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LOWER CURVATURE BOUNDS, TOPONOGOV'S THEOREM, AND BOUNDED TOPOLOGY (1)

BY UWE ABRESCH

Introduction

Classically the theory of non-compact Riemannian manifolds with negative sectional curvature is based on the visibility axiom (*cf.* [9]); heuristically speaking this axiom requires that the curvatures do not decay to zero too quickly. In contrast the theory of manifolds M^n with positive curvature does not require an additional hypothesis in the non-compact case. There are even nice results, if one supposes the curvature only to be non-negative: By the Toponogov splitting theorem such a manifold is isometric to a Riemannian product $N^k \times \mathbb{R}^{n-k}$, where the factor N^k does not contain a line. The soul theorem due to Cheeger and Gromoll claims that any non negatively curved manifold M^n is diffeomorphic to the normal bundle of a compact, embedded submanifold. Moreover Gromov has shown that there is a universal upper bound $C(n)$ on the sum of the Betti numbers $\beta_i(M^n)$.

In this paper we are going to study a larger class of manifolds and include for instance "asymptotically flat manifolds".

DEFINITION. — *A complete Riemannian manifold* (M^n, g) *with base point* 0 *is called asymptotically non-negatively curved, iff there exists a monotone decreasing function* $\lambda: [0, \infty) \to [0, \infty)$ *such that*

(i) $b_0(\lambda) := \int_0^\infty r.\lambda(r)\, dr < \infty$ *and*

(ii) *the sectional curvatures at any point* $p \in M^n$ *are bounded from below by* $-\lambda(d(0, p))$.

The convergence of the integral $b_0(\lambda)$ implies a decay condition on the lower curvature bound λ. This condition is analyzed in more detail in chapter II. For instance it asserts that there is a unique non-negative solution of the Riccati equation $u'(r) = u(r)^2 - \lambda(r)$

which decays to zero for $r \to \infty$. Thus one has another numerical invariant

$$b_1(\lambda) := \int_0^\infty u(r)\, dr$$

Both b_0 and b_1 depend on λ in a monotone way, and they can be regarded as invariants of the manifold M^n by taking the minimal monotone function λ which meets the conditions (i) and (ii); notice that the numbers do not change when the metric g on M^n is scaled with a global factor.

Main results

A. *A generalized triangle comparison theorem of Toponogov type.* — The model spaces will be arbitrary simply-connected surfaces of revolution with non-positive curvature, and the comparison triangle will have one vertex at the pole of the model space (*cf.* I.3.1 and I.3.2).

Rotationally symmetrical model surfaces have already been introduced by Elerath [8]. However, he makes different assumptions on their curvature; his version of the Toponogov theorem has been designed as a tool towards a refined soul theorem, whereas we want to study triangles in asymptotically non-negatively curved manifolds which have one vertex at the base point 0. Employing in addition the analysis done in chapter II, we obtain lower bounds on their angles which are uniform with respect to the size of the triangles (*cf.* III.1). Such uniform bounds can be derived from the standard Toponogov theorem only in the case of non-negative curvature, and in this more special setup they provide an important tool. Similarly our uniform estimates are the key to the following theorem.

B. Theorem. — *For asymptotically non-negatively curved manifolds* M^n *there exist universal upper bounds on the number of ends and on the Betti numbers:*

$$\#\{\text{ends of } M^n\} \leqq 2.\pi^{n-1}.\exp((n-1).b_1(M^n)) \tag{1}$$

$$\sum_i \beta_i(M^n) \leqq C(n).\exp\left(\frac{15n-13}{4}.b_1(M^n)\right). \tag{2}$$

The function $C(n)$ can be effectively estimated by an expression which grows exponentially in n^3.

The proof of B1 is carried out in chapter III, while B2 is deferred to a subsequent paper. Finally theorem B is optimal in the sense that the topology of a surface M^2 is not necessarily bounded, when its integral b_0 diverges. Moreover we can prove:

C. Theorem. — *Suppose that the integral* $b_0(\lambda)$ *of a function* $\lambda : [0, \infty) \to [0, \infty)$ *diverges. Then every non-compact, connected surface* M^2 *with base point* 0 *carries a complete* C^2*-metric whose curvature* κ *obeys:*

$$\kappa(p) = -\lambda(d(0, p)), \qquad p \in M^2.$$

I. — Models and Toponogov's theorem

The standard Toponogov theorem compares triangles in a Riemannian manifold (M^n, g) to the corresponding Alexandrov triangles in suitable spaces of constant curvature (*cf* [5] or [11]). It is worthwhile noticing that the models are essentially two-dimensional. We are going to extend the theorem and allow for any simply-connected surface of revolution with non-positive curvature as a model space. Except for the plane, none of these surfaces can be isometrically embedded into three-dimensional euclidean space in an equivariant way. Elerath in contrast considers embedded surfaces of revolution with non-negative curvature. In fact, in order to control the cut-locus, he has to make even stronger assumptions (*cf.* [8]).

Our generalisation of the triangle comparison theorem does not require any additional condition, since we use models with non-positive curvature. We describe them in terms of continuous functions $k : [0, \infty) \to [0, \infty)$. More precisely each function k uniquely determines a simply connected surface of revolution $M^2(-k)$ which has curvature $-k(d(., p_0))$; here d stands for the Riemannian distance, and p_0 denotes the pole in $M^2(-k)$.

It is convenient to simultaneously consider the approximating functions $k_\varepsilon : [0, \infty) \to [0, \infty)$ which are defined by:

$$k_\varepsilon(r) := \sup\{k(r') \mid r' \geqq 0 \text{ and } |r - r'| \leqq \varepsilon\}, \qquad \varepsilon \geqq 0. \tag{1.1}$$

By notation $k_0 = k$. In polar coordinates (r, φ) the metric of $M^2(-k_\varepsilon)$ looks like:

$$dr^2 + y_\varepsilon(r)^2 . d\varphi^2, \tag{1.2}$$

where the function y_ε is given by the Jacobi field equation:

$$y''_\varepsilon = k_\varepsilon . y_\varepsilon, \qquad y_\varepsilon(0) = 0 \qquad \text{and} \qquad y'_\varepsilon(0) = 1. \tag{1.3}$$

2. We proceed to summarize the elementary properties of our model spaces.

2.1. Lemma. — *The coordinate functions $r(s)$ and $\varphi(s)$ along a unit-speed geodesic $s \mapsto \gamma(s)$ in the model surface $M^2(-k)$ obey the equations:*

(i) $$r'^2 + (y \circ r)^2 . \varphi'^2 = 1,$$

(ii) $$(y \circ r)^2 . \varphi' = \text{Const.} \qquad \text{(Clairaut)},$$

(iii) $$(y \circ r)^2 . (1 - r'^2) = \text{Const.}^2.$$

We skip the obvious proof and recall that by notation p_0 always denotes the pole of the model space. When looking at a geodesic triangle $\Delta = (p_0, p_1, p_2)$ with edges of length $l_i = d(p_{i+1}, p_{i+2})$ —indices taken modulo 3—, formula 2.1 (iii) becomes:

$$y(l_1) . \sin(\sphericalangle \text{at } p_2) = y(l_2) . \sin(\sphericalangle \text{at } p_1).$$

This generalizes the well-known Law of Sines in euclidean geometry ($k \equiv 0$, $y \equiv \text{id}$) and in hyperbolic geometry ($-k \equiv -1$, $y \equiv \sinh$).

