ANNALES SCIENTIFIQUES DE L'É.N.S.

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Local homology of groups of volume preserving diffeomorphisms. I

LOCAL HOMOLOGY OF GROUPS

OF VOLUME PRESERVING DIFFEOMORPHISMS, I

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ABSTRACT. — In this paper we prove the volume preserving analogue of the Mather-Thurston theorem, which relates the group of compactly supported C^∞-diffeomorphisms of $\mathbf{R}^n$ to the n-fold loop space of Haefliger's classifying space for codimension n foliations ([7], [15]). The proof is based on a study of the behaviour of monoids of self-embeddings of manifolds much as in [8] and [9], where a proof is given of the original Mather-Thurston theorem. However, because every volume preserving self-embedding of a compact manifold is a diffeomorphism, we must extend our techniques to non-compact W.

1. Statement of main results

Let Γ_{sl}^n be the groupoid of germs of C^∞-diffeomorphisms of $\mathbf{R}^n$ which preserve the volume form $\omega = dx_1 \wedge \ldots \wedge dx_n$, with the sheaf topology. *See* [3] and [10]. Its classifying space $B\,\Gamma_{sl}^n$ is the Haefliger classifying space for transversely oriented smooth codimension n foliations with transverse volume form. The homomorphism $\Gamma_{sl}^n \to \mathscr{SL}(n, \mathbf{R})$, which takes a germ f at x to its differential df_x at x, induces a map:

$$\nu : B\Gamma_{sl}^n \to B\,\mathscr{SL}(n, \mathbf{R}) \qquad \text{where} \qquad \mathscr{SL}(n, \mathbf{R})$$

is the classical special linear group. We will denote the homotopy fibre of ν by $\overline{B}\,\Gamma_{sl}^n$. This space classifies foliations with transverse volume form and trivialized normal bundle. Haefliger showed in [3] that $\pi_i(\overline{B}\Gamma_{sl}^n) = 0$, $1 \leq i < n$, and that $\pi_n(\overline{B}\Gamma_{sl}^n) \neq 0$. In fact, $\pi_n(\overline{B}\,\Gamma_{sl}^n) \cong \mathbf{R}$. *See* [10], Corollary to Theorem 2.

In [10] and [11] we studied the relation between the groupoid Γ_{sl}^n and the discrete group $\mathrm{Diff}_\omega \mathbf{R}^n$ of all ω-preserving diffeomorphisms of $\mathbf{R}^n$. Our main result was that there is a natural map $B\,\mathrm{Diff}_\omega \mathbf{R}^n \to B\,\Gamma_{sl}^n$ which induces an isomorphism on integer homology, except

possibly if $n = 2$.   This map essentially comes from taking a diffeomorphism to its germ at 0.   See [10], § 2.   It is not hard to construct a commutative diagram:

$$
\begin{array}{ccc}
\text{B Diff}_\omega \mathbf{R}^n & \longrightarrow & \text{B}\Gamma^n_{sl} \\
\downarrow & & \downarrow v \\
\text{B }\mathscr{D}\mathit{iff}_\omega \mathbf{R}^n & \longrightarrow & \text{B}\mathscr{S}\mathscr{L}(n, \mathbf{R})
\end{array}
$$

where $\mathscr{D}\mathit{iff}_\omega \mathbf{R}^n$ is the same group as $\text{Diff}_\omega \mathbf{R}^n$ but taken with the compact-open C^∞-topology.   It follows that the associated map of homotopy fibres $\overline{\text{B}} \, \mathscr{D}\mathit{iff}_\omega \mathbf{R}^n \to \overline{\text{B}} \, \Gamma^n_{sl}$ also induces an isomorphism on integer homology.

In the present paper this result will be extended to the group $\mathscr{D}\mathit{iff}_\omega W$ of all ω-preserving diffeomorphisms of a non-compact oriented smooth manifold W with a smooth nowhere-zero n-form ω.   We will assume that W is connected and without boundary, and that each of its ends is trivial (i. e., a product) and has infinite ω-volume.   We give $\mathscr{D}\mathit{iff}_\omega W$ the compact-open C^∞-topology, and denote its identity component by $\mathscr{D}\mathit{iff}_{\omega 0} W$.   Since the group of orientation-preserving diffeomorphisms of W acts transitively on the set of volume forms which satisfy the above conditions [2], the isomorphism class of the topological group $\mathscr{D}\mathit{iff}_\omega W$ is independent of the choice of ω.   There is a homomorphism Φ, called the *flux*, which takes $\mathscr{D}\mathit{iff}_{\omega 0} W$ to $H^{n-1}(W; \mathbf{R})$ and is defined as follows: if z is an $(n-1)$-cycle in W, then:

$$
\Phi(f)z = \int_c \omega,
$$

where c is an n-chain in W with boundary $f_*(z) - z$.   By [10], Lemma 2, our hypotheses on W and ω imply that Φ is surjective.   We will denote the kernel of Φ by $\mathscr{D}\mathit{iff}^\Phi_{\omega 0} W$; and the contractible abelian group $H^{n-1}(W; \mathbf{R})$ by V.   Then the sequence $\mathscr{D}\mathit{iff}^\Phi_{\omega 0} W \to \mathscr{D}\mathit{iff}_{\omega 0} W \to V$ gives rise to a fibration:

$$
\overline{\text{B}} \, \mathscr{D}\mathit{iff}^\Phi_{\omega 0} W \to \overline{\text{B}} \, \mathscr{D}\mathit{iff}_{\omega 0} W \to \overline{\text{B}} \, V.
$$

(Given a topological group $\mathscr{G}$, the notation $\overline{\text{B}} \, \mathscr{G}$ stands for the homotopy fibre of the map $\text{B}G \to \text{B}\mathscr{G}$, where G is $\mathscr{G}$ considered with the discrete topology.   According to [17], this space $\overline{\text{B}} \, \mathscr{G}$ depends only on the germ of $\mathscr{G}$ at the identity, and its homology $H_*(\overline{\text{B}} \, \mathscr{G})$ is called the *local homology* of $\mathscr{G}$.)

We will relate $\overline{\text{B}} \, \mathscr{D}\mathit{iff}_{\omega 0} W$ to $\text{B} \, \Gamma^n_{sl}$ by the following construction.   Let $E \to \text{B} \, \mathscr{D}\mathit{iff}_{\omega 0} W$ be the canonical fibration with fibre W.   Its pull-back to $\text{B Diff}_{\omega 0} W$ has discrete structural group, and so may be considered to be foliated.   When pulled back further to $\overline{\text{B}} \, \mathscr{D}\mathit{iff}_{\omega 0} W$, the fibration becomes trivial.   Therefore, there is a canonical foliation of the product $(\overline{\text{B}} \, \mathscr{D}\mathit{iff}_{\omega 0} W) \times W$ which is transverse to the fibres $y \times W$.   It has a transverse volume form

which restricts to ω on each fibre.   Clearly, its normal bundle is the product with $\overline{B}\ \mathscr{D}\!\mathit{iff}_{\omega 0}\ W$ of the tangent bundle TW to W.   Therefore, this foliation is classified by the diagram:

$$
\begin{array}{ccc}
(\overline{B}\ \mathscr{D}\!\mathit{iff}_{\omega 0}\ W)\times W & \xrightarrow{\ \ F\ \ } & B\ \Gamma^{n}_{sl} \\
\downarrow & & \downarrow{\scriptstyle v} \\
W & \xrightarrow{\ \ \tau\ \ } & B\ \mathscr{S}\mathscr{L}(n,\ \mathbf{R})
\end{array}
$$

where τ classifies TW.   We will assume that v has been made into a Hurewicz fibration so that this diagram commutes exactly.   Then, corresponding to each point y in $\overline{B}\ \mathscr{D}\!\mathit{iff}_{\omega 0}\ W$, there is a lifting $F\,|\,(y\times W)$ of τ to $B\ \Gamma^{n}_{sl}$.   Equivalently, we may consider this lifting to be a section of the pull-back by τ of the fibration $B\Gamma^{n}_{sl}\to B\ \mathscr{S}\mathscr{L}(n,\ \mathbf{R})$.   We will define $S_{\omega}(W)$ to be the space of all such sections, or liftings, with the compact-open topology.   Then this construction defines a map:

$$
f_{W}\ :\ \ \overline{B}\ \mathscr{D}\!\mathit{iff}_{\omega 0}\ W\to S_{\omega}(W).
$$

Since $\overline{B}\ \Gamma^{n}_{sl}$ is $(n-1)$-connected, $S_{\omega}(W)$ is connected.   In general it will not be simply connected.   In   fact,   because $\pi_{n}(\overline{B}\ \Gamma^{n}_{sl})\cong\mathbf{R}$,   obstruction   theory   shows   that $\pi_{1}(S_{\omega}(W))\cong H^{n-1}(W;\ \mathbf{R})=V$.   Therefore, there is a fibration sequence:

$$
\tilde{S}_{\omega}(W)\to S_{\omega}(W)\to\overline{B}\ \mathscr{V},
$$

where $\tilde{S}_{\omega}$ is the universal cover of S_{ω}.   We will show in Lemma 6.1 below that there is a commutative diagram:

$$
\begin{array}{ccccc}
\overline{B}\ \mathscr{D}\!\mathit{iff}^{\Phi}_{\omega 0}\ W & \longrightarrow & \overline{B}\ \mathscr{D}\!\mathit{iff}_{\omega 0}\ W & \longrightarrow & \overline{B}\ \mathscr{V} \\
\downarrow{\scriptstyle \tilde{f}_{W}} & & \downarrow{\scriptstyle f_{W}} & & \downarrow{\scriptstyle =} \\
\tilde{S}_{\omega}(W) & \longrightarrow & S_{\omega}(W) & \longrightarrow & \overline{B}\ \mathscr{V}.
\end{array}
$$

Our first main result is:

THEOREM 1.1. — *If* $n=\dim\ W\neq 2$, *then*:

$$
\tilde{f}_{W}\ :\ \ \overline{B}\ \mathscr{D}\!\mathit{iff}^{\Phi}_{\omega 0}\ W\to\tilde{S}_{\omega}(W),
$$

induces an isomorphism on integer homology.   Equivalently, f_{W} *is a homology equivalence.*

(A map $f:X\to Y$ is called a *homology equivalence* if it induces an isomorphism $H_{*}(X,\ f^{*}A)\to H_{*}(Y,\ A)$ for all coefficient systems A of abelian groups on Y.   *See* [4], where such maps are called acyclic.)

When $n=1$ these maps $\tilde{f}_{W}$ and f_{W} are both homotopy equivalences.   For in this case $W=\mathbf{R}$, and it is easy to check that both $\overline{B}\ \mathscr{D}\!\mathit{iff}_{\omega 0}\ \mathbf{R}$ and $S_{\omega}(\mathbf{R})\simeq\overline{B}\ \Gamma^{1}_{sl}$ are $K(\pi,\ 1)$'s where π is the discrete group of additive reals.   However when $n>1$ the maps f do not induce an isomorphism on π_{1} and so are not homotopy equivalences.

It is not clear whether Theorem 1.1 is true when $n = 2$. However a slightly weaker version does hold: *see* Theorem 7.3 below.

Our second main result concerns the group $\mathcal{D}\!iff^{\,c}_{\omega 0}\, W$ of compactly supported ω-preserving diffeomorphisms of W, taken with the usual direct limit topology. Notice that the canonical foliation of $(\overline{B}\,\mathcal{D}\!iff^{\,c}_{\omega 0}\, W) \times W$ is the trivial (product) foliation near infinity in W. One can therefore choose the classifying map F so that each section $f_W(y)$ equals a given base section s_0 outside a compact subset of W. *See* [8], § 2 and [9], Appendix. We will denote the space of such compactly supported sections by $S^c_\omega(W)$, and will give it the direct limit topology as in [8]. Then f_W induces a map $\overline{B}\,\mathcal{D}\!iff^{\,c}_{\omega 0}\, W \to S^c_\omega(W)$. Since $\pi_n(\overline{B}\,\Gamma^n_{sl}) \neq 0$, the space $S^c_\omega(W)$ is not connected, and we write $S^c_{\omega 0}(W)$ for the connected component which contains the base point s_0. It is not hard to check that the homotopy type of the space $S^c_\omega(W)$ does not depend on the choice of base section s_0. Further, all the components of $S^c_\omega(W)$ have the same homotopy type. This means for example that when W is parallelizable we may identify $S_\omega(W)$ with the space of maps $W \to \overline{B}\,\Gamma^n_{sl}$ and may take s_0 to be the constant map. Thus $S^c_\omega(\mathbf{R}^n)$ is the space of maps from $S^n = \mathbf{R}^n \cup \infty$ to $\overline{B}\,\Gamma^n_{sl}$ which take a neighbourhood of ∞ to the base point in $\overline{B}\,\Gamma^n_{sl}$, and so it is homotopy equivalent to the n-fold loop space $\Omega^n\,\overline{B}\,\Gamma^n_{sl}$.

THEOREM 1.2. — *The map:*

$$f^{\,c}_W \,:\quad \overline{B}\,\mathcal{D}\!iff^{\,c}_{\omega 0}\, W \to S^c_{\omega 0}(W),$$

is a homology equivalence.

