is homotopic to a constant, so $\zeta(x) = (1-x)^{-1}$. Taking logarithms of these 2 expressions for $\zeta(x)$ gives

$$\sum_{n > 0} \frac{x^n}{n} = \sum_{m \geq 0} (-1)^m \sum_\delta \sum_{k > 0} \frac{1}{k} \varepsilon(\delta)^k x^{o(\delta) k}.$$

Now we take the coefficient of $x^{\mu(\gamma)}$ on both sides. δ^k covers $\gamma \Leftrightarrow o(\delta)k = \mu(\gamma)$ and in this case $\mu(\delta^k) = k$, $\varepsilon(\delta^k) = \varepsilon(\delta)^k$.

<div align="right">Q.E.D.</div>

The idea that the counting argument here is related to the Lefschetz formula of a simplicial map of a simplex is due to Bowen [B2] but we believe this use of Morse-Smale diffeomorphisms and zeta functions is new (the zeta function can be avoided by applying the Lefschetz formula to $\alpha^{\mu(\gamma)}$ directly).

SECTION 2. — Ruelle and Selberg functions for basic sets.

Let φ_t be a flow on a compact space X and E a vector bundle over X. If $\psi_t: E \to E$ is a flow that covers φ_t and is linear on fibers we call ψ a lift of φ. We define the spectral radius $r(\psi)$ by giving E a Riemannian metric (the choice of metric doesn't matter as X is compact) and letting $r(\psi) = \lim_{t \to \infty} \|\psi_t\|^{1/t} < \infty$. Here $\|\psi_t\|$ is the supremum of the operator norms of the maps on fibers $\psi_t \mid E_x: E_x \to E_y$, $y = \varphi_t x$.

To a closed orbit γ through a point $x \in X$ we associate the linear map $\psi_{l(\gamma)}: E_x \to E_x$. Varying x only changes this by a similarity, giving a similarity class ψ_γ that we call the *monodromy* of ψ around γ. Clearly the spectral radii satisfy $r(\psi_\gamma) \leqq r(\psi)^{l(\gamma)}$. An important special case is when Λ is a basic set as in Section 1, $\varphi = \varphi \mid \Lambda$, $E = E^s$ is the stable bundle of Λ and $\psi_t = T\varphi_t \mid E^s$. Here $r(\psi) \in (0, 1)$ and we write S_γ for ψ_γ, which we call the *stable monodromy* around γ.

Suppose the closed orbits γ of φ are such that the number $N_t(\varphi)$ of closed orbits of length $\leqq t$ grows no worse than exponentially. Then we let $\rho = \overline{\lim} t^{-1} \log N_t(\varphi) < \infty$ and we have

LEMMA 1. — *For $r(\psi) < e^{-\rho}$ the series*

$$\sum_\gamma \frac{1}{\mu(\gamma)} \operatorname{Tr} \psi_\gamma$$

converges absolutely, and uniformly in $r(\psi)$.

Proof. — Majorize by $\Sigma_\gamma \exp(r(\psi) + \varepsilon) l(\gamma)$ and take $\varepsilon < e^{-\rho} - r(\psi)$.

<div align="right">Q.E.D.</div>

Given any lift ψ, there is a complex family of lifts ψ^z, $z \in \mathbf{C}$, with $\psi^0 = \psi$ defined by scalar multiplication, $\psi_t^z = e^{-tz} \psi_t$. Thus $r(\psi^z) = r(\psi) e^{-z}$ and we can define the *Ruelle* (zeta) *function* of ψ for $\operatorname{Re} z > \rho + \log r(\psi)$ to be

$$R_\psi(z) = \exp \sum_\gamma \frac{-1}{\mu(\gamma)} \operatorname{Tr} \psi_\gamma^z.$$

Clearly R_ψ is holomorphic on this halfplane by uniform convergence on compacts.

When φ is the flow on a basic set Λ we can incorporate the bundle E^s into our zeta function. Note here that $\rho = h(\varphi)$ is the topological entropy of φ [R2].

LEMMA 2. — *For $r(\psi) < e^{-\rho}$ the series*

$$\sum_\gamma \frac{1}{\mu(\gamma)} \frac{\operatorname{Tr}\psi_\gamma}{\operatorname{Det} I - S_\gamma}$$

converges absolutely, and uniformly in $r(\psi)$.

The same majorization works since $\operatorname{Det} I - S_\gamma \to 1$.

We let the *Selberg* (zeta) *function* of ψ be, for $\operatorname{Re} z > h(\varphi) + \log r(\psi)$,

$$S_\psi(z) = \exp\left[\sum_\gamma \frac{-1}{\mu(\gamma)} \frac{\operatorname{Tr}\psi_\gamma^z}{\operatorname{Det} I - S_\gamma}\right].$$

When E is the trivial bundle $\Lambda \times \mathbf{R}$ and ψ the trivial lift that fixes the $\mathbf{R}$ factor we write $S_\varphi(z)$ and $R_\varphi(z)$ and $R_\psi(z)$.

For example, let $\Lambda = \Gamma\backslash\operatorname{PSL}(2, \mathbf{R})$ where Γ is a discrete cocompact subgroup. Let φ_t be the flow on Λ induced by right multiplication by the one parameter group $\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$. Then φ_t is a transitive Anosov flow, essentially the geodesic flow on the orbifold $\Gamma\backslash H$, H the upper halfplane. As the stable contraction is uniform exponential, we have $h(\varphi) = 1$ and $S_\gamma = e^{-l(\gamma)}$. Thus

$$S_\varphi(z) = \exp\sum_\gamma \frac{-1}{\mu(\gamma)} \frac{\exp - z l(\gamma)}{1 - \exp - l(\gamma)}.$$

Now fix a prime orbit γ_0 with $l(\gamma_0) = l$, say, and note that its iterates contribute to the sum an amount

$$\sum_{m > 0} -\frac{1}{m} e^{-zlm} \sum_{k \geq 0} e^{-klm} = \sum_{k=0}^\infty \log 1 - e^{-(z+k)l}.$$

This gives

$$S_\varphi(z) = \prod_{k=0}^\infty \prod_{\substack{\text{prime} \\ \gamma}} 1 - e^{-(z+k)l(\gamma)}$$

which is just the zeta function $Z_\Gamma(z)$ of Selberg's paper [S].

Note that in terms of Ruelle functions we have in this case the relationships

$$S_\varphi(z) = R_\varphi(z) R_\varphi(z+1) R_\varphi(z+2)\ldots$$

$$R_\varphi(z) = \frac{S_\varphi(z)}{S_\varphi(z+1)}.$$

We will now generalize these formulas to arbitrary basic sets Λ and lifts ψ.

Let us write $\Lambda^j E^s$ for the jth exterior power of the stable bundle E^s and λ^j for the natural lift $\Lambda^j (T \varphi_t \mid E^s)$. We tensor E with $\Lambda^j E^s$ and form the lift $\psi \otimes \lambda^j$. Using the matrix identity $\text{Det} \, I - B = \Sigma (-1)^j \text{Tr} \, \Lambda^j B$ with $B = S_\gamma$ we have the finite factorizations

PROPOSITION 2

$$R_\psi (z) = \prod_{j \geq 0} S_{\psi \otimes \lambda^j} (z)^{(-1)^j}$$

$$R_\varphi (z) = \prod_{j \geq 0} S_{\lambda^j} (z)^{(-1)^j}.$$

Next let $s^j E^s$ denote the jth symmetric power of E^s and s^j the jth symmetric power of $T \varphi_t \mid E^s$. On $E \otimes s^j E^s$ we have the lift $\psi \otimes s^j$. Taking the matrix identity $\det (I - B)^{-1} = \sum_{j \geq 0} \text{Tr} \, s^j B$, valid for $r(B) < 1$, with $B = S_\gamma$ gives the infinite factorizations

PROPOSITION 3

$$S_\psi (z) = \prod_{j \geq 0} R_{\psi \otimes s^j} (z)$$

$$S_\varphi (z) = \prod_{j \geq 0} R_{s^j} (z).$$

These propositions generalize the formulas we gave for Fuchsian groups.

There are product representations for the Ruelle functions.

PROPOSITION 4

$$R_\psi (z) = \prod_\gamma \text{Det} (I - e^{-zl(\gamma)} \psi_\gamma)$$

$$R_\varphi (z) = \prod_\gamma 1 - e^{-zl(\gamma)}$$

where γ runs over the prime closed orbits of φ.

This follows easily by writing $\text{Det} \, I - B = \exp \sum_{n > 0} \frac{-1}{n} Tr \, B^n$ where $B = \psi_\gamma^z$. It implies the product formula for S_ψ.