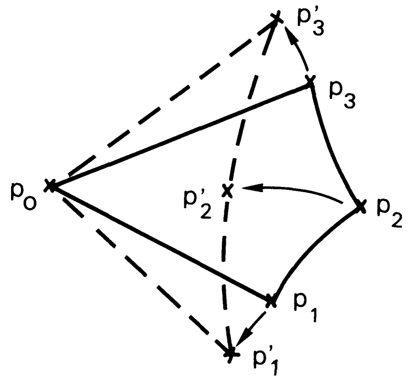
2.2. LEMMA. — *Given triangles* $\Delta=(p_0, p_1, p_2)$ *and* $\Delta'=(p_0, p'_1, p'_2)$ *in a surface* $M^2(-k)$ *such that* $l_0=l'_0$ *and* $l_1=l'_1$, *one has monotonicity:*

$$\sphericalangle at\, p_2 < \sphericalangle at\, p'_2 \quad \Leftrightarrow \quad l_2 < l'_2$$

Proof. — We may rotate Δ' about p_0 and without loss of generality may assume that $p'_2=p_2$. Then the claim becomes obvious, since $M^2(-k)$ is simply-connected and has non-positive curvature.

2.3. LEMMA. — *Let* p_0, p_1, p_2, *and* p_3 *be the vertices of a quadrilateral in* $M^2(-k)$. *Moreover suppose that* $\sphericalangle at\ p_2<\pi$ *and that:*

$$d(p_1, p_2)+d(p_2, p_3)<d(p_3, p_0)+d(p_0, p_1)$$



Then there is a triangle $\Delta=(p_0, p'_1, p'_3)$, *unique up to rotation about* p_0, *such that:*

$$d(p_0, p'_1)=d(p_0, p_1)$$
$$d(p_0, p'_3)=d(p_0, p_3)$$
$$d(p'_1, p'_3)=d(p_1, p_2)+d(p_2, p_3)$$
$$\sphericalangle at\, p'_1 < \sphericalangle at\, p_1$$
$$\sphericalangle at\, p'_3 < \sphericalangle at\, p_3.$$

Proof. — The idea is to bend in the corner at p_2: we move p_2 towards the pole p_0. We keep the length of all edges fixed by moving the vertices p_1 and p_3 in an appropriate way. Obviously:

$$l_{crit} := \max\{d(p_0, p_1)-d(p_1, p_2), d(p_0, p_3)-d(p_2, p_3)\}>0.$$

If $d(p_0, p_2)$ gets as small as l_{crit}, one of the triangles (p_0, p_1, p_2) and (p_0, p_2, p_3) becomes degenerate, and the quadrilateral has $\sphericalangle$at $p_2>\pi$. Now the claim is obvious, since the angle depends continuously on $d(p_0, p_2)$.

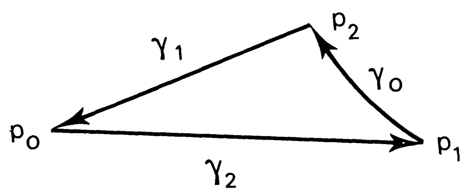
For later use we state another continuity property, which is due to the fact that the functions k_ε converge to k uniformly on compact subsets.

2.4. LEMMA. — *Given triangles* $\Delta=(p_0, p_1, p_2)$ *in* $M^2(-k)$ *and* $\Delta^\varepsilon=(p_0^\varepsilon, p_1^\varepsilon, p_2^\varepsilon)$ *in* $M^2(-k_\varepsilon)$ *which have edges of equal length* $l_i^\varepsilon=l_i$, $i=1, 2, 3$, *then their angles depend continuously on* ε, *e. g.*:

$$\lim_{\varepsilon\to 0} \sphericalangle \text{ at } p_i^\varepsilon = \sphericalangle \text{ at } p_i$$

3. In this section we are going to establish the generalized comparison theorem. Notation will be changed slightly: if there is a bar above a letter this symbol will refer to data in the model space, whereas unbarred letters will refer to data in the Riemannian manifold (M^n, g).

3.1. ASSUMPTIONS. — (i) p_0, p_1, *and* p_2 *are the vertices of a* (*generalized*) *geodesic triangle in a Riemannian manifold* M^n; *the edges* γ_1 *and* γ_2 *are supposed to be minimizing, whereas* γ_0 *is only required to be a geodesic. We continue denoting the length of* γ_i *by* l_i, $i=1, 2, 3$.



(ii) *at any point* $p\in M^n$ *the sectional curvatures shall be bounded from below by* $-k(d(p, p_0))\leqq 0$,

(iii) *the pole* $\bar{p}_0$ *of* $M^2(-k)$ *shall be a vertex of the comparison triangle* $\bar{\Delta}=(\bar{p}_0, \bar{p}_1, \bar{p}_2)$.

3.2. THEOREM. — *Under the assumptions* 3.1 *the following conclusions hold:*

(*a*) *if* $l_i=\bar{l}_i$ *for all the edges, then* $\sphericalangle$ *at* $p_1\geqq\sphericalangle$ *at* $\bar{p}_1$ *and* $\sphericalangle$ *at* $p_2\geqq\sphericalangle$ *at* $\bar{p}_2$.

(*b*) *if* $l_0=\bar{l}_0$, $l_1=\bar{l}_1$, *and* $\sphericalangle$ *at* $p_2\leqq\sphericalangle$ *at* $\bar{p}_2$, *then* $l_2\leqq\bar{l}_2$.

3.3. *Remarks*. — (i) Actually it is sufficient to require condition 3.1. (ii) only for those points $p\in M^n$ which are ε-close to any minimizing geodesic from p_0 to a point on γ_0. These points p obey the conditions:

$$d(p, p_0)+d(p, p_i)\leqq l_0+l_i+\varepsilon; \qquad i=1, 2,$$

and

$$d(p, p_0)+\frac{1}{2}.(d(p, p_1)+d(p, p_2))\leqq l_0+\frac{1}{2}.(l_1+l_2)+\varepsilon.$$

(ii) The Alexandrov triangle $\bar{\Delta}=(\bar{p}_0, \bar{p}_1, \bar{p}_2)$ which is required in part (*a*) of the theorem exists, if and only if $l_0\leqq l_1+l_2$.

(iii) Even if one assumes in addition that the edge γ_0 is minimizing, there is in contrast to the constant curvature case in general no easy way to restore the information on the angles at p_0 and $\bar{p}_0$. The reason is that $\bar{p}_0$ plays the rather special role of the pole in $M^2(-k)$.

Proof (*cf.* [5], [11]). — We give a straightforward extension of the classical argument.

(*a*) ⇒ (*b*): If $l_0 \geqq l_1 + l_2$, the claim is an obvious consequence of the triangle inequality $\bar{l}_0 \leqq \bar{l}_1 + \bar{l}_2$ which holds in $M^2(-k)$. Else there exists an Alexandrov triangle and the claim can be deduced from part (*a*) by means of the monotonicity principal 2.2.