Thus there is a homology equivalence $\overline{B}\,\mathcal{D}\!iff^{\,c}_{\omega 0}\, \mathbf{R}^n \to (\Omega^n\,\overline{B}\,\Gamma^n_{sl})_0$. Using Thurston and Banyaga's computation of $H_1(\overline{B}\,\mathcal{D}\!iff^{\,c}_{\omega 0}\, \mathbf{R}^n;\, \mathbf{Z})$, this yields:

COROLLARY 1.3. — $\pi_{n+1}(\overline{B}\,\Gamma^n_{sl}) = 0$ *for* $n \geq 3$, *and* $\pi_3(\overline{B}\,\Gamma^2_{sl}) \cong \mathbf{R}$.

Obstruction theory now shows that $\pi_1(S^c_{\omega 0}(W))$ is abelian and isomorphic to $H^{n-1}_c(W;\, \mathbf{R})$ when $n \geq 3$. Therefore in this case Theorem 1.2 is equivalent to the statement that the map:

$$\tilde{f}^{\,c}_W \,:\quad \overline{B}\,\mathcal{D}\!iff^{\,c\Phi}_{\omega 0}\, W \to \tilde{S}^c_{\omega 0}(W),$$

is a homology equivalence, where $\tilde{S}_{\omega 0}$ is the universal cover of $S_{\omega 0}$ and $\mathcal{D}\!iff^{\,c\Phi}_{\omega 0}\, W$ is the kernel of the flux homomorphism. When $n = 2$ it follows from Rousseau [12], that $\pi_1 S^c_{\omega 0}(W)$ is nilpotent but in general not abelian. *See* Theorem 7.7.

The proof of Theorem 1.1 is based on a study of the behaviour of monoids of self-embeddings of manifolds as in [8] and [9]. Because every volume preserving self-embedding of a manifold W is a diffeomorphism unless W is non-compact, we will deal exclusively with non-compact manifolds. This introduces certain complications into the argument. For example, one cannot induct over an ordinary handle decomposition of W since this involves compact pieces. One must use "handles with spikes". Also, when X has non-empty boundary, one cannot work simply with the monoid of all self-embeddings of X since this may not have good properties.

In § 2 we formulate the properties which our monoids are required to have, and then show by an inductive argument that, if such monoids exist, the map f_W in Theorem 1.1 is a **Z**-homology equivalence. As we will see in § 6, Theorems 1.1 and 1.2 for $n \geq 3$ follow fairly easily from this. The hard work lies in establishing that monoids with the required properties do in fact exist. This occupies § 3-§ 5. The most difficult step is to show that the monoids can be "localized", for instance that the group $\mathscr{D}iff_\omega W$ can be replaced by the monoid $\mathscr{D}iff_\omega(W, X)$ which, roughly speaking, consists of all diffeomorphisms of W which take the submanifold X into itself. The arguments of [9], § 4, which show that $\overline{B}\ \mathscr{D}iff(W, X) \cong \overline{B}\ \mathscr{D}iff\ W$, fail when X is non-compact since one cannot then find a suitable *open* subset $\mathscr{U}$ of $\mathscr{D}iff\ W$. However, they may be elaborated to prove Proposition 4.5 below, which is half the localization lemma. The other half uses some results of [10] and [11] which are not valid when $n = 2$. The case $n = 2$ is discussed separately in § 7. The proof of Theorem 1.2 for $\mathbf{R}^2$ is written up on its own as Theorem 7.2. This is the shortest complete proof in the whole paper, since there is no need for an induction here. Once the basic notation is understood, it should be possible to read it, especially if one accepts (7.2) (i) temporarily.

Perhaps it is worth remarking that almost everything in this paper is valid for the group $\mathscr{D}iff\ W$ of all diffeomorphisms of W. The only exceptions are the results which involve the flux homomorphism. Also, there are corresponding results for the group of volume preserving diffeomorphisms of a compact manifold. This will be the subject of a later paper.

Finally, a word about notation. We will use the language and notations of [9] wherever practicable. In particular, we will work in Vogt's category $HG(^1)$. (All this means for us is that the topology put on a product may not be the usual product topology.) A map will be called a **Z**-*homology equivalence* if it induces an isomorphism on untwisted integer homology, and a *homology equivalence* if it induces an isomorphism on homology for all local coefficients. Thus $f : X \to Y$ is a homology equivalence if and only if the induced map $\tilde{f} : f^*\tilde{Y} \to \tilde{Y}$ is a **Z**-homology equivalence, where $\tilde{Y}$ is the universal cover of Y. In particular, these concepts are the same when the target space is simply connected. We will call a sequence $F \to E \to B$ a *homology* (resp. *homotopy*) *fibration sequence* if there is an associated map of F into the homotopy fibre of $E \to B$ which is a **Z**-homology (resp. homotopy) equivalence. If in addition $\pi_1(B)$ acts trivially on $H_*(F; \mathbf{Z})$, the sequence will be said to be *simple*. Finally, a space is called *acyclic* if its reduced integer homology vanishes.

(^{1}) The statement made in [9], p. 107 that one can consider a group of germs such as Top_n to be a space in the category HG is false, since these groups are not quasi-topological spaces in the sense of Vogt. In fact, the set of continuous maps from a compact space X to Top_n (which is described by Haefliger in [3] for instance) does not inject into the set of set-theoretical maps from X to Top_n. Hence in the proof given on p. 107 of [9] that $\overline{B}\ \mathscr{E}mb\ \mathbf{R}^n \simeq \overline{B}\ \Gamma_0^n$ where $\mathscr{E}mb$ consists of C^0-embeddings, one must either think of the spaces as quasi-topological spaces in Haefliger's sense or use the semi-simplicial models of § 3 below. This problem does not arise when considering C^r-embeddings for $r \geq 1$.

2. Sketch of the proof of Theorem 1.1'

The inductive argument which proves Theorem 1.1 is very like that of [8], § 3. However, because we can only deal with spaces which are well-behaved at infinity and which have no compact pieces, it is important to describe carefully the admissible pairs over which we induct. We will assume from now on that W is as in § 1. Then W is the interior of a compact manifold whose boundary will be denoted by $\hat{W}$. We will choose, once and for all, a product structure $\hat{W} \times [0, \infty)$ in a neighbourhood of infinity in W. Further, we will put a Riemannian metric on W which is a product in $\hat{W} \times [0, \infty)$, and will assume that ω is the corresponding volume form. If A is a closed subset of W, we will write A_ε for its closed ε-neighbourhood.

A (possibly empty) closed subset A of W will be called *nice* if it is a product at infinity. (This means that $A \cap (\hat{W} \times [0, \infty)) = \hat{A} \times [0, \infty)$ for some $\hat{A}$ in $\hat{W}$.) Further, a submanifold X of W will be called *admissible* if it is a nice top dimensional submanifold of W whose boundary contains no compact components. Thus both X and $\overline{W - X}$ have finitely many components, none of which are compact. In distinction to [8] we do not allow X to have corners. Note also that if $\varepsilon > 0$ is sufficiently small, X_ε will be an admissible submanifold of W which is diffeomorphic to X. Finally, a pair (X, A) consisting of an admissible submanifold X of W together with a nice subset A of W will be called *admissible* if no component of $W - A$ or of $\partial X - A$ has compact closure. It follows that no component of $X - A$ will have compact closure either. Examples of such pairs are (W, X) and (X, ∂X), where X is an admissible submanifold of W. As will soon become clear, these non-compactness assumptions are needed so that we do not run up against volume obstructions. Note also that an admissible pair is determined by its intersection with the compact manifold $W - \hat{W} \times (0, \infty)$.

Let $\mathcal{D}iff_\omega$ (W, rel A) be the direct limit over all $\varepsilon > 0$ of the groups $\{g \in \mathcal{D}iff_\omega W; g = \mathrm{id}$ on $A_\varepsilon\}$, where these are given the compact-open C^∞-topology. The identity component of this group is denoted by $\mathcal{D}iff_{\omega 0}$ (W, rel A). Recall that $\overline{B} \, \mathcal{D}iff_{\omega 0}$ (W, rel A) $\simeq \overline{B} \, \mathcal{D}iff_\omega$ (W, rel A) by [9], Lemma 3.5. Further, let S_ω(W, rel A) be the direct limit over all $\varepsilon > 0$ of the subspaces $\{s \in S_\omega(W) : s = s_0$ on $A_\varepsilon\}$. By [8], § 2 and [9], Appendix, f_W maps $\overline{B} \, \mathcal{D}iff_{\omega 0}$ (W, rel A) into S_ω(W, rel A). We will prove:

THEOREM 2.1. — *For every admissible pair* (W, A) *with* dim $W \geq 3$, *the natural map*:

$$f_W : \quad \overline{B} \, \mathcal{D}iff_{\omega 0}(W, \text{rel A}) \to S_\omega(W, \text{rel A}),$$

is a **Z***-homology equivalence.*

COROLLARY 2.2. — *If* X *is an admissible submanifold of* W *where* dim $W \geq 3$, *then*:

$$f_X : \quad \overline{B} \, \mathcal{D}iff_{\omega 0}(X, \text{rel } \partial X) \to S_\omega(X, \text{rel } \partial X),$$

is a **Z***-homology equivalence.*

In order to prove this we must investigate how the functors $\overline{B} \, \mathcal{D}iff_{\omega 0}$ and S_ω behave under restriction. The following lemma is elementary.

LEMMA 2.3. — *Suppose that* (X, A) *and* (Y, A) *are admissible pairs with* $Y \subset \text{Int } X$. *Then:*

$$S_\omega(X, \text{ rel } Y \cup A) \to S_\omega(X, \text{ rel } A) \to S_\omega(Y, \text{ rel } Y \cap A),$$

is a simple homotopy fibration sequence.

Proof. — The fact that this is a fibration sequence may be proved as in [8], Lemma 3. However, because $\overline{B} \Gamma_{sl}^n$ is $(n-1)$-connected rather than n-connected, the argument of that lemma does not suffice to show that the action of π_1 (base) is trivial. To prove this, let l_t, $0 \le t \le 1$, be a loop in $S_\omega(Y, \text{ rel } Y \cap A)$ and choose $\eta > 0$ so that $\overline{Y_\eta - Y} \cong Y \times [0, \eta]$ is contained in $\text{Int } X$. Because A is a product near infinity, we may replace A by a slightly larger set with the property that $A \cap \overline{Y_\eta - Y} = (\partial Y \cap A) \times [0, \eta]$. It is easy to see that l_t may be extended to a path $\tilde{l}_t$ in $S_\omega(X, \text{ rel } A)$ with $\tilde{l}_0 = \text{id}$ and with support in Y_ε, for any given ε, $0 < \varepsilon < \eta$. In particular, we may suppose that $\tilde{l}_1$ has support in $Y_\varepsilon - (Y \cup A) \cong (\partial Y - A) \times (0, \varepsilon)$. Also, one can check that the action of $\tilde{l}_t$ on $H_*(S_\omega(X, \text{ rel } Y \cup A))$ is induced by the map $s \mapsto s \amalg \tilde{l}_1$ of $S_\omega(X, \text{ rel } Y_\varepsilon \cup A)$ into $S_\omega(X, \text{rel } Y \cup A)$, where $\amalg$ denotes the "addition" of sections with disjoint supports. However, this map is homotopic to the identity, since the space $S_\omega(Y_\varepsilon, \text{ rel } A \cup \partial Y_\varepsilon \cup Y)$ which contains $\tilde{l}_1$ is connected because $\partial Y - A$ contains no compact components. $\square$

Before we can discuss the behaviour of the functor $\overline{B} \mathcal{D}iff_\omega$ under restriction we must extend this functor to manifolds with boundary. In [8] and [9] we did this by considering the space $\overline{B} \mathscr{E}mb_0 X$, where $\mathscr{E}mb_0 X$ is the identity component of the monoid of all self-embeddings of X. Further we defined $\mathscr{E}mb_0(X, Y)$ to be the identity component of the submonoid of $\mathscr{E}mb_0 X$ consisting of all embeddings which take Y into itself, and looked at the restriction map $\overline{B} \mathscr{E}mb_0(X, Y) \to \overline{B} \mathscr{E}mb_0 Y$. Unfortunately, when X and Y are non-compact, one cannot prove anything useful about these spaces and maps. Moreover, the definitions which do work are rather complicated. Therefore, we will postpone their details until § 3 and here will simply state their properties.

Let X, Y and A satisfy the hypotheses of Lemma 2.3. Then we will define spaces $\overline{B} E_0(X, Y, \text{ rel } A)$ and maps:

$$\overline{B} E_0(X, Y, \text{ rel } A) \overset{i_X}{\hookrightarrow} \overline{B} E_0(X, \text{ rel } A) \overset{f_X}{\to} S_\omega(X, \text{ rel } A),$$

which depend only on X, Y and $A \cap X$ and not on the ambient manifold W, and which have the following properties.

(2.4) (Normalization). — $\overline{B} E_0(W, \text{ rel } A) \simeq \overline{B} \mathcal{D}iff_{\omega 0}(W, \text{rel } A)$, *and* f_W *is the map of* (2.1).

