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ASYMPTOTIC COMPLETENESS IN LONG RANGE SCATTERING II

By PL. MUTHURAMALINGAM AND KALYAN B. SINHA

ABSTRACT. — Asymptotic completeness is proved for $-\Delta/2+W_S(Q)+W_L(Q)$ on $L^2(R^\nu)$, $\nu \geqq 3$. Here W_S is a short range potential while W_L is a C^m long range potential for m large enough and W_L behaves like $(1+|x|)^{-\alpha}$ at ∞ for $1/2<\alpha<1$.

1. Introduction

The existence of modified wave operators in long range scattering has been known for some time ([1] to [5]). The completeness for these wave operators for Coulomb potential was shown by Dollard [6] using an eigenfunction expansion. Various spectral properties have been obtained by Lavine [7], Ikebe [8] and Saito [9]. The completeness problem for a large class of long range potentials was studied by Weidmann [10] and Georgescu [11] for the spherically symmetric case and by Ikebe and Isozaki [12] and Kitada [13] for more general situation. Also Agmon [14] has given a proof of completeness using eigenfunction expansion method. In the context of the algebraic theory of scattering (*see* [1], Chap. 13), completeness for spherically symmetric potential was established by Thomas [15].

More recently Enss ([16], [17]) and Perry [18] have proved completeness and absence of singular continuous spectrum for long range Schrodinger operators on $L^2(R^\nu)$ using time dependent methods. While Enss obtained his results for $\alpha>(2\nu+2)/(2\nu+3)$, Perry assumed a dilation analytic long range part as well as $\alpha>1/2$. In this paper we prove the existence and completeness of the wave operators for $\alpha>1/2$ without assuming dilation analyticity.

In section 2 we introduce the notations and state the assumptions made on the potentials. We also collect here some results on asymptotic evolution of certain observables and on smooth perturbations from [19] and [7] respectively. These are used in section 6 to get higher order estimates in the asymptotic evolution of the observables. In section 3, the generalised coherent states are introduced and their properties stated. The section 4 is devoted to the proof of the main result while the more technical parts in it are relegated to sections 5 and 6. In the appendix the existence of the wave operators are obtained as a simple corollory.

2. Notations and preliminaries

Let $Q=(Q_1, \ldots, Q_\nu)$, $P=(P_1, \ldots, P_\nu)$ $P_j=-iD_j=-i\partial/\partial x_j$ be the self adjoint operator families on $L^2(R^\nu)$, $\nu \geqq 3$ representing the position and momentum observables respectively. Let H_0 be the unique self adjoint extension of $-\Delta/2 \equiv 1/2\, P^2$, the free Hamiltonian and let

$$(2.1) \qquad \begin{cases} H' = H + W_S(Q), \\ H = H_0 + W(Q). \end{cases}$$

In (2.1) the real potentials W_S and W have the following properties:

$$(2.2) \qquad \begin{cases} W_S(x) = (1+|x|)^{-1-\varepsilon} V_S(x) \text{ for some } \varepsilon \text{ in } (0,1) \text{ where } V_S, \\ \quad \text{is in } L^p(R^\nu) + L^\infty(R^\nu) \text{ for some } p > \max\{\nu/2, 1\}. \end{cases}$$

Furthermore, there exists α in $(1/2, 1)$ and an integer m with $m > 4 + 2\nu$ such that W is a $C^m(R^\nu)$ function and

$$(2.3) \qquad |D^n W(x)| \leqq K_n (1+|x|)^{-|n|-\alpha}$$

for all multi-indices $n=(n_1, \ldots, n_\nu)$ of length $|n| = n_1 + n_2 + \ldots + n_\nu \leqq m$.

Remark. — Note that $W_S(Q)(H_0+1)^{-1}$ is compact. For $\nu=3$ the Coulomb potential $W_c(x) = k|x|^{-1}$, k constant, can be written in the form $W_S + W$ where W_S and W satisfy (2.2) and (2.3) respectively (*see* e. g. [1], p. 531).

It is well known [1], [20], [21] that H', H are self adjoint on $D(H') = D(H) = D(H_0)$; also $(H'+i)^{-1}(H+i)$, $(H+i)^{-1}(H_0+i)$, $(H'+i)^{-1}(H_0+i)$, $(H+i)^{-1}(H'+i)$, $(H_0+i)^{-1}(H+i)$, $(H_0+i)^{-1}(H'+i)$ are all bounded. Since $(H'+i)^{-1} - (H+i)^{-1}$, $(H'+i)^{-1} - (H_0+i)^{-1}$, $(H+i)^{-1} - (H_0+i)^{-1}$ are compact we have that $\varphi(H') - \varphi(H)$, $\varphi(H') - \varphi(H_0)$, $\varphi(H) - \varphi(H_0)$ are compact for any bounded continuous function on R vanishing at $\pm\infty$.

For any self adjoint operator B on $\mathscr{H} = L^2(R^\nu)$ let $\mathscr{H}_{ac}(B)$, $\mathscr{H}_c(B)$, $\mathscr{H}_p(B)$ be the absolutely continuous, continuous, point subspaces of B and let $E_{ac}(B)$, $E_c(B)$, $E_p(B)$ be the corresponding orthogonal projections. Also let $B_c = BE_c(B)$, $B_{ac} = BE_{ac}(B)$. We adopt the following notations:

$$V'_t = \exp[-itH'],$$

$$V_t = \exp[-itH],$$

$$U_t = \exp[-itH_0],$$

$$X(t, P) = \int_0^t d\tau\, W(\tau P).$$

$X(t, P)$ will also be denoted by $X(t)$ when no confusion is possible. It is established in [1] and [2] that the modified wave operators

$$(2.4) \qquad \Omega'_{\pm} = s - \lim_{t \to \pm\infty} V_t'^{*} U_t \exp[-iX(t, P)],$$

exist; are isometries and satisfy the intertwining property

$$(2.5) \qquad V_t \Omega'_{\pm} = \Omega'_{\pm} U_t.$$

$$\Omega_{\pm} = s - \lim_{t \to \pm\infty} V_t^{*} U_t \exp[-iX(t, P)]$$

(2.6) and

$$\Omega_{\pm}(H', H) = s - \lim_{t \to \pm\infty} V_t'^{*} V_t E_{ac}(H).$$

The first type of wave operators is known to exist ([1], [2]) and we shall give an independent proof in the appendix. The second wave operators for relative scattering also exist and are complete by virtue of the results of [7]. Thus by the chain rule of wave operators it suffices to show that $\Omega_{\pm}$ are complete in order to conclude the completeness of $\Omega'_{\pm}$. This is precisely the content of section 4. Here we state the result of [7] which takes care of relative scattering and which serves as an input to the calculations of section 6.

THEOREM 2.1. — *Let* H, H′ *be as above. Then*

(i) $\mathscr{H}_{ac}(H') = \mathscr{H}_p(H')^{\perp}$, $\mathscr{H}_{ac}(H) = \mathscr{H}_p(H)$,

(ii) *The open interval* $(0, \infty)$ *does not contain any eigenvalue of* H.

(iii) *For* ψ *in* $C_0(0, \infty)$, f *in* $L^2(R^{\nu})$ *and* $\gamma > 1/2$ *there exists a constant depending only on* ψ *and* γ *such that*

$$\int_{-\infty}^{\infty} dt \, \| (1 + |Q|)^{-\gamma} \psi(H) V_t f \|^2 \leqq K \| f \|^2.$$

(iv) $\Omega_{\pm}(H', H)$ *exist and are complete i.e. both* $\Omega_{+}(H', H)$ *and* $\Omega_{-}(H', H)$ *have the same range* $\mathscr{H}_{ac}(H')$.

Proof. — (ii) is the Kato-Agmon-Simon Theorem of [23]. The rest of the conclusions are essentially the content of the Theorem 1 of [7].

Q.E.D.

We need also a result of [19] which we state as a theorem.

THEOREM 2.2. — *Let* A *be the generator of the dilation group* Y_{θ} *given by* $(Y_{\theta} g)(x) = \exp(-\nu\theta/2)\ g(xe^{-\theta/2})$ *for* $-\infty < \theta < \infty$ *so that* $A = 1/4(P.Q + Q.P)$ *on*

$\mathcal{S} \equiv \mathcal{S}(R^\nu)$ *and* $Y_\theta = \exp(-i\theta A)$. *Then for any f in* $\mathcal{H}_{ac}(H)$ *and* φ *any bounded continuous function on* R

$$s-\lim_{t \to \pm\infty} V_t^* \varphi(A/t) V_t f = \varphi(H_{ac}) f.$$

We shall use the notation F(M) for the operator of multiplication by the characteristic function of the Borel set $M \subseteq R^\nu$.

3. Generalised coherent states and their properties

To prove Range $\Omega_\pm = \mathcal{H}_c(H)$, we use the idea of generalised coherent states ([25], [26], [27]).

Choose and fix c in (0, 1/3), $\eta \in \mathcal{S}(R^\nu)$, the Schwarz space of rapidly decreasing functions, such that $\hat{\eta}$ — the Fourier transform of η has

$$\text{supp}\,\hat{\eta} \subseteq \{ k \in R^\nu : |k| \leq c/8 \}. \tag{3.1}$$

We further normalize η as

$$\|\eta\|^2 = \int_{R^\nu} dx\, |\eta(x)|^2 = 1. \tag{3.2}$$

Define, for (x, k) in $R^\nu \times R^\nu$, η_{xk} by

$$(\hat{\eta}_{xk})(p) = (\exp(-ix.p))\, \hat{\eta}(p-k), \tag{3.3}$$

so that

$$\eta_{xk}(y) = (\exp(ik.(y-x)))\, \eta(y-x), \tag{3.4}$$

η_{xk} is called a generalised coherent state.

For any Borel subset M of $R^\nu \times R^\nu$ define an operator R(M) on $L^2(R^\nu)$ by

$$R(M) = (2\pi)^{-\nu} \int_M dx\, dk\, (\eta_{xk}, .)\, \eta_{xk}, \tag{3.5}$$

where the integral is to be interpreted in the weak sense. Then R is a positive operator valued measure defined on the Borel subsets of $R^\nu \times R^\nu$. Furthermore,

$$0 \leq R^2(M) \leq R(M) \leq R(R^\nu \times R^\nu) = 1. \tag{3.6}$$

Of special interest is when $M = B \times R^\nu$ or $R^\nu \times B$, B Borel in R^ν. In such a case, $R(B \times R^\nu)$, $R(R^\nu \times B)$] is a multiplication operator in the position [momentum space] respectively and is given by,

$$R(B \times R^\nu) = (\chi_B * |\eta|^2)(Q), \tag{3.7}$$

(3.8) $$R(R^{\nu}\times B)=(\chi_B * |\hat{\eta}|^2)(P).$$

In (3.7), (3.8) $*$ stands for the convolution operator and χ_B the indicator function of B. All these results are contained in [25] and [27].

Using (3.8), (3.1) it is easy to see that

(3.9) $$0\leqq R\{(x,k):|k|\notin[2c,(2c)^{-1}]\}\leqq F(|P|\notin[3c,(3c)^{-1}]),$$

(3.10) $$0\leqq R\{(x,k):|k|\in[2c,(2c)^{-1}]\}\leqq F(|P|\in[c,c^{-1}]).$$

Proceeding as in [25], p. 43 and using (3.7), we have that their exist constants t_0, K_0, a function $\psi:R^{\nu}\to[0,\infty)$ with $\psi(x)(1+|x|)^2$ bounded such that for $t\geqq t_0$,

(3.11) $$0\leqq R\{(x,k):|x|\leqq|t|\}\geqq F(|Q|\leqq 2t)+t^{-1}\psi(Q),$$

(3.12) $$0\leqq R\{(x,k):|x|\geqq t\}\leqq F\left(|Q|\geqq\frac{1}{2}t\right)+t^{-1}K_0.$$

4. Main result

From the definition and existence of $\Omega_\pm$ it follows that

(4.1) $$V_t\Omega_\pm=\Omega_\pm U_t$$

and

(4.2) $$\operatorname{Range}\Omega_\pm\subset\mathscr{H}_{ac}(H).$$

Then we have

LEMMA 4.1. — (i) *Range* $\Omega_\pm=\{f\in\mathscr{H}: s-\lim_{t\to\pm\infty}(\Omega_\pm\exp[iX(t)]-1)V_tf=0\}$,

(ii) *Let* $D=\{f\in\mathscr{H}_{ac}(H): H$ *spectral support of f is compact in* $(0,\infty)\}$.