(*a*) We pick $r_\Delta > 0$ and $K > 0$ such that the edge γ_0 is contained in the ball $B(p_0, r_\Delta)$ and that the sectional curvatures in $B(p_0, r_\Delta + 1)$ are bounded from above by K. Next we put the comparison triangles into the model spaces $M^2(-k_\varepsilon)$ instead of $M^2(-k)$. By a limiting argument based on lemma 2.4 it is sufficient to prove the result (*a*) for all surfaces $M^2(-k_\varepsilon)$ with $0 < \varepsilon < \min\{1, \pi/\sqrt{K}\}$. We fix the value of ε, subdivide γ_0 into pieces shorter than $\varepsilon/2$, and pick minimizing geodesics from p_0 to all the partition points on γ_0. Provided that the claimed comparison result holds for all the small triangles, the deformation lemma 2.3 extends the inequalities to Δ and $\bar{\Delta}$. In view of the monotonicity principle 2.2 we have reduced the proof to showing:

(*) If $l_1 = \bar{l}_1 < r_\Delta$, $l_0 = \bar{l}_0 < \varepsilon/2$ and $\measuredangle \text{at } p_2 = \measuredangle \text{at } \bar{p}_2$, then $l_2 \leqq \bar{l}_2$

In order to see this, we extend $-\gamma_0'(l_0)$ and $-\bar{\gamma}_0'(l_0)$ to parallel vector fields along the edges γ_1 and $\bar{\gamma}_1$ respectively. They give rise to ruled surfaces c and $\bar{c}$. For example $c : \mathbb{R} \times [0, l_1] \to M^n$, $(s, t) \mapsto c(s, t)$ is characterized by the formulae:

$$c(0, t) = \gamma_1(t), \qquad c'(0, 0) = -\gamma_0'(l_0),$$
$$\nabla_{\dot{c}} c' \big|_{(0, t)} = 0, \qquad \nabla_{c'} c' = 0.$$

Here as usual a prime denotes a derivative with respect to s and a dot denotes a derivative with respect to t.

Observations. — (i) $\bar{\gamma}_2$ is contained in the image of $[0, \infty) \times [0, l_1]$ under $\bar{c}$. By notation $\bar{\gamma}_2(\bar{l}_2) = \bar{p}_1$ and $\bar{\gamma}_2(0) = \bar{p}_0$. Because of the Gauss-Bonnet theorem $\measuredangle(\bar{\gamma}_2', \bar{c}')$ is non-decreasing along $\bar{\gamma}_2$. Hence $\bar{\gamma}_2$ is contained in $\bar{V} := \bar{c}(U)$, where U stands for the cube $[0, \varepsilon/2) \times [0, l_1]$.

(ii) $c(U)$ is contained in $B(p_0, r_\Delta + 1)$, and therefore our choices above imply that in U there are no focal points on the geodesics $s \mapsto c(s, t)$. By construction the inequality

$$-k(d(p_0, c(s, t))) \geqq -k_{\varepsilon/2}(d(p_0, c(0, t))) = -k_{\varepsilon/2}(d(\bar{p}_0, \bar{c}(0, t))) \geqq -k_\varepsilon(d(\bar{p}_0, \bar{c}(s, t)))$$

holds for all $(s, t) \in U$. Hence Rauch's comparison theorem yields:

$$|\dot{c}^\perp| \leqq |\dot{\bar{c}}^\perp| \quad \text{on U.}$$

Here $\perp$ denotes the component orthogonal to the unit vectors c' resp. $\bar{c}'$. We conclude that the map $c \circ \bar{c}^{-1} : \bar{V} \to M^n$ is distance non-increasing.

It follows that $t \mapsto c \circ \bar{c}^{-1} \circ \bar{\gamma}_2(t)$ defines a curve in M^n which joins p_0 and p_1 and is not longer than $\bar{\gamma}_2$. This proves (*).

II. — Analyzing the decay condition

Throughout this chapter we assume $\lambda : [0, \infty) \to [0, \infty)$ to be a monotone non-increasing function. Roughly speaking the integral $b_0(\lambda)$ converges, iff $\lambda(r)$ decays a little quicker than r^{-2} for $r \to \infty$. We start making this observation more precise.

1.1. LEMMA. — *Whenever $b_0(\lambda)$ converges, there exist monotone non-increasing functions:*

$$\lambda_1 : r \mapsto \int_r^\infty \lambda(\rho)\, d\rho,$$

$$\lambda_2 : r \mapsto \int_r^\infty \lambda_1(\rho)\, d\rho,$$

$$\lambda_b : r \mapsto \int_r^\infty \rho.\lambda(\rho)\, d\rho = \lambda_2(r) + r.\lambda_1(r).$$

Moreover the following estimates hold: $(r \geqq 0)$

$$r^2.\lambda(r) \leqq 2.b_0(\lambda),$$

$$r.\lambda_1(r) \leqq b_0(\lambda),$$

$$\tilde{b}_0(\lambda) := \int_0^\infty \inf\{\lambda_1(r), \sqrt{\lambda(r)}\}\, dr \leqq \lambda_2(0) = b_0(\lambda).$$

Proof. — The expressions $\lambda_1(r)$ and $\lambda_b(r)$ obviously converge. The existence of $\lambda_2(r)$ follows from the theorems by Fubini and Tonelli:

$$\lambda_2(r) = \int_r^\infty \int_t^\infty \lambda(\rho)\, d\rho\, dt = \int_r^\infty (\rho - r).\lambda(\rho)\, d\rho = \lambda_b(r) - r.\lambda_1(r).$$

The remaining estimates are due to the computations:

$$r^2.\lambda(r) = 2.\lambda(r).\int_0^r \rho\, d\rho \leqq 2.\int_0^r \rho.\lambda(\rho)\, d\rho \leqq 2.b_0(\lambda)$$

and

$$r.\lambda_1(r) = \lambda_1(r).\int_0^r d\rho \leqq \int_0^r \lambda_1(\rho)\, d\rho \leqq b_0(\lambda).$$

1.2. *Remarks.* — (i) Almost the same computations give rise to the formulae:

$$\lim_{r \to \infty} r^2.\lambda(r) = 0$$

and:

$$\lim_{r \to \infty} r.\lambda_1(r) = 0.$$

(ii) Observe that for $T_c\lambda(r) := c^2 . \lambda(c . r)$, $r \geqq 0$, $c > 0$, one has:

$$(T_c\lambda)_1(r) = c . \lambda_1(c . r)$$

and

$$(T_c\lambda)_2(r) = \lambda_2(c . r).$$

Therefore the invariant $b_0(M^n)$ does not change, when the metric of the asymptotically non-negatively curved manifold is scaled with some global factor.

(iii) We point out that it does not depend on the choice of the base point 0 in a Riemannian manifold (M^n, g) whether the integral b_0 converges. However its numerical value is very sensitive with respect to the position of 0. This is related to the fact that b_0 does not detect certain curvature singularities at 0. Such a task would require much more refined numerical invariants.

Later on we shall need information about the models $M^2(-\lambda)$. This means basically that we have to study the Jacobi field equation:

$$(*) \qquad y''(r) = \lambda(r) . y(r).$$

In particular we are interested in monotone decaying solutions z_l with boundary values $z_l(0) = 1$ and $z_l(l) = 0$ (l any positive number) and in their limit z_∞, which is a positive, monotone decaying solution of (*).

2.1. Lemma. — *The following conditions are equivalent:* (i) $b_0(\lambda) < \infty$,

(ii) *for any solution* y *of equation* (*) *there exists* $y'(\infty) := \lim_{r \to \infty} y'(r)$.

