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SINGULARITIES OF ENVELOPES OF FAMILIES OF SUBMANIFOLDS IN $\mathbb{R}^N$

BY MÁRIO JORGE DIAS CARNEIRO

Introduction

In his article "Sur la Théorie des Enveloppes" [18], published in 1962, R. Thom defined envelope of families of submanifolds as the image of the singular set in a diagram of C^∞ mappings of type $N \xleftarrow{\Pi} M \xrightarrow{g} P$ where Π is a submersion.

Contrary to other generalizations of Mather's Theory of Singularities of Mappings, which are based mostly on extensions of the Preparation Theorem, there is not such extension for this situation (*see* [8]) making it even more attractive.

In this hesis and other papers ([8], [9]), J. P. Dufour discusses the general theory of stability of diagrams of mappings. Among other things, Dufour studies the classification and unfoldings of germs of the type $(\mathbb{R}, 0) \leftarrow (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ and classifies generic germs of diagrams $(\mathbb{R}, 0) \xleftarrow{f} (\mathbb{R}^{n+1}, 0) \xrightarrow{g} (\mathbb{R}^{n+1}, 0)$ such that g has a fold singularity at 0.

Also Arnold in [1] studies some diagrams, although one of the most relevant results for the study of envelopes is stated without proof.

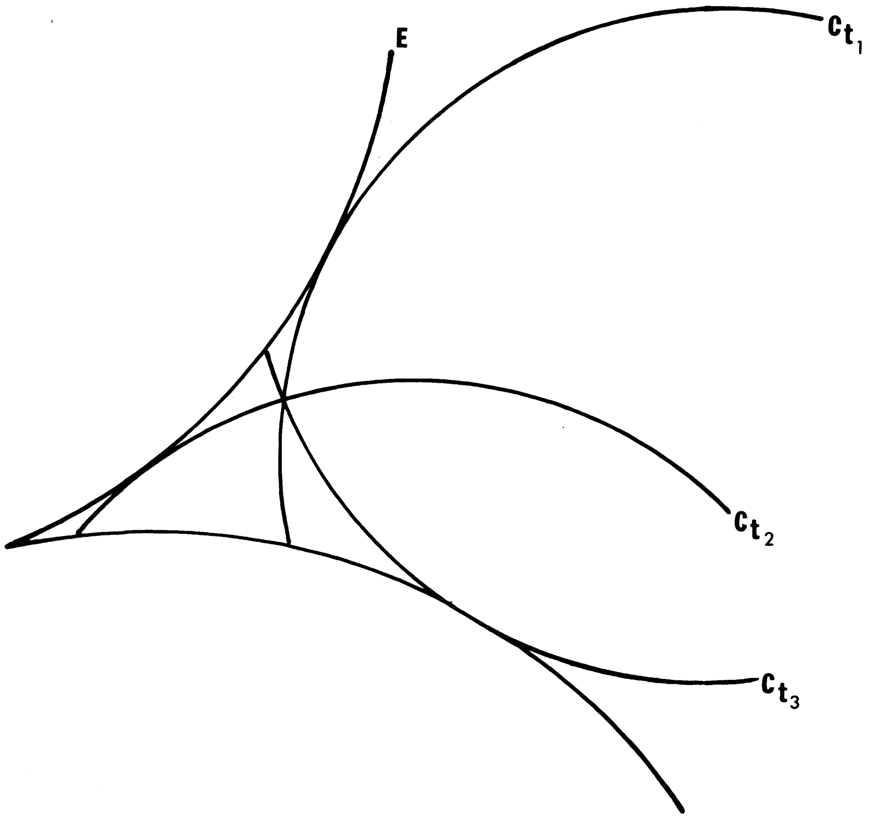
This paper deals with diagrams of C^∞ maps germs at 0 $\mathbb{R} \xleftarrow{\Pi} (\mathbb{R}^n, 0) \xrightarrow{\bar{f}} \mathbb{R}^n$ which define a one-parameter family of submanifolds of codimension one in $\mathbb{R}^n$, for $n \geqq 2$ (*see* Definition 1).

The most interesting generic case is $0 \in S_{1_n}(\bar{f})$, since for $0 \in S_{1_k}(\bar{f})$ and $1 \leqq k \leqq n-1$ it is possible to obtain normal forms for stable cases (Thm. 1).

For $0 \in S_{1_n}(\bar{f})$ the family is always unstable but we may characterize in Theorem 2 the equivalence between two germs of family in terms of relations in the parameter space (weak equivalence $\Leftrightarrow$ equivalence). We do this by constructing an invariant set associated to each family and by showing that once we have equivalence on this set we have equivalence of the families.

This construction in the case $n=2$ is illustrated by the Figure 1:

If C_t a family of curves such that its envelope has a cusp singularity at the origin then for each point in the interior of the cusp there exists exactly *three* curves passing through it.



In this way we obtain a germ of a smooth surface $\Sigma \subset \mathbb{R}^3$ formed by the triple (t_1, t_2, t_3) of parameters such that the corresponding curves C_{t_1}, C_{t_2}, C_{t_3} intersect. Σ is invariant under permutation of the coordinates and our result is that Σ characterizes the family in the following sense: two germs of family of curves are equivalent (*see* Def. 2) if and only if there exists a diffeomorphism $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\varphi \times \varphi \times \varphi(\Sigma) = \Sigma'$ for Σ and Σ' surfaces associated to the germs respectively.

This also shows the relation between envelopes and webs and allows to show the topological unstability of these diagrams [10].

The results presented here are contained in the author's thesis [6] and we would like to thank professor J. Mather for his most valuable suggestions and Professor S. Chern for his helpful exposition of some results of web theory.

1. Definitions and examples

We start giving the concept of envelope of a family of submanifolds following Thom's article [18] and we define equivalence of such families. Since we are going to treat the local situation we define *germs* of families.

DEFINITION 1. — Let X be a C^∞ manifold of dimension n and p be a point in X. *A germ at p of a q-parameter family of submanifolds of codimension $m-n+q$ in $\mathbb{R}^m$* is a diagram of C^∞ map germs of the form $\mathbb{R}^q \xleftarrow{\Pi} (X, p) \xrightarrow{\bar{f}} \mathbb{R}^m$ that satisfies the following conditions:

(*a*) Π is a germ of a fibration;

(*b*) $\bar{f}$ *restricted to each fiber* $\Pi^{-1}(w)$, $w \in \mathbb{R}^q$ is a germ of an one to one immersion.

If $S(\bar{f}) = \{ x \in X : d\bar{f}(x) \text{ is not surjective} \}$ then its image $E = \bar{f}(S(\bar{f}))$ is called the *envelope* of the family.

Comments. — If we look at the classical concept of envelope (*see* for example [5], [13]) we see that E is the set of *Characteristic points* and by imposing the condition (*b*) we are avoiding singularities of the elements of the family and self-intersections. If we put the additional condition that E intersects any member of the family not transversally we have the classical envelope which is called by Thom the geometric envelope.

DEFINITION 2. — Two germs of family $\mathbb{R}^q \xleftarrow{\Pi_1} (X, p) \xrightarrow{\bar{f}_1} \mathbb{R}^m$ and $\mathbb{R}^q \xleftarrow{\Pi_2} (X', p') \xrightarrow{\bar{f}} \mathbb{R}^m$ are *equivalent* if there exist germs of C^∞ diffeomorphisms:

$$\Phi : \quad (X, p) \to (X', p'), \qquad \Psi : \quad (\mathbb{R}^m, \bar{f}_1(p)) \to (\mathbb{R}^m, \bar{f}(p'))$$

and:

$$\varphi : \quad (\mathbb{R}^q, \Pi_1(p)) \to (\mathbb{R}^q, \Pi_2(p')),$$

such that $\Psi \circ \bar{f}_1 \circ \Phi^{-1} = \bar{f}$ and $\varphi \circ \Pi_1 \circ \Phi^{-1} = \Pi_2$. That is, two germs of family are equivalent if and only if their diagram are equivalent (in the sense of [9]).

At this point, we may choose different notions of stability for germs of family. We chose to work with the one used by Dufour in [9].

DEFINITION 3. — Homotopic Stability of Germs of Family:

A germ of a family at $p \in X$ defined by the diagram:

$$\mathbb{R}^q \xleftarrow{\Pi} (X, p) \xrightarrow{\bar{f}} \mathbb{R}^m,$$

is *homotopically stable* if for any diagram of C^∞ map germs at $(p, 0)$:

$$\mathbb{R}^q \xleftarrow{G} (X \times \mathbb{R}, (p, 0)) \xrightarrow{F} \mathbb{R}^m,$$

satisfying $F(x, 0) = \bar{f}(x)$ and $G(x, 0) = \Pi(x)$ there exist C^∞ map germs:

$$\Phi : \quad (X \times \mathbb{R}, (0, p)) \to X, \qquad \varphi : \quad (\mathbb{R}^q \times \mathbb{R}, (\Pi(p), 0)) \to \mathbb{R}^q$$

and

$$\Psi : \quad (\mathbb{R}^m \times \mathbb{R}, (\bar{f}(p), 0)) \to \mathbb{R}^m$$

and a real number $\varepsilon > 0$ so that (1.1) φ_λ, Φ_λ, Ψ_λ are germs of diffeomorphism for $|\lambda| < \varepsilon$:

(1.2) $\Phi_0 = \mathrm{id}_X$, $\varphi_0 = \mathrm{id}_{\mathbb{R}^q}$, $\Psi_0 = \mathrm{id}_{\mathbb{R}^m}$ (id_Y = germ of the identity of Y).