If $s-\lim_{t\to\pm\infty}(\Omega_\pm\exp[iX(t)]-1)V_tf=0$ *for each f in* D *then* $\mathscr{H}_{ac}(H)=$ *Range* $\Omega_\pm$.

Proof. — The proof of (i) is obvious. For (ii) note that D is dense in $\mathscr{H}_{ac}(H)$, and use (4.2), (i).

Q.E.D.

We now verify (ii) of Lemma 4.1 and this we split into two parts, one a result in the norm operator topology and the other one in the strong operator topology. We prove the result for $t\to+\infty$ only, the other case being similar.

Let $f\in D$ be such that H-spectral support of f is compact in $(9c^2/2,(18c^2)^{-1})$ for some c in $(0,1/3)$. For $t\geqq 0$ define

$$M(t)=\{(x,k)\in R^{\nu}\times R^{\nu}:|x|\leqq t,|k|\in[2c,(2c)^{-1}]\},$$

$$\tilde{M}(t)=\text{the complement of } M(t) \text{ in } R^{\nu}\times R^{\nu}.$$

Next, the two results referred above, are stated and their proofs are postponed till sections 5 and 6.

THEOREM 4.2. — *Let* W *and* H *be as in section* 2. *Then for every* β *satisfying* $1-\alpha<\beta<\min\{\alpha,(m-2-\nu/2)/(m+\nu)\}$

$$\lim_{t\to\infty}\|\{\Omega_+\exp[iX(t)]-1\}U_tR(M(t^\beta))\|=0.$$

THEOREM 4.3. — *Let* $\beta>1/2$. *Then for every* g *in* $\mathscr{H}_{ac}(H)$ *and for every* ψ, *a bounded continuous function on* $[0,\infty)$ *with* $\psi(0)=0$,

$$s-\lim_{t\to\pm\infty}\psi(|Q|.|t|^{-\beta})U_t^*V_tg=0.$$

A consequence of Theorem 4.3 *is the following.*

LEMMA 4.4. — *Let* $\tilde{M}(t)$, f *be as above. Then for* $\beta>1/2$

$$s-\lim_{t\to\infty}R(\tilde{M}(t^\beta))U_t^*V_tf=0.$$

Proof. — Choose φ in $C_0^\infty(0,\infty)$ so that $\varphi(H)f=f$, $0\leqq\varphi\leqq1$ and $\varphi=0$ outside $(9c^2/2,(18c^2)^{-1})$. Since $\varphi(H)-\varphi(H_0)$ is compact and since $f\in\mathscr{H}_{ac}(H)$ we have

$$(4.3)\qquad \lim_{t\to\infty}\{\|\{1-\varphi(H_0)\}U_t^*V_tf\|=\lim_{t\to\infty}\|\{\varphi(H)-\varphi(H_0)\}V_tf\|=0.$$

Also by (3.6), (3.9)

$$(4.4)\qquad 0\leqq R^2\{(x,k):|k|\notin[2c,(2c)^{-1}]\}\leqq1-\varphi(H_0).$$

Combining (4.3) and (4.4)

$$(4.5)\qquad \lim_{t\to\infty}\|R\{(x,k):|k|\notin[2c,(2c)^{-1}]\}U_t^*V_tf\|=0.$$

By using Theorem 4.3 it is easy to see that

$$(4.6)\qquad \lim_{t\to\infty}\|F(|Q|\geqq t^\beta/2)U_t^*V_tf\|=0.$$

As in the derivation of (4.5) we have using (3.6), (3.12) and (4.6)

$$(4.7)\qquad \lim_{t\to\infty}\|R\{(x,k):|x|\geqq t^\beta\}U_t^*V_tf\|=0.$$

The result follows from (4.5), (4.7) and the property

$$R(M_1\cup M_2)\leqq R(M_1)+R(M_2).$$

Q.E.D.

Now we have the main result as:

THEOREM 4.5. — (i) *Range* $\Omega_\pm=\mathscr{H}_{ac}(H)$.
(ii) *Range* $\Omega'_\pm=\mathscr{H}_{ac}(H')$.

Proof. — (i) (For positive sign only). Choose β such that $1-\alpha<1/2<\beta<\min\{\alpha,(m-2-\nu/2)/(m+\nu)\}$ which is possible since $\alpha>1/2$ and since $m>4+2\nu$ implies $(m-2-\nu/2)/(m+\nu)>1/2$. Let f have H spectral support as in Lemma 4.4. Then

$$\begin{aligned}\|\{\Omega_+\exp[iX(t)]-1\}V_tf\| &= \|\{\Omega_+\exp[iX(t)]-1\}U_tR(M(t^\beta))U_t^*V_tf\\ &\quad+\{\Omega_+\exp[iX(t)]-1\}U_tR(\tilde{M}(t^\beta))U_t^*V_tf\|\\ &\leqq\|\{\Omega_+\exp[iX(t)]-1\}U_tR(M(t^\beta))\|\ \|f\|+2\|R(\tilde{M}(t^\beta))U_t^*V_tf\|\end{aligned}$$

which converges to 0 as $t\to\infty$ by Theorem 4.2 and Lemma 4.4. Thus Lemma 4.1 (ii) is verified and the result follows.

(ii) Follows from (i), Theorem 2.1 (iv) and the chain rule for wave operators ([1], [2].

Q.E.D.

5. Proof of Theorem 4.2

The proof is in a sense similar to that of the existence of the wave operator Ω_+ given in [1] and [2]. All absolute constants will be denoted by the same letter K.

Since $\exp[-iX(t)]$ is feebly oscillating [1], i. e.

$$s\text{-}\lim_{t\to\infty}\exp[-iX(t+s)]-\exp[-iX(t)]=0 \text{ for each } s,$$

we have

$$\Omega_+=s\text{-}\lim_{t\to\infty}V_t^*U_t\exp[-iX(t+s)].$$

Thus

$$\begin{aligned}&\{\Omega_+\exp[iX(s)]-1\}U_sR(M(s^\beta))\\ &\qquad=s\text{-}\lim_{t\to\infty}\{V_t^*U_t\exp(-i[X(t+s)-X(s)])-1\}U_sR(M(s^\beta)),\end{aligned}$$

and

$$\begin{aligned}(5.1)\quad&\|\{\Omega_+\exp[iX(s)]-1\}U_sR(M(s^\beta))\|\\ &\leqq\sup_{t\geqq 0}\|\{V_t^*U_t\exp(-i[X(t+s)-X(s)])-1\}U_sR(M(s^\beta))\|\\ &\leqq\int_0^\infty dt\left\|\frac{d}{dt}V_t^*U_t\exp(-i[X(t+s)-X(s)])U_sR(M(s^\beta))\right\|\\ &\leqq\int_0^\infty dt\|\{W(Q)-W((t+s)P)\}U_{t+s}\exp(-i[X(t+s)-X(s)])R(M(s^\beta))\|.\end{aligned}$$

By [1] and [2]

$$(5.2)\quad U_t^*\{W(Q)-W(tP)\}U_t = \int_0^1 d\rho\, U_{t/\rho}^*(\nabla W)(\rho Q)\, U_{t/\rho} Q + it/2\int_0^1 d\rho\, U_{t/\rho}^*(\Delta W)(\rho Q)\, U_{t/\rho}.$$

We have by (2.3), (5.1), (5.2)

$$(5.3)\quad \|\{\Omega_+ \exp[iX(s)]-1\}U_s R(M(s^\beta)\| \leqq K\sum_{j=1}^{v}\int_0^\infty dt\int_0^1 d\rho\,\|(1+|\rho Q|)^{-1-\alpha} \times U_{(t+s)/\rho} Q_j \exp(-i[X(t+s)-X(s)])\, R(M(s^\beta))\| + K\int_0^\infty dt\,(t+s)\int_0^1 d\rho\,\|(1+|\rho Q|)^{-2-\alpha} \times U_{(t+s)/\rho}\exp(-i[X(t+s)-X(s)])\, R(M(s^\beta))\|.$$

The result will follow if every term of the right hand side of (5.3) converges to zero as s tends to ∞.

First, we consider the second term.

The t-integrand of the second term of the right hand side of (5.3)

$$\leqq (t+s)\int_0^1 d\rho\,\|(1+|\rho Q|)^{-2-\alpha} F(|Q|\geqq (t+s)c/(8\rho))\| + (t+s)\int_0^1 d\rho\,\|F(|Q|\leqq (t+s)c/(8\rho))\, U_{(t+s)/\rho}\exp(-i[X(t+s)-X(s)])\, R(M(s^\beta))\|.$$

Using Lemma 5.1 (ii) (*see* below) for the second term in the above expression and the fact $m-1/2\,v>2$ from (2.3) we get that the t-integrand of the second term of the right hand side of (5.3)

$$\leqq K(t+s)^{-1-\alpha}+K(t+s)\int_0^1 d\rho\,\{(t+s)/\rho\}^{-m+1/2\,v}(t+s)^{\beta(m+v)} \leqq K\{(t+s)^{-1-\alpha}+(t+s)^{1+v(1/2+\beta)-m(1-\beta)}\}.$$

By virtue of the choice of β, the second term of right hand side of (5.3) converges to 0 as $s\to\infty$. Similarly, the t integrand of the j-th terms of the first summand of the right hand side of (5.3)

$$
\begin{aligned}
&\leqq \int_0^1 d\rho \,\| (1+|\rho\, Q|)^{-1-\alpha} F(|Q| \geqq (t+s)\, c/(8\,\rho)) \\
&\qquad \times U_{(t+s)/\rho} [Q_j, \exp(-i\{X(t+s)-X(s)\})] \\
&\qquad \times R(M(s^\beta)) \| + \int_0^1 d\rho \,\| (1+|\rho\, Q|)^{-1-\alpha} F(|Q| \geqq (t+s)\, c/(8\,\rho)) \\
&\qquad \times U_{(t+s)/\rho} \exp(-i[X(t+s)-X(s)]) \\
&\qquad \times Q_j R(M(s^\beta)) \| + \int_0^1 d\rho \,\| F(|Q| \leqq (t+s)\, c/(8\,\rho)) [U_{(t+s)/\rho}, Q_j] \\
&\qquad \times \exp(-i[X(t+s)-X(s)]) R(M(s^\beta)) \| \\
&\qquad + \int_0^1 d\rho \,\| F(|Q| \leqq (t+s)\, c/(8\,\rho))\, Q_j \\
(5.4) &\qquad \times U_{(t+s)/\rho} \exp(-i[X(t+s)-X(s)]) R(M(s^\beta)) \| \\
&\leqq K \int_0^1 d\rho\, (t+s)^{-1-\alpha} \| \{ (\partial/\partial P_j) \exp(-i[X(t+s)-X(s)]) \} R(M(s^\beta)) \| \\
&\qquad + K \int_0^1 d\rho\, (t+s)^{-1-\alpha} \| \, |Q| R(M(s^\beta)) \| \\
&\qquad + K \int_0^1 d\rho\, \rho^{-1} (t+s) \| F(|Q| \leqq (t+s)\, c/(8\,\rho)) \\
&\qquad \times U_{(t+s)/\rho} P_j \exp(-i[X(t+s)-X(s)]) R(M(s^\beta)) \| \\
&\qquad + K \int d\rho\, \rho^{-1} (t+s) \| F(|Q| \leqq (t+s)\, c/(8\,\rho)) \\
&\qquad \times U_{(t+s)/\rho} \exp(-i[X(t+s)-X(s)]) R(M(s^\beta)) \|.
\end{aligned}
$$

Now using Lemma 5.1 (iv), (v), (iii), (ii) for 1st, 2nd, 3rd, 4th terms respectively we have that the right hand side of (5.4)

$$
\leqq K(t+s)^{-1-\alpha} t^{1-\alpha} + K(t+s)^{-1-\alpha} s^\beta + K(t+s)^{1+\nu(1/2+\beta)-m(1-\beta)}.
$$

Thus the first term of the right hand side of (5.3) also converges to 0 as $s \to \infty$.