Proof. — We shall only show that (i) implies (ii). We assume that $r > r_1 > 0$ and compute:

$$\begin{aligned} |y'(r) - y'(r_1)| &\leqq \int_{r_1}^{r} \lambda(\rho) . |y(\rho)| \, d\rho \\ &\leqq \lambda_1(r_1) . |y(r_1)| + \lambda_2(r_1) . |y'(r_1)| \\ &\qquad + \lambda_2(r_1) . \max\{|y'(\rho) - y'(r_1)| \,|\, r_1 \leqq \rho \leqq r\}. \end{aligned}$$

Provided that r_1 is sufficiently large, we know that $\lambda_2(r_1) \leqq 1/2$, and hence we obtain for $r > r_1 \gg 0$ that:

$$|y'(r) - y'(r_1)| \leqq 2 . \lambda_1(r_1) . |y(r_1)| + 2\lambda_2(r_1) . |y'(r_1)| =: C(r_1)$$

This already shows that y' remains bounded. We iterate the inequality and conclude that for $r > r_2 > r_1 \gg 0$ we have:

$$|y'(r) - y'(r_2)| \leqq 2 . \lambda_1(r_2) . |y(r_1)| + 2 . (r_2 . \lambda_1(r_2) + \lambda_2(r_2)) . (|y'(r_1)| + C(r_1)).$$

The right-hand side converges to zero for $r_2 \to \infty$.

2.2. Lemma:

$$(1 + b_0(\lambda)) . z_\infty(\infty) \leqq 1.$$

This estimate is an obvious consequence of the following formulae:

$$z'_{\infty}(r)=\int_r^{\infty}\lambda(\rho).z_{\infty}(\rho)\,d\rho,$$

$$z_{\infty}(r)=z_{\infty}(\infty)+\int_r^{\infty}(\rho-r).\lambda(\rho).z_{\infty}(\rho)\,d\rho\geqq z_{\infty}(\infty).(1+\lambda_2(r)).$$

The Jacobi field z_{∞} is closely related to the invariant b_1; observe that the function $-z_{\infty}(r)^{-1}.z'_{\infty}(r)$ converges to zero for $r\to\infty$, and that it obeys the Riccati equation:

$$(**)\qquad u'(r)=u(r)^2-\lambda(r).$$

2.3. LEMMA. — *Let $b_0(\lambda)<\infty$; then there is a unique non-negative solution u of $(**)$ such that $u(r)\to 0$ for $r\to\infty$. Moreover one has the estimate:*

(i) $$0\leqq u(r)\leqq\min\{\lambda_1(r),\sqrt{\lambda(r)}\},$$

(ii) $$b_1(\lambda):=\int_0^{\infty}u(r)\,dr\leqq\tilde{b}_0(\lambda)\leqq b_0(\lambda).$$

Proof. — Consider the continuous functions u_l which vanish identically on $[l,\infty)$ and solve for $(**)$ on $[0,l]$. Since $0'=0\geqq-\lambda$ and $\lambda_1'=-\lambda\leqq\lambda_1^2-\lambda$, standard monotonicity arguments yield the estimate

$$0\leqq u_l(r)\leqq\lambda_1(r).$$

Therefore the limits

$$u(r):=\lim_{r\to\infty}u_l(r),$$

exist and the function u meets the desired conditions.

(i) It is also easy to verify that the functions u_l are monotonic and that hence $u_l(r)\leqq\sqrt{\lambda(r)}$.

(ii) This inequality is clear from the definitions.

2.4. *Remarks.* — (i) $z_{\infty}(r)=\exp\left(-\int_0^r u(\rho)\,d\rho\right)\geqq z_{\infty}(\infty)>0.$

(ii) By lemma 2.2 and lemma 2.3 (ii) it is clear that all our invariants associated to a function λ are *equivalent* in some non-linear sense:

$$b_1(\lambda)\leqq\tilde{b}_0(\lambda)\leqq b_0(\lambda)\leqq\exp(b_1(\lambda))-1.$$

Moreover all the invariants depend on the function λ in a monotone way.

In chapter III there will be a situation where some uniform control on a family of model spaces $M^2(-\lambda)$ is required. This estimate can be done comparing the solutions z_l of $(*)$ to the function z_{∞}. For the sake of brevity we shall use the notation:

$$\beta:=z_{\infty}(\infty)=\exp(-b_1(\lambda)).$$

2.5. LEMMA:

(i) $$\beta \leqq z_\infty(r) \leqq 1, \qquad 0 \leqq r < \infty,$$

(ii) $$\left(1-\frac{r}{l}\right).z_\infty(l) \leqq z_l(r), \qquad 0 \leqq r \leqq l,$$

(iii) $$\frac{\beta}{l} \leqq -z_l'(l) \leqq -z_l'(0) \leqq -z_\infty'(0)+\frac{1}{l}.z_\infty(l) \leqq \frac{1}{l}+\lambda_1(0).$$

Proof. — (i) *cf.* remark 2.4 (i).

(ii) The difference $\Delta z := z_\infty - z_l$ also solves the differential equation $(*)$. As it is non-negative on $[0, \infty)$, it is a convex function:

$$\Delta z(r) \leqq \frac{r}{l}.\Delta z(l)+\left(1-\frac{r}{l}\right).\Delta z(0)=\frac{r}{l}.z_\infty(l)$$

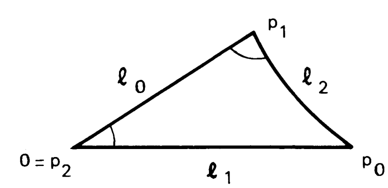
$$\Rightarrow \quad z_l(r) \geqq z_\infty(r)-\frac{r}{l}.z_\infty(l) \geqq \left(1-\frac{r}{l}\right).z_\infty(l).$$

(iii) Using part (ii) and monotonicity, the first, the second, and the last inequality are obvious. In order to obtain the third inequality, we use the convexity of Δz and compute:

$$z_\infty'(0)-z_l'(0)=\Delta z'(0) \leqq \frac{1}{l}.\Delta z(l)=\frac{1}{l}.z_\infty(l).$$

III. — Geodesic triangles and the number of ends in asymptotically non-negatively curved manifolds

The generalized triangles which have one of their vertices at the base point 0 of M^n form a rather distinguished class of objects. They might serve as tools to study the properties of M^n; in view of packing arguments their angle at 0 deserves special interest. At a first look this very angle seems to cause difficulties: there might be conjugate points which prevent one from controling the Jacobi fields along a family of geodesics emanating from 0; moreover there is no hypothesis on the cut-locus, and hence one does not know the lower curvature bound along these geodesics explicitly. Nevertheless the generalized Toponogov theorem allows for some rough estimates.