(2.5) (Normalization). — *If* D *is a closed* $(n-1)$-*disc and* $n \ge 3$, *then:*

$$f_D : \quad \overline{B} E_0(D \times \mathbf{R}) \to S_\omega(D \times \mathbf{R}) \simeq \overline{B} \Gamma_{sl}^n,$$

is a homology equivalence.

(2.6) (Behaviour with respect to A) :

(i) $\overline{B} E_0(X, Y, \text{ rel } A) = \overline{B} E_0(X, Y, \text{ rel } A \cap X)$;

(ii) *if* $A \subset A'$, *then* $\overline{B} E_0 (X, Y, \text{rel } A) \supset \overline{B} E_0 (X, Y, \text{rel } A')$;

(iii) *if* $A_1 \supset A_2 \ldots$ *and* $A = \cap A_i$, *then* $\overline{B} E_0 (X, Y, \text{rel } A) = \varinjlim \overline{B} E_0 (X, Y, \text{rel } A_i)$.

(2.7) (Triviality). — *If Z is a compact connected manifold then:*

(i) $\overline{B} E_0 (\partial Z \times [0, 1] \times \mathbf{R}, \text{rel } \partial Z \times 0 \times \mathbf{R})$ *is contractible, and*

(ii) *if either* $\dim Z \geqq 2$ *or* $\partial Z \neq 0$, *then*

$$\overline{B} E_0 (Z \times [0, \infty), \text{rel } \partial Z \times [0, \infty) \cap Z \times 0)$$

is acyclic.

(2.8) (Localization) *If* $\dim X \geq 3$, *then:*

$$i_X : \quad \overline{B} E_0 (X, Y, \text{rel } A) \subsetneqq \overline{B} E_0 (X, \text{rel } A),$$

is a **Z**-*homology equivalence.*

(2.9) (Fibration). — *There is a simple homology fibration sequence:*

$$\overline{B} E_0 (X, \text{rel } Y \cup A) \to \overline{B} E_0 (X, Y, \text{rel } A) \to \overline{B} E_0 (Y, \text{rel } Y \cap A).$$

Moreover, this sequence maps by f_X *to the sequence of section spaces in Lemma* 2.3.

It is worth remarking now that the non-compactness conditions in the definition of admissible pair are essential in the proof of (2.8). For example, if:

$$X = S^2 \times \mathbf{R}, \quad Y = S^2 \times [0, \infty) \quad \text{and} \quad A = S^2 \times (-\infty, -1],$$

then any ω-preserving diffeomorphism of X which is the identity on A and which takes Y into itself must in fact take Y *onto* itself. This implies that $\overline{B} E_0 (X, Y, \text{rel } A)$ is homotopic to $\overline{B} \mathscr{G}$, where $\mathscr{G}$ is the subgroup of $\mathscr{Diff}_\omega (X, \text{rel } A)$ consisting of diffeomorphisms which take Y onto Y. Now $\overline{B} E_0 (X, \text{rel } A)$ is acyclic by (2.7). However, it is not at all clear that $\overline{B} \mathscr{G}$ is acyclic.

The remainder of this section is devoted to a proof of the following theorem.

THEOREM 2.10. — *For every admissible pair* (X, A) *with* $\dim X \geqq 3$:

$$f_X : \quad \overline{B} E_0 (X, \text{rel } A) \to S_\omega (X, \text{rel } A),$$

is a **Z**-*homology equivalence.*

By (2.4), Theorem 2.1 is the special case X = W. Observe also that (2.10) holds in the situations considered in (2.7). Theorem 2.10 is significantly easier to prove when (X, A) is a product of the form $(\hat{X} \times \mathbf{R}, \hat{A} \times \mathbf{R})$ since one can then argue by induction over a handle decomposition of the pair $(\hat{X}, \hat{A})$. In the general case, one must use "handles with spikes" in order to satisfy the non-compactness assumptions. Note also that because we have no results on $\pi_1 (\overline{B} E_0 (X, \text{rel } A))$ which are comparable to [9], Lemmas 5.3 and 5.4, we do not claim that f_X is ever a homotopy equivalence.

The proof of (2.10) is based on the following easy lemma.

LEMMA 2.11. — *Suppose that* (X, A) *and* (Y, A) *are admissible pairs with* $Y \subset \text{Int} \, X$ *and* $\dim X \geqq 3$. *Then*:

(a) *If* (2.10) *holds for any two of* (X, Y $\cup$ A), (X, A) *and* (Y, Y $\cap$ A), *then it holds for the third also, and*

(b) *if* (2.10) *holds for* (X, Y) *and* (X, A) *and for either* (X, Y $\cup$ A) *or* (X, Y $\cap$ A), *then it holds for all four pairs.*

Proof. — Part (a) follows immediately from (2.8), (2.9) and the spectral sequence comparison theorem. To prove (b) one compares the diagram:

$$\bar{B} \, E_0(X, \text{rel} \, Y \cup A) \longrightarrow \bar{B} \, E_0(X, Y, \text{rel} \, A) \longrightarrow \bar{B} \, E_0(Y, \text{rel} \, A)$$
$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow =$$
$$\bar{B} \, E_0(X, \text{rel} \, Y) \longrightarrow \bar{B} \, E_0(X, Y, \text{rel} \, Y \cap A) \longrightarrow \bar{B} \, E_0(Y, \text{rel} \, Y \cap A)$$

with the corresponding diagram for S and uses (2.6), (2.8), (2.9) and part (a) above. *See* [8], Prop. 4.

LEMMA 2.12. — *Theorem* 2.10 *holds for* $(D^{n-1} \times \mathbf{R}, \partial D^{n-1} \times \mathbf{R})$, $n \geqq 3$.

Proof. — Consider the pairs $(D^{n-1} \times \mathbf{R}, S^k \times \mathbf{R})$, $k = -1, 0, \ldots, n-2$, where D^{n-1} is the unit $(n-1)$-disc in $\mathbf{R}^{n-1}$ and S^k is the image in D^{n-1} of the sphere of radius $1/2$ in $\mathbf{R}^{k+1} \subset \mathbf{R}^{n-1}$. Then (2.10) holds for $(D^{n-1} \times \mathbf{R}, S^{-1} \times \mathbf{R}) = (D^{n-1} \times \mathbf{R}, \emptyset)$ by (2.5). By (2.7) (i), (2.10) also holds for $(D^{n-1} \times \mathbf{R}, D' \times \mathbf{R})$ where D' is an $(n-1)$ disc in $\text{Int} \, D^{n-1}$. Hence, an application of (2.11) (b) shows that (2.10) holds for $(D^{n-1} \times \mathbf{R}, (D_1 \cup D_2) \times \mathbf{R})$ where D_1 and D_2 are disjoint $(n-1)$-discs in D^{n-1}. Therefore, by (2.6), (2.10) holds for the pair $(D^{n-1} \times \mathbf{R}, S^0 \times \mathbf{R})$. Continuing in this way, one easily sees that (2.10) holds for $(D^{n-1} \times \mathbf{R}, S^k \times \mathbf{R})$, $k = 1, \ldots, n-2$. At each stage one should thicken $(D^{n-1} \times \mathbf{R}, S^k \times \mathbf{R})$ slightly so that it has the form $(D^{n-1} \times \mathbf{R}, Y \cup A)$, where Y and A are nice manifolds in $\text{Int} \, (D^{n-1} \times \mathbf{R})$, which are both diffeomorphic to $D' \times \mathbf{R}$ and whose intersection $Y \cap A$ retracts onto $S^{k-1} \times \mathbf{R}$. One can then apply (2.11) (b). Finally, using (2.7) (i), (2.11) (a) and the fact that (2.10) holds for $(D^{n-1} \times \mathbf{R}, S^{n-2} \times \mathbf{R})$, one proves that (2.10) holds for $(D^{n-1} \times \mathbf{R}, \partial D^{n-1} \times \mathbf{R})$. $\square$

COROLLARY 2.13. — *Theorem* 2.10 *holds for* $(S^{n-1} \times \mathbf{R}, \emptyset)$, *if* $n \geqq 3$.

Proof. — This follows from (2.12) using (2.5) and (2.11) (a). $\square$

In fact, Corollary (2.13) is the only case of Theorem 2.10 which is needed in the proof of Corollary 1.3, and the reader who is interested primarily in that may skip to §6, taking $W = S^{n-1} \times \mathbf{R}$ in (6.2).

LEMMA 2.14. — *Theorem* 2.10 *holds for admissible pairs* (W, A) *of dimension* $\geqq 3$ *where A is an admissible submanifold of* W *which intersects every end of* W.

*Proof.* — Let $W^0 = W - (\hat{W} \times (0, \infty))$, and let:

$$A \cap W^0 = X_0^0 \subset X_1^0 \subset \ldots \subset X_p^0 = W^0,$$

be a handle decomposition of the pair $(W^0, A \cap W^0)$ in which the index of the handles is non-decreasing and $< n$. The hypotheses on A imply that every component of ∂X_i^0, $0 \leq i \leq p$, meets ∂W^0. Now let Z be a nice submanifold of W such that $W^0 \cup A$ is contained in Int Z as a deformation retract. We will alter the above handle decomposition of $(W^0, A \cap W^0)$ to a decomposition:

$$A = X_0 \subset X_1 \subset \ldots \subset X_p = Z,$$

of the pair (Z, A) by "handles with spikes" in the following way.

Put $X_0 = A$ and suppose that $H_1 = \overline{X_1^0 - X_0^0}$ is a handle of index $k < n$. Thus $H_1 \cong D^k \times D^{n-k}$ is attached to X_0^0 along $\partial D^k \times D^{n-k}$. Let $\gamma(t), 0 \leq t \leq 1$, be an embedded arc in ∂X_0^0 with its first endpoint $\gamma(0)$ on the boundary of $H_1 \cap X_0^0$ and the other on ∂W^0. (*See* figure. Note that such γ exists by our hypotheses on the X_i^0.) We may suppose that $\gamma(t) \in \partial X_0^0 - H_1$ for $t > 0$. Extend γ to an infinite arc $\gamma : [0, \infty) \to X_0$ by setting $\gamma(t) = \gamma(1) \times (t-1)$ in $\hat{W} \times (0, \infty)$ for $t > 1$. Then we will call a suitably smoothed set of the form $H_1 \cup \overline{(\delta\text{-}nbhd \text{ of } \gamma)}$ a *handle with one spike*. We will require that the handle $H_1' = \overline{X_1 - X_0}$ have exactly two spikes. Also each component of $(\partial H_1') - X_0$ should have
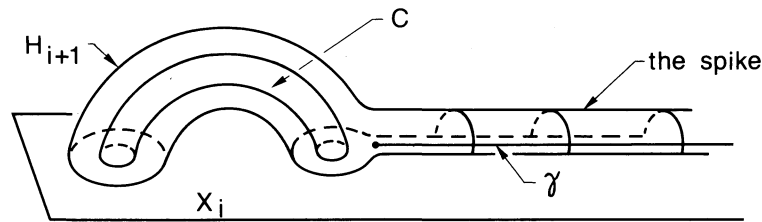
non-compact closure. Therefore, if $k = n-1$ we will attach one spike to each component of the boundary of $H_1 \cap X_0$. This should be done in such a way that $X_1 = X_0 \cup H_1'$ is an admissible submanifold of W which contains $X_0 \cup H_1$ as a deformation retract. We now repeat this construction to get $X_2, \ldots, X_p$. Notice that, because $X_i \cap W^0$ retracts onto X_i^0, we may suppose at each stage that the handle H_{i+1} is attached to the boundary of X_i rather than to X_i^0. Finally note that X_p will be a deformation retract of Z. Therefore, we may redefine Z, setting it equal to X_p.

We now prove that Theorem 2.10 holds for the pairs (W, X_i) where i goes down from p to 0. Observe that it holds when $i = p$ by (2.7) (ii). So suppose inductively that it holds for the pair (W, X_{i+1}). We will show that it holds for the pair (W, X_i) by the argument of [9], Lemma 2.8. Thus, let C be the core $D^k \times \frac{1}{2} D^{n-k}$ of H_{i+1}. Further, let $Y = W - \text{Int } H_{i+1}'$. We aim to apply (2.11)(b) with this Y, and with the set A equal to $B = X_i \cup C$. We should slightly alter Y along $Y \cap \partial X_i$ in order to make it smooth. Then the pairs (W, B) and (Y, B) are admissible. (Note in particular that $\partial Y - B$ has no compact components.) Therefore, we can apply (2.11)(b). Observe that (2.10) holds for the pair (W, B) because it holds for (W, X_{i+1}) by hypothesis, and because X_{i+1} retracts onto

$B = X_i \cup C$. [The details may be filled in using (2.6).] Therefore, if we show that (2.10) holds for both (W, Y) and (W, Y $\cup$ B), it will follow from (2.11)(b) that it holds for the pair:

$$(W, Y \cap B) = (W, Y \cap (X_i \cup C)) = (W, X_i),$$

as well. This will complete the inductive step.