PROPOSITION 5

$$S_\psi (z) = \prod_\gamma \prod_{j \geq 0} \text{Det} (I - e^{-zl(\gamma)} \psi_\gamma \otimes S_\gamma^j)$$

$$S_\varphi (z) = \prod_\gamma \prod_{j \geq 0} \text{Det} (I - e^{-zl(\gamma)} S_\gamma^j).$$

Now choose a Markov family as in Section 1. We obtain symbolic flows f_m, $m = 0, \ldots,$ card $|I|$, and semiconjugacies π_m from f_m to φ. For a fixed ψ we pull back E to a bundle $E_m = \pi_m^* E$ over Γ_m and we pull back the lift ψ to a lift ψ_m of f_m. For $E = E^s$ with its standard lift we write S_δ for the monodromy of ψ_m around δ. We define,

for a fixed ψ, the m-th *component* of S_ψ to be

$$S_{\psi_m}(z) = \exp \sum_\delta \frac{-1}{\mu(\delta)} \frac{\varepsilon_m(\delta) \, T r \, \psi_{m\delta}^z}{\mathrm{Det} \, I - S_\delta}$$

where δ runs over the closed orbits of f_m. Since the flows f_m have entropies $h(f_m) \leq h(\varphi)$, we have from Proposition 1

PROPOSITION 6

$$S_\psi(z) = \prod_{m \geq 0} S_{\psi_m}(z)^{(-1)^m}.$$

Proposition 6 carries the combinatorial information of the closed cover of the basic set determined by our Markov partition. Proposition 2 refines this by taking the exterior powers of our bundle into account. Together they are reminiscent of the Cech-DeRham spectral sequence [BT] which is based on an open cover of a manifold and the differential forms on the finite intersections of these open sets.

We now turn to questions of meromorphic extension of these zeta functions. Ruelle showed they have a meromorphic continuation about $z = h(\varphi)$ with a pole at that point. Examples of Pollicott show that the function $R_\varphi(z)$ may have an essential singularity at $z = h(\varphi) - \varepsilon$ [Po]. On the other hand, his work with Parry [PPo] shows $R_\varphi(z)$ is regular on a neighborhood of the line $\mathrm{Re}\, z = h(\varphi)$, aside from the pole at $h(\varphi)$ and except when φ is a constant time suspension [c.f. Section 5.A. for a related result for $S_\varphi(z)$].

An important case where R_φ is meromorphic on the entire complex plane is given in [R1], namely when φ is Anosov and the Anosov foliations are analytic. We will now turn to generalizing this theorem of Ruelle's.

SECTION 3. — Meromorphic continuation.

We will now impose a variety of analyticity assumptions. Recall that if M_1 and M_2 are analytic manifolds and $X \subset M_1$ is a closed set a map $f : X \to M_2$ is analytic if at each $x \in X$ there is a neighborhood N_x and an analytic map $f_x : N_x \to M_2$ with $f_x = f$ on $N_x \cap X$. Thus we may speak of analytic vector bundles, etc., over closed sets in an analytic manifold.

We suppose Λ is an *unstable-analytic* basic set, meaning that the manifold M, the flow φ and above all the unstable lamination W^u are analytic. (There is an obvious analogous notion of *stable-analytic*.) Several classes of examples of unstable-analytic basic sets are known. The usual construction of the solenoid and horseshoe can be so chosen near Ω (which suffices). For Anosov flows, see [T1], [T2] where Anosov flows that are stable- and unstable-analytic are produced from Lie groups. These include the examples discussed in Section 5 and also some examples on torus bundles over locally symmetric spaces of negative curvature.

Let $W^u(\Lambda) = \bigcup_{x \in \Lambda} W^u(x)$ be the unstable set of Λ. We assume that E is an analytic vector bundle over some compact neighborhood N of Λ in $W^u(\Lambda)$ with a polarization, meaning that for all $x \in \Lambda$ the restriction $E \mid W^u(x) \cap N$ of E to an unstable leaf has a

preferred flat connection. Thus for nearby points on an unstable leaf the E-fibers are identified in some fixed way. We assume this polarization is analytic, i. e. in an analytic lamination chart with 2 analytic transversals D_1, D_2 to the lamination the bundle isomorphism $E \mid D_1 \cap N \to E \mid D_2 \cap N$ induced by the polarization is real analytic. We suppose that the lift ψ is compatible with the polarization, i. e. the flat connections on $N \cap W^u(x)$ and $N \cap W^u(\varphi^t x)$ correspond under ψ_t. To be concise, we will say ψ is a *polar-analytic* lift if all these assumptions on E, ψ hold.

These assumptions essentially mean that E is pulled back from an analytic bundle over the leaf-space of the unstable lamination, this bundle admitting an analytic lifted flow. This leaf space being pathological one must work in local charts as here. Polarizations of line bundles are an ingredient of geometric quantization theory. Note that our lift ψ is just a polarization of E along the foliation defined by the flow.

For example, suppose Λ is an unstable-analytic repeller with, say, $\bigcap_{t < 0} \varphi_t M = \Lambda$. For any foliation $\mathscr{F}$ the normal bundle $TM/T\mathscr{F}$ carries a canonical flat connection. Here we take $\mathscr{F}$ to be the weak-unstable foliation tangent to $E^u \oplus E^c$, whose leaves are $\bigcup_{t \in R} \varphi_t W^u(x)$. Clearly $\mathscr{F}$ is real analytic. This makes $E^s \cong TM/T\mathscr{F}$ into a polarized analytic bundle and $D\varphi_t \mid E^s$ is then a polar-analytic lift. (It is not necessary to assume W^s analytic here.) It follows that the exterior powers λ^j are also polar-analytic lifts.

Our main results in this paper are the following generalizations of Ruelle's Theorem 3 in [R 1].

THEOREM 1. — *Let Λ be an unstable-analytic basic and ψ a polar-analytic lift. Then $S_\psi(z)$ has a meromorphic continuation to* C.

THEOREM 2. — *Let Λ be an unstable-analytic hyperbolic repeller and ψ a polar-analytic lift. Then $R_\psi(z)$ has a meromorphic continuation to* C.

Note that by Proposition 2 and our observations on λ^j, Theorem 2 is a consequence of Theorem 1. By Proposition 6, Theorem 1 will follow when it is shown that, for some Markov family, each m-th component $S_{\psi_m}(z)$ is an entire function.

To do this, we will eventually exhibit a holomorphic family of operators $\mathscr{L}_m(z)$, $z \in C$, on a certain Banach space B_m each of which is a nuclear operator of order 0. By the theory of Fredholm determinants, $\det I - \mathscr{L}_m(z)$ is an entire function. We will choose $\mathscr{L}_m$ so that $S_{\psi_m}(z) = \det I - \mathscr{L}_m(z)$ for Re z large.

The operators $\mathscr{L}_m$ will be pullback or transfer operators like those used in statistical mechanics. In general one has a set U partitioned into finitely many subsets U_k, $k \in K$, with maps $F_{kh}: U_k \to U_h$ for certain pairs $(k, h) \in B \subset K \times K$. The transfer operator on functions take $s: U \to C$ to $\mathscr{L}s$ where for $x \in U_k$, $\mathscr{L}s(x) = \sum_h s(F_{kh}x)$ and where h runs over the indices with $(k, h) \in B$. Given a bundle $E = \amalg E_k$ over U and bundle maps $L_{kh}: F_{kh}^* E_h \to E_k$ for $(k, h) \in B$ there is also a transfer acting on sections of E. Here one has

$$(\mathscr{L}s) \mid X_k = \sum_h L_{kh} F_{kh}^* (s \mid X_h).$$

In our application, each U_k is the closure of a bounded open set $\mathcal{O}_k \subset \mathbf{C}^d$, for some d, $F_{kh} U_k \subset \mathcal{O}_h$ and F_{kh} is holomorphic. We then say that $F = \bigcup F_{kh}$ is a *holomorphic correspondence* on U. Suppose that E is holomorphic and Hermitian and the bundle maps L_{kh} are holomorphic. Let B be the Banach space of continuous sections of E that are holomorphic on $\mathcal{O} = \amalg \mathcal{O}_k$ with the sup norm. Clearly $\mathscr{L}$ acts on B as a bounded operator that we call the *holomorphic transfer* of F and the given bundle data. Note that if E and F are fixed, $\mathscr{L}$ depends linearly on the L_{kh}.

We will now begin to define the $\mathscr{L}_m$. First choose the transverse discs D_i that support our Markov family to lie in small lamination charts of form $\mathbf{B}^{d_u} \times \mathbf{B}^{d_s+1}$, where the factors are balls of the indicated dimension, so that D_i has the form $\mathbf{B}^{d_u} \times D_i^V$, where V_i^V is analytic of dimension d_s. Then we can take R_i^V as a compact subset of D_i^V. Also D_i is an analytic manifold, so all return maps and return time functions are analytic (by the analytic implicit function theorem). Note that the return time along an unstable leaf is constant, so we obtain an analytic return time function $t_{ij}: R_i^V \to (0, \infty)$. Also the vertical component of our return maps C_{ij} define $V_{ij}: R_i^V \to R_j^V$ that are analytic. The lift ψ gives bundle maps $\psi_{ij}: E \mid R_i \to E \mid R_j$. The polarization allows one to collapse E along W^u leaves and obtain bundles $\bar{E}_i$ over R_i^V. Then ψ_{ij} induces $\bar{\psi}_{ij}: \bar{E}_i \to \bar{E}_j$, a bundle map.