(1.3) $\Psi_\lambda \circ F_\lambda \circ \Phi_\lambda^{-1} = \bar{f}$ and $\varphi_\lambda \circ G_\lambda \circ \Phi_\lambda^{-1} = \Pi$ (where we are setting $h_\lambda(x) = h(x, \lambda)$).

We see then that $\bar{f}$ must be stable in the sense of the theory of singularities of mappins (Mather [14]) and we call this a right-left stable map germ. Because Π is the germ of a submersion we may assume that $\Pi(u, x) = u$ for suitable coordinates (u, x) about $p \in X$. Using this, we may rewrite the above definition by requiring that for any germ $F: (X \times \mathbb{R}, (p, 0)) \to \mathbb{R}^m$ with $F(x, 0) = \bar{f}(x)$, $\forall x \in X$ there exists C^∞ map germs:

$$\Phi : \quad (X \times \mathbb{R}, (p, 0)) \to X, \qquad \varphi : \quad (\mathbb{R}^q \times \mathbb{R}, (\Pi(p), 0)) \to \mathbb{R}^q$$

and

$$\Psi : \quad (\mathbb{R}^m \times \mathbb{R}, (f(p), 0)) \to \mathbb{R}^m$$

and a number $\varepsilon > 0$ such that the first two conditions above are satisfied but the third is substituted by

$$(1.3') \qquad \begin{cases} \Psi_\lambda \circ F_\lambda \circ \Phi_\lambda^{-1} = \bar{f}, \\ \varphi_\lambda \circ \Pi \circ \Phi_\lambda^{-1} = \Pi. \end{cases}$$

In other words, Φ_λ is fiber preserving. We call such change of coordinates in the domain *an admissible change of coordinates*. Let $C_p^\infty(X) = \{$ germs at p of C^∞ functions from X to $\mathbb{R}\}$ and choose local coordinates $(u_1, \ldots, u_q, x_1, \ldots, x_{n-q}, \lambda)$ for $X \times \mathbb{R}$ in a neighborhood of $(p, 0)$ and $(y_1, \ldots, y_m)$ for $\mathbb{R}^m$ in a neighborhood of $\bar{f}(p)$. As usual, taking the derivative with respect to λ in (1.3′) we obtain the linearized equations:

$$\begin{cases} dF.\left(X + \dfrac{\partial}{\partial \lambda}\right) = Y \circ \tilde{F}, \\ d\Pi . X = Z \circ \tilde{\Pi}. \end{cases}$$

Where

$$X = \frac{\partial \Phi_\lambda^{-1}}{\partial \lambda} \circ \Phi_\lambda = \sum_{i=1}^{n-q} X_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{q} U_j \frac{\partial}{\partial u_j},$$

for $X_i's$ and $U_j's$ belonging to $C_{p,0}^\infty(X \times \mathbb{R})$:

$$Y = \frac{\partial \Psi_\lambda^{-1}}{\partial \lambda} \circ \Psi_\lambda = \sum_{i=1}^{m} Y_i \frac{\partial}{\partial y_i},$$

$$Y_i \in C_{\bar{f}(p), 0}^\infty(Y \times \mathbb{R}), \qquad Z = \frac{\partial \varphi_\lambda^{-1}}{\partial \lambda} \circ \varphi_\lambda = \sum_{i=1}^{q} Z_i \frac{\partial}{\partial u_i}, \qquad Z_i \in C_{(\Pi(p'), 0)}^\infty(\mathbb{R}^q \times \mathbb{R})$$

and we are setting $\tilde{F}(u, x, \lambda)=(F(u, x, \lambda), \lambda)$ and $\tilde{\Pi}(u, x, \lambda)=(u, \lambda)$. Since we are taking $\Pi(u, x)=u$ we get $U_j=Z_j \circ \tilde{\Pi}$ and it all comes to solve the equation:

$$dF.\left(X+\frac{\partial}{\partial\lambda}\right)=Y \circ \tilde{F} \tag{1.4}$$

for X germ of vector field of the form:

$$X(u, x, \lambda)=\sum_{i=1}^{n-q} X_i(u, x, \lambda)\frac{\partial}{\partial x_i}+\sum_{i=1}^{q} U_j(u, \lambda)\frac{\partial}{\partial u_j}.$$

Integrating these vector fields we obtain that to prove homotopic stability is equivalent to show that we can solve (1.4) for any germ F.

Example 1. — Let $h_1, \ldots, h_k, \alpha_1, \ldots, \alpha_s$ belong to $C_0^\infty(\mathbb{R}^q)$ and J be the ideal of $C_0^\infty(\mathbb{R}^q)$ generated by $h_1, \ldots, h_k$. Suppose that the images of $1, \alpha_1, \ldots, \alpha_s$ in $C_0^\infty(\mathbb{R}^q)/J$ form a basis for this real vector space. It follows from a corollary of the Preparation Theorem ([12], p. 109), that the germ of family:

$$(\mathbb{R}^q, 0) \xleftarrow{\Pi} (\mathbb{R}^q \times \mathbb{R}^{n-q}, 0) \xrightarrow{\bar{f}} (\mathbb{R}^m, 0)$$

for $n=ks+q$ and $m=k(s+1)$ defined by $\bar{f}(u, x)=(x, f_1(u, x), \ldots, f_k(u, x))$ where:

$$f_j(u, x)=h_j(u)+\sum_{i=(j-1)s+1}^{js} \alpha_{i-(j-1)s}(u)x_i, \qquad \Pi(u, x)=u$$

is homotopically stable (*see* [6] for details). This is a particular case of what we call in [6] a strongly infinitesimally stable germ of family. For these germs we may take $\varphi(u)=u$.

2. Envelopes of one parameter families of submanifolds of codimension one in $\mathbb{R}^n$

We turn our attention to the case $q=1$, $m=n$. Since we are doing a local study we take $X=\mathbb{R}^n$ and $p=0$. So we are going to study generic germs of family which are defined by a diagram $(\mathbb{R}, 0) \xleftarrow{\Pi} (\mathbb{R} \times \mathbb{R}^{n-1}, 0) \xrightarrow{\bar{f}} (\mathbb{R}^n, 0)$ satisfing conditions (*a*) and (*b*) of Definition 1.

The case $n=1$ is studied by Dufour in [8], so we also assume $n \geqq 2$. In view of condition (*b*), generically the singularities of $\bar{f}$ are of the type S_{1_k} for $1 \leqq k \leqq n$ (Morin Singularities [12]).

In [6] we prove the following.

THEOREM 1. — *A germ at $p \in X$ of a one parameter family of codimension one submanifolds of $\mathbb{R}^n$, $p \in S_{1_k}(\bar{f})$, is homotopically stable if and only if $1 \leqslant k \leqslant n-1$ and it is equivalent to the germ*

$(\mathbb{R}, 0) \xleftarrow{\Pi} (\mathbb{R}^n, 0) \xrightarrow{\bar{f}} (\mathbb{R}^n, 0)$ *such that for* $2 \leqq k \leqq n-1$:

$$\begin{cases} \Pi(t, x) = t, \\ \bar{f}(t, x) = \left(x, t^{k+1} + \sum_{i=1}^{k} x_i t^i\right), \end{cases}$$

and for $k=1$:

$$\begin{cases} \Pi(t, x) = t, \\ \bar{f}(t, x) = (x, t^2 + t x_1), \end{cases}$$

or

$$\begin{cases} \Pi(t, x) = t, \\ \bar{f}(t, x) = (x, t^2 + t\left(\sum_{i=1}^{n-1} \pm x_i^2\right), \end{cases}$$

where $(t, x_1, \ldots, x_{n-1})$ *are local coordinates of* $\mathbb{R}^n$ *about* 0.

Comments. — Except for the case $k=1$ the proof of this theorem follows standard arguments of the theory of Singularities of Mappings.

Also, it is not difficult to see that for $k=n$ one cannot solve the linearized equation (1.4).

The second normal form in the case $k=1$ involves an analysis of the set of points in the parameter space corresponding to submanifolds that do not intersect in general position. Once we get a normal form for this set the proof follows by using the Preparation Theorem. □

As we said before, we are going to study here the case $k=n$. Stable cases, including a complete proof of the above theorem will be analysed in other article.

First we are going to make some calculations in order to obtain an initial normal form for a generic germ of family having singularity of type S_{1_n}.