First consider the pair (W, Y). Because $\mathscr{D}\!iff_{\omega 0}$ (W, rel Y) is isomorphic $\mathscr{D}\!iff_{\omega 0}$ ($D^{n-1} \times \mathbf{R}$, rel $\partial D^{n-1} \times \mathbf{R}$), Theorem 2.10 holds for (W, Y) by (2.4) and Lemma 2.12. Similarly, if H_{i+1} has index k, $\overline{B} E_0$ (W, rel Y $\cup X_i \cup C$) is just $\overline{B} E_0$ (K_{n-k}, rel ∂K_{n-k}), where K_j is a nice submanifold of W which is diffeomorphic to ($D^{n-j+1} \times S^{j-1}$) $\cup$ (two spikes).

Note that K_1 is the disjoint union of two copies of $D^n \cup$ (one spike). Therefore, (2.10) holds for ($K_1, \partial K_1$) by (2.7) (ii). One now shows that (2.10) holds for the other K_j by induction on j, as in [9], Lemma 2.7. In fact, Lemma 2.11(b) implies that if (P, Q) is an admissible pair of submanifolds of W, then (2.10) holds for (P $\cup$ Q, ∂ (P $\cup$ Q)) provided that it hold for (P, ∂P), (Q, ∂Q) and (P $\cap$ Q, ∂ (P $\cap$ Q)). Therefore, it suffices to check that K_j may be written as the union P $\cup$ Q, where P and Q both have the form $D^n \cup$ (two spikes) $\cong D^{n-1} \times \mathbf{R}$, and where P $\cap$ Q $= K_{j-1}$. To do this, think of K_j as a thickening of the set $S^{j-1} \cup$ (two spikes), where the spikes are attached to the equatorial sphere S^{j-2}. Then take P and Q to be suitable neighbourhoods of the upper and lower hemispheres, which intersect in a neighbourhood of $S^{j-2} \cup$ (two spikes). *See* figure. $\square$
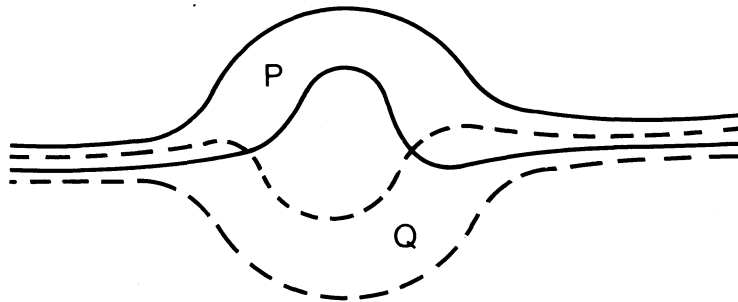
(2.15) PROOF OF THEOREM 2.10 FOR GENERAL (X, A).

Put $W' = X \cup (\partial X \times [0, \infty))$. We may suppose that $\omega|X$ is the restriction of a volume form on W' which is a product near the ends of W'. Then, it follows from (2.7) that Theorem 2.10 holds for the pair (W', X). Hence, by (2.11)(a), Theorem 2.10 will hold for (X, A) provided that it holds for (W', A). By (2.6) we may suppose that A is either empty or is a top dimensional submanifold of W'. Then it is easy to find sets A' and A'' in W' whose intersection is A and which are such that Lemma 2.14 applies to the three pairs (W', A'), (W', A'') and (W', A' $\cup$ A''). Since one can also arrange for (A', A) to be an admissible pair, the conclusion follows by applying (2.11)(b). $\square$

3. The monoids E_U (X, Y, rel A)

These are essentially the same as the monoids $\mathscr{E}mb_0^q$ (X, Y, rel A) of [9], §5. However, there is no isotopy extension theorem for non-compact manifolds. For example, any embedding of the x_1-axis into $\mathbf{R}^3$ which is the identity on the negative x_1-axis and which knots up the positive x_1-axis is isotopic to the identity in the compact-open topology. But no such embedding can be extended to the whole of $\mathbf{R}^3$. Since the proof of the fibration lemma in [9] depends heavily on the isotopy extension theorem, we will build into the definition of E_U the fact that isotopies can be extended. Thus these monoids will be rather artificial. However, the localization theorem (2.8) together with the normalization conditions (2.4) and (2.5) tell us that the restrictions imposed do not really matter.

First, here are some general notational conventions. We will use bold face letters **E, Diff**, ... to denote semi-simplicial Kan monoids, and will write $\mathbf{E}_U$, $\mathbf{Diff}_U$, ... for their identity components. Then the thin realization $|\mathbf{E}|$ of $\mathbf{E}$ is a topological monoid both in Vogt's category $\mathscr{HG}$ and its more familiar subcategory of compactly generated spaces. Corresponding to $|\mathbf{E}|$ is the discrete monoid E formed by the zero-simplices of $\mathbf{E}$. The inclusion $E \hookrightarrow |\mathbf{E}|$ induces a map $BE \hookrightarrow B|\mathbf{E}|$ whose homotopy fibre will be called $\overline{B}\,E$. For example, we will define $\mathbf{E}(W)$ to be the total singular complex of $\mathscr{Diff}_\omega W$. Then $E(W)$ is the discrete group $\mathrm{Diff}_\omega W$ and $\overline{B}\,E(W) \simeq \overline{B}\,\mathscr{Diff}_\omega W$. It will sometimes be convenient to write $\mathbf{Diff}_\omega W$ instead of $\mathbf{E}(W)$.

DEFINITION 3.1. – Let (X, A) be an admissible pair, and suppose first that A is a product near ∂X. (This means that for some $\eta > 0$:

$$A \cap (\partial X)_\eta = (A \cap \partial X) \times [-\eta, \eta],$$

where $\partial X \times [-\eta, \eta]$ is the product structure on $(\partial X)_\eta$ induced by the metric on W. Also, if $\delta < 0$ we will write X_δ for $X - \partial X \times (\delta, 0]$.) Then $E(W, X, \mathrm{rel}\,A)$ is defined to be the submonoid of $E(W)$ which consists of all singular p-simplices $\sigma : \Delta^p \to \mathscr{Diff}_\omega(W)$ which satisfy the following conditions for some ε with $0 < \varepsilon < \eta$:

(i) For each $t \in \Delta^p$ and all δ, $-\varepsilon \leqq \delta \leqq \varepsilon$, $\sigma(t)$ takes X_δ into itself and equals the identity on A_ε.

(ii) There is a p-simplex $\tilde{\sigma}$ in $E(W)$ such that $\tilde{\sigma}(t)$ equals $\sigma(t)$ on X_ε and has support in $\overline{X_\eta - A_\varepsilon}$ for all t. Moreover, if $\sigma(v) = \mathrm{id}$ for some vertex $v \in \Delta^p$, one can assume that $\tilde{\sigma}(v) = \mathrm{id}$ as well.

Note that if these conditions are satisfied by one ε, they are satisfied for any $\varepsilon' < \varepsilon$. One can also check that η may be replaced by any $\eta' < \eta$. In fact, because every component of $\partial X - A$ has non-compact closure, each component of $(\partial X - A) \times [0, \eta]$ has infinite volume and one can change the support of $\tilde{\sigma}$ to $X_{\eta'} - A_{\varepsilon'}$ by conjugation. Therefore the precise values of ε and η do not matter here.

Note further that if $A_1 \supseteqq A_2 \supseteqq \dots$ is a nested family of nice subsets of W with $A = \cap A_i$, then for all $\varepsilon > 0$ there is i such that $A_i \supseteqq \mathrm{Int}\,A_\varepsilon$. It follows that $E(W, X, \mathrm{rel}\,A) = \varinjlim E(W, X,$ rel $A_i)$ whenever both sides are defined. Therefore, if A as an arbitrary nice subset of W

we may define $\mathbf{E}(W, X, \operatorname{rel} A)$ to be $\varinjlim \mathbf{E}(W, X, \operatorname{rel} A_i)$, where the A_i tend to A as above and are products near ∂X.

As mentioned above, we will denote the identity component of $\mathbf{E}(W, X, \operatorname{rel} A)$ by $\mathbf{E}_0(W, X, \operatorname{rel} A)$. It follows easily from the second part of condition (ii) above that if σ is in $\mathbf{E}_0(W, X, \operatorname{rel} A)$ one can choose $\tilde{\sigma}$ to be in $\mathbf{E}_0(W, X, \operatorname{rel} A)$ also. Recall from [9], §3 that $\overline{\mathbf{B}}\,\mathbf{E}(W, \operatorname{rel} A) \simeq \overline{\mathbf{B}}\,\mathbf{E}_0(W, \operatorname{rel} A)$ because $\mathbf{E}(W, \operatorname{rel} A)$ is a group.

If Y is an admissible submanifold in W which is contained in Int X, we define $\mathbf{E}_0(W, X, Y, \operatorname{rel} A)$ to be the identity component of $\mathbf{E}_0(W, X, \operatorname{rel} A) \cap \mathbf{E}_0(W, Y, \operatorname{rel} A)$. Further, we define $\mathbf{E}_0(X, Y, \operatorname{rel} A)$ to be the quotient of $\mathbf{E}_0(W, X, Y, \operatorname{rel} A)$ by the relation $\sim$, where $\sigma \sim \sigma'$ if and only if there is $\delta > 0$ such that $\sigma(t) = \sigma'(t)$ on X_δ for all t. Thus $\mathbf{E}_0(X, Y, \operatorname{rel} A)$ is a semi-simplicial Kan monoid of germs of self-embeddings of X whose simplices have representatives defined on the uniform neighbourhood X_o for some $\delta > 0$. The definition has been designed so that the restriction map $\mathbf{E}_0(X, Y, \operatorname{rel} A) \to \mathbf{E}_0(Y, \operatorname{rel} A)$ is well-behaved: *see* Lemma 3.4. Notice, in particular, that because $\mathbf{E}_0(Y)$ consists of germs of self-embeddings of Y taken along some Y_o rather than along an arbitrary neighbourhood of Y, the kernel of the restriction $\mathbf{E}_0(X, Y) \to \mathbf{E}_0(Y)$ consists of embeddings which are the identity on Y_o for some $\delta > 0$ and so is a submonoid of $\mathbf{E}(X, \operatorname{rel} Y)$. This gives us plenty of room to work in: *see* for example the proof of (2.7) in (3.3) below.

There is an obvious inclusion i_X of $\overline{\mathbf{B}}\,\mathbf{E}_0(X, Y, \operatorname{rel} A)$ into $\overline{\mathbf{B}}\,\mathbf{E}_0(X, \operatorname{rel} A)$. Also we will define the map f_X to be the composite:

$$\overline{\mathbf{B}}\,\mathbf{E}_0(X, \operatorname{rel} A) \xrightarrow{f_1} \overline{\mathbf{B}}\,\mathscr{E}mb_0^g(X, \operatorname{rel} A) \xrightarrow{f_2} S_\omega(X, \operatorname{rel}_D A) \xrightarrow{f_3} S_\omega(X, \operatorname{rel} A),$$

where f_1 is induced by the natural map $|\mathbf{E}_0| \to \mathscr{E}mb_0$, and where f_2 and f_3 are constructed in [9], Appendix and [8] §2. Then in the situation of (2.9) there is a strictly commutative diagram:

$$
\begin{array}{ccccc}
\overline{\mathbf{B}}\,\mathbf{E}_0(X, \operatorname{rel} Y \cup A) & \to & \overline{\mathbf{B}}\,\mathbf{E}_0(X, Y, \operatorname{rel} A) & \to & \overline{\mathbf{B}}\,\mathbf{E}_0(Y, \operatorname{rel} Y \cap A) \\
\downarrow & & \downarrow & & \downarrow \\
S_\omega(X, \operatorname{rel}_D Y \cup A) & \to & S_\omega(X, \operatorname{rel}_D A) & \to & S_\omega(Y, \operatorname{rel}_D Y \cap A) \\
\uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq \\
S_\omega(X, \operatorname{rel} Y \cup A) & \to & S_\omega(X, \operatorname{rel} A) & \to & S_\omega(Y, \operatorname{rel} Y \cap A).
\end{array}
$$

Hence one can compare the spectral sequences of the rows as in (2.11).

(3.2) VERIFICATION OF PROPERTIES (2.4)-(2.9).

(2.4) and (2.6) follow immediately from the construction. The proofs of (2.7) and (2.9) are relatively easy and are given in this section. We will prove the localization property (2.8) in §4. The proof of (2.5) uses (2.8), and so is postponed until §5.