Fix $m \geqq 0$. Let $K = I_m$, $B = A_m$. For each $k \in K$ let $X_k \subset \mathcal{M}^V$ be the image of the cylinder set $\{(k_n) \mid k_0 = k\} \cap \Sigma_B$ under the map $\Sigma_B \overset{\pi_m}{\to} \mathcal{M} \to \mathcal{M}^V$. If $k = (i, P)$ then X_k is a compact subset of R_i^V. To each $(k, h) \in B$ there is a return map $F_{kh}: X_k \to X_h$ that is a composite of V_{ij}'s. By adding the return times and composing the $\bar{\psi}_{ij}$'s en route from X_k to X_h we obtain a return time function t_{kh} on X_k and a bundle map $\bar{\psi}_{kh}: \bar{E}_k \to \bar{E}_h$, where $\bar{E}_{(i, P)} = \bar{E}_i \mid X_{(i, P)}$. These maps are all analytic.

Now we complexify. We choose bounded open neighborhoods N_k of X_k in $\mathbf{C}^{d_s}$, where $X_k \subset R_i^V \subset D_i^V \subset \mathbf{R}^{d_s} \subset \mathbf{C}^{d_s}$. If N_k is small enough, we can extend each F_{kh} to a holomorphic map $N_k \to \mathbf{C}^{d_s}$, that we again denote F_{kh}. We will show that arbitrarily small N_k can be chosen, for $k \in K$, so that $F_{kh} N_k \subset \text{int } N_h$, for all $(k, h) \in B$: then $\mathcal{O}_k = \text{int } N_k$, $U_k = \bar{\mathcal{O}}_k$ gives a holomorphic correspondence.

Choose an analytic metric on M, and give each D_i^V a metric by representing it as a slice of $D_i \subset M$. Then the uniform stable contraction and the analyticity of the D_i imply that any composition of $\geqq n$ return maps has a differential that is a contraction, for some n. The same then holds for our holomorphic extension, if N_k's are taken sufficiently small. This implies that some smaller neighborhoods N_k' have the property that they stay within $\bigcup N_k$ under arbitrarily long compositions of F_{jk}'s. On $\bigcup TN_k'$ define Finsler metrices $d_0, \ldots, d_n$ to be $d_p(v) = \sup \| T(F_{k_p 1} \ldots F_{k_1 k_2} F_{k_0 k_1}) v \|$ over $(i_l, i_{l+1}) \in B$. Then $d_p(TF_{k_0 k_1} v) \leqq d_{p-1}(v)$ for $p = 0, \ldots, n-1$ and $d_n(TF_{k_0 k_1} v) < d_0(v)$ by our choice of n. Thus $d = d_0 + \ldots + d_n$ is a Finsler contracted by all the branches F_{jk}. (This is a modification of Mather's trick.) We see then that the ε-ball around X_k in this Finsler will suffice for our N_k.

Now we have the holomorphic correspondence and we note that if U_k are small enough then the t_{kh}'s extend analytically to U_k as does the bundle $\bar{E}_k$ and the isomorphism $\bar{\psi}_{kh}$. We denote the extension of $\bar{E}_k$ by E_k and give it some Hermitian metric. We

define $L_{kh}^z \colon F_{kh}^* E_h \to E_k (z \in \mathbf{C})$ so that $(L_{kh}^z)^{-1} \colon E_k \to F_{kh}^* E_h$ is the map of bundles over U_k associated to the bundle map $\varepsilon_{kh} e^{-ztkh} \psi_{kh}$. We denote the resulting transfer $\mathscr{L}_m(z)$: clearly it depends holomorphically on z.

From our computations in Section 4 it will follow that a holomorphic transfer $\mathscr{L}$ has a well defined trace $\mathrm{T}r\,\mathscr{L} = \Sigma\,\lambda_i$ and Fredholm determinant $\mathrm{Det}\,(\mathrm{I} - \mathscr{L}) = \Pi\,(1 - \lambda_i)$ where λ_i are the eigenvalues of $\mathscr{L}$. We now compute $\mathrm{T}r\,\mathscr{L}$ in a special case.

LEMMA 5. — *Suppose that* $|\mathrm{K}| = |\mathrm{B}| = 1$ *and that the map* $\mathrm{F} \colon \mathrm{U} \to int\,\mathrm{U}$ *is a contraction. Then if* p *denotes the fixed point of* F

$$\mathrm{T}r\,\mathscr{L} = \frac{\mathrm{T}r_c\mathrm{L} \mid \mathrm{E}_p}{\mathrm{Det}_c(\mathrm{I} - \mathrm{T}_p\mathrm{F})}.$$

Proof. — If we pass from U to a smaller compact neighborhood $\mathrm{V} \subset \mathrm{U}$ with $\mathrm{FV} \subset int\,\mathrm{V}$, the two transfers are shift equivalent and so have the same nonzero eigenvalues and the same trace. Thus we may suppose U connected and E trivial, say $\mathrm{E} = \mathrm{U} \times \mathbf{C}^n$. In these coordinates, $(\mathscr{L}s)(u) = \mathrm{M}(u) \cdot s(\mathrm{F}u)$ where $\mathrm{M} \colon \mathrm{U} \to \mathrm{M}_n\mathbf{C}$ is a holomorphic matrix valued function and $s \colon \mathrm{U} \to \mathbf{C}^n$.

Now filter $\mathrm{B} = \mathrm{B}(0) \supset \mathrm{B}(1) \supset \mathrm{B}(2) \dots$ where $\mathrm{B}(j)$ is the sections of E vanishing to order j at p. This filtration is invariant by $\mathscr{L}$. The quotient $\mathrm{B}(j)/\mathrm{B}(j+1)$ is the homogeneous n-tuples of degree j, so that the induced map is $\mathrm{M}(p) \otimes s^j\mathrm{T}$, where $\mathrm{T} = (\mathrm{T}_p\mathrm{F})^*$ is the complex dual of $\mathrm{T}_p\mathrm{F}$ and $s^j\mathrm{T}$ is the j-th exterior power of T. Thus $\mathrm{T}r\,\mathscr{L} = \sum_{j \geq 0} \mathrm{T}r\,\mathrm{M}(p) \cdot \mathrm{T}r\,s^j\mathrm{T}$. One has $\mathrm{M}(p) = \mathrm{L} \mid \mathrm{E}_p$ and, as in Proposition 3, $\sum_{j \geq 0} \mathrm{T}r\,s^j\mathrm{T} = \mathrm{Det}_c(\mathrm{I} - \mathrm{T})^{-1}$.

Q.E.D.

This Lemma is proven for $\mathrm{E} = \lambda^j$ in [R1] by evaluation of an integral kernel for $\mathscr{L}$. It is also a part of the proof of the Atiyah-Bott fixed point formula for holomorphic maps [AB].

Now consider a general $\mathscr{L}$. It is geometrically obvious that $\mathrm{T}r\,\mathscr{L}^p$, $p > 0$, is a sum over the fixed points of $\sigma_\mathrm{B}^p \colon \Sigma_\mathrm{B} \to \Sigma_\mathrm{B}$ of traces of transfer operators of the p-fold compositions of the F_{ij}'s. When $\mathscr{L} = \mathscr{L}_m(z)$, these summands are computed in Lemma 5. Here $x \in \mathrm{Fix}\,\sigma_{\mathrm{A}_m}^p$ corresponds to a point p_x in U that is periodic of period p for the correspondence F. We find that for $\mathrm{Re}\,z$ large

$$\mathrm{T}r\,\mathscr{L}_m(z)^p = \sum_x \varepsilon(\delta)\,e^{-sl(\delta)}\,\mathrm{T}r\,\psi_\delta \cdot \mathrm{Det}\,(\mathrm{I} - \mathrm{S}_\delta)^{-1}$$

where δ denotes the orbit of f_m determined by the cycle $x, fx, \dots, f^p x = x$ in Σ_B. The $l(\delta)$ arises as a sum of t_{kh}'s, the $\varepsilon(\delta)$ is a product of ε_{kh}'s and ψ_δ is the composition of ψ_{kh}'s. The complex determinant and trace in Lemma 5 give the real determinant and trace $\mathrm{Det}\,\mathrm{I} - \mathrm{S}_\delta$, $\mathrm{T}r\,\psi_\delta$ because our F_{kh}'s are the complexification of the real analytic maps that compose to S_δ and our $\bar{\psi}_{kh}$'s are the complexification of bundle maps that compose to ψ_δ.

Now note that each δ corresponds to $p/u(\delta)$ distinct x's. This gives, for $\operatorname{Re} z$ large,

$$\sum \frac{1}{p} \operatorname{T} r \, \mathscr{L}_m(z)^p = \sum_\delta \frac{1}{\mu(\delta)} \varepsilon(\delta) e^{-sl(\delta)} \frac{\operatorname{T} r \, \psi_\delta}{\operatorname{Det}(\operatorname{I} - \operatorname{S}_\delta)}.$$

Thus $\operatorname{S}_{\psi_m}(z) = \operatorname{Det} \operatorname{I} - \mathscr{L}_m(z)$ for $\operatorname{Re} z$ large.