Without loss of generality we take a representative for such germ of the form $\Pi(t, x) = t$, $\bar{f}(t, x) = (x, f(t, x))$ for $(t, x) = (t, x_1, \ldots, x_{n-1})$, coordinates in a neighborhood of 0 in $\mathbb{R}^n$. The conditions $0 \in S_{1_n}(\bar{f})$ and $j^k \bar{f} \pitchfork S_{1_n}$ $1 \leqq k \leqq n$, are equivalent to:

$$\frac{\partial f}{\partial t}(0) = \ldots = \frac{\partial^n f}{\partial t^n}(0) = 0$$

and the differentials:

$$d\left(\frac{\partial^i f}{\partial t^i}\right)(0), \qquad i=1, \ldots, n$$

are linearly independent (*see* [12] p. 176). This also implies that it is right-left stable. Making the following change of coordinates in the range:

$$\begin{cases} \bar{y}_i = y_i \quad \text{for } i=1, \ldots, n-1, \\ \bar{y}_n = y_n - f(0, y_1, \ldots, y_{n-1}). \end{cases}$$

We may assume that $f(0, x) = 0$.

Using the transversality hypothesis we get that the matrix:

$$\left(\frac{\partial^{i+1} f}{\partial x_j \partial t^i}(0)\right)_{\substack{1 \leqq i \leqq n-1 \\ 1 \leqq j \leqq n-1}}$$

is non singular.

Denoting by M_y the ideal in $C_0^\infty(\mathbb{R}^n)$ of germs of functions vanishing at 0 and $\bar{f}^*(M_y)$ = ideal of $C_0^\infty(\mathbb{R}^n)$ generated by the coordinate functions $y_1 \circ \bar{f}, \ldots, y_n \circ \bar{f}$ then the hypothesis imply that the local algebra $Q_0(\bar{f}) = C_0^\infty(\mathbb{R}^n)/\bar{f}^*(M_y)$ is isomorphic to $\mathbb{R}[t]/(t^{n+1})$. Therefore by a corollary of the Preparation Theorem ([12]) there exist C^∞ functions $A_0, \ldots, A_n$ such that $t^{n+1} = \sum_{i=0}^{n} A_i \circ \bar{f}(t, x) t^i$. And taking Taylor expansions at 0 we may check that:

$$\frac{\partial A_0}{\partial y_i}(0) = 0 \qquad \text{for} \quad i = 1, \ldots, n-1, \qquad \frac{\partial A_0}{\partial y_n}(0) \neq 0$$

and the matrix:

$$\left(\frac{\partial A_i}{\partial y_j}(0)\right), \qquad i = 1, \ldots, n-1; \quad j = 1, \ldots, n-1,$$

is non singular.

Hence we can perform the admissible change of coordinates in the domain:

$$\Phi(t, x) = (t, -A_1 \circ \bar{f}(t, x), \ldots, -A_{n-1} \circ \bar{f}(t, x)) = (t, \bar{x})$$

and:

$$\Psi(y) = (-A_1(y), \ldots, -A_{n-1}(y), A_0(y)) = \tilde{y},$$

in the range to get:

$$\begin{cases} \tilde{y}_i \circ \bar{f}(t, x) = -A_i \circ \bar{f}(t, x) = \bar{x}_i, \qquad i = 1, \ldots, n-1, \\ \tilde{y}_n \circ \bar{f}(t, x) = A_0 \circ \bar{f}(t, x) = t^{n+1} - \sum_{i=1}^{n} A_i \circ \bar{f}(t, x) t^i = t^{n+1} + \sum_{i=1}^{n-1} \bar{x}_i t^i - t^n A_n \circ \bar{f}(t, x). \end{cases}$$

That is:

$$\Psi \circ \bar{f} \circ \Phi^{-1}(t, \bar{x}) = (\bar{x}, t^{n+1} + \sum_{i=1}^{n-1} \bar{x}_i t^i + t^n L(t, \bar{x})),$$

with $L(0) = 0$ and $(\partial L/\partial t)(0) = 0$. (Notice that $\Pi \circ \Phi^{-1}(t, \bar{x}) = t$). From now on we are assuming that the germ of family is represented by mappings in the above form.

The basic problem to get normal forms for these type of germs comes from the fact that arbitrarily near 0 there are points where $(n+1)$ submanifolds of the family intersect (*see* what happens for example the cusp in $\mathbb{R}^2$). These submanifolds correspond to $(n+1)$-uples of parameter and this relation in the parameter space must be preserved by admissible changes of coordinates in the domain. We are going to study in detail this relation.

We *remark* first that if $g:(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a right-left stable map germ such that $0 \in S_{1_k}(g)$ then the closure in $\mathbb{R}^n$ of $\{p \in \mathbb{R}^n / \exists\, q \in \mathbb{R}^n,\ q \neq p \text{ with } g(p)=g(q)\}$ is equal to the image of a right-left stable germ $g_1 : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $0 \in S_{1_{k-1}}(g_1)$. Moreover if p_1 is a point such that there exist k distinct points $p_2, \ldots, p_{k+1}$ with $g(p_i)=g(p_1)$ for $i=2, \ldots, k+1$, then there exist k distinct points $p'_2, \ldots, p'_{k+1}$ such that $g_1(p'_i)=p_1$ for $i=2, \ldots, n+1$. We use this in the following.

PROPOSITION 1. — *Let* $(\mathbb{R}, 0) \xleftarrow{\Pi} (\mathbb{R}^n, 0) \xrightarrow{\bar{f}} (\mathbb{R}^n, 0)$ *be a germ of a family such that* $\bar{f}$ *is right-left stable and* $0 \in S_{1_n}(\bar{f})$.

There exists a germ of a C^∞ *function* $h:(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *with* $dh(0) = -(1, \ldots, 1)$, *satisfying: If we denote by* $M_t = \bar{f}(\Pi^{-1}(t))$ *the submanifold corresponding to the parameter* t *then* $t_1, \ldots, t_{n+1}$ *are distinct values such that* $\bigcap_{i=1}^{n+1} M_{t_i} \neq \emptyset$ *if and only* $t_{n+1} = h(t_1, \ldots, t_n)$.

Proof. — As we noted before we may choose a representative for such germ of the form:

$$\bar{f}(t, x) = \left(x,\ t^{n+1} + \sum_{i=1}^{n-1} x_i t^i + t^n L(t, x)\right),$$

with $L(0)=0$ and $\partial L/\partial t(0)=0$ and $\Pi(t, x)=t$. We define inductively right-left stable map germs $e_i:(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $0 \in S_{1_i}(e_i)$ for $i=1, \ldots, n-1$ as follows: e_{n-1} is the map germ obtained by applying the previous remark to $\bar{f}$ so that the image of e_{n-1} is the closure in $\mathbb{R}^n$ of the set $\{p \in \mathbb{R}^n / \exists\, q \neq p \text{ with } \bar{f}(p) = \bar{f}(q)\}$. Assuming e_{n-k} constructed we get e_{n-k-1} by applying the remark to e_{n-k}. We have, the image of e_{n-k-1} = closure in $\mathbb{R}^n$ of the set $\{p \in \mathbb{R}^n / \exists\, q \neq p \text{ with } e_{n-k}(p) = e_{n-k}(q)\}$. Using the above expression for the germ of family we obtain a representative for e_{n-k} of the form:

$$\begin{aligned} &e_{n-k}(t_1, \ldots, t_{k+1}, x_{k+1}, \ldots, x_{n-1}) \\ &\qquad = (t_1, \ldots, t_k, \bar{e}_{n-k}(t_1, \ldots, t_{k+1}, x_{k+1}, \ldots, x_{n-1}), x_{k+1}, \ldots, x_{n-1}) \end{aligned}$$

with:

$$\begin{aligned} &\bar{e}_{n-k}(t_1, \ldots, t_{k+1}, x_{k+1}, \ldots, x_{n-1}) \\ &\qquad = -\left[P_{n+1, k+1}(t_1, \ldots, t_{k+1}) + \sum_{i=k+1}^{n-1} x_i P_{i, k+1}(t_1, \ldots, t_{i, k+1})\right] \\ &\qquad\qquad + R_k(t_1, \ldots, t_{k+1}, x_{k+1}, \ldots, x_{n-1}) \end{aligned}$$

where:

$$P_{i, k+1}(t_1, \ldots, t_{k+1}) = \sum_{\beta_1, \ldots, \beta_{k+1}} t_1^{\beta_1} \ldots t_{k+1}^{\beta_{k+1}},$$

for β_i integers such that:

$$0 \leqq \beta_1 \leqq i-k, \qquad 0 \leqq \beta_j \leqq i-k-\beta_1 - \ldots - \beta_{j-1}, \qquad j=2, \ldots, k$$

and $\beta_{k+1}=i-k-\beta_1-\ldots-\beta_k$ and $R_k \in M^{n-k+2}$, the ideal of functions vanishing at 0, together with all the derivatives up to order $n-k+2$. We obtain also that R_k is invariant under permutations of $t_1, \ldots, t_{k+1}$.