(3.3) PROOF OF (2.7)

The semi-simplicial monoid:

$$\mathbf{M}_1 = \mathbf{E}_0(\partial Z \times [0, 1] \times \mathbf{R}, \operatorname{rel} \partial Z \times 0 \times \mathbf{R})$$

and group:

$$M_\infty = E_0 (Z \times [0, \infty), \text{ rel } \partial Z \times [0, \infty) \cup Z \times 0)$$

are contractible, and so it will suffice to show that BM_1 is contractible and that BM_∞ is acyclic, where M_1 and M_∞ are the corresponding discrete objects. The first statement is true by the argument of [8], Lemma 4. One has to check that for any m_1, m_2 in M_1 there is an element m in M_1 such that $m_1 m = m_2 m = m$. For then the category $C(M_1 \diagdown \star)$ will be filtering and so contractible. But because both m_1 and m_2 are the identity on $\partial Z \times [0, \varepsilon] \times \mathbf{R}$ for some $\varepsilon > 0$, it suffices to choose m so that it maps $\partial Z \times [0, 1] \times \mathbf{R}$ into $\partial Z \times [0, \varepsilon] \times \mathbf{R}$. (This argument would not work if we knew only that m_1 and m_2 were the identity on some arbitrary neighbourhood U of $\partial Z \times 0 \times \mathbf{R}$, since there might not be enough room to fit $\partial Z \times [0, 1] \times \mathbf{R}$ into U.)

When $\partial Z \neq 0$, one shows that BM_∞ is acyclic by applying [10], Proposition 1.3. To do this, one has to check that M_∞ is the direct limit of groups which are isomorphic to the group H which occurs there. This may be done by considering "fat" neighbourhoods of $\partial Z \times \mathbf{R}$ as in [10], Proposition 1.2. When $\partial Z = \emptyset$ and $\dim Z \geq 2$, the desired result is exactly Theorem 1 of [10] and [11]. $\square$

The first step in the proof of the fibration property is the following lemma.

LEMMA 3.4. — *If* $Y \subset \text{Int } X$, *there is a Kan fibration:*

$$K \to E_\upsilon (X, Y, \text{ rel } A) \overset{\rho}{\to} E_\upsilon (Y, \text{ rel } Y \cap A),$$

where $K = E(X, \text{ rel } Y \cup A) \cap E_\upsilon (X, Y, \text{ rel } A)$ *is the kernel of* ρ.

Proof. — We may suppose that A is a product in $\partial Y \times [-\eta, \eta]$, where $Y_\eta \subset \text{Int } X$. Then, given $\sigma \in E_\upsilon (Y, \text{ rel } Y \cap A)$, condition (ii) of (3.1) implies that one can choose the extension σ of $\sigma \mid Y_0$, so that its restriction $\bar\sigma$ to X_0 is the identity near ∂X and on A_ε and so is in $E_\upsilon (X, Y, \text{ rel } A)$. Hence ρ is surjective. Also, if σ' is any other lifting of σ to $E_\upsilon (X, Y, \text{ rel } A)$, one has $\sigma'(t) = \bar\sigma (t) (\bar\sigma^{-1} (t) \sigma'(t))$ for all t. Observe that $t \mapsto \bar\sigma^{-1} (t) \sigma'(t)$ is a simplex in $\mathbf{K}$. It is now easy to check that ρ is a Kan fibration with fibre $\mathbf{K}$. $\square$

(3.5) PROOF OF (2.9)

Let $\mathscr{K}' \to \mathscr{M}' \to \mathscr{N}'$ be the thin realization of the Kan fibration of Lemma 3.4, and let K, M, N be the corresponding discrete monoids. Thus $M = E_0(X, Y, \text{ rel } A)$, and so on. By arguing as in Lemma 3.4 above, one can easily check that the sequence $K \to M \to N$ satisfies the conditions of [9], Proposition 3.6 and 3.7. Hence, by [9], Proposition 3.8, $\overline{B} \mathscr{K}' \to \overline{B} \mathscr{M}' \to \overline{B} \mathscr{N}'$ is an (integer) homology fibration sequence.

It remains to show that:

(*a*) $\overline{B} \mathscr{K}' \simeq \overline{B} \mathscr{K}'_0 \simeq \overline{B} E_0(X, \text{ rel } Y \cup A)$, and:

(*b*) $\pi_1(\overline{B} \mathscr{N}')$ acts trivially on $H_*(\overline{B} \mathscr{K}')$.

The proof of (*a*) is almost identical to the corresponding proof in [9], Lemma 5.6 and will not be repeated here. (Notice that, because the elements of $\mathscr{N}'$ now consist of germs of

embeddings there is no difference between $\mathscr{K}$ and $\mathscr{K}'$ in the present situation.) However, we must rework the proof of (b) given in [9], Lemma 5.6, since we cannot now reduce to the case when $\pi_1 (BN) = 0$. We will use the notation of that lemma without further comment.

Let U be the interior of $(Y \cup A)_\varepsilon$ for some $\varepsilon > 0$, and put $\mathscr{M}^U = | E_0 (Y_\varepsilon, Y, \mathrm{rel}\, (\partial Y_\varepsilon) \cup A) |$. Then $\mathscr{M}^U$ and $M^U = M \cap \mathscr{M}^U$ are made from diffeomorphisms with support in U. Since the elements of K_U and $\mathscr{K}_U$ are the identity on U, they commute with the elements of M^U and $\mathscr{M}^U$. Hence, there is a functor:

$$C (K_U \diagdown \mathscr{K}_U) \times C (M^U \diagdown \mathscr{M}^U) \quad \to \quad C (M \diagdown \mathscr{M}),$$

given by multiplication. This induces a map:

$$(K_U \diagdown \mathscr{K}_U) \times (M^U \diagdown \mathscr{M}^U) \quad \to \quad M \diagdown \mathscr{M}'.$$

Now notice that by Definition (3.1) (ii) any loop l in $N \diagdown \mathscr{N}'$ may be lifted to a path $\tilde{l}$ in $M^U \diagdown \mathscr{M}^U$. The argument of [9], Lemma 5.6 now shows that there is a homotopy commutative diagram:

$$
\begin{array}{ccc}
K_U \diagdown \mathscr{K}_U & \hookrightarrow & F \\
{\scriptstyle \text{mult. by} \atop \scriptstyle \tilde{l}(1)} \downarrow & & \downarrow {\scriptstyle \text{action of } l} \\
K \diagdown \mathscr{K}' & \hookrightarrow & F.
\end{array}
$$

(Here one uses the fact that if $z \in K_U \diagdown \mathscr{K}_U$, the products $z . \tilde{l} (t)$ are well-defined elements of $M \diagdown \mathscr{M}'$.) Since $\tilde{l}(1)$ is homotopic in $(K \diagdown \mathscr{K}') \cap (M^U \diagdown \mathscr{M}^U)$ to an element of $\mathscr{K}' \cap \mathscr{M}^U$ which is in the image of $K \cap M^U$, it follows as before that right multiplication by $\tilde{l}(1)$ is homotopic to the inclusion map $K_U \diagdown \mathscr{K}_U \hookrightarrow K \diagdown \mathscr{K}'$. The result now follows because $K \diagdown \mathscr{K}'$ is the direct limit of the spaces $K_U \diagdown \mathscr{K}_U$. $\square$

4. Localization

As we will see, property (2.8) is an easy consequence of the following special case.

THEOREM 4.1. — *Suppose that* $n = \dim W \geqq 3$ *and that* (X, A) *is an admissible pair in* W. *Then the inclusion:*

$$\overline{B} E_U (W, X, \mathrm{rel}\, A) \hookrightarrow \overline{B} E_U (W, \mathrm{rel}\, A),$$

is a **Z**-*homology equivalence.*

In fact, it is easier to prove this in a slightly different form. Let V be the total singular complex of the topological group $H^{n-1} (W, A; \mathbf{R}) \cong \mathscr{V} \cong \mathbf{R}^p$. Then the flux homomorphism Φ induces a homomorphism $E_U (W, \mathrm{rel}\, A) \to V$, whose kernel will be denoted by $D_U (W, \mathrm{rel}\, A)$. Similarly, we write $D_U (W, X, \mathrm{rel}\, A)$ for the kernel $D_U (W) \cap E_U (W, X, \mathrm{rel}\, A)$ of the restriction of Φ to $E_U (W, X, \mathrm{rel}\, A)$.

LEMMA 4.2. — *In order to prove Theorem* 4.1, *it suffices to show that the inclusion:*

$$\overline{B}\,\mathbf{D}_{\upsilon}(W, X, \operatorname{rel} A) \hookrightarrow \overline{B}\,\mathbf{D}_{\upsilon}(W, \operatorname{rel} A),$$

is a **Z**-*homology equivalence.*

Proof. — We will adapt the proof of [10], Lemma 2. One can find disjoint, oriented, smooth and possibly non-compact 1-manifolds $L_1, \ldots, L_p$ in $W - A$ which are dual to a set of generators for the vector space $H_{n-1}(W, A; \mathbf{R})$ and so which form a basis of $H_1(W - A; \mathbf{R}) \cong H^{n-1}(W, A; \mathbf{R})$.

Moreover one can assume that the L_i do not meet ∂X. To see this, note first that one can homotop the L_i so that there are only finitely many intersections. Because each component of (∂X)-A is non-compact, one can at any intersection point x choose a path γ_x in (∂X)-A which goes to infinity. Then one can eliminate the intersection at x by separating L_i into
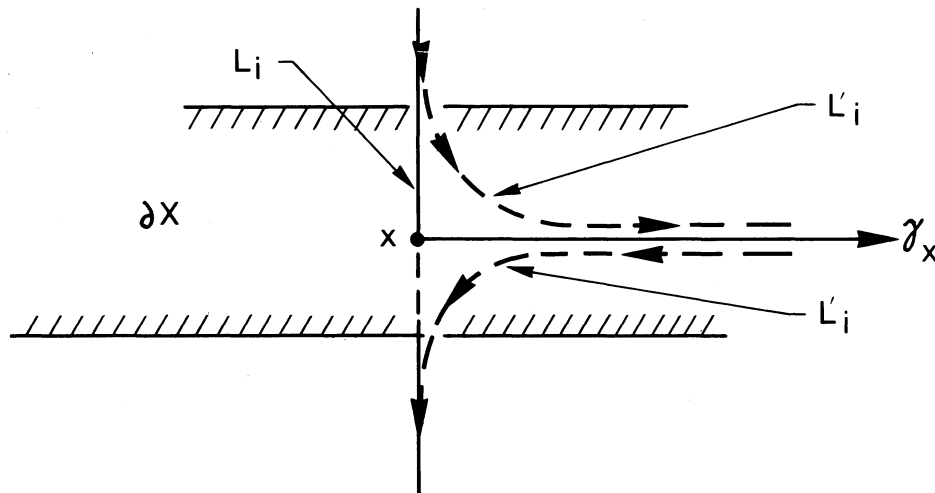
two strands L'_i, as indicated in the figure. After a finite number of such steps we will obtain a manifold $L = \cup L_i$ which we may assume to lie outside $(\partial X \cup A)_\varepsilon$. Now let U be a closed neighbourhood of L which does not meet $(\partial X \cup A)_\varepsilon$ and whose ends all have infinite volume. Then it is easy to construct a continuous homomorphism:

$$s: \quad \mathscr{V} = H^{n-1}(W, A; \mathbf{R}) \to \mathscr{D}\!\mathit{iff}_{\omega_0}(U, \operatorname{rel} \partial U),$$

which is a right inverse to Φ. For example, $s(rc_1)$ will be a diffeomorphism which pushes forward by an amount proportional to $r \in \mathbf{R}$ in a neighbourhood of L_1 and is the identity elsewhere in U. Since the diffeomorphisms in the image of s are the identity on $(\partial X \cup A)_\varepsilon$, they belong to $E_0(W, X, \operatorname{rel} A)$. Hence s gives rise to a splitting homomorphism:

$$V \to E_0(W, X, \operatorname{rel} A) \subseteq E_0(W, \operatorname{rel} A).$$

By [9], §3 this implies that the sequence:

$(\star)$ $\qquad$ $\overline{B} \, \mathbf{D}_0(W, X, \text{rel } A) \to \overline{B} \, \mathbf{E}_0(W, X, \text{rel } A) \to \overline{B} \, V,$

is a homotopy fibration sequence.   Since:

$$\overline{B} \, \mathbf{D}_0(W, \text{rel } A) \to \overline{B} \, \mathbf{E}_0(W, \text{rel } A) \to \overline{B} \, V,$$

comes from an exact sequence of groups, it is also a homotopy fibration sequence.   The desired conclusion follows immediately. $\square$

Before beginning the proof of Theorem 4.1 we introduce the following definitions.

DEFINITIONS 4.3. $-$ Let $\hat{W}_1, \ldots, \hat{W}_k$ be the connected components of the "boundary at infinity" $\hat{W}$ of W (*see* §2), and for each subset $S \subseteq J = \{1, \ldots, k\}$, let $W^S = \bigcup_{j \in S} \hat{W}_j$. Thus $W^J = \hat{W}$.   Similarly, if X is an admissible submanifold of W which is the product $\hat{X} \times [0, \infty)$ in $\hat{W} \times [0, \infty)$, we write X^S for $\hat{X} \cap W^S$.   Then we define $\mathbf{D}(W, X^S)$ to be the submonoid of $\mathbf{D}(W)$ which consists of all p-simplices σ which satisfy the following conditions for some η, ε and λ, where $0 < \varepsilon < \eta$:

(i) For all $t \in \Delta^p$ and all δ, $-\varepsilon \le \delta \le \varepsilon$, $\sigma(t)$ takes $X_0^S \times [\lambda, \infty)$ into X_0.