Q.E.D.

We are left to prove Lemma 5.1 for which we use the method of stationary phase.

LEMMA 5.1. — (i) (Stationary phase). *Let $\mathscr{F}$ be a compact subset of an open set* G *of* R^ν. *Let for each real t*

$$C(x_0, t)=\{x_0+kt : k\in G\},$$

be the classically allowed region for particles starting at x_0 *with velocities in* G. *Then for any positive integer m*, there exists a constant K_m *such that*

$$|(U_t g)(x)| \leqq K_m [1+\operatorname{dist}(x, C(x_0, t))]^{-m} \sum_{|n|\leqq m} \| D_p^n (\hat{g}(p)\exp(i x_0 p))\|_\infty$$

for all g in $L^2(R^\nu)$ *with supp* $\hat{g}\subset \mathscr{F}$ *and all* $x\notin C(x_0, t)$.

(ii) *For* $0\leqq\rho\leqq 1$, t, $s\geqq 0$ *and m any positive integer, there exists a constant* K_m *such that*

$$\| F(|Q|\leqq(t+s)c/(8\rho))\, U_{(t+s)/\rho}\exp(-i[X(t+s)-X(s)])\, R(M(s^\beta))\| \leqq K_m((t+s)/\rho)^{-m+1/2\,\nu} s^{\beta\nu}(t+s)^{m\beta}.$$

(iii) *The same estimate as in* (ii) *holds when* $U_{(t+s)/\rho}$ *is replaced by* $U_{(t+s)/\rho}\, P_j$.

(iv) $\|\{(\partial/\partial P_j)\exp(-i[X(t+s)-X(s)])\}\, R(M(s^\beta))\|\leqq K\, t^{1-\alpha}$,

(v) *There exist constants* s_0 *and* K *such that for* $s\geqq s_0$, $\| |Q| R(M(s^\beta))\|\leqq K s^\beta$.

Proof. — (i) See the proof of Lemma 2, [22], p. 336.

(ii) We follow [26] and use (i). Writing out the integral for $R(M(s^\beta))$ and recalling from (3.4) that $\|\eta_{xk}\|=1$, we see that

$$(5.5)\quad \| F(|Q|\leqq(t+s)c/(8\rho))\, U_{(t+s)/\rho}\exp(-i[X(t+s)-X(s)])\, R(M(s^\beta))\| \leqq K\left(\frac{(t+s)c}{8\rho}\right)^{\nu/2}\times$$

$$\sup_{|q|\leqq(t+s)c/8\rho}\int_{M(s^\beta)} dx\, dk\, |\{U_{(t+s)/\rho}(\exp(-i[X(t+s)-X(s)]))\,\eta_{xk}\}(q)|,$$

where K is a constant. Now choose

$$\mathscr{F}=\{x\in R^\nu : 2c-(c/8)\leqq|x|\leqq(2c)^{-1}+(c/8)\},$$
$$G=\{x\in R^\nu : 2c-(c/4)<|x|<(2c)^{-1}+(c/4)\},$$
$$g=\{\exp(-i[X(t+s)-X(s)])\}\,\eta_{xk}$$

and apply (i) with $x_0=0$ to get, when $|q|\leqq(t+s)c/(8\rho)$ and $(x, k)\in M(s^\beta)$,

$$(5.6)\quad |\{U_{(t+s)/\rho}\exp(-i[X(t+s)-X(s)])\,\eta_{xk}\}(q)|$$
$$\leqq K_m\{(t+s)/\rho\}^{-m}\sum_{|n|\leqq m}\| D_p^n\{(\exp(-i[x.p+\int_s^{s+t} d\tau\, W(\tau P)]))\,\hat{\eta}(p-k)\}\|_\infty$$
$$\leqq K_m\{(t+s)/\rho\}^{-m}[s^\beta+(t+s)^{1-\alpha}-s^{1-\alpha}]^m$$
$$\leqq K_m\{(t+s)/\rho\}^{-m}(t+s)^{m\beta}\ \text{since } 1-\alpha<\beta.$$

Substituting (5.6) in (5.5) the result follows.

(iii) Similar to (ii).

(iv) Using monotonicity of R, (3.10) and the inequality $R^2 \leqq R$ we have

$$0 \leqq R^2(M(s^\beta)) \leqq F(|P| \in [c, c^{-1}]). \tag{5.7}$$

Thus

$$\begin{aligned}
&\| \{ (\partial/\partial P_j) \exp(-i[X(t+s)-X(s)]) R(M(s^\beta)) \| \\
&\qquad = \| R(M(s^\beta)) (\partial/\partial P_j) (\exp(i[X(t+s)-X(s)])) \| \\
&\qquad \leqq \| F(|P| \in [c, c^{-1}]) (\partial/\partial P_j) \exp(-i[X(t+s)-X(s)]) \| \quad \text{by (5.7)} \\
&\qquad \leqq K \| F(|P| \in [c, c^{-1}]) \int_s^{s+t} d\tau (1+|\tau P|)^{-\alpha} \| \\
&\qquad \leqq K \int_0^t d\tau (\tau+s)^{-\alpha} \leqq K t^{1-\alpha}.
\end{aligned}$$

(v) Similar to (iv). By (3.11), there exists $s_0 \geqq 1$ such that for $s \geqq s_0$, $0 \leqq R^2(M(s^\beta)) \leqq F(|Q| \leqq 2 s^\beta) + s^{-\beta} \psi(Q)$, where $|Q|^2 \psi(Q)$ is a bounded operator.

Therefore,

$$\begin{aligned}
&\| \, |Q| R(M(s^\beta)) \|^2 = \| \, |Q| R^2(M(s^\beta)) |Q| \, \| \\
&\qquad \leqq \| \, |Q| \{ F(|Q| \leqq 2 s^\beta) + s^{-\beta} \psi(Q) \} |Q| \, \| \leqq K s^{2\beta}.
\end{aligned}$$

Q.E.D.

6. Proof of Theorem 4.3

A formal calculation shows that

$$(V_t^* U_t Q_j U_t^* V_t - Q_j) f = \int_0^t ds\, s\, V_s^* (D_j W)(Q) V_s f. \tag{6.1}$$

If $\|(1+|Q|)^{-3/2} V_s f\| \leqq K(1+|s|)^{-3/2}$ for f in $D(|Q|)$ then by (6.1) we get

$$\operatorname*{s\text{-}lim}_{t \to \infty} |Q| \, |t|^{-\beta} U_t^* V_t f = 0 \qquad \text{for} \quad \beta > 1/2$$

and so by Lemma 2.1 of [19] we get Theorem 4.3. In [18] the author derives a similar result using dilation analyticity for the potential W. We do not assume dilation analyticity, though, the potential W is C^m for sufficiently large m. If his method can be compared to Taylor's series, our method is like Taylor's formula with remainder.

We give a heuristic argument how one can expect these results. By Corollory 2.7 of [19], for suitable set of vector f in $\mathscr{H}_{ac}(H)$, $V_s^*(s/A) V_s f$ behaves like $H^{-1} f$ as $|s| \to \infty$ and so $|s| \cdot \|(1+|A|)^{-1} V_s f\|$ is bounded in s. Similarly we expect $|s|^{3/2} \|(1+|A|)^{-3/2} V_s f\|$ to be bounded in s. These two results we prove in Lemma 6.6 and 6.7 respectively in a rigorous manner.

We use the following notations: For any two operators X, Y the operator [X, Y] is given by

$$[X, Y]=XY-YX, \quad (\mathrm{ad}_X)Y=[X, Y], \quad (\mathrm{ad}_X^j)Y=\mathrm{ad}_X\,\mathrm{ad}_X^{j-1} \quad \text{for } j\geqq 1, \quad \mathrm{ad}_X^0 Y=Y.$$

Note that $\mathrm{ad}_A^j W$ are all bounded operators for $j=0, 1, 2, 3$.

LEMMA 6.1. — (i) $[A, H](H+i)^{-1}$, $(H+i)^{-1}[A, H]$ *are both bounded operators,*

(ii) $\|[A, V_s](H+i)^{-1}\|\leqq K(1+|s|)$,

(iii) $[A, \psi(H)](H+i)^{-1}$ *is bounded for each* ψ *in* $\mathscr{S}(R)$,

(iv) $[A, \psi(H)]$ *is bounded for* ψ *in* $\mathscr{S}(R)$,

(v) $H^j\{(\mathrm{ad}_A^k)(W)\}(H+i)^{-j}$ *is bounded for* $k=0, 1, 2$; $j=0, 1, 2$,

(vi) $\|\{(\mathrm{ad}_A^2)(V_s)\}(H+i)^{-2}\|\leqq K(1+|s|)^2$,

(vii) $\{(\mathrm{ad}_A^2)(\psi(H))\}(H+i)^{-2}$ *is bounded for* ψ *in* $\mathscr{S}(R)$,

(viii) $(H+i)[A, H](H+i)^{-2}$ *is bounded,*

(ix) $\{(\mathrm{ad}_A^2)(H+i)^2\}(H+i)^{-2}$, $\{(\mathrm{ad}_A)(H+i)^2\}(H+i)^{-2}$ *are both bounded,*

(x) $(\mathrm{ad}_A^2)(\psi(H))$ *is bounded for* ψ *in* $\mathscr{S}(R)$,

(xi) $\|\{(\mathrm{ad}_A^3)(V_s)\}(H+i)^{-3}\|\leqq K(1+|s|)^3$,

(xii) $(H+i)\psi(H)\{(\mathrm{ad}_A^3)(H+i)^2\}(H+i)^{-3}$ *is bounded for* ψ *in* $\mathscr{S}(R)$,

(xiii) $(\mathrm{ad}_A^3)(\psi(H))$ *is bounded for* ψ *in* $\mathscr{S}(R)$,

(xiv) $A\psi(H)(A+i)^{-1}$ *is bounded for* ψ *in* $\mathscr{S}(R)$,

(xv) $A^2\psi(H)(A+i)^{-2}$ *is bounded for* ψ *in* $\mathscr{S}(R)$,

(xvi) $A^3\psi(H)(A+i)^{-3}$ *is bounded for* ψ *in* $\mathscr{S}(R)$,

(xvii) $[A, (H+i)^{-1}](H+i)$ *is bounded,*

(xviii) $(\mathrm{ad}_A^2)(H+i)^{-1}$ *is bounded,*

(xix) $(H+i)^{-1}A(1+|Q|)^{-2}$, $(H+i)^{-1}A^2(1+|Q|)^{-2}$ *are both bounded,*

(xx) $A^2(H+i)^{-1}(1+|Q|)^{-2}$ *is bounded,*

(xxi) $(A+i)^2\psi(H)(1+|Q|)^{-2}$ *is bounded for* ψ *in* $\mathscr{S}(R)$.

Proof. — (i) and (ii) are obvious.

(iii) Follows from (ii) and $[A, \psi(H)]=\int ds\,\tilde{\psi}(s)[A, V_s]$.