From this construction it follows that $t_1, \ldots, t_{n+1}$ are distinct values of the parameter such that the corresponding submanifolds intersect if and only if:

$$e_1(t_1, \ldots, t_{n-1}, t_n)=e_1(t_1, \ldots, t_{n-1}, t_{n+1}).$$

Using the above expression for e_1 we obtain an equation of the form:

$$\sum_{i=1}^{n+1} t_i+R(t_1, \ldots, t_{n+1})=0,$$

with $R \in M^2_{(t_1, \ldots, t_{n+1})}$ invariant under permutations of $t_1, \ldots, t_{n+1}$. The proposition follows from the Implicit Function Theorem. □

DEFINITION 4. — Given a germ of family $(\mathbb{R}, 0) \xleftarrow{\Pi} (\mathbb{R}^n, 0) \xrightarrow{\bar{f}} (\mathbb{R}^n, 0)$ with $\bar{f}$ right-left stable and $0 \in S_{1_n}(\bar{f})$ the germ of the set:

$$S_{\bar{f}}=\{(t_1, \ldots, t_{n+1}) \in \mathbb{R}^{n+1} / M_{t_1} \cap \ldots \cap M_{t_{n+1}} \neq \emptyset\},$$

is called *the hypersurface associated to* $\bar{f}$.

Proposition 1 says that we may represent $S_{\bar{f}}$ by the graph of a smooth function $h(t_1, \ldots, t_n)$. We note also that $S_{\bar{f}}$ is invariant under permutations of $t_1, \ldots, t_{n+1}$.

If $(\bar{f}_1, \Pi_1)$ and $(\bar{f}_2, \Pi_2)$ are equivalent germs of family of type S_{1_n} with $\bar{f}_1, \bar{f}_2$ right-left stable, then whenever $M_{t_1} \cap \ldots \cap M_{t_{n+1}}$ is non empty we have $M'_{T_1} \ldots M'_{T_{n+1}}$ is non empty where $M_{t_i}=\bar{f}_1(\Pi^{-1}(t_i))$, $M'_{T_i}=\bar{f}_2(\Pi_2^{-1}(T_i))$ and $T_i=\varphi(t_i)$ for $\varphi: (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ germ of diffeomorphism. This motivates the following.

DEFINITION 5. — Two generic germs of family with singularity of type S_{1_n} are *weakly equivalent* if there exists a germ of diffeomorphism $\varphi: (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\varphi \times \ldots \times \varphi (S_{\bar{f}_1})=S_{\bar{f}_2}$ for $S_{\bar{f}_i}$ = hypersurface associated to the germ $(\bar{f}_i, \Pi_i)$.

We just observed that this is a necessary condition for equivalance of germs of family. The following theorem gives the converse.

THEOREM 2. — *Two generic germs of one parameter family of submanifolds of* $\mathbb{R}^n$, *having singularity of type* S_{1_n} *are equivalent if and only if they are weakly equivalent.*

Proof. — Without loss of generality we may assume that the given germs $(\bar{f}_1, \Pi_1)$ and $(\bar{f}_2, \Pi_2)$ have the same associated hypersurface $A \subset \mathbb{R}^{n+1}$.

From Proposition 1 we know that $T_0 A$, the tangent plane to A at 0 has equation $\sum_{i=1}^{n+1} t_i=0$ and that A is invariant under the action of $S(n+1)$ = group permutations of the coordinates $t_1, \dots, t_{n+1}$ of $\mathbb{R}^{n+1}$. Hence the exponential map $\delta : A \to T_0 A$ is a local equivariant diffeomorphism where the action on $T_0 A$ is given by the differential of the action on A (*see* Bredon [4], p. 304), that is, the permutations of $t_1, \dots, t_{n+1}$ restricted to $\sum_{i=1}^{n+1} t_i = 0$.

Associated to each germ, we construct C^∞ map germs at $0, g_1, g_2 : (A, 0) \to (\mathbb{R}^n, 0)$ defined by $g_i(t_1, \dots, t_{n+1})$ equals the point where the corresponding submanifolds $M^i_{t_1}, \dots, M^i_{t_{n+1}}$ intersect. Parametrizing a neighborhood of 0 in A by the exponential map we obtain an expression for these map germs:

$$g_i \circ \delta^{-1} = \bar{f}_1 \circ e^i_{n-1} \circ \dots \circ e^i_1 \circ \tau,$$

with $\tau = \hat{\Pi} \circ \delta^{-1}$, local diffeomorphism, $\hat{\Pi}(t_1, \dots, t_{n+1}) = (t_1, \dots, t_n)$ and e^i_j ($i=1,2$; $j=1, \dots, n-1$) are the germs obtained in Proposition 1 associated to $\bar{f}_1$ and $\bar{f}_2$ respectively. If we consider the action of $S(n+1)$ in $\mathbb{R}^n$ given by:

$$\alpha.(t_1, \dots, t_n) = \hat{\Pi}\left(\alpha.\left(t_1, \dots, t_n, -\sum_{i=1}^{n} t_i\right)\right),$$

for $\alpha \in S(n+1)$ then each $g_i \circ \delta^{-1}$ is invariant and the embedding of the hyperplane:

$$k(t_1, \dots, t_n) = \left(t_1, \dots, t_n, -\sum_{i=1}^{n} t_i\right),$$

is equivariant. We will use this invariance to get an equivalence between the two germs of family in the image of g_i.

To do this we let $\rho = (\rho_0, \dots, \rho_n) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be defined by ρ_i = elementary symmetric polynomial of degree $i+1$. It follows from the Preparation Theorem (*see* [16] or [12]) that $\rho^* : C^\infty_0(\mathbb{R}^{n+1}) \to C^\infty_0(\mathbb{R}^{n+1})^{S(n+1)}$ is surjective. ($C^\infty_0(\mathbb{R}^{n+1})^G$ denotes the algebra of C^∞ germs at 0 of functions invariant under the action of the group G and as usual $\rho^*(\lambda) = \lambda \circ \rho$).

By taking averages and using the surjectivity of k^* we also get that $k^* : C^\infty_0(\mathbb{R}^{n+1})^{S(n+1)} \to C^\infty_0(\mathbb{R}^n)^{S(n+1)}$ is surjective. Hence if we let $\sigma_i = \rho_i \circ k$ and $\sigma = (\sigma_1, \dots, \sigma_n)$, since each coordinate function of $\bar{f}_i \circ e^i_{n-1} \circ \dots \circ e^i_1 \circ \tau \circ k$ belongs to $C^\infty_0(\mathbb{R}^n)^{S(n+1)}$ and $\rho_0 \circ k = 0$, we obtain C^∞ map germs $H_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $i=1, 2$ such that $H_i \circ \sigma = \bar{f}_i \circ e^i_{n-1} \circ \dots \circ e^i_1 \circ \tau \circ k$. If we show that each H_i is a germ of a diffeomorphism at 0 then we will have:

$$\bar{f}_1 \circ e^1_{n-1} \circ \dots \circ e^1_1 \circ \tau \circ k = H_1 \circ \sigma = H_1 \circ H_2^{-1} \circ \bar{f}_2 \circ e^2_{n-1} \circ \dots \circ e^2_1 \circ \tau \circ k,$$

which implies, since $\tau \circ k$ is a local diffeomorphism,

$$\bar{f}_1 \circ e^1_{n-1} \circ \dots \circ e^1_1 = H_1 \circ H_2^{-1} \circ \bar{f}_2 \circ e^2_{n-1} \circ \dots \circ e^2_1.$$

In other words, proving that H_i is a local diffeomorphism we get an equivalence between the two germs of family restricted to the closure of the set of points of the range where $n+1$ submanifolds intersect.

Proof that H_2 is a germ of diffeomorphism at 0 : In order to simplify the notation we write $\bar{f}_2=\bar{f}$, $e_i^2=e_i$ and $H_2=G$. As before we may assume that a representative of $\bar{f}$ is of the from:

$$\bar{f}(t, x)=\left(x,\ t^{n+1}+\sum_{i=1}^{n-1} x_i t^i+t^n R(x, t)\right),$$

with $R(0)=0$, $(\partial R/\partial t)(0)=0$. We have $G\circ\sigma=(G_1\circ\sigma, \ldots, G_n\circ\sigma)=\bar{f}\circ e_{n-1}\circ\ldots\circ e_1\circ\tau$ which implies:

$$G_n\circ\sigma=\tau_n^{n+1}+\sum_{i=1}^{n+1}(G_i\circ\sigma)\tau_1^i+\tau_1^n R(\tau_1, G_1\circ\sigma, \ldots, G_{n-1}\circ\sigma). \tag{2.1}$$

Using the definition of the elementary symmetric polynomials we have:

$$0=\tau_1^{n+1}+\tau_1^{n-1}\sigma_1\circ\tau+\ldots+(-1)^{n+1}\sigma_n\circ\tau. \tag{2.2}$$

Combining (2.1) and (2.2) we get:

$$G_n\circ\sigma\pm\sigma_n\circ\tau=\sum_{i=1}^{n-1}[G_i\circ\sigma\pm\sigma_i\circ\tau]\tau_1^i+\tau_1^n R(\tau_1, G_1\circ\sigma, \ldots, G_n\circ\sigma).$$

Since permutations of $t_1, \ldots, t_n$ give permutations of $\tau_1, \ldots, \tau_n$ using the invariance of σ_i we obtain a system of equations:

$$G_n\circ\sigma\pm\sigma_n\circ\tau=\sum_{i=1}^{n-1}(G_i\circ\sigma\pm\sigma_i\circ\tau)\tau_j^i+\tau_j^n R(\tau_j, G_1\circ\sigma, \ldots, G_n\circ\sigma),$$

for $j=1, \ldots, n$. If $\tau_i\neq\tau_j$ for $i\neq j$ we may "solve" this system to get

$$G_i\circ\sigma\pm\sigma_{n-i}\circ\tau\in M_{(t_1, \ldots, t_n)}^{n+2-i}$$

for $i=1, \ldots, n-1$.