But $\operatorname{Det} \operatorname{I} - \mathscr{L}$ depends holomorphically on $\mathscr{L}$ [Gr1]. Thus $\operatorname{S}_{\psi_m}(z)$ is an entire function, proving Theorem 1.

Replacing φ_t by φ_{-t}, we obtain meromorphicity results for stable-analytic basic sets. In particular we have the following corollary of Theorem 2.

THEOREM 2'. — *Let* Λ *be a hyperbolic attractor that is stable-analytic* (*i. e.* M, *the flow* φ *and the stable foliation* W^s *are analytic*). *Suppose* ψ *is a polar-analytic lift. Then* $\operatorname{R}_\psi(z)$ *has a meromorphic continuation to* **C**.

For a diffeomorphism $f \colon \mathrm{N} \to \mathrm{N}$ with an Axiom A basic set Ω, corresponding results hold. Define Ω to be unstable- or stable-analytic if its time one suspension flow φ is (or by mimicking the flow definitions). Given a bundle E over $\mathrm{W}^u \Omega$, an analytic polarization and a compatible analytic bundle map $\operatorname{L} \colon f^* \operatorname{E} \to \operatorname{E}$, one can define Ruelle and Selberg functions

$$\operatorname{R}_{\operatorname{L}}(t) = \exp \sum_\delta \frac{1}{\mu(\delta)} t^{l(\delta)} \operatorname{T} r \, \operatorname{L}_\delta$$

$$\operatorname{S}_{\operatorname{L}}(t) = \exp \sum_\delta \frac{1}{\mu(\delta)} \frac{t^{l(\delta)} \operatorname{T} r \, \operatorname{L}_\delta}{|\operatorname{Det} \operatorname{I} - \operatorname{S}_\delta|}$$

where δ runs over the closed orbits of $f \mid \Omega$. Here if δ has multiplicity μ and contains p points $x, fx, \ldots, f^{p-1} x \in \Omega$ then for $l(\delta) = p\mu$, $\operatorname{S}_\delta = \operatorname{Det} \operatorname{I} - \operatorname{T} f^{l(\delta)} \mid \operatorname{E}_x^s$. Here L_δ is the shift equivalence class of the map

$$\operatorname{E}_x \xrightarrow{f^{l(\delta)*}} f^{*l(\delta)} \operatorname{E}_x \to f^{*l(\delta)-1} \operatorname{E}_x \to \ldots \to f^* \operatorname{E}_x \to \operatorname{E}_x$$

where L is used $l(\delta)$ times. If L is invertible, E and L give a polarized bundle and a polar-analytic lift on the suspension Λ. Thus the above theorems with $t = e^{-z}$ give that $\operatorname{S}_{\operatorname{L}}$ is meromorphic on **C** and so is $\operatorname{R}_{\operatorname{L}}$ when Ω is a repeller. Even when L isn't invertible, one can set up transfer operators as before and prove the same results.

If P is a compact manifold and $g \colon \operatorname{P} \to \operatorname{P}$ is an expanding map, then the solenoid construction produces a disc bundle N over P with a lift $h \colon \mathrm{N} \to \mathrm{N}$ that contracts fibres uniformly. Then $\bigcap_{n \geq 0} f^n \operatorname{N} = \Omega$ is a hyperbolic attractor with W^s the given disc bundle. If P and g are analytic one can choose N, h analytic as well, so Ω is stable-analytic. Using f^{-1} to get an unstable-analytic repeller, the discussion of the preceding paragraph shows how to generate meromorphic zeta functions for g. The necessary data is an analytic bundle E over P and an analytic bundle map $\operatorname{L} \colon g^* \operatorname{E} \to \operatorname{E}$. (The polarization arises for free here when one pulls back the data to N.) One obtains, with the same formulas as above for $\operatorname{R}_{\operatorname{L}}$ and $\operatorname{S}_{\operatorname{L}}$, that each is

meromorphic on **C**. Taking $E = P \times \mathbf{C}^n$ and L to be multiplication by an analytic matrix-valued function $\alpha: P \to M_n \mathbf{C}$ gives that

$$R_\alpha(t) = \exp \sum_{p > 0} a_p \frac{t^p}{p}$$

is meromorphic on **C**, where

$$a_p = \sum_{g_p \cdot x = x} T r \alpha(x) \alpha(gx) \ldots \alpha(g^{p-1} x).$$

For $n = 1$, this is Theorem 2 in [R1].

In this expanding map case, however, the proof can be simplified. One can avoid Markov partitions entirely by complexifying P directly and setting up the transfer operator globally. This shows that $S_L(t)$ is *entire*, not just meromorphic.

We remark that in statistical mechanics the transfer operators arise from the partition of a one sided subshift of finite type into cylinder sets. The F_{kh} are then branches of the inverse of the shift. The E_{kh} are trivial and the Banach space is a class of Hölder continuous functions. The L_{kh} are multiplication by a positive function. Here the transfer operator is positive (but not trace class). One uses Perron-Frobenius theory (not Fredholm theory) to define Gibbs states, pressure and entropy, including the invariant measures of dynamical interest. Here one only has control over one eigenvector, the positive one, and there may be continuous spectrum.

SECTION 4. — Growth estimates and the length spectrum.

Using Grothendieck's work on nuclear operators, we will estimate the zeroes and poles of the zeta functions S_ψ, R_ψ of Section 3, much as in [R1]. Then we will give a formula for the length spectrum of a stable-analytic hyperbolic attractor.

Grothendieck defined a class of nuclear spaces. We omit the definition but note that it includes the space $\Omega(E)$ of holomorphic sections of a holomorphic vector bundle over a complex manifold with the topology of uniform convergence on compacts. For N nuclear and B a Banach space, a bounded linear map $G: N \to B$ has the form

$$\sum_{i=1}^{\infty} \lambda_i x_i' \otimes y_i,$$ where $y_i \in N$, $x_i' \in B$, $\lambda_i > 0$, such that λ_i is a sequence of rapid decrease (i. e. $\lambda_i = O(i^{-n})$, any $n > 0$), the y_i are bounded and the x_i' are equicontinuous. When N is quasicomplete (like $\Omega(E)$) and $H: N \to N$ is bounded the eigenvalues μ_i of N are also of rapid decrease ([Gr2], p. 61) so that $\det I - H = \Pi(1 - \mu_i)$ and $T r H = \Sigma \mu_i$ ([Gr2], p. 19).

For $n = \Omega(E)$ more can be shown about the λ_i and μ_i although not quite as much as asserted in [Gr2], pp. 62-64. First the λ_i can be chosen to be $O(k^{d\sqrt{i}})$ where $d = \dim V$, for some $k \in (0, 1)$. Grothendieck asserted one could use $O(k^i)$ here but this is not so. Let B_d be the open unit ball in $\mathbf{C}^d$ and let $N = \Omega(B_d \times \mathbf{C})$. Let B be the Banach space of continuous functions on $\bar{B}_d$ that are holomorphic on B_d in the sup norm. For $k \in (0, 1)$ let $G = \mathscr{L}_k: N \to B$ be the transfer for the map $F(z) = kz$, $\mathscr{L}_k s(z) = s(kz)$. As each eigenfunction $z_1^{i_1} \ldots z_d^{i_d}$ has eigenvalue $k^{i_1 + \cdots + i_d}$, the sequence λ_i cannot converge

faster than the eigenvalue sequence $\mu_1, \mu_2 \ldots = 1, k, \ldots, k, k^2, \ldots, k^2, k^3, \ldots$ where k^i occurs with multiplicity $\binom{d+i-1}{i}$. Thus the smallest i with $\mu_i = k^{n+1}$ is $\binom{d+n}{d} \sim \dfrac{n^d}{d!}$, whence $\mu_i = O(k^{i\alpha})$ only for $\alpha \leqq \dfrac{1}{d}$. Substituting $O(k^{d\sqrt{i}})$ for $O(k^i)$ on pp. 62-64 of [Gr2] gives the correct statement made above.

Second one can estimate the Fredholm determinant $\operatorname{Det} I - z \, H$ more sharply than for ordinary nuclear operators of order zero, for which one knows it to be an entire function of order zero. Write $H = \sum_{i > 1} \lambda_i x_i' \otimes x_i$ where $|\langle x_i', x_i \rangle| \leqq 1$ for all i and where $|\lambda_i| \leqq C k^{d\sqrt{i}}$, some $C > 0$. Then we show

LEMMA 6. — $\operatorname{Log} |\operatorname{Det} I + H| = O(1) + O(\log_+ C)^{d+1}$, where the implied constants depend only on d, k.