(iv) is a consequence of (iii) and the identity

$$[A, \psi(H)(H+i)](H+i)^{-1}=[A, \psi(H)]+\psi(H)[A, H](H+i)^{-1}.$$

(v) Follows by the boundedness of the derivatives of W and the commutation rules between P and Q.

(vi) Follows from (ii) on using (i), (v).

(vii) Similar to (iii).

(viii) and (ix) follow from the commutation rules.

(x) Similar to (iv) by noting that

$$\{(\mathrm{ad}_A^2)(\psi(H)(H+i)^2)\}(H+i)^{-2}$$
$$=\{(\mathrm{ad}_A^2)(\psi(H)\}+2[A,\psi(H)][A,(H+i)^2](H+i)^{-2}+\psi(H)\{(\mathrm{ad}_A^2)(H+i)^2\}(H+i)^{-2}$$

and using (iv), (ix).

(xi) Similar to (vi), by using (vi), (ii), (v).

(xii) Obvious by the commutation rules.

(xiii) Similar to (x) by using (x), (iv), (ix), (xii).

(xiv) Follows from (iv) on using the equality

$$A\psi(H)(A+i)^{-1}=[A,\psi(H)](A+i)^{-1}+\psi(H)A(A+i)^{-1}.$$

(xv) It is easy to verify that

$$A^2\psi(H)(A+i)^{-2}=\{(\mathrm{ad}_A^2)\psi(H)\}(A+i)^{-2}+[A,\psi(H)]A(A+i)^{-2}+A\psi(H)A(A+i)^{-2}.$$

The result easily follows from (x), (iv), (xiv)

(xvi) Similar to (xv).

(xvii) and (xviii) are obvious.

(xix) For $(H+i)^{-1}A(1+|Q|)^{-2}$ note that A is a linear combination of $P_jQ_j(j=1, 2, \ldots, \nu)$ and 1 and that $(H+i)^{-1}P_j$, $Q_j(1+|Q|)^{-2}$ are both bounded.

For $(H+i)^{-1}A^2(1+|Q|)^{-2}$ the result follows as in previous case by noting that A^2 is a linear combination of $P_jP_kQ_jQ_k$, $P_jQ_j(j, k=1, 2, \ldots, \nu)$ and 1.

(xx) It is easily seen that

$$A^2(H+i)^{-1}(1+|Q|)^{-2}=\{(\mathrm{ad}_A^2)(H+i)^{-1}\}(1+|Q|)^{-2}$$
$$+2[A,(H+i)^{-1}](H+i)(H+i)^{-1}A(1+|Q|)^{-2}+(H+i)^{-1}A^2(1+|Q|)^{-2}.$$

The result follows from (xviii), (xvii), (xix).

(xxi) Follows from (xiv), (xv), (xx) on using the identity

$$(A+i)^2\psi(H)(1+|Q|)^{-2}$$
$$=\{(A+i)^2\psi(H)(H+i)(A+i)^{-2}\}.\{(A+i)^2(H+i)^{-1}(1+|Q|)^{-2}\}.$$

Q.E.D.

We shall need an interpolation result in the following form.

LEMMA 6.2 (Interpolation). — *Let* $\mathscr{H}$ *be a Hilbert space*, T *a bounded operator on* $\mathscr{H}$; X, Y *positive selfadjoint operators on* $\mathscr{H}$. *Furthermore assume that for real* β, δ *with* $0\leqq\beta<\delta$ *the operators* $Y^\beta TX^{-\beta}$ *and* $Y^\delta TX^{-\delta}$ *are bounded. Then for each* γ *in* (β, δ) *the operator* $Y^\gamma TX^{-\gamma}$ *is bounded and*

$$\|Y^\gamma TX^{-\gamma}\|\leqq\|Y^\beta TX^{-\beta}\|^{(\delta-\gamma)/(\delta-\beta)}\|Y^\delta TX^{-\delta}\|^{(\gamma-\beta)/(\delta-\beta)}.$$

Proof. — Similar to Proposition 09, page 44, in [21].

Q.E.D.

Next Theorem combines Lemma 6.1 and Lemma 6.2.

THEOREM 6.3. — *Let* ψ *be in* $\mathcal{S}(\mathrm{R})$. *Then*

(i) *the operators* $(\mathrm{ad}_{\mathrm{A}}^{j})(\psi(\mathrm{H}))$ *are bounded for* $j=1, 2, 3$,

(ii) *the operators* $\mathrm{A}^{j}\psi(\mathrm{H})(\mathrm{A}+i)^{-j}$ *are bounded for* $j=1, 2, 3$,

(iii) *the operators* $(1+|\mathrm{A}|)^{j}\psi(\mathrm{H})(1+|\mathrm{Q}|)^{-j}$ *are bounded for* $j=1/2, 1, 3/2$.

Proof. — (i) Refer Lemma 6.1 (iv), (x), (xiii).

(ii) Refer Lemma 6.1 (xiv), (xv), (xvi).

(iii) Follows from Interpolation Lemma 6.2 and Lemma 6.1 (xxi).

Q.E.D.

Remark. — All the calculations in the proof of Lemma 6.1 and Theorem 6.3 are to be done as a form on $\mathcal{S}\times\mathcal{S}$ first and then extended by density to the whole $\mathcal{H}$ as a bounded operator. All the objects like [A, H], $[\mathrm{A}, \mathrm{V}_s]$, $[\mathrm{A}, \psi(\mathrm{H})]$, etc. are to be looked upon as closure of the form on $\mathcal{S}\times\mathcal{S}$.

From Theorem A.1 of [19], we have for $f, g\in\mathcal{S}$, $(\mathrm{V}_t f, \mathrm{A}g)=(\mathrm{V}_t\mathrm{A}f, g)+(t\mathrm{V}_t\mathrm{H}f, g)-\int_0^t ds(\mathrm{V}_{t-s}\mathrm{W}_1\mathrm{V}_s f, g)$, where $\mathrm{W}_1=\mathrm{W}+i[\mathrm{A}, \mathrm{W}]$. Multiplying both sides of the above by $\tilde{\psi}(t)$ and integrating (as a Riemann Integral), one has

$$(\psi(\mathrm{H}), f, \mathrm{A}g)=(\psi(\mathrm{H})\mathrm{A}f, g)+i(\psi'(\mathrm{H})\mathrm{H}f, g)+(\mathrm{G}(\psi)f, g),$$

where $\mathrm{G}(\psi)=-\int dt\,\tilde{\psi}(t)\int_0^t ds\,\mathrm{V}_{t-s}\mathrm{W}_1\mathrm{V}_s$ (a strong Bochner integral). Clearly $\mathrm{G}(\psi)\in\mathcal{B}(\mathcal{H})$ and $\|\mathrm{G}(\psi)\|\leqq\|\mathrm{W}_1\|\left(\int|t|\,|\tilde{\psi}(t)|\,dt\right)$. Thus the second equation can be extended as a form on D(A) and this also shows that $\psi(\mathrm{H})\mathrm{D}(\mathrm{A})\subseteq\mathrm{D}(\mathrm{A})$. A long but elementary calculation, similar to the above, shows that in fact $\psi(\mathrm{H})\mathrm{D}(\mathrm{A}^j)\subseteq\mathrm{D}(\mathrm{A}^j)$ for $j=1, 2, 3$. Thus for example Theorem 6.3 (ii) states that for $j=1, 2, 3$, $\mathrm{A}^j\psi(\mathrm{H})(\mathrm{A}+i)^{-j}$ is defined everywhere and is bounded. It also follows from the above equations that V_t maps $\mathrm{D}(\mathrm{A})\cap\mathrm{D}(\mathrm{H})$ into D(A) and

$$\mathrm{V}_t^*(\mathrm{A}-t\mathrm{H})\mathrm{V}_t f=\mathrm{A}f-\int_0^t ds\,\mathrm{V}_s^*\mathrm{W}_1\mathrm{V}_s f, \forall f\in\mathrm{D}(\mathrm{A})\cap\mathrm{D}(\mathrm{H}).$$

Now we shall expand $\mathrm{V}_t^*\varphi(\mathrm{A}/t)\mathrm{V}_t$, for φ in $\mathrm{C}_0^\infty(\mathrm{R})$, around $t=\infty$ and using Theorems 2.1 and 2.2 obtain the inequality

$$\|(1+|\mathrm{Q}|)^{-1/2}\psi(\mathrm{H})\mathrm{V}_t(\mathrm{A}+i)^{-1}\|\leqq\mathrm{K}(1+|t|)^{-1/2} \quad \text{for} \quad \psi \text{ in } \mathrm{C}_0(0, \infty).$$

This is fed back into the calculations to get the inequality

$$\|(1+|\mathrm{Q}|)^{-1}\psi(\mathrm{H})\mathrm{V}_t(\mathrm{A}+i)^{-2}\|\leqq\mathrm{K}(1+|t|)^{-1}$$

in Lemma 6.6. One more interation leads to the desired inequality

$$\|(1+|Q|)^{-3/2}\psi(H)V_t.(A+i)^{-3}\|\leqq K(1+|t|)^{-3/2}$$

in Lemma 6.7.

For this we begin with a Lemma.

LEMMA 6.4. — (i) *For* φ *in* $\mathscr{S}(R)$ *define* $\tilde{\varphi}(u)=(2\pi)^{-1}\int dt\,\varphi(t)e^{iut}$.

Then for any selfadjoing operator B

$$\varphi(B)=\int du\,\tilde{\varphi}(u)\,e^{-iuB},$$

$$i\varphi'(B)=\int du\,u\,\tilde{\varphi}(u)\,e^{-iuB},$$

$$-\varphi''(B)=\int du\,u^2\,\tilde{\varphi}(u)\,e^{-iuB}.$$

(ii) $[A, H]=i(H-W_1)$ *where* $W_1=W+i[A, W]$,

(iii) $[A, [A, H]]=-H+W_2$ *where* $W_2=W_1-i[A, W_1]$,

(iv) $V_t^*(A-tH)V_t=A-\int_0^t ds\,V_s^*W_1V_s$ *on* $D(A)\cap D(H)$.

(v) *Define* Y_u, $E(2,u)$, $E(3,u)$ *by*

$$Y_u=\exp(-iuA),$$

$$E(2,u)=(H+i)^{-1}Y_u\int_0^u d\lambda_1\int_0^{\lambda_1} d\lambda_2\,Y_{\lambda_2}^*\{\mathrm{ad}_A^2(H)\}Y_{\lambda_2}(H+i)^{-1},$$

$$E(3,u)=(H+i)^{-1}Y_u\int_0^u d\lambda_1\int_0^{\lambda_1} d\lambda_2\int_0^{\lambda_2} d\lambda_3\,Y_{\lambda_3}^*\{\mathrm{ad}_A^3(H)\}Y_{\lambda_3}(H+i)^{-1}.$$

Then

$$\|E(2,u)\|\leqq K|u|^2 \quad \textit{for all } u \textit{ in } R$$

and

$$\|E(3,u)\|\leqq K|u|^3 \quad \textit{for all } u \textit{ in } R.$$

(vi) *For* $t\neq 0$ *and* φ *in* $\mathscr{S}(R)$ *define* $E(2,\varphi,t)$, $E(3,\varphi,t)$ *by*

$$E(2,\varphi,t)=\int du\,\tilde{\varphi}(u)\,E(2,u/t),$$

$$E(3,\varphi,t)=\int du\,\tilde{\varphi}(u)\,E(3,u/t).$$

Then

$$\| E(2, \varphi, t) \| \leqq K(\varphi) |t|^{-2},$$

and

$$\| E(3, \varphi, t) \| \leqq K(\varphi) |t|^{-3},$$

where $K(\varphi)$ *is a constant depending only on* φ.