Writing $G_i(y_1, \ldots, y_n)=\sum_{j=1}^{n}\alpha_j^i y_j+\sum_{s,k}T_{sk}^i(y)\,y_s y_k$ for T_{sk}^i smooth functions, we get:

$$G_i\circ\sigma\pm\sigma_{n-i}\circ\tau=\sum_{j=1}^{n}\alpha_j^i\sigma_j+G_{n-i}\circ\tau+\sum_{sk}T_{sk}^i(\sigma)\,\sigma_s\sigma_k$$

belongs to $M_{(t_1, \ldots, t_n)}^{n+2-i}$. Taking now the homogeneous component of degree $l\,(2\leqq l\leqq n-i+1)$ of the Taylor expansion at 0 of the above expression and using the fact that $d\tau(0)=\mathrm{Id}_{\mathbb{R}^n}$ we obtain:

$$\text{for}\quad 2\leqq l\leqq n-i,\qquad \alpha_{l-1}^i\sigma_{l-1}+\sum_{sk}P_{sk}^{i,l}(t_1, \ldots, t_n)\,\sigma_s\sigma_k=0,$$

where $P_{s,k}^{i,l}$ is a polynomial of degree $l-s-k-2$ and:

$$\alpha_{n-i}^i\sigma_{n-i}\pm\sigma_{n-i}+\sum_{sk}Q_{sk}^{i,l}(t_1,\ldots, t_n)\,\sigma_s\sigma_k=0,$$

with Q_{sk}^{il} polynomial of degree $n-i-s-k-1$.

Taking the average of the above expressions over the elements of $S(n+1)$ acting on $\mathbb{R}^n$ we obtain:

$$\begin{cases} \alpha_{l-1}^i \sigma_{l-1} \text{ equals a polynomial in } \sigma_1, \ldots, \sigma_s \text{ with } s<l-1, \\ (\alpha_{n-i}^i \pm 1)\sigma_{n-i} \text{ equals a polynomial in } \sigma_1, \ldots, \sigma_s \text{ with } s<n-1. \end{cases}$$

Since the $\sigma_i's$ are algebraically independent this implies:

$$\begin{cases} \alpha_{l-1}^i = 0 \quad \text{for} \quad 2 \leqq l \leqq n-i, \\ \alpha_{n-i}^i = \pm 1. \end{cases}$$

For $i=n$ we look at the equation:

$$G_n \circ \sigma \pm \sigma_n \circ \tau = \sum_{i=1}^{n-1} (G_i \circ \sigma \pm \sigma_{n-i} \circ \tau) \tau_1^i + \tau_1^n R(\tau_1, G_1 \circ \sigma, \ldots, G_{n-1} \circ \sigma)$$

and use the same process as before with $(G_i \circ \sigma \pm \sigma_{n-i} \circ \tau) \tau_1^i \in M_{(t_1, \ldots, t_n)}^{n+2}$ and $R(t, 0) \in M_t^2$ to obtain $\alpha_j^n = 0$ for $j=1, \ldots, n-1$ and $\alpha_n^n = \pm 1$.

The conclusion is that the matrix (α_j^i) is invertible and hence G is a local diffeomorphism.

Let's return to the notation of the proof of the Theorem and write:

$$(2.5) \qquad \bar{f}_1 \circ e_{n-1}^1 \circ \ldots \circ e_1^1 = H_1 \circ H_2^{-1} \circ \bar{f}_2 \circ e_{n-1}^2 \circ \ldots \circ e_1^2.$$

Assuming, without loss of generality, that for suitable local coordinates we have $\bar{f}_1(t, x) = (x, f_1(t, x))$ and $\bar{f}_2(t, u) = (u, f_2(t, u))$ and writing $M = H_1 \circ H_2^{-1} = (M_1, \ldots, M_n)$, it follows from the proof that H_i is a local diffeomorphism that the matrix:

$$\left(\frac{\partial M_i}{\partial y_j}(0)\right) i=1, \ldots, n-1; j=1, \ldots, n-1,$$

is non-singular.

Thus, defining:

$$\Phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \text{ by } \Phi(t, x) = (t, M_1 \circ \bar{f}_2, \ldots, M_{n-1} \circ \bar{f}_2),$$

germ of diffeomorphism, we obtain $\bar{g} = M \circ \bar{f}_2 \circ \Phi^{-1}$ another germ of family weakly equivalent to $\bar{f}_1$ (and obviously equivalent to $\bar{f}_2$) such that $\bar{g}(t, x) = (x, g(t, x))$ and if $\tilde{e}_{n-1}, \ldots, \tilde{e}_1$ are the mappings associated to $\bar{g}$ as constructed in the begining of this section it is easy to check that:

$$\bar{g} \circ \tilde{e}_{n-1} \circ \ldots \circ \tilde{e}_1 = \bar{f}_1 \circ e_{n-1}^1 \circ \ldots \circ e_1^1.$$

Theorem 2 follows now from the following lemmas:

LEMMA 1. — *Let $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a C^∞ right-left stable map germ such that $0 \in S_{1_n}(g)$.*

If $g^* : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$ is given by $g^*(\lambda) = \lambda \circ g$ then a sufficient condition for a map germ v to be in the image $g^*(C_0^\infty(\mathbb{R}^n))$ is that for any representatives $\tilde{g} : U \to \mathbb{R}^n$ and $\tilde{v} : U \to \mathbb{R}$ and for any pair of points $(p, q) \in U \times U$ whenever $\tilde{g}(p) = \tilde{g}(q)$ we have $\tilde{v}(p) = \tilde{v}(q)$.

Proof. — This lemma is a consequence of Glaeser's article [11].

There is no loss of generality if we write a representative of g in the form $\tilde{g}(x, t)=\left(x, t^{s+1}+\sum_{i=1}^{s-1} x_i t^i\right)$ for $(x, t)=(x_1, \ldots, x_n, t)$ coordinates in a neighborhood U of 0 in $\mathbb{R}^n$. This mapping satisfies all hypothesis of Theorem 1 of [11] so $\tilde{g}^*(C^\infty(\mathbb{R}^n))$ is closed in $C^\infty(U)$.

Therefore if we take $\tilde{v}$ a representative for v defined in U it suffices to that $\tilde{v}\in\overline{\tilde{g}(C^\infty(\mathbb{R}^n))}$. And for this we use Proposition VIII of [11] that tells us that we need only to verify that $\tilde{v}$ belongs bipointwise to $\tilde{g}^*(C^\infty(\mathbb{R}^n))$. In other words, we need only to check that for each pair p, q of points in $U\times U$ there exists a function $\Gamma_{p,q}\in C^\infty(\mathbb{R}^n)$ such that $\tilde{v}-\Gamma_{p,q}\circ\tilde{g}$ is flat at p and q. (Has 0 Taylor expansion at these points.)

Let us check first that $\tilde{v}$ belongs pointwise to $g^*(C^\infty(\mathbb{R}^n))$. We have two possibilities:

(1) $p\notin S(\tilde{g})$ (p is a regular point). We let $U_p\subset U$ be a neighborhood of p such that $\tilde{g}|_{U_p}$ is a diffeomorphism and define $\lambda^p\in C^\infty(\mathbb{R}^n)$ such that $\lambda^p=\tilde{v}\circ(\tilde{g}|_{U_p})^{-1}$ in a neighborhood of $\tilde{g}(p)$. Hence $T_p\tilde{v}=T_p(\lambda^p\circ\tilde{g})$. (Here $T_p h$ means the Taylor expansion of h at the point p.)

(2) $p\in S(\tilde{g})$: In this case $p\in S_{1_k}(\tilde{g})$ for some $k\leqq s$. If $p=(x_0, t_0)$ by the Preparation Theorem we may write $\tilde{v}=\sum_{i=0}^{k}(A_i\circ\tilde{g})(t-t_0)^i$ in a small open neighborhood of p, $U_p\subset U$. Since $\tilde{g}$ is right left stable and $p\in S_{1_k}(\tilde{g})$ there exist $k+1$ distinct point, $p_1, \ldots, p_{k+1}$ arbitrarily near of p such that $\tilde{g}(p_1)=\tilde{g}(p_j)$ for $j=2, \ldots, k+1$. By hypothesis we also have $\tilde{v}(p_1)=\tilde{v}(p_j)$. Therefore, if we write $p_j=(x, t_j)$ and $\tilde{g}(p_j)=q$ the polynomial $p(t)=\sum_{i=0}^{k}A_i(q)(t-t_0)^i$ satisfies $p(t_1)=p(t_j)=\tilde{v}(p_j)$ for $j=2, \ldots, k+1$. That means that it is constant and $A_i(q)=0$ for $i=1, \ldots, k$. In other words, A_i vanishes in the open subset $\{q\in\mathbb{R}^n/\#\{\tilde{g}^{-1}(q)\cap U_p=k+1\}\}\cap V_p$ where V_p is a small open neighborhood of $\tilde{g}(p)$. Since $\tilde{g}(p)$ is in the boundary of this set we obtain A_j flat at $\tilde{g}(p)$ for $j=1, \ldots, k$. Hence $T_p\tilde{v}=T_p(A_0\circ\tilde{g})$, concluding the proof that $\tilde{v}$ belongs pointwise to $\tilde{g}^*(C^\infty(\mathbb{R}^n))$. This also takes care of pairs of points p, q such that $\tilde{g}(p)\neq\tilde{g}(q)$ (it is enough to use a partition of unity).