(ii) There is $\bar{\sigma} \in \mathbf{D}(W)$ such that each $\bar{\sigma}(t)$ equals $\sigma(t)$ on $X_\varepsilon^S \times [\lambda, \infty)$ and equals the identity on $(\hat{W} - X_\eta^S) \times [\lambda, \infty)$.   Moreover, if $\sigma(v) = \text{id}$ for some vertex $v \in \Delta^p$, one can choose $\bar{\sigma}$ so that in addition $\bar{\sigma}(v) = \text{id}$.

As usual, $\mathbf{D}_0(W, X^S)$ is the identity component of $\mathbf{D}(W, X^S)$.   Note that, just as in Definition 3.1, the precise values of η, ε and λ do not matter here.   Also, if σ is in $\mathbf{D}_0(W, X^S)$, one can choose the extension $\bar{\sigma}$ in (ii) so that it also belongs to $\mathbf{D}_0(W, X^S)$.

Further, we will denote by $\mathbf{D}_0^S(W, X)$ the monoid of germs at W^S of the elements of $\mathbf{D}_0(W, X^S)$.   Thus there is a quotient map $q : \mathbf{D}_0(W, X^S) \to \mathbf{D}_0^S(W, X)$, whose kernel consists of simplices which are the identity in some neighbourhood $W^S \times (\lambda, \infty)$ of W^S.   We denote the identity component of Ker q by $\mathbf{D}_0(W, \text{rel } W^S)$.   We will also write $\mathbf{D}_0(W, X, \text{rel } W^S)$ for the identity component of $\mathbf{D}_0(W, X) \cap \mathbf{D}_0(W, \text{rel } W^S)$, and $\mathbf{D}_0(W, X^T, \text{rel } W^S)$ for the identity component of $\mathbf{D}_0(W, X^T) \cap \mathbf{D}_0(W, \text{rel } W^S)$.

The proof of Theorem 4.1 is based on the following lemmas.   Observe that the first two hold for all dimensions n.

LEMMA 4.4. $-$ *For any* $S \subseteq J = \{1, \ldots, k\}$:

$$\overline{B} \, \mathbf{D}_0(W, X^J, \text{rel } W^S) \to \overline{B} \, \mathbf{D}_0(W, X^J) \xrightarrow{q} \overline{B} \, \mathbf{D}_0^S(W, X),$$

is a simple (homotopy) fibration sequence.

PROPOSITION 4.5. $-$ *If* X_i^{J-S} *is not empty for any component* X_i *of* X, *then:*

$$\overline{B} \, \mathbf{D}_0(W, X, \text{rel } W^S) \hookrightarrow \overline{B} \, \mathbf{D}_0(W, X^J, \text{rel } W^S),$$

is a homotopy equivalence.

(Note that if X^{J-S} is empty then $\mathbf{D}_0(W, X, \text{rel } W^S)$ is a *group* and the result is false.)

PROPOSITION 4.6. — *Let* Z *be a compact connected manifold without boundary, and suppose that* V *is a top dimensional compact submanifold of* Z. *If either* dim $Z \geq 2$ *or* $V \neq \emptyset$, *then* $\overline{\mathbf{B}} \mathbf{D}_0(Z \times \mathbf{R}, V \times \mathbf{R}, \text{rel}(Z \times \mathbf{R})^-)$ *is acyclic.*

Here $(Z \times \mathbf{R})^-$ is the end $Z \times (-\infty)$ of $Z \times \mathbf{R}$.

The proofs of Lemma 4.4 and Proposition 4.6 are fairly straightforward. However Proposition 4.5 requires more work. It is based on [9], § 4 where we showed how to localize over a *compact* manifold (in the non-volume preserving case). The ideas of that proof work here because we are extending the localization over a compact piece, that is from X^J to X, but they must be adapted to cope with the fact that $\mathbf{D}_0(W, X^J, \text{rel } W^S)$ is not a group. Before going into the details of this, we will prove the simplest case of Theorem 4.1.

(4.7) PROOF OF THEOREM 4.1 WHEN $A = \emptyset$. — By (4.2) and (4.5), Theorem 4.1 will follow if we prove that the inclusion $i_1 : \overline{\mathbf{B}} \mathbf{D}_0(W, X^J) \hookrightarrow \overline{\mathbf{B}} \mathbf{D}_0(W)$ is a **Z**-homology equivalence. To do this, consider the diagram:

$$\overline{\mathbf{B}} \mathbf{D}_0(W, \text{rel } W^J) \to \overline{\mathbf{B}} \mathbf{D}_0(W, X^J) \to \overline{\mathbf{B}} \mathbf{D}_0^J(W, X)$$
$$\downarrow = \qquad\qquad \downarrow i_1 \qquad\qquad \downarrow i_2$$
$$\overline{\mathbf{B}} \mathbf{D}_0(W, \text{rel } W^J) \to \overline{\mathbf{B}} \mathbf{D}_0(W) \to \overline{\mathbf{B}} \mathbf{D}_0^J(W).$$

We may apply Lemma 4.4 to both rows. Therefore, it suffices to show that the inclusion at infinity i_2 is a **Z**-homology equivalence. To do this, observe that the group $\mathbf{D}_0^J(W)$ is isomorphic to the group $\mathbf{D}_0^+(\hat{W} \times \mathbf{R})$ of germs at the end $(\hat{W} \times \mathbf{R})^+ = \hat{W} \times \infty$ of the product manifold $\hat{W} \times \mathbf{R}$. Similarly, $\mathbf{D}_0^J(W, X) \cong \mathbf{D}_0^+(\hat{W} \times \mathbf{R}, \hat{X} \times \mathbf{R})$. Now consider the diagram:

$$\overline{\mathbf{B}} \mathbf{D}_0(\hat{W} \times \mathbf{R}, \text{rel}(\hat{W} \times \mathbf{R})^{\pm}) \to \overline{\mathbf{B}} \mathbf{D}_0(\hat{W} \times \mathbf{R}, (\hat{X} \times \mathbf{R})^+, \text{rel}(\hat{W} \times \mathbf{R})^-) \to \overline{\mathbf{B}} \mathbf{D}_0^+(\hat{W} \times \mathbf{R}, \hat{X} \times \mathbf{R})$$
$$\downarrow = \qquad\qquad\qquad \downarrow i_3 \qquad\qquad\qquad \downarrow i_2$$
$$\overline{\mathbf{B}} \mathbf{D}_0(\hat{W} \times \mathbf{R}, \text{rel}(\hat{W} \times \mathbf{R})^{\pm}) \to \overline{\mathbf{B}} \mathbf{D}_0(\hat{W} \times \mathbf{R}, \text{rel}(\hat{W} \times \mathbf{R})^-) \to \overline{\mathbf{B}} \mathbf{D}_0^+(\hat{W} \times \mathbf{R})$$

Applying Lemma 4.4 again, we see that it suffices to prove that i_3 is a **Z**-homology equivalence. However Propositions 4.5 and 4.6 imply that both:

$$\overline{\mathbf{B}} \mathbf{D}_0(\hat{W} \times \mathbf{R}, \text{rel}(\hat{W} \times \mathbf{R})^-) \qquad \text{and} \qquad \overline{\mathbf{B}} \mathbf{D}_0(\hat{W} \times \mathbf{R}, (\hat{X} \times \mathbf{R})^+, \text{rel}(\hat{W} \times \mathbf{R})^-)$$

are acyclic. $\square$

(4.8) PROOF OF (4.4). — We first show that for any p-simplex σ in $\mathbf{D}_0^S(W, X)$ and any $\lambda > 0$, there is a simplex $s(\sigma) \in q^{-1}(\sigma)$ with support in $W^S \times (\lambda, \infty)$. To see this, note that it suffices to consider σ such that $\sigma(v) = \text{id}$ for some vertex $v \in \Delta^p$. Choose μ so that $\sigma(t)(W^S \times [\mu, \infty)) \subseteq W^S \times (\lambda, \infty)$ for all $t \in \Delta^p$. Then use Krygin's isotopy extension theorem [5] to extend the isotopy $t \mapsto \sigma(t) | W^S \times [\mu, \infty)$ to be the identity outside

$W^S \times (\lambda, \infty)$. No volume obstructions arise because σ is in the kernel of the flux homomorphism Φ. Using this fact, one can now repeat the proof of (2.8) given in (3.4) and (3.5) above. In order to see that one has a fibration sequence rather than a homology fibration sequence it suffices to check that the sequence:

$$BD_0(W, X^J, rel\ W^S) \to BD_0(W, X^J) \xrightarrow{q} BD_0^S(W, X),$$

is a fibration sequence. This follows by [9] Proposition 3.6. For, as mentioned above, given $n \in D_0^S(W, X)$ we may choose $s(n) \in q^{-1}(n)$ to have support in $W^S \times [\lambda, \infty)$. Then the endomorphism c_n of $K = D_0(W, X^J, rel\ W^S)$ is given by $k \mapsto s(n)^{-1} ks(n)$, and so is in fact an isomorphism. Hence c_n acts by homotopy equivalences on BK, as required. $\square$

(4.9) PROOF OF (4.6)

Because $D_0(Z \times R, V \times R, rel(Z \times R)^-)$ is contractible, and because $D_0(Z \times R, V \times R, rel(Z \times R)^-)$ is the direct limit of the groups $D_0(Z \times R, V \times R, rel\ Z \times (-\infty, -\lambda])$, it suffices to show that $BD_0(Z \times R, V \times R, rel\ Z \times (-\infty, 0])$ is acyclic. If $V = \emptyset$ this is Theorem 1 of [10]. (Note that $D_0(Z \times R, rel\ Z \times (-\infty, 0])$ is the full group $Diff_\omega(Z \times [0, \infty), rel\ Z \times 0)$ because the flux homomorphism vanishes here.) The general case may be deduced as follows.

LEMMA 4.10. — *If* $V \neq \emptyset$, *then* $BD_0(Z \times R, V \times R, rel\ Z \times (-\infty, 0])$ *is acyclic.*

Proof. — For simplicity, let us write $Q(Z, V)$ for the discrete monoid $D_0(Z \times R, V \times R, rel\ Z \times (-\infty, 0])$ and so on. Consider the sequence:

$$BQ(Z, rel\ V) \to BQ(Z, V) \to BQ(V).$$

This is a homology fibration sequence by [9], Proposition 3.6. Also $BQ(Z, rel\ V)$ is acyclic by the second half of (2.7). To see that $BQ(V)$ is acyclic, look at the sequence:

$$BQ(V_\varepsilon, rel\ V \cup \partial(V_\varepsilon)) \to BQ(V_\varepsilon, V, rel\ \partial(V_\varepsilon)) \to BQ(V),$$

where $\varepsilon > 0$. This is a homology fibration sequence in the usual way. Also both:

$$BQ(V_\varepsilon, rel\ V \cup \partial(V_\varepsilon)) \qquad and \qquad BQ(V_\varepsilon, rel\ \partial(V_\varepsilon))$$

are acyclic by (2.7). Hence it suffices to prove that the inclusion:

$$BQ(V_\varepsilon, V, rel\ \partial(V_\varepsilon)) \hookrightarrow BQ(V_\varepsilon, rel\ \partial(V_\varepsilon)),$$

is a equivalence. We do this by applying the "easy localization lemma":

LEMMA 4.11. — *Let* M_0 *be a submonoid of* M *with the property that for each finite subset* T *of* M *there is an invertible element* s *of* M *such that* $sT \subset M_0$. *Then the inclusion* $BM_0 \hookrightarrow BM$ *is a homotopy equivalence.*

Proof. — Every finite subcomplex of $M /\!\!/ M_0$ contains only a finite set T of vertices, and hence lies in $s^{-1} M_0 /\!\!/ M_0$ for some s. But $s^{-1} M_0 /\!\!/ M_0$ is contractible. Hence $M /\!\!/ M_0$ is contractible. For more details *see* [13], Proposition 2.5 or [9], § 3. $\square$

In our case, given a finite subset T of $M = Q(V_\varepsilon, \text{rel } \partial(V_\varepsilon))$, choose α, $0 < \alpha < \varepsilon$, so that each element of T has support in $V_{\varepsilon - 2\alpha} \times [\alpha, \infty)$. Then let s be any element of M_0 which takes $V_{\varepsilon - \alpha + \delta} \times [\alpha, \infty)$ into $V_\delta \times [0, \infty)$ whenever $|\delta| < \alpha$. It is easy to check that the elements of sT take $V_\delta \times [0, \infty)$ into itself for these δ. They also satisfy the extension condition (3.1)(ii) since they are the identity near $\partial(V_\varepsilon) \times [0, \infty)$. Hence $sT \in M_0$. Finally note that s is invertible in M, since M is a group. Thus:

$$BQ(V_\varepsilon, V, \text{rel } \partial(V_\varepsilon)) \hookrightarrow BQ(V_\varepsilon, \text{rel } \partial(V_\varepsilon))$$

is an equivalence. This completes the proof of (4.10) and hence of (4.6). $\square$

It remains to prove (4.5). For simplicity, let us first consider the case when $S = \emptyset$. Then we must show:

PROPOSITION 4.12. — *The inclusion:*

$$\overline{B} \, D_0(W, X) \hookrightarrow \overline{B} \, D_0(W, X^J),$$

is a homotopy equivalence.