Let φ, t *as in* (vi). *Then as a form on* $D(H)$,

(vii) $i[H, \varphi(A/t)] = -it^{-1} \varphi'(A/t) [A, H] - i(H+i) E(2, \varphi, t)(H+i)$,

(viii) $i[H, \varphi(A/t)] = -it^{-1} \varphi'(A/t) [A, H] + 1/2\, it^{-2} \varphi''(A/t) \{ ad_A^2(H) \} + (H+i) E(3, \varphi, t)(H+i)$,

(ix) $\dfrac{d}{dt}\{ V_t^* \varphi(A/t) V_t \} = -t^{-1} V_t^* \varphi'(A/t) \{ A t^{-1} + i[A, H] \} V_t - i V_t^* (H+i) E(2, \varphi, t) . (H+i) V_t$,

(x) $\dfrac{d}{dt.}\{ V_t^* \varphi(A/t) V_t \} = -t^{-1} V_t^* \varphi'(A/t) \{ A t^{-1} + i[A, H] \} V_t + \dfrac{1}{2} it^{-2} V_t^* \varphi''(A/t) \{ ad_A^2(H) \} V_t + V_t^* (H+i) E(3, \varphi, t)(H+i) V_t$.

(xi) *If further* $\varphi = 0$ *in a neighbourhood of* 0 *then*

$$\| (H+i)^{-1} \varphi(A/t)(1+|Q|)^{-\gamma} \| \leqq K(1+|t|)^{-\gamma} \quad \text{for} \quad \gamma \text{ in } [0, 1].$$

Proof. — Observations in the Remark preceeding the Lemma 6.4 leads to an easy verification of (i), (ii), (iii) and (iv). Next we note that

$$\| (H_0+1)^{-1} Y_{\lambda_2-u}^* H_0 Y_{\lambda_2-u} \| \quad \text{and} \quad \| Y_{\lambda_2}^* H_0 Y_{\lambda_2} (H_0+1)^{-1} \|$$

are uniformly bounded for $0 \leqq \lambda_2 \leqq \lambda_1 \leqq u$ and for $u \leqq \lambda_1 \leqq \lambda_2 \leqq 0$ respectively and thus (v) follows from (iii).

(vi) is obvious from the definition and (v).

(vii) Since Y_α leaves $D(H) = D(H_0)$ invariant, one has as a form on $D(H_0)$ the identity:

$$[H, Y_{u/t}] = iut^{-1} Y_{u/t} [A, H] - Y_{u/t} \int_0^{u/t} d\lambda_1 \int_0^{\lambda_1} Y_{\lambda_2}^* \{ ad_A^2(H) \} Y_{\lambda_2} d\lambda_2.$$

Now the result follows from (i) and the definitions in (vi).

(viii) Similar to (vii) by noting

$$[H, Y_{u/t}] = iut^{-1} Y_{u/t} [A, H] - 1/2 (u/t)^2 Y_{u/t} \{ ad_A^2(H) \} - i Y_{u/t} \int_0^{u/t} d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \int_0^{\lambda_2} d\lambda_3 Y_{\lambda_3}^* \{ ad_A^3(H) \} Y_{\lambda_3}.$$

(ix) Since

$$\frac{d}{dt}\{V_t^* \varphi(A/t) V_t\} = V_t^* i [H, \varphi(A/t)] V_t - t^{-1} V_t^* \varphi'(A/t)(A/t) V_t,$$

we apply (vii) to get the result.

(x) Similar to (ix) where instead of (vii) use (viii),

(xi) Clearly $\| [(H_0+1)^{-1}, Y_{u/t}] \| \leq K \mid u/t \mid$ for all u and t and so:

$$(6.2) \qquad \| [(H_0+1)^{-1}, \psi(A/t)] \| \leq K(\psi)(1+|t|)^{-1} \quad \text{for} \quad \psi \text{ in } \mathscr{S}(R).$$

By the commutation rules $(i+A)(H_0+1)^{-1}(1+|Q|)^{-1}$ is bounded and thus by the interpolation Lemma 6.2, we see that

$$(6.3) \qquad (1+|A|)^{\gamma}(H_0+1)^{-1}(1+|Q|)^{-\gamma} \text{ is bounded.}$$

Further,

$$(6.4) \qquad \| \varphi(A/t)(1+|A|)^{-\gamma} \| \leq K(1+|t|)^{-\gamma}.$$

The result follows from (6.2), (6.3), (6.4) and the identity

$$(H+i)^{-1} \varphi(A/t)(1+|Q|)^{-\gamma} = (H+i)^{-1}(H_0+1)[(H_0+1)^{-1}, \quad \varphi(A/t)](1+|Q|)^{-\gamma}$$
$$+ (H+i)^{-1}(H_0+1)\varphi(A/t)(1+|A|)^{-\gamma}(1+|A|)^{\gamma}(H_0+1)^{-1}(1+|Q|)^{-\gamma}.$$

Q.E.D.

In what follows we shall often use the big 0 notation. We write $B(t) = C(t) + 0(|t|^{\gamma})$ to mean $\| B(t) - C(t) \| \leq K |t|^{\gamma}$ for some constant K.

LEMMA 6.5. — (i) *Let* ψ *be in* $C_0^{\infty}(0, \infty)$ *and* V *any measurable function on* R^{ν} *such that* $|V(x)| \leq K(1+|x|)^{-\alpha}$. *Then*

$$\left\| \int_0^t ds\, V_s^* V(Q) V_s \psi(H) \right\| \leq K(1+|t|)^{1/2},$$

Let $t \neq 0$. *Then for* φ *in* $\mathscr{S}(R)$ *with* $\varphi = 1$ *in a neighbourhood of* 0 *and* ψ *in* $C_0(0, \infty)$, *we have*

$$\| (H+i)^{-1} \frac{d}{dt}\{V_t^* \varphi(A/t) V_t\} \psi(H)(A+i)^{-1} \| \leq K(1+|t|)^{-3/2},$$

(iii) *For* φ, ψ *as in* (ii)

$$\| (H+i)^{-1} V_t^* \{\varphi(A/t) - \varphi(H)\} V_t \psi(H)(A+i)^{-1} \| \leq K(1+|t|)^{-1/2},$$

(iv) $\| (1+|Q|)^{\prime 1/2} V_t \psi(H)(A+i)^{-1} \| \leq K(1+|t|)^{-1/2}$ *for* ψ *in* $C_0(0, \infty)$.

Proof. — (i) Let $f \in L^2(R^{\nu})$ be such that $\| f \| = 1$. Then

$$\left\| \int_0^t ds\, V_s^* V(Q) V_s \psi(H) f \right\|$$

$$\leqq \int_0^t ds \| V(Q) V_s \psi(H) f \|$$

$$\leqq \left\{ \int_0^t ds \right\}^{1/2} \left\{ \int_0^\infty ds \| V(Q) V_s \psi(H) f \|^2 \right\}^{1/2} \text{ by Cauchy-Schwartz inequality}$$

$$\leqq K(1+|t|)^{1/2} \text{ by Theorem 2.1 (iii).}$$

(ii) Using Lemma 6.4 (ix), (ii), (iv), (vi), (xi) and Theorem 6.3 (ii) and the fact that $\psi(H)\, D(A) \nsubseteq D(A)$ we get

$$(H+i)^{-1} \frac{d}{dt} \{ V_t^* \varphi(A/t) V_t \} \psi(H)(A+i)^{-1}$$

$$= t^{-2} V_t^* \varphi'(A/t) V_t \int_0^t ds\, V_s^* W_1 V_s \psi(H)(A+i)^{-1} + O(1+|t|)^{-3/2}).$$

Now the result follows by applying (i) to the first term.

(iii) Since $\psi(H) = \psi(H_{ac})$, by Theorem 2.2 $\operatorname*{s-lim}_{t \to \infty} V_t^* \varphi(A/t) V_t \psi(H) = \varphi(H) \psi(H)$.

The derivative in (ii) is strongly continuous for $t \neq 0$ and thus the result follows by integrating (ii).

(iv) We deduce (iv) from (iii) by an appropriate choice of φ. For the given ψ in $C_0^\infty(0, \infty)$ choose φ such that $0 \leqq \varphi \leqq 1$, $\varphi = 1$ in a neighbourhood of 0 and $\varphi = 0$ on support of ψ. For such a φ we have $\varphi(H)\psi(H) = 0$ and thus by (iii) we have

$$(6.5) \qquad \| (H+i)^{-1} \varphi(A/t) V_t \psi(H)(A+i)^{-1} \| \leqq K(1+|t|)^{-1/2}$$

and clearly

$$(6.6) \qquad \| (1+|A|)^{-1/2} \{ 1-\varphi(A/t) \} \| \leqq K(1+|t|)^{-1/2}.$$

Choose ψ_1 in $C_0^\infty(0, \infty)$ such that $0 \leqq \psi_1 < 1$, $\psi_1 = 1$ on supp ψ. Then we get

$$\begin{aligned} \| (1+|Q|)^{-1/2} V_t \psi(H)(A+i)^{-1} \| &\leqq \| (1+|Q|)^{-1/2} \psi_1(H)(H+i) \| \\ &\times \| (H+i)^{-1} \varphi(A/t) V_t \psi(H)(A+i)^{-1} \| + \| (1+|Q|)^{-1/2} \psi_1(H)(1+|A|)^{1/2} \| \\ &\times \| (1+|A|)^{-1/2} \{ 1-\varphi(A/t) \} \| \, \| V_t \psi(H)(A+i)^{-1} \|. \end{aligned}$$

Now we deduce (iv) by using (6.5), (6.6) and Theorem 6.3 (iii).

Q.E.D.

Remark. — If $W=0$, then Lemma 6.4 (X) reduces to

$$\frac{d}{dt} \{ U_t^* \varphi(A/t) U_t \} = -t^{-2} U_t^* \varphi'(A/t) U_t A - \frac{i}{2} t^{-2} U_t^* \varphi''(A/t) U_t H_0$$

$$+ U_t^* (H_0+i) E(3, \varphi, t)(H_0+i) U_t,$$

since $i[A, H_0] = -H_0$ and since $(At^{-1} - H_0) U_t = U_t A t^{-1}$.

Thus in such a case, Lemma 6.5 (iii) looks like

$$\|(H_0+i)^{-1} U_t^* \{ \varphi(A/t)-\varphi(H_0) \} U_t \psi(H_0)(A+i)^{-1}\| = O(t^{-1}).$$

The lower rate of t-decrease in the general case is therefore due to the fact that $V_t^*(A t^{-1}+i[A, H]) V_t - A t^{-1}$ contains the potential W and its derivatives, and turns out to be of the order $t^{-1/2}$.