We will proceed now by induction in s: For $s=1$ (fold) if $\tilde{g}(p)=\tilde{g}(q)$ then both are regular points. If $U_p\subset U$ is an open neighborhood of p such that $\tilde{g}\,|\,U_p$ is a diffeomorphism, we take $\Gamma\in C^\infty(\mathbb{R}^n)$ such that $\Gamma|_{\tilde{g}(U_p)}: \tilde{g}(U_p)\to\mathbb{R}^n$ is given by $\Gamma(y)=\tilde{v}\circ(\tilde{g}|_{U_p})^{-1}(y)$. Clearly $\tilde{v}|_{U_p}=\Gamma\circ\tilde{g}|_{U_p}$ and if we take $V_q=\tilde{g}^{-1}(\tilde{g}(U_p))\cap U_q$, where $U_q\subset U$ is a neighborhood of q disjoint from U_p, then for all $p'\in V_q$ there exists a $q'\in U_p$ such that $\tilde{g}(p')=\tilde{g}(q')$. Hence $\tilde{v}(p')=\tilde{v}(q')=\Gamma\circ\tilde{g}(q')=\Gamma\circ\tilde{g}(p')$ and so $\tilde{v}-\Gamma\circ\tilde{g}$ vanishes on $U_p\cup V_q$ (and is flat at p and q).

Let us assume the lemma for $s<k$ and prove it for $s=k$. We consider two cases for $\tilde{g}(p)=\tilde{g}(q)$. If one of the points is regular then we proceed as in the proof the case $s=1$. If both p and q are singular points, then the right − left stability of $\tilde{g}$ implies that we must have $p\in S_{1_l}(\tilde{g})$ and $q\in S_{1_r}(\tilde{g})$ with $r, l<k$. Choosing small enough disjoint open neighborhood U_p, U_q of p and q respectively and using the induction hypothesis for $\tilde{g}|_{U_p}$, $\tilde{g}|_{U_q}$, $\tilde{v}|_{U_p}$, $\tilde{v}|_{U_q}$

we obtain two functions A and B satisfying: $\tilde{v}\,|\,\mathrm{U}_p=\mathrm{A}\circ\tilde{g}\,|\,\mathrm{U}_p$ and $\tilde{g}\,|\,\mathrm{U}_q=\mathrm{B}\circ\tilde{g}\,|\,\mathrm{U}_q$. But if b belongs to $\tilde{g}(\mathrm{U}_p)\cap\tilde{g}(\mathrm{U}_q)$ then $b=\tilde{g}(p')=\tilde{g}(q')$ for $(p', q')\in\mathrm{U}_p\times\mathrm{U}_q$ and $\mathrm{A}(b)=\mathrm{A}\ (\tilde{g}(p'))=v(p')=v(q')=\mathrm{B}(\tilde{g}(q'))=\mathrm{B}(b)$. In other words, $\mathrm{A}-\mathrm{B}$ vanishes on $\tilde{g}(\mathrm{U}_p)\cap\tilde{g}(\mathrm{U}_q)\neq\emptyset$. The right-left stability of $\tilde{g}$ implies that $\mathrm{A}-\mathrm{B}$ is flat at $\tilde{g}(p)=\tilde{g}(q)$, so $\tilde{v}-\mathrm{A}\circ\tilde{g}$ is flat at p and q.

This concludes the induction step and the proof of Lemma 1. $\square$

LEMMA 2. — *Suppose* $(\bar{f}, \Pi)$ *and* $(\bar{g}, \Pi)$ *are two germs of family satisfying the hypothesis of Theorem* 2 *and:*

$$(a)\ \bar{f}\,(t, x)=(x, f\,(t, x)); \qquad \bar{g}(t, x)=(x, g\,(t, x)),$$

$$(b)\quad \bar{g}\circ\tilde{e}_{n-1}\circ\ldots\circ\tilde{e}_1=\bar{f}\circ e_{n-1}\circ\ldots\circ e_1,$$

then:

(i) *there exist germs of diffeomorphisms* $\alpha_1, \ldots, \alpha_{n-1}$ *such that* $\alpha_i\circ e_i=\tilde{e}_i\circ\alpha_{i-1}$;

(ii) $(\bar{f}, \Pi)$ *is equivalent to* $(\bar{g}, \Pi)$.

Proof of part (i). — We will construct the $\alpha_i's$ by induction.

First we notice that by the choice of coordinates we have made and by the definition of e_i we may write:

$$(2.6)\quad e_i(t_1, \ldots, t_{n-i+1}, x_{n-i+1}, \ldots, x_{n-1})=(t_1, \ldots, t_{n-i}, \bar{e}_i, x_{n-i+1}, \ldots, x_{n-1}),$$

with $\bar{e}_i$ invariant under permutation of the variables $t_1, \ldots, t_{n-i+1}$ for $i=1, \ldots, n-1$. For $\tilde{e}_i$ we obtain a similar expression. By hypothesis:

$$\begin{aligned}\bar{f}\circ e_{n-1}\circ\ldots\circ e_1&=(\bar{e}_{n-1}\circ e_{n-2}\circ\ldots\circ e_1, \ldots, \bar{e}_1, f\circ e_{n-1}\circ\ldots\circ e_1)\\ &=\bar{g}\circ\tilde{e}_{n-1}\circ\ldots\circ\tilde{e}_1=(\tilde{e}_{n-1}\circ\tilde{e}_{n-2}\circ\ldots\circ\tilde{e}_1, \ldots, \bar{\tilde{e}}_1, g\circ\tilde{e}_{n-1}\circ\ldots\circ\tilde{e}_1).\end{aligned}$$

Therefore $\bar{e}_1=\bar{\tilde{e}}_1$ and we may take $\alpha_1=$identity.

We are going to construct inductively α_i, $i=2, \ldots, n-1$ satisfying (i) of the statement of the lemma with the following additional property:

$$\alpha_i=\mathrm{I}_{n-i}\times\bar{\alpha}_i \qquad \text{with} \quad \mathrm{I}_{n-i}=\text{identity of } \mathbb{R}^{n-i},$$

$\bar{\alpha}_i$ invariant under permutation of the first $n-i$ variables. That is:

$$\alpha_i(t_1, \ldots, t_{n-i}, x_{n-i}, \ldots, x_{n-1})=(t_1, \ldots, t_{n-i}, \alpha_i^1, \ldots, \alpha_i^{i-1}, x_{n-1}),$$

with α_i^j invariant under permutations of $t_1, \ldots, t_{n-i}$.

Let us suppose α_i constructed for $i\leqq k$ and let us obtain α_{k+1}. If $d_{k+1}=e_{k+1}-\tilde{e}_{k+1}\circ\alpha_k$ then, using the induction hypothesis and equation (2.6) we get:

$$\tilde{e}_{k+1}\circ\alpha_k\circ e_k\circ\ldots\circ e_1=\tilde{e}_{k+1}\circ\tilde{e}_k\circ\alpha_{k-1}\circ\ldots\circ e_1=\tilde{e}_{k+1}\circ\ldots\circ\tilde{e}_1,$$

so $d_{k+1}\circ e_k\circ\ldots\circ e_1=0$.

If we denote by $F_k = e_k \circ \ldots \circ e_1(\mathbb{R}^n)$, $F_{k+1} = e_{k+1}(F_k)$ and $J_i = \{\lambda \in C^\infty(\mathbb{R}^n)/\lambda$ restricted to F_i is equal to $0\}$ for $i = k, k+1$ (observe that $F_k = e_{k+1}^{-1}(F_{k+1})$) we just saw that each coordinate function d_{k+1}^j of d_{k+1} belongs to J_k. Actually, from the expression for α_k we obtain:

$$d_{k+1}(t_1, \ldots, t_{n-k}, x_{n-k}, \ldots, x_{n-1})$$
$$= (0, \ldots, 0, \bar{e}_{k+1} - \bar{\bar{e}}_{k+1} \circ \alpha_k, x_{n-k} - \alpha_k^1, \ldots, x_{n-2} - \alpha_k^{k-1}, 0).$$

We want to show first that each d_{k+1}^j satisfies the hypothesis of Lemma 1 (for $g = e_{k+1}$ in that lemma), so we need to check that whenever $e_{k+1}(p_1) = e_{k+1}(p_2)$ for $p_1 \neq p_2$ then $d_{k+1}(p_1) = d_{k+1}(p_2)$.