The proof of (4.12) is rather long, and occupies (4.13)-(4.24). We will use the notation of [9], § 4 without further comment. Also we will write $\hat{X}$ instead of X^J, and will call a homotopy equivalence simply an equivalence. To begin, here is a sketch of the proof.

Let $N = D_0(W, \hat{X})$ and $M = D_0(W, X)$. Then $N = D_0(W, \hat{X})$ is a union of discrete monoids N_ε, $\varepsilon > 0$, where $n \in N_\varepsilon$ if it satisfies Definition 4.3 for this ε. Thus:

$$\overline{B} \, D_0(W, \hat{X}) = \varinjlim N_\varepsilon \diagdown |M|.$$

Similarly:

$$\overline{B} \, D_0(W, X) = \varinjlim M_\varepsilon \diagdown |N|.$$

Therefore, it will suffice to show that $M_\varepsilon \diagdown |M| \cong N_\varepsilon \diagdown |N|$ for small $\varepsilon > 0$. We will do this in the following way.

First, we will enlarge M_ε to a monoid M_ε' which has a simpler relation to N_ε. This monoid M_ε' is the identity component of the submonoid:

$$\{ n \in N_\varepsilon : n(X_\varepsilon) \subseteq X_\varepsilon \} \text{ of } N_\varepsilon.$$

(By definition, the identity component of a submonoid Q of N consists of all q for which there is a 1-simplex $t \mapsto q_t$ in $|N|$ with $q_0 = \text{id}$, $q_1 = q$ and $q_t \in Q$ for all t.) Next, we will enlarge the subcomplex $|M|$ of $|N|$ to a connected open subset $\mathcal{U}$ of $|N|$. This set $\mathcal{U}$ is the component of:

$$\{ (\sigma, t) : \sigma(t)(X) \subset \text{Int } X_\varepsilon \},$$

which contains the base point. It is open because every simplex of $|N|$ takes $X - (cpct)$ into Int X_ε. Note that it is not a subcomplex of $|N|$. However, it does contain $|M|$ as a

deformation retract. Further, both M_ε and M'_ε act on $\mathcal{U}$ by multiplication on the left, and we will see that:

LEMMA 4.13. — $M_\varepsilon \backslash\!\backslash |\mathbf{M}| \;\simeq\; M_\varepsilon \backslash\!\backslash \mathcal{U} \;\simeq\; M'_\varepsilon \backslash\!\backslash \mathcal{U}.$

Now let N^*_ε be the group of invertible elements in N_ε, and put $M^*_\varepsilon = M'_\varepsilon \cap N^*_\varepsilon$. Then an element $n \in N_\varepsilon$ belongs to N^*_ε if its germ at infinity takes X_δ onto itself for $-\varepsilon \leq \delta \leq \varepsilon$. In particular, we will have:

$$\hat{X}_\delta \times [\mu, \infty) \;\supset\; n(\hat{X}_\delta \times [\lambda, \infty)) \;\supset\; \hat{X}_\delta \times [\nu, \infty)$$

for some $\mu < \lambda < \nu$. Note also that N^*_ε and M^*_ε need not be connected. We now consider the following diagram:

$$
\begin{array}{ccc}
M^*_\varepsilon \backslash\!\backslash_{\Gamma} \mathcal{U} & \overset{i}{\hookrightarrow} & N^*_\varepsilon \backslash\!\backslash_{\Gamma} |\mathbf{N}| \\
\downarrow & & \downarrow \\
M'_\varepsilon \backslash\!\backslash \mathcal{U} & \overset{j}{\hookrightarrow} & N_\varepsilon \backslash\!\backslash |\mathbf{N}|.
\end{array}
$$

We will prove:

LEMMA 4.14. — i is an equivalence.

LEMMA 4.15. — The homotopy fibres of i and j are equivalent.

Clearly, these results imply that j is an equivalence, and hence, by (4.13), that Proposition 4.12 holds.

(4.16) PROOF OF (4.13)

We first show that M_ε is the identity component M''_ε of the monoid:

$$\left\{ n \in N_\varepsilon : n(X_\delta) \subseteq X_\delta, \; -\varepsilon \leq \delta \leq \varepsilon \right\}.$$

This is not quite obvious, because the extension condition (4.3)(ii) which is satisfied by the elements of M''_ε is not the same as condition (3.1)(ii), which is satisfied by M_ε. Therefore, we must show that for each $n \in M''_\varepsilon$ and each $\eta > \varepsilon$ there is $m \in N$ which equals n on X_ε and has support in X_η. This may be seen as follows. Since M''_ε is connected, there is a 1-simplex $t \mapsto n_t$ in N with $n_0 = \mathrm{id}$, $n_1 = n$ and $n_t \in M''_\varepsilon$ for all t. By (4.3)(ii), one can find a 1-simplex $t \mapsto \tilde{n}_t$ with $\tilde{n}_0 = \mathrm{id}$. and such that for some $\lambda > 0$, $\tilde{n}_t = n_t$ on $\hat{X}_\varepsilon \times [\lambda, \infty)$ and $\tilde{n}_t = \mathrm{id}$. on $(\hat{W} - \hat{X}_\eta) \times [\lambda, \infty)$. Now choose $\mu > \lambda$ so that $\tilde{n}_t (\hat{W} \times (\mu, \infty))$ is disjoint from $n_t(X_\varepsilon \cap W^\lambda)$ and $W^\mu - X_\eta$, where W^λ is $W - \hat{W} \times (\lambda, \infty)$. By [5], there is a 1-simplex $t \mapsto m_t$, such that $m_t = n_t$ on X_ε, $m_t = \tilde{n}_t$ on $\hat{W} \times (\mu, \infty)$ and $m_t = \mathrm{id}$ outside X_η. Thus we may take $m = m^1$. [Note that the trick here, as elsewhere, is to use (4.3)(ii) to extend isotopies over noncompact regions and then to use [5] to adjust the result on a compact set.]

The next step is to show that the inclusion $BM_\varepsilon = BM''_\varepsilon \hookrightarrow BM'_\varepsilon$ is an equivalence. This follows by the easy localization lemma 4.11. The details are straightforward, and will be left to the reader.

Finally consider the diagram:

$$\begin{array}{ccccc}
\mathscr{U} \to \mathrm{M}_\varepsilon & \searrow & \mathscr{U} \to \mathrm{BM}_\varepsilon \\
\parallel & & \downarrow & & \downarrow \\
\mathscr{U} \to \mathrm{M}'_\varepsilon & \searrow & \mathscr{U} \to \mathrm{BM}'_\varepsilon.
\end{array}$$

Since M_ε and M'_ε are connected, these monoids act on $\mathscr{U}$ by homotopy equivalences.   Hence the rows are fibration sequences by [9], Lemma 3.1 and the result follows.   □

The next lemma contains most of the technical part of the proofs of (4.14) and (4.15).   If $j : \mathrm{N} \hookrightarrow |\mathrm{N}|$ is the canonical inclusion, we will write U for $j^{-1}(\mathscr{U})$.   Thus U consists of those elements g of N which take X into $\mathrm{Int}\,\mathrm{X}_\varepsilon$ and are isotopic to the identity through such elements.   Also, recall that $\mathrm{W}^\lambda = \mathrm{W} - \hat{\mathrm{W}} \times (\lambda, \infty)$.

LEMMA 4.17. − (a) *For every* $n \in \mathrm{N}$ *and* $\lambda > 0$ *there is* $g \in \mathrm{N}^*_\varepsilon$ *such that* $g = n$ *on* $\mathrm{W}^\lambda, g^{-1}\,n \in \mathrm{U}$ *and :*

$$n(\mathrm{X}) \subseteq g(\mathrm{X}) \subseteq n(\mathrm{X}) \cup (\mathrm{Int}\,\hat{\mathrm{X}}_\varepsilon \times [\lambda, \infty)).$$

(b) *If* $n \in \mathrm{N}_\varepsilon$ *takes* $\hat{\mathrm{X}}_\varepsilon \times [\lambda, \infty)$ *into* X, *there is* $g \in \mathrm{N}^*_\varepsilon$ *such that* $g = n$ *on* $\mathrm{W}^\lambda, g^{-1}\,n \in \mathrm{U}$ *and:*

$$n(\mathrm{X}_\varepsilon) \subseteq g(\mathrm{X}_\varepsilon) \subseteq n(\mathrm{X}_\varepsilon) \cup (\mathrm{Int}\,\hat{\mathrm{X}}_\varepsilon \times [\lambda, \infty)).$$

(c) *We may assume in* (a) *and* (b) *that* $g^{-1}\,n$ *is isotopic to the identity by an isotopy in* U *which equals the identity on* W^λ.

(4.18) PROOF OF (4.14).

Set $\mathscr{G} = |\mathrm{N}|$, $\mathrm{G} = \mathrm{N}^*_\varepsilon$, $\mathscr{U} = \mathscr{U}$ and $\mathrm{M} = \mathrm{M}^*_\varepsilon$ in the proof of [9], Lemma 4.1.   Clearly (4.14) will hold if we show that these $\mathscr{G}$, G, $\mathscr{U}$ and M have all the properties needed for the proof to go through.   First note that the subsets $g\,\mathscr{U}$, $g \in \mathrm{N}^*_\varepsilon$, of $|\mathrm{N}|$ are open because $\mathscr{U}$ is open and because the elements of N^*_ε are invertible.   Moreover, Lemma 4.17 (a) implies that the sets $g\,\mathscr{U}$, $g \in \mathrm{N}^*_\varepsilon$, cover $|\mathrm{N}|$.   Next, note that $g\,\mathscr{U} \subseteq \mathscr{U}$ if and only if g takes X_ε into itself and is isotopic to the identity through such elements.   Since the elements of M^*_ε are isotopic to the identity in M'_ε, it follows that $g\,\mathscr{U} \subseteq \mathscr{U}$ if and only if $g \in \mathrm{M}^*_\varepsilon$.   Therefore, it remains to check that condition (iii) of [9], Lemma 4.2 holds.   Thus, given $(\sigma, t) \in g_1\,\mathscr{U} \cap g_2\,\mathscr{U}$, we must find $\tilde{f} \in \mathrm{N}^*_\varepsilon$ such that $(\sigma, t) \in \tilde{f}\,\mathscr{U}$ and $g_i^{-1}\,\tilde{f} \in \mathrm{M}^*_\varepsilon$ for $i = 1, 2$.

Let $n = \sigma(t)$.   Then $(\sigma, t) \in f\,\mathscr{U}$ if and only if $f^{-1}\,n \in \mathrm{U}$.   Note that in order for an element h to be in U two conditions must be satisfied: we need $h(\mathrm{X}) \subseteq \mathrm{Int}\,\mathrm{X}_\varepsilon$ and we need h to be isotopic to the identity through such elements.   On the other hand, we did not specify that M^*_ε be connected.   Therefore an element h of N^*_ε will be in M^*_ε if $h(\mathrm{X}_\varepsilon) \subseteq \mathrm{X}_\varepsilon$.   We will find the desired $\tilde{f} \in \mathrm{N}^*_\varepsilon$ in the following way.   First choose λ so that:

$$\hat{\mathrm{X}}_\delta \times [\lambda, \infty) \subseteq g_i(\mathrm{X}_\delta),$$

for $i = 1, 2$ and all δ, $-\varepsilon \leq \delta \leq \varepsilon$.   By Lemma 4.17 (a) there is $f \in \mathrm{N}^*_\varepsilon$ such that $f^{-1}\,n \in \mathrm{U}$ and:

$$n(\mathrm{X}) \subseteq f(\mathrm{X}) \subseteq n(\mathrm{X}) \cup (\mathrm{Int}\,\hat{\mathrm{X}}_\varepsilon \times [\lambda, \infty)).$$

Since $n \in g_i \cup$ we have $n(X) \subseteq g_i(\text{Int } X_\varepsilon)$. Therefore our choice of λ implies that:

$$(\star) \qquad n(X) \subseteq f(X) \subseteq g_i(\text{Int } X_\varepsilon).$$

We must now alter f so that $g_i^{-1} \tilde{f} \in M_\varepsilon^*$. For this we need $g_i^{-1} \tilde{f}(X_\varepsilon) \subseteq X_\varepsilon$, or equivalently $\tilde{f}(X_\varepsilon) \subseteq g_i(X_\varepsilon)$. Since f and the g_i are in N_ε^*. there is a compact subset K of W such that $f(X_\varepsilon - K) \subseteq g_i(X_\varepsilon)$. Therefore $(\star)$ implies that:

$$f(X_\alpha \cap W^\mu) \cup f(\check{X}_\varepsilon \times [\mu, \infty)) \subseteq g_i(X_\varepsilon),$$

for some $\alpha > 0$, μ and both i. Now put $\tilde{f} = fk$, where $k \in N_\varepsilon^*$ is the identity on X and takes X_ε into $(X_\alpha \cap W^\mu) \cup (\check{X}_\varepsilon \times [\mu, \infty))$. Then $g_i^{-1} \tilde{f} \in M_\varepsilon^*$. Also because $f^{-1} n(X) \subseteq X$ by $(\star)$, it is easy to see that $\tilde{f}^{-1} n = k^{-1} f^{-1} n$ will be in U, as long as k is isotopic to the identity through elements which are the identity on X. $\square$