Thus for $|z| \leqq r$, $\log |\operatorname{Det} I + z \, H| = O(1) + O(\log_+ r)^{d+1}$. The criterion for order zero is $O(r^\alpha)$ for all $\alpha > 0$, which is much weaker. Here one can compose with $z = e^{-s}$ and obtain an entire function $g(s)$ of finite order. Indeed we have that $\log |g|$ is bounded in each right halfplane $\operatorname{Re} s \geqq \sigma$ with a bound $O(|\sigma|^{d+1})$ for $\sigma \leqq -1$. We will say that an entire function with such growth properties has *right order* $\leqq d+1$: such functions clearly form a subalgebra of the entire functions of order $\leqq d+1$.

Proof of Lemma 6. — We begin as on [Gr2], pp. 16-17. $\operatorname{Det}(I + z \, H) = \sum_{n=0}^{\infty} \alpha_n z^n$ where

$$\alpha_n = \sum_{i_1 < \ldots < i_n} \lambda_{i_1}, \ldots, \lambda_{i_n} \det \langle x_{i_\alpha}', x_{i_\beta} \rangle_{1 \leqq \alpha, \beta \leqq n}.$$

A matrix of order n with entries in the unit disc has determinant $\leqq n^n$. Letting $f_k(z) = \Sigma \beta_n(k) z^n = \prod_{i=1}^{\infty} 1 + k^{d\sqrt{i}} z$, we have

$$|\alpha_n| \leqq C^n \beta_n(k) n^{n/2}, \qquad \operatorname{Det} I + H | \leqq \Sigma |\alpha_n|.$$

To estimate the $\beta_n = \beta_n(k)$, note that Cauchy's estimate gives $\beta_n \leqq r^{-n} M(r)$, where $M(r) = \max_{|z|=r} |f_k(z)|$. We can estimate $M(r)$ in terms of the number $n(r)$ of zeroes of f_k in $|z| \leqq r$, as follows. Let $N(r) = \displaystyle\int_0^r t^{-1} n(t) \, dt$ and $Q(r) = r \displaystyle\int_r^{\infty} t^{-2} n(t) \, dt$. Since f_k has genus zero and $f(0) = 1$ ([Bo], p. 47) gives $\log M(r) \leqq N(r) + Q(r)$.

Here $n(r) \leqq a(\log_+ r)^d$, where $a = |\log k|^{-d} > 0$. Evaluating $\int_1^r t^{-1} (\log t)^d \, dt$ and $r \displaystyle\int_r^{\infty} t^{-2} (\log t)^d \, dt$ with the substitution $u = \log t$, we get for $r \geqq 1$

$$\log M_k(r) \leqq a_k \, P(\log r)$$

where $P(x) = \sum_{j=0}^{d+1} \frac{d!}{j!} x^j$ is a polynomial of degree $d+1$ and we use M_k, a_k to denote our dependence on $k \in (0, 1)$.

Thus for $r = r_n = \exp(\sqrt[d]{n}/a + 1)$ we find

$$\log 1/\beta_n \geq n \log r_n - a\, P(\log r_n) \geq bn^\alpha - cn$$

with $\alpha = 1 + (1/d)$ and positive constants b, c depending on d, k. Thus $|\operatorname{Det} I + H| \leq 1 + \sum_{n=1}^{\infty} C^n n^n e^{cn} e^{-bn^\alpha}$. Choose β, $\delta > 0$ so that $n \log n + cn - bn^\alpha \leq \delta - \beta n^\alpha$ for all $n \geq 1$. Then

$$|\operatorname{Det} I + H| \leq e^\delta \sum_{n > 0} e^{-\beta n^\alpha} C^n.$$

Now we use the obvious lower bound $\beta_n(\lambda) \geq \lambda^{s_n}$ where $s_n = \sqrt[d]{1} + \ldots + \sqrt[d]{n} \leq \int_1^{n+1} x^{1/d} dx \leq \gamma n^\alpha$, for some $\gamma > 0$. For $\lambda = e^{-\beta/\gamma} \in (0, 1)$, this gives $e^{-\beta n^\alpha} \leq \beta_n(\lambda)$. Thus

$$\log |\operatorname{Det} I + H| \leq \delta + \log f_\lambda(C).$$

By our bound on $\log M_\lambda$, this gives for $C \geq 1$

$$\log |\operatorname{Det} I + H| \leq \delta + |\log \lambda|^{-d} P(\log C).$$

<div align="right">Q.E.D.</div>

Note that for $d = 1$, $\beta_n(\lambda) = \lambda^{(n(n+1)/2)}(1-\lambda)^{-1} \ldots (1-\lambda^n)^{-1}$ [E]. So here $\log 1/\beta_n \sim n^2/2 |\log \lambda|$, as expected, and the main contribution comes from the single term $\lambda \cdot \lambda^2 \ldots \lambda^n$.

Consider the holomorphic transfer $\mathscr{L}$ of Section 3. For each $(k, h) \in B$, the pullback operator from sections of E_h to sections of $F_{kh}^* E_h$ is nuclear, since it extends to a map from $\Omega(E \mid \mathcal{O}_k) \to \Omega(F_{kh}^* E_h)$. The operator $\Omega(F_{kh}^* E_h) \to \Omega(E_k)$ induced by L_{kh} is bounded by $\|L_{kh}\|$, the sup of the operator norm on the fibers. Let $\|L\|$ denote the sup of the norms $\|L_{kh}\|$ over all $(k, h) \in B$. Then for a fixed holomorphic correspondence and a fixed E, Lemma 6 shows that the transfer $\mathscr{L}$ satisfies

$$\log |\det I - \mathscr{L}| = O(\log_+ \|L\|)^{d+1} + O(1).$$

Now we take $\mathscr{L} = \mathscr{L}_m(z)$ for some $m \geq 0$. Then $\|L_{kh}^z\| \leq b e^{-a \operatorname{Re}(z)}$ for some b, $a > 0$. Thus $S_{\psi_m}(z) = \det I - \mathscr{L}_m(z)$ has right order $\leq d_s + 1$. The example considered above shows this order estimate is best possible.

With $S_\psi^+(z) = \prod_{m \text{ even}} S_{\psi_m}(z)$, $S_\psi^-(z) = \prod_{m \text{ odd}} S_{\psi_m}(z)$ we then have, by Proposition 6,

THEOREM 3. — *For an unstable-analytic basic set Λ and a polar-analytic lift ψ, $S_\psi = S_\psi^+/S_\psi^-$ where $S_\psi^\pm$ have right order $\leq d_s + 1$.*

Using Proposition 2 we find, with

$$R_\psi^+ = \prod_{m+j \text{ even}} S_{\psi \otimes \lambda^j, m}, \qquad R_\psi^- = \prod_{m+j \text{ odd}} S_{\psi \otimes \lambda^j, m}.$$

THEOREM 4. — *Let* Λ *be an unstable-analytic hyperbolic repeller and* ψ *a polar-analytic lift. Then* $R_\psi = R_\psi^+/R_\psi^-$ *where* $R_\psi^\pm$ *are entire of right order* $\leq d_s + 1$.
Also we have.

THEOREM 4'. — *Let* Λ *be a stable-analytic hyperbolic attractor. Then* $R_\varphi(z) = R_\varphi^+(z)/R_\varphi^-(z)$ *where* $R_\varphi^\pm$ *have right order* $\leq d_u + 1$.

When $\Lambda = M$ is Anosov and W^u is analytic as well. Theorem 4' was proven by Ruelle [R1], whose method we have followed here. Due to the error in [Gr2], all the estimates in [R1] should have a $\log^{d_s+1}$ instead of a $\log^2$ and he mistakenly states $R_\varphi^\pm$ has order ≤ 2. As will be seen in part II, the geodesic flow φ on a closed hyperbolic d-manifold gives an R_φ such that in any expression of R_φ as a ratio of entire functions each has order $\geq d = d_s + 1$. For d odd, S_φ is known to be entire of order exactly d [G].

Of course the corresponding growth theorems hold for basic sets of diffeomorphisms and hence for expanding maps. One finds $S_L(t) = S_L^+(t)/S_L^-(t)$ where $S_L^\pm(t)$ are Fredholm determinants in t and as such satisfy

$$\log |S_L^\pm(t)| = O(1) + O(\log_+^{d+1} |t|).$$

We now study the length spectrum of a stable analytic hyperbolic attractor Λ, i. e. the sequence of lengths $l(\gamma)$ where γ runs over the closed orbits (with multiplicities). Here the primary object is a mass distribution μ^φ concentrated on this set of lengths in $(0, \infty)$ by assigning the mass $1/\mu(\gamma)$ to the point $l(\gamma)$ and summing over γ. We call μ^φ the *length distribution of* φ. To take the lift ψ into account, we consider also the *weighted length distribution* $\mu^\psi = \sum_\gamma \dfrac{1}{\mu(\gamma)}(\mathrm{Tr}\, \psi_\gamma)\, \delta_{l(\gamma)}$ on $(0, \infty)$.

$$\mu^\varphi[0, T] = N_p(T) + \frac{1}{2} N_p(T/2) + \frac{1}{3} N_p(T/3) + \ldots$$

where $N(T)$ is the number of prime closed orbits of length $\leq T$. This can be inverted to give $N(T)$ from μ^φ. Using in turn

$$N(T) = N_p(T) + N_p(T/2) + N_p(T/3) + \ldots$$

where $N(T)$ is the number of closed orbits of length $\leq T$ (prime or not), we see that μ_i^φ, N, and N_p are equivalent descriptions of the length spectrum

To proceed further we average μ^ψ as follows. Let

$$M_0^\psi(T) = \int_0^{\log T} t\, d\mu^\psi, \qquad T > 1$$

$$M_{i+1}^\psi(T) = \int_1^T M_i^\psi(t)\, dt, \qquad i \geq 0.$$

For ψ the trivial lift on $\Lambda \times \mathbf{R}$, we write M_i^φ or M_i for M_i^ψ. As $M_{i+1}' = M_i$, knowing the *ith mean* M_i is clearly equivalent to knowing the length spectrum. The sharpest estimates of the asymptotics of the length spectrum arise by studying M_i for i sufficiently large: one can pass back to information about M_0 or μ_φ by differencing (Tauberian) techniques as in [Sa], [Ra2] but some information is lost on the size of the error term.