But if we write $p_i = (t_1, \ldots, t_{n-k-1}, t_{n-k}^i, x_{n-k}, \ldots, x_{n-1})$ then by definition $e_{k+1}(p_1) = e_{k+1}(p_2)$ implies that:

$$x_{n-k} = \bar{e}_k(t_1, \ldots, t_{n-k-1}, t_{n-k}^1, t_{n-k}^2, x_{n-k+1}, \ldots, x_{n-1})$$

and:

$$\alpha_k(p_i) = (t_1, \ldots, t_{n-k-1}, t_{n-k}^i, \alpha_k^1 \circ e_k, \ldots, \alpha_k^{k-1} \circ e_k, x_{n-1}).$$

By the induction hypothesis $\alpha_k \circ e_k = \tilde{e}_k \circ \alpha_{k-1}$ hence:

$$\alpha_k(p_i) = (t_1, \ldots, t_{n-k-1}, t_{n-k}^i, \bar{\bar{e}}_k \circ \alpha_{k-1}, \alpha_{k-1}^1, \ldots, \alpha_{k-1}^{k-2}, x_{n-1}).$$

But from the invariance of each α_{k-1}^i, for $i = 1, \ldots, k-2$ and the definition of $\bar{\bar{e}}_k$ we get $\tilde{e}_{k+1} \circ \alpha_k(p_1) = \tilde{e}_{k+1} \circ \alpha_k(p_2)$. Therefore $d_{k+1}(p_1) = d_{k+1}(p_2)$ and applying Lemma 1 to each non zero component of d_{k+1} we obtain $d_{k+1} = \Gamma_{k+1} \circ e_{k+1}$ with

$$\Gamma_{k+1} = (0, \ldots, 0, \Gamma_{k+1}^1, \ldots, \Gamma_{k+1}^k, 0).$$

We also obtain that $\Gamma_{k+1} \in J_{k+1}$ and is flat at 0. So if we let $\tilde{\alpha}_{k+1} = I - \Gamma_{k+1}$ a local diffeomorphism in a nbhd of 0 then:

$$\tilde{\alpha}_{k+1}(t_1, \ldots, t_{n-k-1}, x_{n-k-1}, \ldots, x_{n-1}) = (t_1, \ldots, t_{n-k-1}, \tilde{\alpha}_{k+1}^1, \ldots, \tilde{\alpha}_{k+1}^k, x_{n-1})$$

and:

$$\tilde{\alpha}_{k+1} \circ e_{k+1} = e_{k+1} - \Gamma_{k+1} \circ e_{k+1} = e_{k+1} - d_{k+1} = \tilde{e}_{k+1} \circ \tilde{\alpha}_k.$$

In other to obtain the invariance of α_{k+1}^i we just average:

$$\alpha_{k+1}^i = \frac{1}{(n-k-1)!} \sum_\sigma \tilde{\alpha}_{k+1}^i \circ \sigma,$$

sum over all permutations of $t_1, \ldots, t_{n-k-1}$ and use the invariance of $\bar{\bar{e}}_{k+1}$ and $\bar{e}_{k+1}$ to get α_{k+1}.

This completes the induction and the proof of (i).

Proof of (ii). — The proof of this part is analogous to the induction step. We notice that α_{n-1} obtained before is an admissible change of coordinates, therefore we define $h=\bar{f}-\bar{g}\circ\alpha_{n-1}$ to obtain $h\circ e_{n-1}\circ\ldots\circ e_1=0$. Furthermore, if $\bar{f}(p_1)=\bar{f}(p_2)$ then $\bar{g}\circ\alpha_{n-1}(p_1)=\bar{g}\circ\alpha_{n-1}\circ(p_2)$ so, applying lemma 1 as before we get Γ such that $h=\Gamma\circ\bar{f}$ (Γ vanishes on the image $e_{n-1}\circ\ldots\circ e_1(\mathbb{R}^n)$). We define $\beta=\mathrm{I}-\Gamma$ local diffeomorphism in a neighborhood of 0 such that $\beta\circ\bar{f}=\bar{f}-\Gamma\circ\bar{f}=\bar{f}-h=\bar{g}\circ\alpha_{n-1}$.

In other words the pair of diffeomorphisms (β, α_{n-1}) gives an equivalence between the two germs of family.

This concludes the proof of Theorem 2. □

3. Weak equivalence and invariants for germs of family

In view of Theorem 2 we turn our attention to the study of weak equivalence. In Proposition 1 we obtained the hypersurface associated to a germ of family, it is a germ at 0 of a codimension one submanifold of $\mathbb{R}^{n+1}$, invariant under permutation of a given system of coordinates of $\mathbb{R}^{n+1}$. The next proposition tells us that any such germ can always be realized as an hypersurface associated to some germ of family.

PROPOSITION 2. — *Let* $\mathrm{V}\subset\mathbb{R}^{n+1}$ *be a germ at* 0 *of a codimension one submanifold which is invariant under permutations of some system of coordinates* $(t_1,\ldots, t_{n+1})$ *of* $\mathbb{R}^{n+1}$. *There exists a germ of family* $(\mathbb{R}, 0)\xleftarrow{\Pi}(\mathbb{R}^n, 0)\xrightarrow{\bar{g}}(\mathbb{R}^n, 0)$ *such that* $\bar{g}$ *is right-left stable*, $0\in \mathrm{S}_{1_n}(\bar{g})$ *and* $\mathrm{S}_{\bar{g}}=\mathrm{V}$.

Proof. — The hypothesis implies that we may take a representative for V in a neighborhood of 0 in $\mathbb{R}^{n+1}$ of the form:

$$\mathrm{V}=\{(t_1, \ldots, t_{n+1})\in\mathbb{R}^{n+1}/\mathrm{F}(\rho_0(t_1, \ldots, t_{n+1}), \ldots, \rho_n(t_1, \ldots, t_{n+1}))=0\},$$

where, as before, ρ_i=elementary symmetric polynomial of degree $i+1$ and F is a smooth function. In fact, using the Implicit Function Theorem we may suppose $\mathrm{F}(\rho_0, \ldots, \rho_n)=\rho_0-q(\rho_1, \ldots, \rho_n)$.

If we let $\sigma_i=\rho_i|_{\mathrm{V}}$ for $i=0, \ldots, n$ and $\sigma:\mathrm{V}\to\mathbb{R}^n$ defined by $\sigma=(\sigma_1, \ldots, \sigma_n)$ then the equation $t_1^{n+1}=t_1^n\rho_0-t_1^{n-1}\rho_1+\ldots+(-1)^n\rho_n$ together with the equation for V give us.

$$\sigma_n=(-1)^n t_1^{n+1}+\sum_{i=1}^{n-1}(-1)^{2n-1-i}t_1^i\sigma_{n-i}+(-1)^{n-1}t_1^n q(\sigma_1, \ldots, \sigma_n). \tag{2.7}$$

Thus, if we consider the function:

$$\mathrm{Z}(y_1,\ldots, y_n, t)=y_n-(-1)^n t^{n+1}-\sum_{i=1}^{n-1}(-1)^{2n-1-i}t^i y_i-(-1)^{n-1}t^n q(y_1,\ldots, y_n)$$

we obtain from equation (2.7) that $\mathrm{Z}(\sigma, t_1)=0$. But from the Implicit Function Theorem the zero set of Z can be expressed (locally) by the graph of a smooth a function $y_n=\mathrm{Q}(y_1, \ldots, y_{n-1}, t)$.

So (2.7) is equivalent to:

$$\sigma_n = Q(\sigma_1, \ldots, \sigma_{n-1}, t). \tag{2.8}$$

We will show now that the germ of family represented by:

$$\Pi(t, x) = t, \bar{g}(t, x) = (x, Q(x, t)), \text{ with } x = (x_1, \ldots, x_{n-1}),$$

is the one that we are looking for.

In order to verify that $\bar{g}$ is right-left stable and $0 \in S_{1_n}(\bar{g})$ we just check that $(\partial^i Q/\partial t^i)(0) = 0$, $i = 1, \ldots, n$ and the differentials $\{d(\partial^i Q/\partial t^i)(0)\}$, $i = 1, \ldots, n$ are linearly independent. But this follows just by differentiating implicitly at 0 in the equation $Z(y, t) = 0$ and using $q(0) = 0$.

To obtain $S_{\bar{g}} = V$ it is enough to see that if $p = (t_1, \ldots, t_{n+1})$ is a point of V with $t_i \neq t_j$ for $i \neq j$ then $p \in S_{\bar{g}}$.

We let $p_i = (t_i, \sigma_1(p), \ldots, \sigma_{n-1}(p))$, $i = 1, \ldots, n+1$. From the invariance of σ, equation (2.8) gives us:

$$\sigma_n(p_i) = Q(\sigma_1(p), \ldots, \sigma_{n-1}(p), t_i).$$

Therefore:

$$\bar{g}(p_i) = (\sigma_1(p), \ldots, \sigma_{n-1}(p), Q(\sigma_1(p), \ldots, \sigma_{n-1}(p), t_i)) = \sigma(p),$$

for all i, which means that the submanifolds $M_{t_i} = \bar{g}[\Pi^{-1}(t_i)]$ intersect at the point $\sigma(p)$. It follows from the definition of $S_{\bar{g}}$ that $p \in S_{\bar{g}}$. □

This leads to the following.