LEMMA 6.6. — (i) *Let* $V: R^\nu \to R$ *be twice differentiable with bounded derivatives. Furthermore let* $|V(x)| \leqq K(1+|x|)^{-1/2}$ *and* ψ *be in* $C_0^\infty(0, \infty)$. *Then*

$$\left\| \int_0^t ds\, s\, V_s^* [H, V] V_s \psi(H)(A+i)^{-1} \right\| \leqq K(1+|t|)^{1/2},$$

(ii) *Let* $V_1, V_2: R^\nu \to R$ *be measurable such that* $|V_1(x)| + |V_2(x)| \leqq K(1+|x|)^{-\alpha}$ *and* $\psi_1, \psi_2 \in C_0^\infty(0, \infty)$. *Then*

$$\left\| \int_0^t ds\, V_s^* V_1(Q) \psi_1(H) V_s \int_0^s dy\, V_y^* V_2(Q) V_y \psi_2(H) \right\| \leqq K(1+|t|),$$

(iii) $\|V_t^*(A-tH)^2 V_t \psi(H)(A+i)^{-2}\| \leqq K(1+|t|)$ *for* ψ *in* $C_0^\infty(0, \infty)$,

(iv) $\|(A t^{-1}-H) V_t \psi(H)(A+i)^{-2}\| \leqq K(1+|t|)^{-1/2}$

and

$$\|(H+i)(A t^{-1}-H) V_t \psi(H)(A+i)^{-2}\| \leqq K(1+|t|)^{-1/2} \quad \textit{for} \quad \psi \textit{ in } C_0^\infty(0, \infty).$$

(v) *Let* φ *be in* $\mathscr{S}(R)$ *with* $\varphi = 1$ *in a neighbourhood of* 0 *and* ψ *in* $C_0^\infty(0, \infty)$, *then*

$$\|(H+i)^{-1} \frac{d}{dt}[V_t^* \{ \varphi(A/t) - \varphi'(A/t)(A t^{-1}-H) \} V_t] \psi(H)(A+i)^{-2}\| \leqq K(1+|t|)^{-2}.$$

(vi) *For* φ, ψ *as in* (v)

$$\|(H+i)^{-1} \{ \varphi(A/t) - \varphi(H) - \varphi'(A/t)(A t^{-1}-H) \} V_t \psi(H)(A+i)^{-2}\| \leqq K(1+|t|)^{-1},$$

(vii) $\|(1+|Q|)^{-1} V_t \psi(H)(A+i)^{-2}\| \leqq K(1+|t|)^{-1}$ *for* ψ *in* $C_0^\infty(0, \infty)$.

Proof. — (i) Integrating by parts we see that

$$i \int_0^t ds\, s\, V_s^* [H, V] V_s \psi(H)(A+i)^{-1} = \int_0^t ds\, s \frac{d}{ds}(V_s^* V V_s) \psi(H)(A+i)^{-1}$$

$$= t V_t^* V V_t \psi(H)(A+i)^{-1} - \int_0^t ds\, V_s^* V V_s \psi(H)(A+i)^{-1}.$$

Now the result follows from Lemma 6.5 (iv), (i).

(ii) Let $f \in L^2(R^\nu)$ be such that $\|f\| = 1$. Then

$$\left\| \int_0^t ds\, V_s^* V_1 \psi_1(H) V_s \int_0^s dy\, V_y^* V_2 \psi_2(H) f \right\|$$

$$\leqq \int_0^t ds \int_0^s dy \| V_1 \psi_1(H) V_s V_y^* V_2 \psi_2(H) V_y f \|$$

$$= \int_0^t dy \int_y^t ds \| V_1 \psi_1(H) V_s V_y^* V_2 \psi_2(H) V_y f \|$$

by Fubini's theorem

$$\leqq K \left\{ \int_0^t dy (t-y)^{1/2} \right\} \| V_2 \psi_2(H) V_y f \|$$

by Cauchy-Schwartz inequality and Theorem 2.1 (iii).

$$\leqq K \left\{ \int_0^t dy (t-y) \right\}^{1/2}$$

again by Cauchy-Schwartz inequality and Theorem 2.1 (iii).

$$\leqq K(1+|t|).$$

(iii) Choose ψ_1 in $C_0^\infty(0, \infty)$ such that $\psi_1 \psi = \psi$. Then by a previous remark $\psi(H) D(A^2) \subseteq D(A^2)$ and

$V_t^*(A - tH)^2 V_t \psi(H)(A+i)^{-2}$

$$= \int_0^t ds \frac{d}{ds} \{ V_s^*(A - sH)^2 V_s \} \psi(H)(A+i)^{-2} + O(1)$$

by Theorem 6.3 (ii)

$$= -2 \int_0^t ds V_s^* W_1 (A - sH) V_s \psi(H)(A+i)^{-2} - \int_0^t ds V_s^* [A, W_1] V_s \psi(H)(A+i)^{-2}$$

$$+ \int_0^t s V_s^* [H, W_1] V_s \psi(H)(A+i)^{-2} + O(1)$$

by Lemma 6.4 (ii)

$$= -2 \int_0^t ds V_s^* W_1 \psi_1(H) V_s V_s^*(A - sH) V_s \psi(H)(A+i)^{-2}$$

$$-2 \int_0^t ds V_s^* W_1 [A, \psi_1(H)] V_s \psi(H)(A+i)^{-2} + O((1+|t|)^{1/2})$$

by Lemma 6.5 (iv) and Lemma 6.6 (i)

$= 0(1+|t|)$ by Lemma 6.4 (iv), Lemma 6.3 (ii), Lemma 6.6 (ii) and Lemma 6.3 (i).

(iv) By Lemma 6.4 (iv) and Theorem 6.3 (ii) we get

$$V_t^*(A-tH)V_t\psi(H)(A+i)^{-2}=O(1)-\int_0^t ds\,V_s^*W_1V_s\psi(H)(A+i)^{-2}=O((1+|t|)^{1/2})$$

by Lemma 6.5 (iv).

We have proved the first result. We deduce the second result from the first. By commuting $(H+i)$ we have

$$(H+i)(A\,t^{-1}-H)V_t\psi(H)(A+i)^{-2}=-it^{-1}H\psi(H)V_t(A+i)^{-2}$$
$$+it^{-1}W_1\psi(H)V_t(A+i)^{-2}+(A\,t^{-1}-H)V_t(H+1)\psi(H)(A+i)^{-2}.$$

Now the second result is an easy consequence of the first result.

(v) $$(H+i)^{-1}\frac{d}{dt}\{V_t^*[\varphi(A/t)-\varphi'(A/t)(A\,t^{-1}-H)]V_t\}\psi(H)(A+i)^{-2}$$
$$=-(H+i)^{-1}t^{-1}V_t^*\varphi'(A/t)(A\,t^{-1}+i[A,H])V_t\psi(H)(A+i)^{-2}+O(t^{-2})$$
$$-(H+i)^{-1}V_t^*\{\varphi'(A/t)\,i[H,A\,t^{-1}-H]+i[H,\varphi'(A/t)](A\,t^{-1}-H)$$
$$-t^{-1}\varphi''(A/t)A\,t^{-1}(A\,t^{-1}-H)-t^{-1}\varphi'(A/t)A\,t^{-1}\}V_t\psi(H)(A+i)^{-2}$$

by Lemma 6.4 (ix), (vi),

$$=O(t^{-2})+t^{-1}V_t^*(H+i)^{-1}\varphi''(A/t)(A\,t^{-1}+i[A,H])(A\,t^{-1}-H)V_t\psi(H)(A+i)^{-2}$$
$$-\frac{1}{2}it^{-2}V_t^*(H+i)^{-1}\varphi'''(A/t)\{ad_A^2(H)\}(H+i)^{-1}(H+i)(A\,t^{-1}-H)V_t\psi(H)(A+i)^{-2}$$
$$+O(t^{-3})(H+i)(A\,t^{-1}-H)V_t\psi(H)(A+i)^{-2}$$

by Lemma 6.4 (viii).

The desired result follows from Lemma 6.4 (ii), Lemma 6.6 (iii), Lemma 6.4 (xi), Lemma 6.6 (iv) boundedness of $\{ad_A^2(H)\}(H+i)^{-1}$ and Lemma 6.6 (iv).

(vi) For φ, ψ as in (v) we have by Theorem 2.2 that

$$s\text{-}\lim_{t\to\infty} V_t^*\{\varphi(A/t)-\varphi(H)-\varphi'(A/t)(A\,t^{-1}-H)\}V_t\psi(H)=0.$$

As in Lemma 6.5 (iii), an integration of (v) leads to (vi).

(vii) Choose φ, ψ_1 as in the proof of Lemma 6.5 (iv). Than clearly

$$\|(1+|A|)^{-1}\{1-\varphi(A/t)+\varphi'(A/t)(A\,t^{-1}-H)\}\psi(H)\|\leqq K(1+|t|)^{-1}.$$

Now the rest of the proof is similar to that of the Lemma 6.5 (iv) on using Lemma 6.6 (vi).

Q.E.D.

LEMMA 6.7. — *Let ψ, ψ_1, ψ_2 be in $C_0^\infty(0,\infty)$ such that $\psi_1\psi=\psi$ and $\psi_2\psi_1=\psi_1$. Also assume that φ in $\mathscr{S}(R)$ is such that $\varphi=1$ in a neighbourhood of* 0.

Then:

(i) $\|(1+|Q|)^{-\gamma}P_j\psi(H)V_t(A+i)^{-2}\|\leqq K(1+|t|)^{-\gamma}$

for γ in [0, 1] and $j=1,2,\ldots,\nu$,

(ii) $$\|(H+i)(At^{-1}-H)^2V_t\psi(H)(A+i)^{-2}\|\leqq K(1+|t|)^{-1},$$

(iii) $$\left\|\int_0^t ds\, s\, V_s^*(A-sH)V_sV_s^*[H,W_1]V_s\psi(H)(A+i)^{-2}\right\|\leqq K(1+|t|)^{3/2},$$

(iv) $$\left\|\int_0^t ds\, V_s^*(A-sH)[A,W_1]V_s\psi(H)(A+i)^{-2}\right\|\leqq K(1+|t|)^{3/2},$$

(v) $$\int_0^t ds\, V_s^*(A-sH)^2W_1V_s\psi(H)(A+i)^{-2}$$
$$=O((1+|t|)^{3/2})+\int_0^t ds\, V_s^*(A-sH)W_1(A-sH)V_s\psi(H)(A+i)^{-2},$$

(vi) $$\int_0^t ds\, V_s^*(A-sH)W_1(A-sH)V_s\psi(H)(A+i)^{-2}$$
$$=O((1+|t|)^{3/2})+\int_0^t ds\, V_s^*W_1(A-sH)^2V_s\psi(H)(A+i)^{-2},$$

(vii) $$\int_0^t ds\, V_s^*W_1[\psi_1(H),(A-sH)^2]V_s\psi(H)(A+i)^{-3}=O((1+|t|)^{3/2}),$$

(viii) $$\int_0^t ds\, V_s^*W_1V_s\psi_1(H)$$
$$\times\int_0^s dy\, V_y^*[A-yH,W_1]V_y\psi(H)(A+i)^{-3}=O((1+|t|)^{3/2}),$$

(ix) $$\int_0^t ds\, \dot{V}_s^*W_1\psi_2(H)V_s\int_0^s dy\, V_y^*W_1\psi_1(H)V_y$$
$$\times\int_0^y dz\, V_z^*W_1\psi(H)V_z=O((1+|t|)^{3/2}),$$

(x) $$V_t^*(A-tH)^3V_t\psi(H)(A+i)^{-3}=O((1+|t|)^{3/2}),$$

(xi) $$(H+i)^{-1}\frac{d}{dt}\{t^{-1}V_t^*\varphi''(A/t)HV_t\}\psi(H)(A+i)^{-1}$$
$$=-t^{-2}(H+i)^{-1}V_t^*\varphi''(A/t)V_tH\psi(H)(A+i)^{-1}+O((1+|t|)^{-5/2}),$$

(xii) $$(H+i)^{-1}\frac{d}{dt}[V_t^*\{\varphi(A/t)-\varphi'(A/t)(At^{-1}-H)$$

$$+1/2\,\varphi''(A/t)(A\,t^{-1}-H)^2-1/2\,it^{-1}\,\varphi''(A/t)$$

$$H\}\,V_t]\,\psi(H)(A+i)^{-3}=O((1+|t|)^{-5/2}),$$

(xiii) $(H+i)^{-1}\{\varphi(A/t)-\varphi(H)-\varphi'(A/t)(A\,t^{-1}-H)$

$$+1/2\,\varphi''(A/t)(A^2\,t^{-2}-2\,A\,t^{-1}\,H+H^2)\}\,V_t\,\psi(H)(A+i)=O((1+|t|)^{-3/2}),$$

(xiv) $\|(1+|Q|)^{-3/2}\,V_t\,\psi(H)(A+i)^{-3}\|\leqq K(1+|t|)^{-3/2}.$

Proof. — (i) For j, $k=1, 2, \ldots, \nu$ it is clear that $(1+|Q|)^{-1}\,P_j(H+i)^{-1}\,Q_k$ is bounded by using the commutation rules. So $(1+|Q|)^{-1}\,P_j(H+i)^{-1}(1+|Q|)$ is bounded, which with Lemma 6.6 (vii) yields

$$(1+|Q|)^{-1}\,P_j\,V_t\,\psi(H)(A+i)^{-2}=O((1+|t|)^{-1}).$$

Then the result follows by the interpolation Lemma 6.2.