1. PROPOSITION. — *Let a,* $\varepsilon \in (0, 1)$ *and let* $\Delta = (p_0, p_1, p_2)$ *be a generalized geodesic triangle in an asymptotically non-negatively curved manifold* M^n. *Suppose moreover that*

$l_0 \leqq (1-\varepsilon) . l_1$ *and that* p_2 *is the base point* 0 *of* M^n. *Then the following estimates hold:*

(i) $$\cos(\sphericalangle at\, 0) \geqq \sqrt{1-a^2 . \beta^2 . \varepsilon^2} \Rightarrow l_2 \leqq l_1 - l_0 . \sqrt{1-a^2}.$$

(ii) $$\cos(\sphericalangle at\, p_1) \geqq -\sqrt{1-a^2} \Rightarrow l_1 \leqq l_2 + l_0 . \sqrt{1-a^2 . \beta^2 . \varepsilon^2}$$

(iii) $$\sphericalangle at\, 0\ acute, \Rightarrow |\sin(\sphericalangle at\, 0)| \geqq \beta^2 . \varepsilon^2 . |\sin(\sphericalangle at\, p_1)|.$$

Proof. — We put $l := l_1 = d(p_0, 0)$ and $k(r) := \lambda(|l-r|)$, $r>0$. Making use of the triangle inequality and the monotonicity of λ, we see that for all p in M^n:

(1.1) $$\text{curvatures at } p \geqq -\lambda(d(p, 0)) \geqq -k(d(p, p_0)).$$

(i) We can apply the generalized comparison theorem (*cf.* I.3.2.*b*) and reduce things to a problem in the model space $M^2(-k)$, where the radial Jacobi field y is a multiple of the function $z_l(l - .)$ defined in section II.2. We consider the function r along the edge $\bar{\gamma}_0$. This unit speed geodesic joins $\bar{p}_1 = \bar{\gamma}_0(0)$ and $\bar{p}_2 = \bar{\gamma}_0(l_0)$. By monotonicity we may restrict to the case where:

$$r'(l_0) = \cos(\sphericalangle at\, \bar{p}_2) = \sqrt{1-a^2 . \beta^2 . \varepsilon^2}.$$

The conservation law I.2.1 (iii) becomes:

(1.2) $$z_l(l-r)^2 . (1-r'^2) = \text{Const.}$$

Observing that $r(l_0) = \bar{l}_1 = l_1$, we obtain:

$$a^2 . \beta^2 . \varepsilon^2 = z_l(0)^2 . (1 - \cos^2(\sphericalangle \text{at}\, \bar{p}_2))$$
$$= z_l(l - r(s))^2 . (1 - r'(s)^2)$$
$$\geqq \varepsilon^2 . z_\infty(l)^2 . (1 - r'(s)^2), \qquad 0 \leqq s \leqq l_0;$$

here the inequality is due to lemma II.2.5 (ii) and to the fact that $r(s) \geqq l_1 - l_0 \geqq \varepsilon . l$. We conclude that

$$a^2 \geqq 1 - r'(s)^2, \qquad 0 \leqq s \leqq l_0.$$

Therefore the continuous function r' does not vanish in the interval $[0, l_0]$, and there we get:

$$r' \geqq \sqrt{1-a^2}.$$

Hence $\bar{l}_2 = r(0) \leqq r(l_0) - l_0 . \sqrt{1-a^2} = l_1 - l_0 . \sqrt{1-a^2}$.

(ii) Here an indirect proof works: assume that $l_1 > l_2 + l_0 . \sqrt{1-a^2 . \beta^2 . \varepsilon^2}$. Again we make use of the generalized Toponogov theorem. Exchanging the roles of p_1 and p_2, we obtain a triangle $\bar{\Delta} = (\bar{p}_0, \bar{p}_1, \bar{p}_2)$ in the model $M^2(-k)$ such that

$$\bar{l}_0 = l_0, \qquad \bar{l}_1 \geqq l_1, \qquad \bar{l}_2 = l_2, \qquad \cos(\sphericalangle \text{at}\, \bar{p}_1) = -\sqrt{1-a^2}.$$

The function r along $\bar{\gamma}_0$ obeys $r(0)=l_2$ and $r'(0)=\sqrt{1-a^2}$. Since $l-\bar{l}_2 \leqq l_0 \leqq (1-\varepsilon).l$, we can deduce from formula 1.2 that

$$z_l(l-r)^2.(1-r'^2)=z_l(l-\bar{l}_2)^2.(1-r'(0)^2) \geqq a^2.z_l((1-\varepsilon).l)^2 \geqq a^2.\beta^2.\varepsilon^2.$$

As long as $r(s) \leqq l$, we have $z_l(l-r) \leqq 1$, and hence:

$$r(s) \leqq l_2+s.\sqrt{1-a^2.\beta^2.\varepsilon^2}.$$

The standard continuity argument now yields the contradiction

$$\bar{l}_1=r(l_0) \leqq l_2+l_0.\sqrt{1-a^2.\beta^2.\varepsilon^2}<l_1.$$

(iii) Put $a_1:=\sin(\measuredangle \text{at } p_1)$; then we conclude with the aid of part (ii) that

$$l_1 \leqq l_2+l_0.\sqrt{1-a_1^2.\beta^2.\varepsilon^2}.$$

Reversing the implication in (i), we obtain:

$$\cos(\measuredangle \text{at } 0) \leqq \sqrt{1-a_1^2.\beta^4.\varepsilon^4}.$$

The proof is then finished, as the angle at 0 is acute by hypothesis.

2. We recall that two curves $c_1, c_2 : [0, \infty) \to M^n$ are said to be *cofinal*, if and only if for every compact set $K \subset M^n$ there is some $t>0$ such that $c_1(t_1)$ and $c_2(t_2)$ lie in the same connected component of $M^n K$ for all $t_1, t_2 \geqq t$. An equivalence class of cofinal curves is called an *end of* M^n.

2.1. *Elementary Properties.* — Any family of relatively compact open sets $(U_i)_{i \in \mathbb{N}}$ which exhaust M^n, i. e. which obey $U_i \subset\subset U_{i+1}$ and $\bigcup_i U_i=M^n$, defines a bijection:

$$\{\text{ends E of } M^n\} \overset{\cong}{\leftrightarrow} \{(E_i)_{i \in \mathbb{N}} \mid E_{i+1} \subset E_i \text{ and } E_i \text{ is a connected component of } M^n \setminus \bar{U}_i\}.$$

Notice that for each of these inverse systems $E=(E_i)_{i \in \mathbb{N}}$ the sets E_i are non-empty and their closures $\bar{E}_i$ in M^n are non-compact. Moreover, if M^n has only finitely many ends, then there is some $i_0>0$ such that all the inverse systems $(E_i)_{i \in \mathbb{N}}$ stabilize for $i \geqq i_0$, i. e. $E_{i_0} \setminus E_i$ bounded.

2.2. — Given a point $p \in M^n$, then any end E of M^n contains a *ray* γ emanating from p; recall that by definition γ is a geodesic $[0, \infty) \to M^n$ such that each of its segments is shortest and that $\gamma(0)=p$.

2.3. — Given any two distinct ends E^1 and E^2 of M^n, there is a *line* $\gamma : \mathbb{R} \to M^n$ such that the rays $\gamma_\pm : [0, \infty) \to M^n$, $t \mapsto \gamma(\pm t)$ are contained in E^1 and E^2 respectively.

2.4. — As is the case for the *ideal boundary* in the theory of non-compact surfaces, the set of ends carries a natural topology; a basis for the open sets is parametrized by the non-compact closed subsets $C \subset M^n$:

$$U_C := \{\text{ends } (E_i)_{i \in \mathbb{N}} \mid E_i \subset C \text{ for } i \text{ sufficiently large}\}.$$

In this way {ends E of M^n} becomes a compact, separable, totally-disconnected space. (*cf.* [1], [12].)