We will show that for k large M_k^ψ is a power series with complex exponents. This is analogous to the explicit formulas of prime number theory.

THEOREM 5 (The explicit formula for a hyperbolic attractor). — *Let Λ be a stable-analytic hyperbolic attractor and ψ a polar-analytic lift. Then for $k \geq d_u + 1$ and $T > 1$*

$$T^{-k} M_k^\psi(T) = \sum_\rho c_\rho T^\rho$$

where $\rho \in \{0, -1 \ldots -k\}$ or ρ is a zero or pole of $R_\psi(s)$ and $c_\rho \in \mathbf{C}$.

For $\rho \notin \{0, -1, \ldots, -k\}$, $c_\rho = n_\rho/\rho(\rho+1)\ldots(\rho+k)$, where $n_\rho \in \mathbf{Z}$ is the order of R_ψ at ρ. At the $k+1$ exceptional points c_ρ depends on the terms of order -1, 0 in the Laurent expansion of R_ψ'/R_ψ at ρ.

For the case $\Lambda = \Gamma \backslash \mathrm{PSL}(2, \mathbf{R})$ and ψ trivial, this was proven by Randol (here $d_u = 1$) [Ra1], using the detailed properties of Selberg's zeta function (functional equation, Riemann hypothesis). Our proof is motivated by his but shows these properties are not essential here (the location of the zeroes matters for later asymptotic estimates). We thank Randol and Sarnak for suggesting that one could generalize the explicit formula. We begin with a result that will justify certain contour shifts. Let $\int_r$ denote the path integral around the circle $|z| = r$, $r > 0$.

PROPOSITION 7. — *Suppose $Z(s)$ is the ratio of 2 nonzero entire functions of order $\leq n$. Then there is a $C > 0$ such that for arbitrarily large choices of r*

$$\int_r \left| \frac{Z'(s)}{Z(s)} \right| |ds| \leq C r^n \ln r.$$

Proof. — Suppose at first that Z is entire and then, dividing by a power of s, that $Z(0) \neq 0$.

Then there is a canonical product expansion for Z of the form

$$Z(s) = \exp p(s) \prod_\rho \left(1 - \frac{s}{\rho}\right) \exp\left(\sum_{m=1}^{n} \frac{1}{m}\left(\frac{s}{\rho}\right)^m\right),$$

where ρ runs over the zeroes of Z (with multiplicity) and where p is a polynomial of degree $\leq n$. This gives the partial fraction decomposition

$$\frac{Z'(s)}{Z(s)} = p'(s) + \sum_\rho (s/\rho)^n (s-\rho)^{-1}$$

whence, setting $F(\alpha) = \int_1 \dfrac{|dz|}{z - \alpha}$, we have

$$\frac{1}{2\pi} \int_r \left| \frac{Z'(s)}{Z(s)} \right| \, ds \leqq 0 \, (r^n) + \Sigma$$

where

$$\Sigma = \sum_\rho (r/|\rho|)^n \, F(|\rho|/r).$$

Let $N(x)$ be the number of $|\rho| \leqq x$, counted with multiplicity. We know $N(x) \leqq c x^n$ for some $c > 0$. For $r > 0$ let $N_r(x) = N(rx) \leqq (c r^n) x^n$. Then $\Sigma = \displaystyle\int_0^\infty x^{-n} F(x) \, dN_r(x)$. Suppose $|\rho| \neq r$ for all ρ and choose $\sigma < 1 < \tau$ so that $[r\sigma, r\tau]$ is disjoint from $\{|\rho|\}$. Let $\varepsilon > 0$ satisfy $N(\varepsilon) = 0$. Then we can do the integration over the intervals $[\varepsilon/r, \sigma]$ and $[\tau, \infty]$ and use integration by parts on each to get

$$\Sigma \leqq c r^n \left[2 \, F(\sigma) + F(\tau) + n \int_{\varepsilon/r}^\sigma F(x) \frac{dx}{x} + n \int_\tau^\infty F(x) \frac{dx}{x} \right].$$

By viewing $F(\alpha)$ as the arclength of the curve $\ln(z - \alpha)$, $z = e^{i\theta}$, $\theta \in [-\pi, \pi]$, we get the bounds

$$F(\alpha) \leqq \begin{cases} 2\pi + 2\ln\dfrac{1+\alpha}{1-\alpha}, & 0 \leqq \alpha < 1 \\[4mm] 4\sin^{-1}\alpha + 2\ln\dfrac{1+\alpha}{\alpha-1}, & \alpha > 1. \end{cases}$$

Thus F has only a logarithmic singularity at $\alpha = 1$ and $F(x)/x$ is integrable on $(\varepsilon/r, \infty)$.

Since $N(x) \leqq c x^n$, we can divide the interval from $x/2$ to x into $1 + [c x^n]$ equal intervals and be sure (by the pigeonhole principle) that some interval misses $\{\rho\}$. Let r be the midpoint of such an interval. Then $|\sigma - 1|$, $|\tau - 1|$ are bounded below by $c' r^{-n}$, some $c' > 0$. This gives that $2 \, F(\sigma) + F(\tau) = O(\ln r)$. But $\displaystyle\int_{\varepsilon/r}^\infty F(x) \frac{dx}{x} = O(\ln r)$ as well, so $\Sigma = O(r^n \ln r)$ and the estimate on $\displaystyle\int_r |Z'/Z|$ holds.

When Z is not entire, say $Z = Z_1/Z_2$, one chooses r by the above procedure for the entire function $Z^* = Z_1 Z_2$.

<div align="right">Q.E.D.</div>

Proof of Theorem 5. — The logarithmic derivative of R_ψ is given by

$$\frac{R_\psi'(z)}{R_\psi(z)} = \sum_\gamma \frac{l(\gamma)}{\mu(\gamma)} \, \mathrm{Tr} \, \psi_\gamma \, e^{-z l(\gamma)}.$$

For $\mathrm{Re}\, z \geq c > h(\varphi) + \log r(\psi)$, we showed in Section 2 that this Dirichlet series converges to a bounded function. For $c > 0$ ([HR], Theorem 40) gives

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{R'_\psi(z)}{R_\psi(z)} \frac{k!\, T^z\, dz}{z(z+1)\ldots(z+k)} = \sum_\gamma \frac{l(\gamma)}{\mu(\gamma)} \cdot Tr\,\psi_\gamma \cdot G^k(e^{l(\gamma)}/T)$$

where $G(x) = \max(0, 1-x)$ for $x > 0$. One can easily check that the right hand side is $k!\, T^{-k} M_k(T)$.

Now we let Γ_r be the contour formed by the two rays $c+it$, $t \in (-\infty, -r)$, $t \in (r, \infty)$ and the semicircle $C_r = \{|z-c| = r,\ \mathrm{Re}\, z \leq c\}$ traced upwards and we change our contour of integration from Γ_0 to Γ_r. Then applying Proposition 7 to $Z(s) = R_\psi(s-c)$, where the growth properties are given by Theorem 4', we find that we can choose r large so that the integral over C_r is 0 $(\log r \cdot r^{d_u+1} r^{-k-1}) = o(1)$. As the integrals over the rays tend to zero, we obtain $k!\, T^{-k} M_k(T)$ as the sum of the residues of the integrand in $\mathrm{Re}\, z < c$. But there are no zeroes or poles of R_ψ in $\mathrm{Re}\, z \geq c$, and each $\rho \notin \{0, \ldots, -k\}$ contributes $n_\rho k!\, T^\rho / \rho(\rho+1)\ldots(\rho+k)$.

$$\text{Q.E.D.}$$

Note that the explicit formula above would hold for any function $R(z) = \exp - D(z)$, where $D(z)$ is a Dirichlet series $\Sigma a_i e^{-l_i z}$ absolutely convergent in a right halfplane, provided that $R(z)$ has a meromorphic continuation of the form R^+/R^- where $R^\pm$ are entire functions of finite order. One replaces μ^ψ by $\Sigma a_i \delta_{l_i}$ here. In particular there are exact formulas for the Selberg functions of an unstable-analytic basic set.