(ii) By using the commutation rules repeatedly we have, on the range of $\psi(H)(A+i)^{-2}$, $(H+i)(A\,t^{-1}-H)^2$

$$=(A\,t^{-1}-H)^2(H+i)+2\,t^{-1}[H, A](H+i)^{-1}(H+i)(A\,t^{-1}-H)$$
$$+t^{-2}[A, [H, A]](H+i)^{-1}(H+i)^{-1}(H+i)-t^{-1}[H, [H, A]](H+i)^{-1}(H+i).$$

Note that $[H, A](H+i)^{-1}$, $[A, [H, A]](H+i)^{-1}$, $[H, [H, A]](H+i)^{-1}$ are all bounded operators. Now the result is a consequence of Lemma 6.6 (iii) and Lemma 6.6 (iv)

(iii) Note that by Theorem 6.3 (iii), $(A-s\,H)[H_0, W_1]\,\psi(H)$ is defined everywhere and

$$i\int_0^t ds\,s\,V_s^*(A-s\,H)\,V_s\,V_s^*[H, W_1]\,V_s\,\psi(H)(A+i)^{-2}$$
$$=t\,V_t^*(A-t\,H)\,W_1\,V_t\,\psi(H)(A+i)^{-2}-\int_0^t ds\,V_s^*(A-s\,H)\,W_1\,V_s\,\psi(H)(A+i)^{-2}$$
$$+\int_0^t ds\,s\,V_s^*\,W_1^2\,V_s\,\psi(H)(A+i)^{-2}$$

as in Lemma 6.6 (i) by using Lemma 6.4 (iv)

$$=t\,V_t^*\,W_1(A-t\,H)\,V_t\,\psi(H)(A+i)^{-2}+O(1+|t|)$$
$$-\int_0^t ds\,V_s^*\,W_1(A-s\,H)\,V_s\,\psi(H)(A+i)^{-2}+O(1+|t|)+O(1+|t|)$$

by Lemma 6.6 (vii) Lemma 6.7 (i), Lemma 6.6 (i) and Lemma 6.6 (vii) $=O((1+|t|)^{3/2})$ by Lemma 6.6 (iv).

(iv) $\displaystyle\int_0^t ds\,V_s^*(A-s\,H)[A, W_1]\,V_s\,\psi(H)(A+i)^{-2}$

$$=\int_0^t ds\,V_s^*\{[\mathrm{ad}_A^2(W_1)-s[H, [A, W_1]]\}\,V_s\,\psi(H)(A+i)^{-2}$$

$$+\int_0^t ds\, V_s^* [A, W_1](A - sH) V_s \psi(H)(A+i)^{-2}$$
$$= O(1+|t|) + O(1+|t|) + O((1+|t|)^{3/2}$$

by the boundedness of $ad_A^2(W_1)$, Lemma 6.6 (i), and Lemma 6.6 (iv).

(v) The result follows from (iv) and (iii) on using the identity

$$(A - sH)^2 W_1 = (A - sH) W_1 (A - sH) + (A - sH)[A, W_1] - s(A - sH)[H, W_1].$$

(vi) Clearly

$$(A - sH) W_1 (A - sH) = W_1 (A - sH)^2 + [A - sH, W_1](A - sH). \tag{6.7}$$

Furthermore

$$\int_0^t ds\, V_s^* [A - sH, W_1](A - sH) V_s \psi(H)(A+i)^{-2}$$
$$= O((1+|t|)^{3/2}) + i\int_0^t ds\, s \frac{d}{ds}\{V_s^* W_1 V_s\} V_s^* (A - sH) V_s \psi(H)(A+i)^{-2}$$

by Lemma 6.6 (iv)

$$= O((1+|t|)^{3/2}) + it V_t^* W_1 (A - tH) V_t \psi(H)(A+i)^{-2}$$
$$+ i\int_0^t ds\, s V_s^* W_1^2 V_s \psi(H)(A+i)^{-2} - i\int_0^t ds\, V_s^* W_1 (A - sH) V_s \psi(H)(A+i)^{-2}$$

by integration by parts and Lemma 6.4 (iv)

$$= O((1+|t|)^{3/2}) \text{ by Lemma 6.6 (iv), (vii).} \tag{6.8}$$

The result now follows from (6.7) and (6.8).

(vii). On using the Jacobi identity for the commutator bracket we have

$$-[\psi_1(H), (A - sH)^2] \tag{6.9}$$
$$= \{ ad_A^2(\psi_1(H)) \} - s[H, [A, \psi_1(H)]] + 2[A, \psi_1(H)](A - sH)$$
$$= \{ ad_A^2(\psi_1(H)) \} + si[\psi_1(H), W_1] + 2[A, \psi_1(H)](A - sH).$$

By (6.9), Lemma 6.3 (i), Lemma 6.6 (iv) one gets

$$-\int_0^t ds\, V_s^* W_1 [\psi_1(H), (A - sH)^2] V_s \psi(H)(A+i)^{-2}$$
$$= O(1+|t|) + i\int_0^t ds\, s V_s^* \{ W_1 \psi_1(H) W_1 - W_1^2 \psi_1(H) \}$$
$$\times V_s \psi(H)(A+i)^{-2} + O((1+|t|)^{3/2})$$
$$= O((1+|t|)^{3/2}) \text{ by Lemma 6.5 (iv).}$$

(viii) $$\left\| \int_0^t ds\, V_s^* W_1 V_s \psi_1(H) \int_0^s dy\, V_y^* [A - yH, W_1] V_y \psi(H)(A+i)^{-2} \right\|$$

$$\leqq K \int_0^t ds \int_0^s dy \,\| [A, W_1] V_y \psi(H)(A+i)^{-2} \|$$

$$+ K \int_0^t ds \left\| \int_0^s dy\, y\, V_y^* [H, W_1] V_y \psi(H)(A+i)^{-2} \right\|$$

$= O((1+|t|)^{3/2})$ by Lemma 6.5 (iv) and Lemma 6.6 (i).

(ix) Similar to Lemma 6.6 (ii).

(x) For ψ in $C_0^\infty(0, \infty)$ choose ψ_1, ψ_2 in $C_0^\infty(0, \infty)$ such that $\psi_1 \psi = \psi$, $\psi_2 \psi_1 = \psi_1$. Then

$$-V_t^*(A-tH)^3 V_t \psi(H)(A+i)^{-3} = -\int_0^t ds \frac{d}{ds} \{ V_s^*(A-sH)^3 V_s \} \psi(H)(A+i)^{-3} + O(1)$$

by Theorem 6.3 (ii)

$$= \int_0^t ds\, V_s^* \{ W_1 (A-sH)^2 + (A-sH) W_1 (A-sH)$$
$$+ (A-sH)^2 W_1 \} V_s \psi(H)(A+i)^{-3} + O(1)$$

by Lemma 6.4 (ii)

$$= \int_0^t ds\, V_s^* \{ W_1 (A-sH)^2 + 2(A-sH) W_1 (A-sH) \} V_s \psi(H)(A+i)^{-3}$$
$$+ O((1+|t|)^{3/2}) \text{ by (v)}$$

$$= 3 \int_0^t ds\, V_s^* W_1 (A-sH)^2 V_s \psi(H)(A+i)^{-3} + O((1+|t|)^{3/2}) \text{ by (vi)}$$

$$= 3 \int_0^t ds\, V_s^* W_1 \psi_2(H) V_s V_s^* (A-sH)^2 V_s \psi(H)(A+i)^{-3} + O((1+|t|)^{3/2}) \text{ by (vii)}$$

$$= 3 \int_0^t ds\, V_s^* W_1 \psi_2(H) V_s \int_0^s dy \frac{d}{dy} \{ V_y^* (A-yH)^2 V_y \} \psi(H)(A+i)^{-3} + O(1+|t|)^{3/2})$$

by Theorem 6.3 (ii)

$$= -3 \int_0^t ds\, V_s^* W_1 \psi_2(H) V_s \int_0^s dy\, V_y^* \{ (A-yH) W_1$$
$$+ W_1 (A-yH) \} V_y \psi(H)(A+i)^{-3} + O((1+|t|)^{3/2}$$

by Lemma 6.4 (ii)

$$= -6 \int_0^t ds\, V_s^* W_1 \psi_2(H) V_s \int_0^s dy\, V_y^* W_1 (A-yH) V_y \psi(H)(A+i)^{-3} + O((1+|t|)^{3/2})$$

by (viii)

$$= -6 \int_0^t ds\, V_s^* W_1 \psi_2(H) V_s \int_0^s dy\, V_y^* W_1 [A - yH, \psi_1(H)] V_y \psi(H)(A+i)^{-3}$$

$$-6 \int_0^t ds\, V_s^* W_1 \psi_2(H) V_s \int_0^s dy\, V_y^* W_1 \psi_1(H) V_y A \psi(H)(A+i)^{-3}$$

$$+6 \int_0^t ds\, V_s^* W_1 \psi_2(H) V_s \int_0^s dy\, V_y^* W_1 \psi_1(H) V_y$$

$$\times \int_0^y dz\, V_z^* W_1 V_z \psi(H)(A+i)^{-3} + O((1+|t|)^{3/2}$$

by Lemma 6.4 (iv)

$=$ First term $+ O((1+|t|)^{3/2})$ by Lemma 6.6. (ii) and Lemma 6.7 (ix).

As in the proof of Lemma 6.6 (ii) we have that the first term $= O((1+|t|)^{3/2})$. Now the result follows.

(xi)

$$(6.10)\quad \frac{d}{dt}\{t^{-1} V_t^* \varphi''(A/t) V_t H\} = -t^{-2} V_t^* \varphi''(A/t) V_t H + t^{-1}\frac{d}{dt}\{V_t^* \varphi''(A/t) V_t\} H.$$

As in Lemma 6.5 (ii) one can prove

$$(H+i)^{-1} d/dt\{V_t^* \varphi''(A/t) V_t\} H \psi(H)(A+i)^{-1} = O(1+|t|)^{-3/2})$$

and the result easily follows from (6.10).

(xii) As in Lemma 6.6 (v) but using Lemma 6.4 (x), (vi), (ii), (iii) we get, on Range $(\psi(H)(A+i)^{-3})$, after a lengthy calculation

$$\frac{d}{dt}[V_t^*\{\varphi(A/t) - \varphi'(A/t)(A t^{-1} - H) + 1/2\, \varphi''(A/t)(A t^{-1} - H)^2\} V_t]$$

$$= V_t^* \varphi''(A/t)\{-1/2\, it^{-2} H + 1/2\, it^{-2} W_2 + 1/2\, t^{-2}[W_1, A] + 1/2\, t^{-1}[H_0, W_1]\} V_t$$

$$+ V_t^* \varphi'''(A/t)\{-1/2\, it^{-2}(-H + W_2)(A t^{-1} - H)$$

$$-1/2\, t^{-1}(A t^{-1} - H)^3 - 1/2\, t^{-1} W_1 (A t^{-1} - H)^2\} V_t$$

$$+ V_t^* \varphi^{(iv)}(A/t)\{1/4\, it^{-2}[\mathrm{ad}_A^2(H)](A t^{-1} - H)^2\}$$

$$\times V_t + (H+i)\, O((1+|t|)^{-3})(H+i)$$

$$+ (H+i)\, O((1+|t|)^{-3})(H+i)(A t^{-1} - H) V_t$$

$$+ (H+i)\, O((1+|t|)^{-3})(H+i)(A t^{-1} - H)^2 V_t.$$

Now the result is a consequence of (xi), Lemma 6.4 (xi), Lemma 6.6 (vii), Lemma 6.7 (i), Lemma 6.6 (iv), Lemma 6.7 (x), Lemma 6.4 (xi), Lemma 6.6 (iii), Lemma 6.7 (ii), Lemma 6.6 (iv) and Lemma 6.7 (ii).