3. THEOREM. — *Every asymptotically non-negatively curved manifold* M^n *has at most finitely many ends. More precisely:*

$$\#\{\textit{ends}\, E\, \textit{of}\, M^n\} \leqq 2 . \pi^{n-1} . \exp((n-1) . b_1).$$

Here b_1 is the invariant introduced in II. 2. 3.

Proof. — For each end E of M^n we pick a unit-speed ray γ_E which emanates from 0 and is contained in E. We consider the set of unit vectors $v_E := \gamma_E'(0)$ in $T_0 M^n$. It is a consequence of proposition 1 that for any unit vector $v \in T_0 M^n$ which is sufficiently close to some v_E the geodesic $\gamma : [0, \infty) \to M^n$, $t \mapsto \exp_0(t . v)$ is contained in the end E; in some more detail one obtains:

$$\langle v, v_E \rangle = \sqrt{1-a^2 . \beta^2 . \varepsilon^2}, \qquad 0 < a^2, \varepsilon^2 < 1$$
$$\Rightarrow \quad d(\gamma_E(t), \gamma((1-\varepsilon) . t)) \leqq t - (1-\varepsilon) . t . \sqrt{1-a^2} =: q_{a, \varepsilon} . t,$$

where $0 < q_{a, \varepsilon} < 1$; therefore γ is contained in E provided $\langle v, v_E \rangle > \sqrt{1-\beta^2}$. Thus the balls B_E^S in the unit sphere $S^{n-1} \subset T_0 M^n$ with centres v_E and radii 1/2.arcsin (β) are pairwise disjoint. Notice that arcsin $(\beta) \geqq \beta = \exp(-b_1)$; so the claimed bound on the number of ends is a direct consequence of the following well-known packing lemma.

3.1. LEMMA. — *Let* $0 < \rho \leqq \pi/2$; *then the number of disjoint balls* $B^S(\rho) \subset S^{n-1}$ *with radius* ρ *does not exceed*

$$\frac{\operatorname{vol} S^{n-1}}{\operatorname{vol} B^S(\rho)} \leqq 2 . \left(\frac{\pi}{2 . \rho}\right)^{n-1}.$$

IV. — Surfaces and other examples

In this chapter we are going to discuss the hypothesis and conclusions of theorem B. A first set of examples shows that for surfaces the theorem is definitely wrong when the integral b_0 diverges (*cf.* IV. 1). Moreover we shall see that the given bounds on the number of ends and on the Betti numbers are reasonable in a certain sense: in section IV. 2 we construct surfaces with large invariants b_0 and b_1 such that theorem B overestimates the number of ends by not more than a factor of 2π; in section IV. 3 we consider the higher-dimensional case and give a set of examples where the bounds actually grow exponentially in $n . b_1(M^n)$.

1. THEOREM. — *Let* $\lambda : [0, \infty) \to [0, \infty)$ *be a continuous function such that the integral*

$$\int_0^\infty r . \lambda(r)\, dr$$

diverges. Then every non-compact, connected surface M^2 *with base point* 0 *carries a complete* C^2*-metric g with curvature*

$$(*) \qquad \kappa(p) = -\lambda(d(0, p)) \qquad \textit{at any point} \quad p \in M^2.$$

Remarks 1.1. — Obviously $\mathbb{R}^2$ becomes the surface of revolution $M^2(-\lambda)$, which has been described in chapter I.1.

1.2. — Suppose that the curvature of a surface (M^2, g) with base point 0 obeys condition $(*)$ above. Then the complement of the cut-locus of 0 is isometric to a tree-like open subset in $M^2(-\lambda)$; the isometry is given by $\exp_0$ and the obvious identifications.

Moreover the generic cut points, i. e. those cut points which are joined to 0 by precisely two minimizing geodesics, lie on open geodesic segments in (M^2, g).

1.3. — In order to reverse the preceeding observation and construct some more examples, we look at two non-intersecting geodesics γ_1 and γ_2 in $M^2(-\lambda)$ which have equal distance to the base point. Notice that they can be mapped onto each other by an isometry φ of $M^2(-\lambda)$. We take that component of $M^2(-\lambda)\setminus(\gamma_1 \cup \gamma_2)$ which contains the pole. We take its closure and glue the boundary components γ_1 and γ_2 by means of φ. The differentiable structure of the quotient manifold M^2 is conveniently described using normal exponential coordinates around the geodesics γ_1 and γ_2. The quotient metric g on the surface M^2 turns out to be of class C^2; the reason is that the curvature function of $M^2(-\lambda)$ is invariant under the clutching map φ.

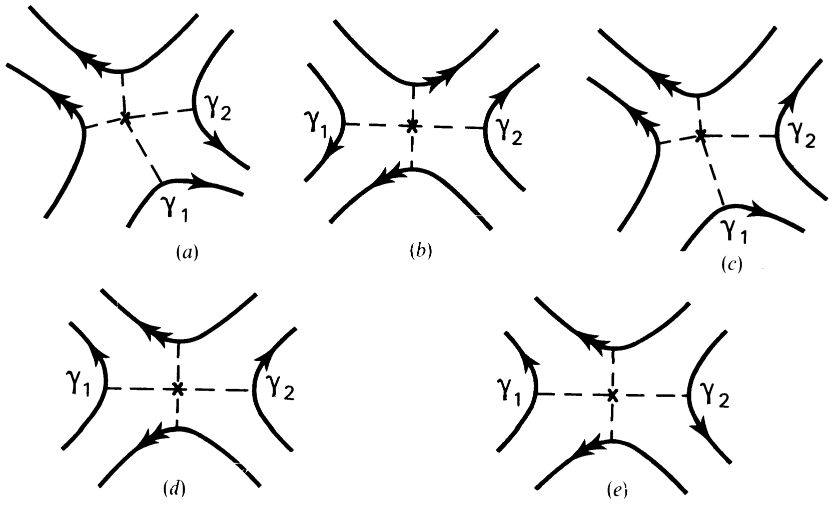
1.4. — This construction can be iterated as long as one can find an appropriate pair of geodesics γ_1, γ_2 in

$$(T_0 M_j^2)^{\text{int}} := \{ x \in T_0 M_j^2 \mid 0 \text{ and } x \text{ are joined by an arc which does not contain a cut point.} \}$$

It gives rise to a surface M_{j+1}^2, which differs from M_j^2 topologically, and the metric g_{j+1} still obeys condition $(*)$. Depending on the position of the geodesics and the orientation of φ there are four distinct cases:

(i) If γ_1 and γ_2 lie in the same end E_j of (M_j^2, g_j), then either E_j is split into two ends E_{j+1}^1 and E_{j+1}^2 (*Fig. a* and *Fig. b*) or a cross cap is attached to E_j (*Fig. c*).

(ii) If γ_1 and γ_2 lie in different ends E_j^1 and E_j^2, then these ends are glued; a handle (*Fig. d*) resp. a Kleinian bottle (*Fig. e*) is attached.



(*a*) (*b*) (*c*) (*d*) (*e*)

1.5. — Basically it is the function λ which determines how often the constructions 1.4 (i) and 1.4 (ii) can be applied. Let us assume that

$$\int_0^\infty r.\lambda(r)\,dr$$

diverges. Then the integral of the curvature over an arbitrary sector in $M^2(-\lambda)$ also diverges, and this surface turns out to be a visibility manifold (*cf.* [9]); the angle $\alpha(\gamma)$ of the sector in which a geodesic γ is seen from the pole 0 decreases to zero when $\mathrm{dist}(0, \gamma) \to \infty$. Hence in any conical end of a surface M_j^2 one can go out far enough and find geodesics γ_1 and γ_2, suitable for the constructions 1.4 (i) and 1.4 (ii). Moreover it is possible to pick these geodesics in such a way that the manifold M_{j+1}^2 has only conical ends, provided M_j^2 had.