Taking ψ trivial, any information about the location of zeroes and poles of R_φ with largest real part gives information about the asymptotics of M_k^ψ for k large. Using Proposition 2, this reduces to the study of $S_{\lambda^j}(z)$, $j = 0, \ldots, d_u$, where $\lambda^j = \lambda^j E^u$. For the geodesic flow on a compact negatively curved locally symmetric space X, the largest zeroes and poles of S_{λ^j} arise from the lowest eigenvalues of the Laplacian on j-forms, as will be shown in part II. For $\dim X > 3$, this implies, as will be seen, that the sharpest estimates on the length spectrum involve the spectrum on differential forms as well as that on functions.

SECTION 5. — Special cases.

A. — We will begin with the case of a transitive Anosov flow φ on a closed 3-manifold M. In this situation, Ratner constructed particularly simple Markov families. Starting with any prime closed orbit γ, one can choose the R_i so $R_i - R_i^0 \subset W^u(\gamma) \cup W^s(\gamma)$ [R]. In particular, the only orbits that contribute to the S_{ψ_m} with $m > 0$ are the multiples of γ.

This implies, by Proposition 6, that

$$S_\varphi(z) = S_{\varphi_0}(z) \prod_{n=1}^{k} \left(\prod_{m=0}^{\infty} 1 - e^{-(z+m)nl(\gamma)} \right)^{a_n}$$

$$R_\varphi(z) = R_{\varphi_0}(z) \prod_{n=1}^{k} (1 - e^{-znl(\gamma)})^{a_n}$$

for some k and integers $a_1, \ldots, a_k$. [Here we replace $1+x$ by $(1-x)^{-1}(1-x^2)$ as needed.] When φ is unstable-analytic and ψ is polar-analytic, $S_{\psi_0}(z)$ is entire (Section 3) and so $S_\psi(z)$ has no poles outside the set $\{az_\gamma - m \mid a, m \in \mathbf{Z}, m \geqq 0\}$ where $z_\gamma = 2\pi i/k!\, l(\gamma)$.

In Ratner's construction the R_i are actual topological rectangles (i. e. R_i^V and R_i^H are closed intervals) and whenever γ touches an R_i at x, R_i fills a full quadrant or halfspace in the local orbit space. Thus the set Q_0 in the proof of Proposition 1 has $\leqq 4$ elements. This implies $k \leqq 12$ (a tetrahedron has 6 edges and if an edge returns with orientation reversed after traversing all edges it contributes to $k = 12$). We may then choose $k = 12$, independent of γ.

Suppose $\rho \neq 0$ is a pole of $S_\varphi(z)$. Then for some $m \geqq 0$, $\rho + m \in i\mathbf{R}$. $l(\gamma)$ is an integral multiple of $2\pi i/(12)!\,(\rho+m) = \alpha$, say. Since this holds for all prime closed γ, φ must be a constant time suspension of a transitive Anosov diffeomorphism $f : K \to K$ [A], with return time some integral multiple of α. Since $\dim K = 2$, f must be topologically conjugate to a hyperbolic linear map on a torus [Fr1]. So we have.

THEOREM 6. — *Let* φ *be an unstable-analytic transitive Anosov flow on a closed* 3-*manifold* M. *Either* φ *has a torus cross-section with constant return time or the meromorphic function* $S_\varphi(z)$ *is regular on* $\mathbf{C} - \{0, -1, -2, \ldots\}$. *In the former case, the poles belong to a set* $\{ai\beta - m \mid a, m \in \mathbf{Z}, a > 0\}$ *where* $\beta > 0$.

This generalizes known results for the Fuchsian group case. Here if Γ is torsionfree, S_φ is entire, but when Γ has torsion the origin is commonly a pole ([S], [H], [F2]).

B. — Now we consider any basic set Λ and a neighborhood U of Λ. If E is a flat bundle over U then there is a natural lift $\psi_t : E \to E$. If U is connected (we may assume so since Λ is connected) then a flat n-plane bundle E has a holonomy representation $\chi : \pi_1 U \to \mathrm{Gl}(n, \mathbf{C})$. Conversely, given χ one constructs $E = \tilde{U} \times_\Gamma \mathbf{C}^n$ where $\tilde{U}$ is the universal cover of U and $\Gamma = \pi_1 U$ acts as deck transformations on $\tilde{U}$ and by χ on $\mathbf{C}^n$. Here we write S_χ, R_χ for S_ψ, R_ψ.

When Λ is unstable-analytic consider the holomorphic transfers $\mathscr{L}_m^\chi(z)$ corresponding to ψ (Section 3). Taking local coordinates, we can view sections as $\mathbf{C}^n$-valued functions and the bundle maps $(L_m^\chi)_{kh}$, $(k, h) \in A_m$, are multiplication by a constant matrix $c_{kh} \in M_n \mathbf{C}$, where c_{kh} is a part of a cocycle for the flat bundle E relative to the cover by $\{\bar{B}_i\}_{i \in I}$ (here we assume our Markov family so small that each B_i lies in a simply connected subset of U). The operator $\mathscr{L}_m(z)$ depends linearly on $c_{kh} \in (M_n \mathbf{C})^{A_m}$, so $\det I - \mathscr{L}_m(z)$ extends to an entire function ξ on $(M_n \mathbf{C})^{A_m} \times \mathbf{C}$. We have, for $G = \mathrm{Gl}(n, \mathbf{C})$

$$
\begin{array}{ccc}
Z^1(\{\bar{B}_i\}, G) \times \mathbf{C} & \xrightarrow{(\pi,\, \mathrm{id})} & \mathrm{Hom}(\Gamma, G) \times \mathbf{C} \\[4pt]
{\scriptstyle(\alpha,\, \mathrm{id})}\downarrow & & \downarrow{\scriptstyle S_{\chi\, m}(z)} \\[4pt]
(M_n \mathbf{C})^{A_m} \times \mathbf{C} & \xrightarrow{\;\xi\;} & \mathbf{C}
\end{array}
$$

where π is the natural projection from cocycles to representations and α restricts a cocyle to pairs in I_m. The fact that ξ is entire here may be loosely described by saying $S_{\chi\, m}(z)$ is entire in (χ, z). Thus the same holds for the $S_\chi^\pm(z)$ of Theorem 3. For a repeller or a stable-analytic attractor, the $R_\chi^\pm(z)$ of Theorems 4.4' are entire in (χ, z) also.

C. — Next we apply this good behavior as χ varies to prove functional equations for S_χ. Suppose X is compact and locally symmetric of negative curvature. Then a unitary representation $\rho: \pi_1 X \to U(n)$ determines a $\chi: \pi_1 SX \to U(n)$, with SX the unit sphere bundle of χ. Here S_χ'/S_χ was studied with Selberg's trace formula by Gangolli [G] who found its poles in terms of the spectrum of the Laplacian on functions on X. He proved the residues are all rational with a common denominator κ and so deduced, in our notation, that $S_\chi(z)^\kappa$ is meromorphic in C. He also proved a functional equation of the form

$$(*) \qquad \frac{S_\chi'(z)}{S_\chi(z)} + \frac{S_\chi'(h-z)}{S_\chi'(h-z)} = n \cdot \text{vol} X \cdot P_G(z)$$

where $P_G(z)$ is an elementary function depending only on the Lie group G of isometries of the universal cover of X and where $h = h_G$ is the entropy of geodesic flow on SX (*see* Part II: Gangolli defined h in terms of the Lie algebra of G).

We have 2 things to add here. First, Theorem 3 shows $S_\chi(z)$ is itself meromorphic, i. e. one may take $\kappa = 1$. Second (*) holds for many *nonunitary* representations χ, even though the trace formula does not apply and one does not know the zeroes/poles of $S_\chi(z)$. For one can integrate (*) to get $S_\chi(z)/S_\chi(h-z) = F(z)$, where F is meromorphic and independent of χ. Writing $S_\chi = S_\chi^+/S_\chi^-$ and $F = F^+/F^-$ where $F^\pm$ are entire, then

$$(**) \qquad S_\chi^+(z) S_\chi^-(h-z) F^-(z) = S_\chi^-(z) S_\chi^+(h-z) F^+(z)$$

is an identity of entire functions. Now pass from χ to a cocycle representative c. Then both sides are entire in $\{c_{kh}\}$, z. It follows that equality continues to hold on the (complex) analytic closure fo the set of cocycle representatives of unitary representations. This closure includes representatives of certain nonunitary representations and for these (*) holds, too.

For instance, suppose $m = 1$. If $\beta = \dim H^1(M; \mathbf{R})$ then one can choose a holomorphic family $(\mathbf{C}^*)^\beta$ of cocycle representatives that span a given connected component of $\text{Hom}(\pi_1 X, \mathbf{C}^*)$. Then (**) gives 2 holomorphic functions on $(\mathbf{C}^*)^\beta \times \mathbf{C}$ that agree on $(\mathbf{S}^1)^\beta \times \mathbf{C}$. It follows that they agree everywhere. Thus (*) holds whenever $m = 1$, and hence for any representation of $\pi_1 X$ that factors through $H_1 X$.