(xiii) A simple calculation shows that

$$(A - t\,H)^2 = A^2 - 2\,t\,AH + t^2\,H^2 + it\,H - it\,W_1, \tag{6.11}$$

and clearly

$$\lim_{|t| \to \infty} \| t^{-1}\,\varphi''(A/t)\,W_1 \| = 0. \tag{6.12}$$

By using Theorem 2.2 we have

$$s\text{-}\lim_{|t| \to \infty} V_t^*\,\varphi(A/t)\,V_t\,\psi(H) = \varphi(H)\,\psi(H), \tag{6.13}$$

$$s\text{-}\lim_{|t| \to \infty} V_t^*\,\varphi'(A/t)(A\,t^{-1} - H)\,V_t\,\psi(H) = 0, \tag{6.14}$$

and

$$s\text{-}\lim_{|t| \to \infty} V_t^*\,\varphi''(A/t)(A^2\,t^{-2} - 2\,A\,t^{-1}\,H + H^2)\,V_t\,\psi(H) = 0. \tag{6.15}$$

As before, we note that the derivative in (xii) is strongly continuous on Range $(\psi(H)\,(A+1)^{-3})$. On integrating (xii) and using (6.11)-(6.15) we have that

$$\begin{aligned} \| (H+i)^{-1} \{ \varphi(A/t) - \varphi(H) - \varphi'(A/t)(A\,t^{-1} - H) \\ + 1/2\,\varphi''(A/t)(A^2\,t^{-2} - 2\,A\,t^{-1}\,H + H^2) \\ - 1/2\,it^{-1}\,\varphi''(A/t)\,W_1 \}\,V_t\,\psi(H)\,(A+i)^{-3} \| \leqq K((1+|t|)^{-3/2}). \end{aligned}$$

Now the result follows from Lemma 6.4 (xi).

(xiv) Choose φ, ψ_1 as in the proof of Lemma 6.6 (vii). Then clearly

$$\begin{aligned} \| (1+|A|)^{-3/2} \{ 1 - \varphi(A/t) + \varphi'(A/t)(A\,t^{-1} - H) \\ - 1/2\,\varphi''(A/t)(A^2\,t^{-2} - 2\,A\,t^{-1}\,H + H^2) \}\,\psi(H) \| \leqq K((1+|t|)^{-3/2}). \end{aligned}$$

Rest of the proof is similar to the proof of Lemma 6.6 (vii).

Q.E.D.

LEMMA 6.8. — (i) *For* ψ *in* $\mathscr{S}(R)$, $(1+|Q|)\,\psi(H)\,(1+|Q|)^{-1}$ *is a bounded operator.*

(ii) *Let* $D = \{ \psi(H)\,f : \psi \in C_0^\infty(0, \infty),\ f \in \mathscr{S}(R^\nu) \}$. *Then* D *is dense in* $\mathscr{H}_{ac}(H)$. *Further* $D \subseteq \mathrm{Dom}(1+|Q|)$.

Proof. — (i) As in Lemma 6.1 (iii) it is easily seen that $[Q_j, \psi(H)]\,(H+i)^{-1}$ is bounded for each j and for each ψ in $\mathscr{S}(R)$. As in Lemma 6.1 (iv), $[Q_j, \psi(H)]$ is bounded for ψ in $\mathscr{S}(R)$. The result is now clear.

(ii) That D is dense in $\mathscr{H}_{ac}(H)$ is obvious. The second assertion follows from (i).

Q.E.D.

THEOREM 6.9. — *Let* $\beta > 1/2$ *and* f *be in* $\mathscr{H}_{ac}(H)$. *Then*

(i) $\displaystyle s\text{-}\lim_{|t| \to \infty} V_t^*\,U_t \exp[-iu.Q\,|t|^{-\beta}]\,U_t^*\,V_t f = f$ *for each* u *in* R^ν,

(ii) *for any bounded continuous function* φ *on* $[0, \infty)$ *with* $\varphi(0)=0$ *we have*

$$s\text{-}\lim_{|t|\to\infty} \varphi(|Q|.|t|^{-\beta})\, U_t^* V_t f=0.$$

Proof. — We prove both the results only for the positive sign.

(i) Clearly it is enough to show for f in D of Lemma 6.8 (ii) that

$$s\text{-}\lim_{t\to\infty} V_t^* U_t Q_j t^{-\beta} U_t^* V_t f=0 \qquad \text{for each } j.$$

Note that for f in D, by Lemma 6.7 (xiv)

$$(1+|Q|)^{-3/2} V_s f = O((1+|s|)^{-3/2}). \tag{6.16}$$

A simple calculation shows that

$$(V_t^* U_t Q_j U_t^* V_t - Q_j) f = (V_t^* Q_j V_t - Q_j - t V_t^* P_j V_t) f = \int_0^t ds\, s\, V_s^* (D_j W)(Q) V_s f.$$

The result follows from (2.3), (6.16) and Lemma 6.8 (ii).

(ii) Follows from (i) and Lemma 2.1 of [19].

Q.E.D.

APPENDIX

We give the proof of the existence of Ω_+ in three steps.

Step 1. — Let $s>0$, set $f_s=\exp(iX(s))\, R(M(s^\beta)) \exp(-iX(s)) f$ for any $f\in\mathscr{H}$, and set $\Omega(t)=V_t^* U_t \exp(-iX(t))$. Then for $\psi\in C_0^\infty(0,)$, $s\text{-}\lim_{\tau\to\infty} \psi(H)\,\Omega(\tau+s) f_s$ exists if

$$\int_1^\infty dt\, \| \{ W(Q)-W(tP) \} U_{t+s} \exp(-i[X(t+s)-X(s)])\, R(M(s^\beta)) \| < \infty.$$

The finiteness of the integral is contained in the proof of Theorem 4.2 and therefore $s\text{-}\lim \psi(H)\, \Omega(\tau) f_s$ exists as $\tau\to\infty$ for every fixed $s>0$.

Step 2. — Since $\psi(H)-\psi(H_0)$ is compact, it follows from Lemma A1 (v) that $s\text{-}\lim \Omega(\tau)\,\psi(H_0) f_s$ exists as $\tau\to\infty$.

Step 3. — Note that the set

$$\bigcup_{0<c<1/3} \left\{ \psi(H_0) f_s \,|\, \operatorname{supp}\psi \subset \left[\frac{9}{2}c^2, (18c^2)^{-1}\right]\right\},$$

is total in $\mathscr{H}$. This is so because, by Lemma A1 (iii)

$$\exp(iX(s))\,\psi(H_0)\, R(\tilde{M}(s^\beta)) \exp(-iX(s))$$

converges strongly to 0 as $s \to \infty$ for every ψ with above mentioned support properties.

Q.E.D.

Lemma A.1. — *Let* $f \in \mathscr{S}(\mathrm{R}^\nu)$ *be such that* $f \in C_0^\infty(\mathrm{R}^\nu \setminus \{0\})$. *Then*

(i) $$\underset{t\to\infty}{s\text{-lim}}\,[\exp(i\mathrm{X}(t))\,u.\mathrm{Q}\exp(-i\mathrm{X}(t))]\,t^{-\beta}f=0,$$

(ii) $$\forall f \in \mathrm{L}^2(\mathrm{R}^\nu),\quad a>0 \quad and \quad \beta>1-\alpha,$$
$$\underset{t\to\infty}{s\text{-lim}}\,\mathrm{F}(|\mathrm{Q}|\geqq at^{\beta})\exp(-i\mathrm{X}(t))f=0,$$

(iii) $\forall f \in \mathrm{L}^2(\mathrm{R}^\nu)$, c *in* $(0, 1/3)$ *and* $\beta>1-\alpha$
$$\underset{t\to\infty}{s\text{-lim}}\,\mathrm{F}(|\mathrm{P}|\in[(3c),(3c)^{-1}])\,\mathrm{R}(\tilde{\mathrm{M}}(t^{\beta}))\exp(-i\mathrm{X}(t))f=0.$$

(iv) *Let* f *be as in* (i) *and* $u \in \mathrm{R}^\nu$. *Then,*

$$\underset{t\to\infty}{s\text{-lim}}\,\mathrm{U}_t^*\exp(i\mathrm{X}(t))\frac{u.\mathrm{Q}}{t}\mathrm{U}_t\exp(-i\mathrm{X}(t))f=(u.\mathrm{P})\,f,$$

(v) $$\underset{t\to\infty}{w\text{-lim}}\,\mathrm{U}_t\exp(-i\mathrm{X}(t))f=0,\qquad \forall f\in\mathrm{L}^2(\mathrm{R}^\nu).$$

Proof. — (i) follows from

$$\|\{\exp(i\mathrm{X}(t))\,u.\mathrm{Q}\exp(-i\mathrm{X}(t))-(u.\mathrm{Q})\}f\|=\|[u.(\nabla\mathrm{X})(t,\mathrm{P})]f\|.$$

(ii) Proceeding as in [19] and using (i) of Lemma 2.1 of [19], we have for every continuous bounded function ψ on [0,) vanishing in a neighbourhood of 0 that

$$\underset{t\to\infty}{s\text{-lim}}\,\psi(|\mathrm{Q}|\,t^{-\beta})\exp(-i\mathrm{X}(t))f=0,\qquad \forall f\in\mathrm{L}^2(\mathrm{R}^\nu).$$

Since every characteristics function is dominated by some such ψ, the result follows.

(iii) Similar to the proof of Lemma 4.4 by using (3.9), (3.12) and (ii).

(iv) This follows from

$$\begin{aligned}&\|\{\mathrm{U}_t^*\exp(i\mathrm{X}(t))(u.\mathrm{Q})\,\mathrm{U}_t\exp(-i\mathrm{X}(t))-(u.\mathrm{Q})-t(u.\mathrm{P})\}\,f\|\\&\qquad=\|\{\exp(i\mathrm{X}(t))(u.\mathrm{Q})\exp(-i\mathrm{X}(t))-(u.\mathrm{Q})\}f\|=\|[u.(\nabla\mathrm{X})(t,\mathrm{P})]f\|.\end{aligned}$$

(v) It suffices to prove the result for $\hat{f}\in C_0^\infty(\mathrm{R}^\nu\setminus\{0\})$. Let $\hat{f}\in C_0^\infty(\mathrm{R}^\nu\setminus\{0\})$ be such that $|\mathrm{P}|$-spectral support of f is in $[a,\infty)$ for some $a>0$. Now choose a continuous bounded function φ on $[0,\infty)$ such that $0\leqq\varphi\leqq 1$, $\varphi(0)=1$ and $\varphi=0$ on an open neighbourhood of $[a,\infty)$. Then by Lemma 2.1, *cf.* [19] and (iv),

(A.1) $$\underset{t\to\infty}{s\text{-lim}}\,\varphi(|\mathrm{Q}|/t)\,\mathrm{U}_t\exp(-i\mathrm{X}(t))f=0.$$

On the other hand for any $g \in L^2(R^\nu)$,

$$s\text{-}\lim_{t \to \infty} [1 - \varphi(|Q|/t)] g = 0. \quad (4.2)$$

Combining (A . 1) and (A . 2) we have the result.

Q.E.D.

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Note. — Since the submission of this paper, some related work has appeared e. g. [28] and [29].

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