Therefore — whenever the above integral of λ diverges — metrical considerations do not impose any conditions on the combinatorial patterns for iterating the constructions 1.4.

Proof of the Theorem. — Standard classification results imply that in remark 1.5 we have constructed all non-compact, orientable and non-orientable surfaces which have finite genus and finitely many ends. Next we consider a sequence of surfaces M_j^2 and perform all the surgery simultaneously. This yields a manifold M_∞^2 which carries a complete C^2-metric g_∞ obeying condition ($*$). Our goal is to employ the classification of surfaces and show that we have constructed representatives for all homeomorphisms types (= diffeomorphism types, since we are in dimension two). We recall that any exhaustion by relatively compact open sets $U_i \subset\subset U_{i+1}$ turns the ideal boundary C of a surface M^2 into the same totally-disconnected, separable, compact topological space and that it moreover singles out nested subspaces $A \subset B \subset C$ which represent the infinitely non-orientable ends and the ends with infinite genus respectively. Richards [12] has shown that the topological type of a surface is determined by the following data:

(i) the triple of totally-disconnected, separable, compact sets $A \subset B \subset C$,

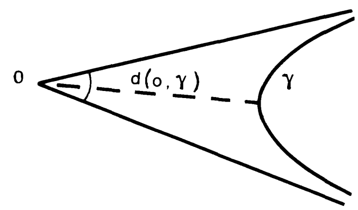
(ii) the orientability type (four choices; dispensible, if $A \neq \emptyset$),

(iii) the genus (dispensible, if $B \neq \emptyset$).

We point out that the particular choice of the exhaustion $(U_i)_{i=1}^\infty$ determines a basis for the topology of A, B, and C. Conversely, fixing such a basis and thinking of M_∞^2 as being exhausted by metrical balls around the base point 0, we get a combinatorial pattern according to which we can iterate the constructions 1.4 and obtain a surface M_∞^2 with the prescribed classificational data.

2. — Next we consider a function λ such that the integral $b_0(\lambda)$ is finite. We are going to construct a surface M^2 which has as many ends as possible. For this purpose we look at the geodesic triangle Δ in $M^2(-\lambda)$ which is given by the pole 0 and an arbitrary geodesic γ and which has two vertices at infinity. The Gauss-Bonnet theorem yields:

$$(**) \qquad \sphericalangle \text{ at } 0 = \pi - \int_\Delta \lambda(d(0, .))\,d\text{vol}.$$



The differential equation $y''=\lambda . y$ and the initial data $y(0)=0$, $y'(0)=1$ as usual describe a radial Jacobi field and determine polar coordinates. Moreover it is possible to define another invariant

$$b_2(\lambda) := \lim_{r \to \infty} \ln y'(r).$$

We can now proceed and estimate the right-hand side of (∗∗):

$$\sphericalangle \text{at } 0 \leqq \pi - \int_0^{d(0,\gamma)} \lambda(r) . y(r)\, dr . \sphericalangle \text{at } 0.$$

Hence $(\sphericalangle \text{at } 0) . y'(d(0,\gamma)) \leqq \pi$, and we can pick at least $2.[y'(d)]$ non-intersecting geodesics γ in $M^2(-\lambda)$, each of them with distance d to the base point 0. Applying the construction from 1.4 (i) as often as we can, we obtain a surface with $[y'(d)]$ ends. Finally we pass to the limit $d \to \infty$:

2.1. Proposition. — *Whenever the invariant integral* $b_0(\lambda)$ *of some function* $\lambda : [0, \infty) \to [0, \infty)$ *is finite, then there exists a complete surface* (M^2, g) *which has at least* $\exp(b_2(\lambda)) - 1$ *ends and whose curvature obeys the condition*

$$\kappa(p) = -\lambda(\text{dist}(p, \textit{base point})) \quad \text{for all } p \in M^2.$$

This proposition shows that for surfaces the previously given upper bound on the number of ends is *sharp* up to a factor of at most 2π; we pick λ to be the characteristic function of $[0, d]$ and compute:

$$b_0(\lambda) = \frac{1}{2} . d^2, \qquad b_1(\lambda) = \ln \cosh(d), \qquad b_2(\lambda) = \ln \sinh(d);$$

asymptotically b_1 and b_2 coincide.

3. — Our last examples shall demonstrate that for asymptotically non-negatively curved manifolds M^n the number of ends and the sum of the Betti numbers can grow exponentially in $n . b_1(M^n)$ each. We point out that Riemannian products of the above surfaces are totally inadequate in either case. Partially this is due to the fact that the function λ changes when passing to products.

We are going to construct some tree-like looking objects. Roughly speaking the desired growth in $n . b_1(M^n)$ is achieved by using building blocks of the same type only. In order to describe these pieces it is convenient to think of a hypersurface in

$\mathbb{R}^{n+1}$ which is obtained by glueing cylinders $\mathbb{R}^+ \times S^{n-1}$ perpendicular onto a hyperplane $\mathbb{R}^n$ where appropriate balls have been removed. The curvature is kept bounded by plugging in some intermediate tubes. Again we use the "same" tube everywhere, and a packing argument assures that the number of ends of a single building block grows exponentially in n.

3.1. *The intermediate tubes.* — We fix some $t_0>0$ and consider the warped products $\mathrm{T}b^n(t_0)=([0, t_0]\times S^{n-1}, ds^2)$, where the metric is defined by:

$$ds^2 = dt^2 + \sinh^{-2}(t_0).\cosh^2(t_0-t).d\omega^2$$

Here $d\omega^2$ denotes the standard metric on S^{n-1}.

PROPERTIES:

(i) $\operatorname{diam} \mathrm{Tb}^n(t_0) \leqq \max\limits_{0\leqq t\leqq t_0} t+\pi.\sinh^{-1}(t_0).\cosh(t_0-t)$

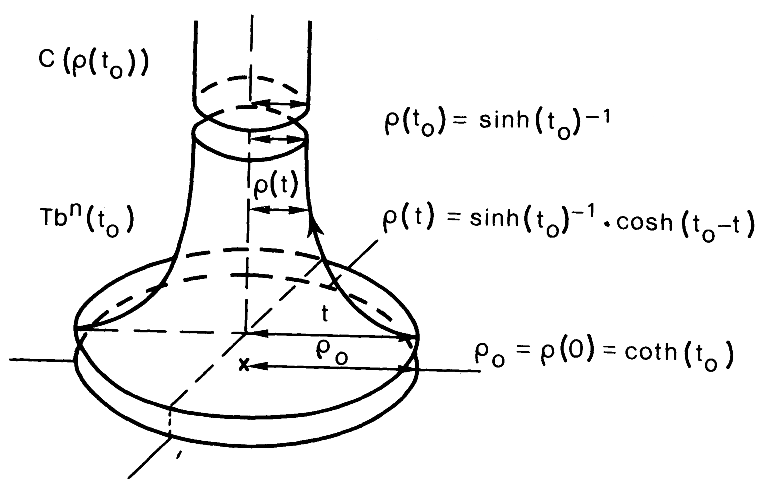
$$=\pi.\coth(t_0)+\max\left\{0,\ t_0-\pi.\tanh\left(\frac{1}{2}.t_0\right)\right\}$$

(ii) the tubes $\mathrm{Tb}^n(t_0)$ can be embedded as rotationally symmetrical hypersurfaces in $\mathbb{R}^{n+1}$.