DEFINITION 5. — Two germs at 0, V_1, V_2, of codimension one submanifolds of $\mathbb{R}^{n+1}$ invariant under permutation of a system of coordinates in $\mathbb{R}^{n+1}$ are C^∞ (C^0) *equivalent* if the exists a germ of a diffeomorphism (resp. homeomorphism) $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\varphi \times \ldots \times \varphi(V_1) = V_2$ (the product is taken $(n+1)$ times).

Theorem 2 and Proposition 2 tell us that in order to classify generic germs of family with $0 \in S_{1_n}(\bar{f})$ we need to classify germs of submanifolds satisfying the symmetry condition under the above equivalence relation.

Applying a linearization Theorem of Sternberg [17] to the invariant subset $V \cap \{(t_1, \ldots, t_{n+1})/t_1 = \ldots = t_n\}$ we obtain a submanifolds V' equivalent to V such that $V' \cap \{t_1 = \ldots = t_n\} = (t_1, \ldots, t, -t, -nt)$; we call V' the osculating manifold of V (*see* [6]). Hence two germs of manifolds are equivalent if and only if their osculating manifolds are equivalent. This implies that there exists a germ ar 0 of a diffeomorphism $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\varphi(-nt) = -n\varphi(t)$. But this means that φ is linear. We remark that Dufour in [10] proves that even in the case of C^0 equivalence between submanifolds the germ φ must be linear.

The conclusion is that two generic germs of family at 0 satisfying the hypothesis of Theorem 2 are equivalent (C^0 or C^∞) if and only if there exists a real number $\lambda \neq 0$ such that

$\lambda . S'_{\bar{g}} = S'_{\bar{f}}$, where $S'_{\bar{g}}$ is the osculating manifold of the hypersurface associated to the germ $(\bar{g}, \Pi)$ and:

$$\lambda . (t_1, \ldots, t_{n+1}) = (\lambda t_1, \ldots, \lambda t_{n+1}).$$

To finish this article we characterize those germs of family equivalent to the one defined by

$$\Pi(t, x) = t$$
$$\bar{f}(t, x) = \left(x, t^{n+1} + \sum_{i=1}^{n-1} x_i t^i\right).$$

This family has the hyperplane $\sum_{i=1}^{n+1} t_i = 0$ as associated hypersurface and we already know that we need just to characterize those germs at 0 of symmetric submanifolds equivalent to this hyperplane. One way of doing this is to use some results of Web Theory (*see* [3] or [7]). If we let $V = \text{graph}(h)$ be a representative of the germ of symmetric manifold and:

$$w_i = \frac{\partial h}{\partial t_i} dt_i \qquad \text{for} \quad i = 1, \ldots, n$$

and:

$$w_{n+1} = \sum_{i=1}^{n} - \frac{\partial h}{\partial t_i} dt_i$$

then we obtain a $(n+1)$ – web of codimension one submanifolds of $\mathbb{R}^n$ defined $w_i = 0$, for $i = 1, \ldots, n+1$ (*see* [7] for définition). According to [7] a $(n+1)$-web is called *octahedral* $(n \geqq 3)$ or *hexagonal* (case $n = 2$) if it is equivalent to one formed by $n+1$ families of parallel hyperplanes. The characterization we are looking for follows from:

PROPOSITION A [3] CASE $n = 2$. – *If* Π *is a* 1-*form defined by* $dw_i = \Pi \wedge w_i$ *for* $i = 1, 2$ *the web is* hexagonal *if and only if* $d\Pi = 0$.

PROPOSITION B [7], CASE $n \geqq 3$. – *A web is octahedral if and only if there exists a* 1-*form* Π *such that* $dw_i = \Pi \wedge w_i$ *for* $i = 1, 2, \ldots, n$. *Actually, computing these condition explicity for our web we get:*

(A') *case* $n = 2$. – Denoting by $h_i = \partial h / \partial t_i$ we obtain:

$$w_i = h_i \, dt_i \qquad \text{and} \qquad \Pi = \frac{h_{12}}{h_1 h_2} . (w_1 + w_2),$$

with:

$$h_{12} = \frac{\partial^2 h}{\partial t_1 \partial t_2}.$$

Therefore $d\Pi = 0$ if and only if:

$$\frac{\partial^2}{\partial t_1 \partial t_2} \log \left(\frac{h_1}{h_2}\right) = 0.$$

(Notice that $h_i(0) = -1$ so this expression makes sense in a neighborhood of 0.)

(B') $n \geqq 3$. — $dw_i = d(\log h_i) \wedge w_i$ and there exists Π with $dw_i = \Pi \wedge w_i$ if and only if:

$$(a) \qquad \frac{h_{jk}}{h_i} = \frac{h_{jk}}{h_j}, \qquad k \neq i, j;$$

$$(b) \qquad \frac{\partial^2}{\partial t_i \partial t_j} \log\left(\frac{h_i}{h_j}\right) = 0, \qquad \forall_{i,j}.$$

In any case it is easy to obtain n smooth functions $U_1, \ldots, U_n$ with $U_i(0)=0$ and $U'_j(0)=U'_i(0)\neq 0$ so that the mapping:

$$H(t_1, \ldots, t_n) = \left(h(t_1, \ldots, t_n), \sum_{i=1}^{n} U_i(t_i)\right)$$

has constant rank 1.

Therefore there exists a smooth function W defined in a neighborhood of 0 such that:

$$\sum_{i=1}^{n} U_i(t_i) = W(h(t_1, \ldots, t_n)).$$

From this we also get $W'(0) = -U'_i(0)$. This means that another equation for V is:

$$\sum_{i=1}^{n} U_i(t_i) - W(t_{n+1}) = 0;.$$

But since V is symmetric we can average this equation to obtain:

$$V = \left\{ (t_1, \ldots, t_{n+1}) \Big/ \sum_{j=1}^{n+1} \left(\sum_{i=1}^{n=1} U_i - W \right)(t_j) = 0 \right\}.$$

It is enough to check that:

$$\Psi = \sum_{i=1}^{n} U_i - W.$$

is a local diffeomorphism. But since:

$$\Psi'(0) = \sum_{i=1}^{n} U'_i(0) - W'(0) = n\, U'_1(0) + U'_1(0) = (n+1)\, U'_1(0) \neq 0$$

Ψ provides an equivalence between V and the hyperplane. □

REFERENCES

[1] V. I. ARNOLD, *Wave Front Evolution and Equivariant Morse Lemma* (*Comm. on Pure and Applied Math.*, Vol. XXIX, 1976, pp. 557-582).

[2] W. BLASCHKE, *Topological Questions of Differential Geometry* (*Mimmeographed Notes*, University of Chicago 1932).

[3] W. Blaschke, *Geometrie der Gewebe*, Springer-Verlag, Balin, 1938.

[4] G. E. Bredon, *Introduction to Compact Transformation Group*, Academic Press, 1972.

[5] V. G. Boltyanskii, *Envelopes*, Pergamon Press, Macmillan, New York, 1964.

[6] M. J. D. Carneiro, *On the Envelope Theory* (*Thesis*, Princeton University, 1980).

[7] S. Chern and P. Griffiths, *Abel's Theorem and Webs*, *Jahusbericht der Deutschen Mathematiker Vereinigung*, Vol. 80, 1978, pp. 13-110.

[8] J. P. Dufour, *Diagrammes d'applications differentiables* (*Thèse*, Université du Languedoc, 1979).

[9] J. P. Dufour, *Sur la stabilité des diagrammes d'applications differentiables*, (Ann. scient. Éc. Norm. Sup., 4^e^ série, t. 10, 1977, pp. 153-174).

[10] J. P. Dufour, *Triplets de fonctions et stabilité des enveloppes*, preprint, 1981.

[11] G. Glaeser, *Fonctions composées différentiables* (*Annals of Mathematics*, Vol. 77, N° 1, January 1963, pp. 193-208).

[12] M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, (*G.T.M.*, Springer-Verlag, 1973).

[13] G. Julia, *Cours de géometrie infinitesimal*, Gauthier-Villars, Paris, 1953.

[14] J. N. Mather, *Stability of* C^∞ *Mappings II: Infinitesimal Stability Implies Stability* (*Ann. Math.*, Vol. 89, 1969, pp. 254-291).

[15] J. N. Mather, *Stability of* C^∞ *Mappings III: Finitely Determined Map Germs* (*I.H.E.S., Publications Mathematiques*, No. 35, 1968).

[16] V. Pœnaru, *Singularities* C^∞ *in Presence de Symmetrie*, Springer-Verlag (*Lecture Notes*, Berlin, 1976).

[17] S. Sternberg, *Local* C^n *Transformations of the Real Line* (*Duke Math. J.*, Vol. 24, 1957, pp. 97-112).

[18] R. Thom, *Sur la theorie des enveloppes* (*J. Math. pure et appl.* T. XLI, fasc. 2, 1962).

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