D. — To better understand such functional equations, we will here describe a new, simple example of one. Suppose N is a simply connected nilpotent Lie group, Γ a discrete cocompact subgroup and $\alpha: N \to N$ an automorphism with $\alpha(\Gamma) \subset \Gamma$. If $A = T_e \alpha: T_e N \to T_e N$ is hyperbolic then the induced map $f: \Gamma \backslash N \to \Gamma \backslash N$ is an *Anosov automorphism* of the nilmanifold $\Gamma \backslash N$. The time one suspension flow $\varphi_t: X \to X$ is an analytic Anosov flow, both stable- and unstable-analytic, with $\pi_1 X = \Gamma \times_\alpha \mathbf{Z}$.

Suppose that $\chi: \pi_1 X \to Gl(m, \mathbf{C})$ factors through $\mathbf{Z}$. Then we may identify χ with its value $B \in Gl(m, \mathbf{C})$ at $1 \in \mathbf{Z}$. We then have

$$S_\chi(z) = \exp \sum_{n=1}^\infty \frac{-1}{n} \frac{Tr(B^n) e^{-nz}}{\text{Det} I - A_s^n} N_n$$

where N_n is the number of fixed points of f^n and where α_s denotes the stable part of $A = A_s \oplus A_u$. Here we use that A_s^n and $T_x f^n$ are similar for any $x \in \mathrm{Fix}\,(f^n)$. (Of course we can write $S_\chi(z) = S_L(e^{-z})$ where $L : (\Gamma \backslash N) \times \mathbf{C}^m \to (\Gamma \backslash N) \times \mathbf{C}^m$ is defined by multiplication by B.)

The following is the Selberg functional equation for the flow φ and representation χ.

THEOREM 7. — *If χ^* is the contragredient representation of χ, h the topological entropy of φ, $\omega = \Lambda^{d_u} E^u$ the unstable orientation bundle for φ and $\varepsilon_u = \mathrm{sign}\,\det A_u$ then*

$$S_\chi(h - z) = S_{\chi^* \otimes w}(z)^{(-1)^{d_u}} a^m$$

where $a = \varepsilon_u e^h$ if $d_u = 1$, $a = 1$ if $d_u > 1$.

Proof. — By upper triangulating B, we can suppose $n = 1$, say $B = \rho \in \mathbf{C}^*$.

Say $F(t) = \exp -\Sigma_n t^n \mathrm{Det}(A_u^n - I)$. If we let $\lambda_1, \ldots, \lambda_{d_u}$ denote the unstable eigenvalues, and write λ_J for the product of λ_j, $j \in J \subset \{1, \ldots, d_u\}$, we can expand this determinant as $\Pi(\lambda_j - 1)$ and obtain $F(t) = \prod_J (1 - \lambda_J t)^{(-1)^{d_u} + |J|}$. If $c = \lambda_1 \ldots \lambda_u = \varepsilon_u e^h$ we find that $F(1/ct) = \prod_J (1 - 1/\lambda_J t)^{(-1)^{|J|}}$ by pairing J with its complement. Clearing denominators gives

$$F(1/ct) = \prod_j (-1 \lambda_J t)^{(-1)^{|J|}} \cdot \prod_J (\lambda_J)^{(-1)^{|J|+1}} \cdot (-t)^{\Sigma_J(-1)^{|J|}} = \begin{cases} F(t)^{(-1)^{d_u}}, & \text{for } d_u > 1 \\ F(t)^{-1} c, & \text{for } d_u = 1. \end{cases}$$

Now we use the Lefschetz formula for an iterate f^n. All fixed points have index $\varepsilon_u^n(-1)^{d_u}$, so $L(f^n) = N_n \varepsilon_u^n(-1)^{d_u}$. But a cohomology computation ([Ma2], [F3]) gives $L(f^n) = \mathrm{Det}\,I - A^n = \mathrm{Det}\,(I - A_s^n)\,\mathrm{Det}\,(I - A_u^n)$. Thus, eliminating $N_n/\mathrm{Det}\,I - A_s^n$, we find

$$S_\chi(z) = \exp \Sigma \frac{-1}{n} \rho^n e^{-nz} \mathrm{Det}\,(I - A_u^n) \varepsilon_u^n(-1)^{d_u} = F(\varepsilon_u \rho\, e^{-z}).$$

Thus $S_\chi(h - z) = F(\rho\, e^z/c)$. With $t^{-1} = \rho\, e^z$ and the above functional equation for F, $S_\chi(h - z) = F(e^{-z} \rho^{-1})^{(-1)^{d_u}}$ for $d_u > 1$. With $\rho' = \rho^{-1} \varepsilon_u$, this is $F(\varepsilon_u \rho'\, e^{-z})^{(-1)^{d_u}} = S_{\chi'}(z)^{(-1)^{d_u}}$. But $\varepsilon_u \rho^{-1}$ corresponds to $w \otimes \chi^*$. If $d_u = 1$, we have an extra factor of $c = \varepsilon_u e^h$ on the right.

Q.E.D.

From these examples, it is plain that a functional equation for an S_ψ should express its values at $h(\varphi) - z$ in terms of Selberg functions $S_{\hat\psi}(z)$ where $\hat\psi$ is related to ψ by some simple involution.

E. — To interpret Theorem 7 we consider another functional equation for a different zeta function. Let K be a closed oriented m-manifold, $f : K \to K$ a map of degree $d \neq 0$. Suppose E is a flat n-plane bundle over K whose holonomy representation $\rho : \pi_1 K \to Gl(n, \mathbf{C})$ is invariant under f_*. Then we can define induced maps on twisted cohomology $f_{E_i} : H^i(K; E) \to H^i(K; f^*E) = H^i(K, E)$. Via the natural nonsingular

pairing

$$H^i(K; E) \otimes H^{m-i}(K, E^*) \to C$$

the operators $f_{E^*, m-i}$ and $d(f_{E, i})^{-1}$ are adjoint. We define the zeta function of any graded linear map $L = (L_i)_{i \in z}$ to be $\zeta(L, t) = \Pi_i (\det I - t L_i)^{(-1)^{i+1}}$, provided that $\Sigma \, rk(L_i) < \infty$.

Our functional equation is:

PROPOSITION 8. — $\zeta(\alpha f_E, 1/|d|t) = \zeta(\alpha \varepsilon f_{E^*}, t)^{(-1)^m} c t^a$, where $c \in C^*$, $a = m \chi(K) \in Z$, $\varepsilon = \text{sgn} \, d = \pm 1$, $\alpha = \pm 1$.

Proof. — Fix an eigenvalue λ of f_{E_i}. It contributes a term $(1 - (\alpha\lambda/|d|t))^{(-1)^{i+1}}$ to the left. Write this as $(1 - \alpha\varepsilon \, dt/\lambda)^{(-1)^{i+1}} (|d|t/\alpha\lambda)^{(-1)^i}$ and note that d/λ is an eigenvalue of $f_{E^*, m-i}$. Multiply over λ. One finds $a = \Sigma_i (-1)^i \dim H^i(K, E)$. Considering a chain complex for K, one sees $a = m \chi(K)$.

<div align="right">Q.E.D.</div>

For E trivial, $\alpha = 1$, and $|d| = 1$, one finds $c = \pm 1$ since the various products over λ are determinants of invertible integer matrices. This case is due to Franks [Fr2]. Of course it is closely related to the functional equation for the Weil zeta function of an m-dimensional algebraic variety V over a field with q elements: here one takes $|d| = q^m$, $\alpha = \varepsilon = 1$, E trivial, f the Frobenius endomorphism of V.

To compare Proposition 8 with Theorem 7, let $Z_\rho^\alpha(s) = \zeta(\alpha f_E, e^{-s})$ for $\alpha = \pm 1$ and let $H = \log |d|$. Proposition 8 gives $Z_\rho^1(H - z) = Z_{\rho^*}^\varepsilon(z) c (e^{-z \chi(K)})^m$ which resembles Theorem 7 very much. In this analogy, one sees that the Selberg functional equation reflects Poincaré duality for the weak-unstable foliation. Also the analogy of e^h and $|d|$ arose in studying the Ruelle-Sullivan classes of a transitive Anosov diffeomorphism [F4] where e^h can be viewed as the degree on the stable leaf space. Both h and H are measuring expansion, H the net topological expansion, h the expansion of unstable leaves.

The resemblances go further. One can write

$$\zeta(L, t) = \exp \Sigma_{p > 0} t^p/p \, \Sigma_i (-1)^i \, Tr \, L_i^p.$$

With $L_i = \alpha f_{E_i}$, one can use the Lefschetz fixed point formula (with coefficients in E) to get an expression for $\log Z_\rho^\alpha$ as a sum over the periodic points of f. When f is transitive Anosov, we find that $(Z_\rho^1)^{(-1)^{d_u}}$ is a Ruelle function R_ψ for the time one suspension flow φ: the lift ψ is just the tensor product of the flat bundle corresponding to ρ with the orientation line bundle of W^u. We omit the routine verification.

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