(iii) $\mathrm{Tb}^2(t_0)$ has constant curvature equal to -1, and for $n>2$ the $\mathrm{Tb}^n(t_0)$ have sectional curvatures $\geqq -1$.

(iv) the boundary components $\{0\}\times S^{n-1}$ and $\{t_0\}\times S^{n-1}$ are spheres with constant curvature $\tanh^2(t_0)$ and $\sinh^2(t_0)$ respectively ($n>2$). As submanifolds in $\mathrm{Tb}^n(t_0)$ they have principal curvatures $\tanh(t_0)$ and 0 respectively.

(v) the tube $\mathrm{Tb}^n(t_0)$ can be doubled in an analytical way along the boundary component $\{t_0\}\times S^{n-1}$. The same boundary component of the tube can be glued isometrically to the boundary of a cylinder $C(\sinh^{-1}(t_0)) := \sinh^{-1}(t_0).(\mathbb{R}^+\times S^{n-1})$ with radius $\sinh^{-1}(t_0)$; this time curvature is only bounded, but non-continuous.



(vi) at $\{0\} \times S^{n-1}$ the tube $Tb^n(t_0)$ can be glued with bounded, but non-continuous curvature to $\mathbb{R}^n \setminus B(x, \coth(t_0))$, $x \in \mathbb{R}^n$ arbitrary.

3.2. *The building blocks* A^n. — Let $t_0 > 0$, $r_0 \geqq 2$, $\rho_0 := \coth(t_0)$, and let B_0 be the ball $B(0, \rho_0)$ in $\mathbb{R}^n$. We pick a maximal family of mutually disjoint open balls $B_1, \ldots, B_N$ with radius ρ_0 in the subset $B(0, (r_0+1).\rho_0) \setminus B_0$. We remove all $N+1$ balls $B_0, \ldots, B_N$ and—as described in 3.1 (vi)—attach tubes $Tb^n(t_0)$ to the boundaries $\partial B_0, \ldots, \partial B_N$. The boundary of the resulting manifold A^n consists of the spheres $\{t_0\} \times S^{n-1}$ in the attached tubes. For latter use it is convenient to single out the boundary of the central tube which has been glued to ∂B_0; we shall call it a^n.

PROPERTIES:

(i)
$$\#\{\text{ends of } A^n\} = 1,$$
$$\#\{\text{boundary components of } A^n\} = N+1;$$

(ii) in $\mathbb{R}^n$ the enlarged balls $2.B_j$, $0 \leqq j \leqq N$, cover $B(0, r_0.\rho_0)$; hence:

$$N+1 \geqq \left(\frac{1}{2}.r_0\right)^n;$$

(iii) in $\mathbb{R}^n$:

$$\operatorname{dist}(B_0, B_j) = (r_0-2).\rho_0; \qquad 1 \leqq j \leqq N.$$

Moreover for each j there exists a curve in $\mathbb{R}^n \setminus \bigcup_{\nu \geqq 0} B_\nu$ which joins ∂B_0 and ∂B_j and which has length $\geqq (r_0-2).\rho_0$;

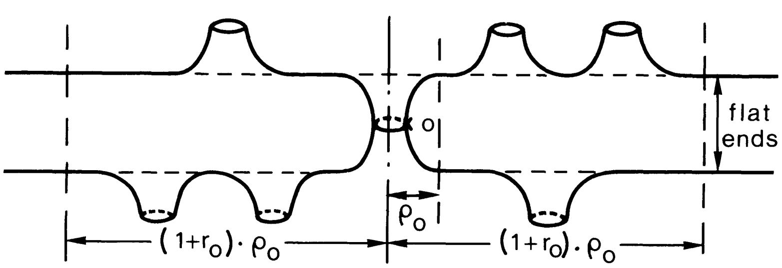
(iv) for any point $p \in A^n$ which is non-flat and for any point $p \in \partial A^n$ one has:

$$d(p, a^n) \leqq \frac{\pi}{2}.(r_0-2).\rho_0 + \operatorname{diam} Tb^n(t_0) + t_0;$$

(v) the sectional curvatures of A^n are $\geqq -1$.

3.3. *The trees* $A^n(\mu)$. — We use the following inductive construction:

(i) We glue two copies of the manifold A^n by identifying their boundary spheres a^n. On this sphere we pick a base point 0 for the quotient $A^n(1)$.



(ii) We assume that $A^n(\mu)$ has been constructed and that all its $2.N^\mu$ boundary components are totally geodesic spheres with diametre equal to $\pi.\sin h^{-1}(t_0)$. Then we can attach to each of these spheres the central boundary a^n of a new copy of A^n, and thus we can glue $2.N^\mu$ copies of A^n to $A^n(\mu)$. We define this larger manifold to be the $\mu+1^{st}$ generation object $A^n(\mu+1)$.

3.4. *The manifolds* $M^n(\mu)$. — We obtain non-compact, complete Riemannian manifolds $M^n(\mu)$ by glueing to each boundary sphere of $A^n(\mu)$ a cylinder $C(\sinh^{-1}(t_0))$.

PROPERTIES:

(i) $$\#\{\text{ends of } M^n(\mu)\}=2.\frac{N^{\mu+1}-1}{N-1},$$

$$\beta_{n-1}(M^n(\mu))=\#\{\text{ends of } M^n(\mu)\}-1\geqq 2.N^\mu\geqq\left(\left(\frac{1}{2}.r_0\right)^n-1\right)^\mu;$$

(ii) the sectional curvatures of $M^n(\mu)$ at any point p are bounded from below by $-\lambda_{(\mu)}(d(p, 0))$, where $\lambda_{(\mu)}$ is the characteristic function of the interval $[0, d_\mu]$, and d_μ is given by:

$$d_\mu := \pi.\sinh^{-1}(t_0)+\mu.\left(\frac{\pi}{2}.r_0.\coth(t_0)+t_0+\max\left\{0,\ t_0-\pi.\tanh\left(\frac{1}{2}.t_0\right)\right\}\right);$$

(iii) for any integer $\mu\geqq 1$ the following inequalities hold:

$$b_1(M^n(\mu))\leqq d_\mu+\ln(1+e^{-2})-\ln(2),$$

$$\ln\beta_{n-1}(M^n(\mu))\geqq\mu.\left(n.\ln\left(\frac{r_0}{2}\right)+\ln\left(1-\left(\frac{r_0}{2}\right)^{-n}\right)\right).$$

Specializing to the case $r_0=7$ and $t_0=2.5$, we obtain:

$$\ln\beta_{n-1}(M^n(\mu))\geqq\frac{6}{5}.n.\mu\geqq\frac{1}{2}.n.b_1(M^n(\mu)).$$

Notice that we still have the freedom to pick μ large. Therefore, when working in terms of the invariant $b_1(M^n)$ and the dimension, any estimate on the number of ends or on the $n-1^{st}$ Betti number has to grow at least exponentially in $n.b_1(M^n)$. Such a result has been achieved in Theorem B.

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