(4.19) PROOF OF (4.15)

Factor i as

$$M_\varepsilon^* \backslash\!\!\backslash \mathscr{U} \;\overset{i_1}{\to}\; M_2^* \backslash\!\!\backslash |N| \;\overset{i_2}{\to}\; N_\varepsilon^* \backslash\!\!\backslash |N|$$

and j as $M_\varepsilon' \backslash\!\!\backslash \mathscr{U} \;\overset{j_1}{\to}\; M_\varepsilon' \backslash\!\!\backslash |N| \;\overset{j_2}{\to}\; N_\varepsilon \backslash\!\!\backslash |N|$. Then the homotopy fibres of i_1 and j_1 are equivalent. In fact, if P is any submonoid of M_ε' the homotopy fibre of the map $P \backslash\!\!\backslash \mathscr{U} \to P \backslash\!\!\backslash |N|$ is the same as the fibre of the map $\mathscr{U} \to |N|$ as may be seen by looking at the diagram:

$$
\begin{array}{ccccc}
\mathscr{U} \to & P \backslash\!\!\backslash \mathscr{U} & \to & BP \\
\downarrow & \downarrow & & \| \\
|N| \to & P \backslash\!\!\backslash |N| & \to & BP.
\end{array}
$$

Now consider i_2 and j_2. A similar argument shows that i_2 (resp. j_2) has the same homotopy fibre as the map $BM_\varepsilon^* \to BN_\varepsilon^*$ (resp. $BM_\varepsilon' \to BN_\varepsilon$). Since N_ε^* is a group, left multiplication by $g \in N_\varepsilon^*$ induces an equivalence $N_\varepsilon^* /\!\!/ M_\varepsilon^* \to N_\varepsilon^* /\!\!/ M_\varepsilon^*$. Hence, by [9], Lemma 3.3, the fibre of $BM_\varepsilon^* \to BN_\varepsilon^*$ is $N_\varepsilon^* /\!\!/ M_\varepsilon^*$. Similarly, the fibre of $BM_\varepsilon' \to BN_\varepsilon$ will be $N_\varepsilon /\!\!/ M_\varepsilon'$, provided that left multiplication by $n \in N_\varepsilon$ induces an equivalence on $N_\varepsilon /\!\!/ M_\varepsilon'$. Therefore (4.16) will follow if we prove:

(4.20) $N_\varepsilon^* /\!\!/ M_\varepsilon^* \simeq N_\varepsilon /\!\!/ M_\varepsilon'$, and:

(4.21) *Left multiplication by $n \in N_\varepsilon$ induces an equivalence*

$$N_\varepsilon /\!\!/ M_\varepsilon' \to N_\varepsilon /\!\!/ M_\varepsilon'.$$

(4.22) PROOF OF (4.20)

Order the elements of N_ε by setting $n \leq n'$ if and only if $n = n' m$ for some $m \in M_\varepsilon'$. (This is not quite a partial order since the conditions $n \leq n'$ and $n' \leq n$ do not imply that $n = n'$.) Then

the space $N_\varepsilon /\!\!/ M_\varepsilon'$ has simplices $(n_0, n_1, \ldots, n_k)$ where $n_0 \geqq n_1 \geqq \ldots \geqq n_k$. Similarly, because $M_\varepsilon^* = N_\varepsilon^* \cap M_\varepsilon'$, the space $N_\varepsilon^* /\!\!/ M_\varepsilon^*$ is the subcomplex with simplices $(n_0, \ldots, n_k)$ where each $n_i \in N_\varepsilon^*$. We must show that any finite subcomplex of $N_\varepsilon /\!\!/ M_\varepsilon'$ retracts into $N_\varepsilon^* /\!\!/ M_\varepsilon^*$. Let C be the subcomplex of $N_\varepsilon /\!\!/ M_\varepsilon'$ spanned by the vertices $n_1, \ldots, n_p$. It will suffice to show that there are $n_1'', \ldots, n_p'' \in N_\varepsilon$ and $g_1, \ldots, g_p \in N_\varepsilon^*$ such that:

(a) $n_i'' \leqq n_i$ for all i, and $n_i'' \leqq n_j''$ whenever $n_i \leqq n_j$; and:

(b) $n_i'' \leqq g_i$ for all i, and $g_i \leqq g_j$ whenever $n_i \leqq n_j$.

For (a) implies that C is homotopic to the subcomplex C'' spanned by the vertices $n_1'', \ldots, n_p''$, and (b) implies that C'' is homotopic to the subcomplex of $N_\varepsilon^* /\!\!/ M_\varepsilon^*$ with vertices $g_1, \ldots, g_p$. Further, we may suppose that $\leqq$ is a partial order on the set $n_1, \ldots, n_p$ since any C retracts onto such a subcomplex.

We will choose the n_i'' so that for some $\nu > 0$ we have $n_i''(\hat{X}_\varepsilon \times (\nu, \infty)) \subseteq X$. This may be done by setting $n_i'' = k n_i$ for all i, where $k \in M_\varepsilon'$ takes $\hat{X}_\varepsilon \times [0, \infty)$ into X. We now construct the g_i by repeated applications of Lemma 4.17 (b). Observe that when $g \in N_\varepsilon^*$ the conditions $n(X_\varepsilon) \subseteq g(X_\varepsilon)$ and $g^{-1} n \in U$ are equivalent to $n \leqq g$. We may suppose inductively that suitable g_j have been found for all $j \neq i_0$ such that $n_j \geqq n_{i_0}$. Choose $\lambda \geqq \nu$ so that $\hat{X}_\varepsilon \times [\lambda, \infty) \subseteq g_j(X_\varepsilon)$ for all such j. Then it is not hard to check that one may take g_{i_0} to be the element $g \in N_\varepsilon^*$ which satisfies the conditions of Lemma 4.17(b) with $n = n_{i_0}''$. □

(4.23) PROOF OF (4.21)

Since each $g \in N_\varepsilon^*$ acts by equivalences on $N_\varepsilon^* /\!\!/ M_\varepsilon^* \simeq N_\varepsilon /\!\!/ M_\varepsilon'$, it suffices to prove that if C is the finite subcomplex of $N_\varepsilon^* /\!\!/ M_\varepsilon^*$ with vertices $g_1, \ldots, g_p$ and if $n \in N_\varepsilon$, there are elements $n' \in N_\varepsilon$ and $g \in N_\varepsilon^*$ such that $n g_i \geqq n' g_i$ and $g g_i \geqq n' g_i$ for all i. For, as above, this will imply that $n C \sim n' C \sim g C$, and hence that multiplication by n is an equivalence $N_\varepsilon /\!\!/ M_\varepsilon' \xrightarrow{\sim} N_\varepsilon /\!\!/ M_\varepsilon'$.

As above, we will take $n' = k n$ where $k \in N_\varepsilon$ takes $\hat{X}_\varepsilon \times [0, \infty)$ into X. Next, choose λ so that:

$$g_i(X) \subseteq W^\lambda \cup X, \qquad n'(\hat{X}_\varepsilon \times [\lambda, \infty)) \subseteq X$$

and:

$$\hat{X}_\varepsilon \times (\lambda, \infty) \subseteq g_i(X_\varepsilon) \subseteq W^\lambda \cup X_\varepsilon.$$

By Lemma 4.17 (b) one can find $g \in N_\varepsilon^*$ such that $g = n'$ on W^λ:

$$g^{-1} n' \in U \qquad \text{and} \qquad n'(X_\varepsilon) \subseteq g(X_\varepsilon).$$

Hence $n' g_i(X_\varepsilon) \subseteq g g_i(X_\varepsilon)$ for all i. Using (c) one can easily check that $g_i^{-1} g^{-1} n' g_i \in U$ for all i. Thus $n' g_i \leqq g g_i$ as required. □

(4.24) PROOF OF (4.17)

We will first prove (a). Since N is connected, there is a 1-simplex $t \mapsto n_t$ in N with $n_0 = \mathrm{id}$ and $n_1 = n$. By Definition 4.3 (ii), one can find a 1-simplex $t \mapsto \tilde{n}_t$ in N with $\tilde{n}_0 = \mathrm{id}$ and such that $\tilde{n}_t = n_t$ on $\hat{X} \times [\mu_1, \infty)$ and $\tilde{n}_t = \mathrm{id}$ on $(\hat{W} - \hat{X}_\alpha) \times [\mu_1, \infty)$ for some $\mu_1 > \lambda$, $\alpha < \varepsilon$ and all t. Choose $\mu_2 > \mu_1$ so that W^{μ_2} contains $\tilde{n}_t^{-1} n_t(W^{\mu_1})$ for all t. It follows from [5] that there is

a 1-simplex $t \mapsto h_t$ in $\mathbf{N}$ such that $h_t = \tilde{n}_t^{-1} n_t$ on $W^{\mu_1} \cup X$ and $h_t = \mathrm{id}$ on $\hat{W} \times [\mu_2, \infty)$. Let $g_1 = \tilde{n}_1 h_1$. Then $g_1 = n_1 = n$ on $W^{\mu_1} \cup X$, and $g_1 = \mathrm{id}$ on $(\hat{W} - \hat{X}_\alpha) \times [\mu_2, \infty)$. Also, one can easily check that $g_1^{-1} n \in U$. Therefore, all we need do is alter g_1 on $\hat{X}_\varepsilon \times [\lambda, \infty)$ to a diffeomorphism g which is the identity on $(X_\varepsilon - X_{-\varepsilon}) \times (\lambda', \infty)$ for some $\lambda' > \lambda$. This will imply that $g \in \mathbf{N}_\varepsilon^*$.

Care is required in order to ensure that $n(X) \subseteq g(X)$.

Here are the details of this construction. Choose $\mu_3 > \mu_2$ so that:

(a) $\qquad\qquad\qquad\qquad n(X) \subset W^{\mu_3} \cup \hat{X} \times (\mu_3, \infty).$

Next, choose $\mu_5 > \mu_4 > \mu_3$ so that:

(b) $g_1(\hat{X}_\delta \times (\mu_4, \infty)) \supseteq \hat{X}_\delta \times (\mu_5, \infty)$ when $\alpha \leq \delta \leq \varepsilon$.

This is possible since $g_1 = \mathrm{id}$ on $(\hat{W} - \hat{X}_\alpha) \times [\mu_2, \infty)$. Now, choose $k_1 \in \mathit{Diff}_\omega W$ so that, for some β, γ with $\alpha < \beta < \gamma < \varepsilon$.

(c) supp $k_1 \subseteq \hat{X}_\gamma \times (\mu_3, \infty)$.

(d) $X \subseteq k_1(X)$.

(e) $\hat{X}_\beta \times [\mu_4, \infty) \subseteq k_1(\hat{X} \times [\mu_4, \infty)) \subseteq \hat{X}_\beta \times [\mu_3, \infty)$; and:

(f) $k_1(\hat{X}_{-\varepsilon} \times [\mu_4, \infty)) \supseteq \hat{X}_\alpha \times [\mu_4, \infty)$.

Finally, choose $k_2 \in \mathit{Diff}_\omega W$ so that:

(g) supp $k_2 \subset \hat{X}_\varepsilon \times (\mu_5, \infty)$;

(h) $X \subset k_2(X_\beta)$; and:

(j) $k_2 = k_1^{-1}$ on $(\hat{X}_\gamma - \hat{X}_\alpha) \times [\mu_6, \infty)$, for some $\mu_6 > \mu_5$.

There are no volume obstructions to finding k_2 because both components of the complement of $\partial(X_\beta) \cup (\hat{X}_\gamma - \hat{X}_\alpha) \times [\mu_6, \infty)$ in $\hat{X}_\varepsilon \times (\mu_5, \infty)$ have infinite volume.

Now, let $g_2 = k_2 g_1 k_1$. Then $g_2 = \mathrm{id}$ on $(X_\gamma - X_{-\varepsilon}) \times [\mu_7, \infty)$ for some $\mu_7 > \mu_6$, by $(c), (f)$ and (j). Also:

$$g_2(X) \cap W^{\mu_5} = k_2 g_1 k_1(X) \cap W^{\mu_5} = g_1 k_1(X) \cap W^{\mu_5} \supseteq g_1(X) \cap W^{\mu_5} \supseteq n(X) \cap W^{\mu_5} \quad \text{by } (g), (d).$$

Further:

$$g_2(X) \cap \hat{W} \times [\mu_5, \infty) = k_2 g_1 k_1(X) \cap \hat{W} \times [\mu_5, \infty)$$
$$\supseteq k_2 g_1 k_1(\hat{X} \times [\mu_4, \infty)) \cap \hat{W} \times [\mu_5, \infty)$$
$$\supseteq k_2 g_1(\hat{X}_\beta \times [\mu_4, \infty)) \cap \hat{W} \times [\mu_5, \infty) \text{ by } (e)$$
$$\supseteq k_2(\hat{X}_\beta \times [\mu_5, \infty)) \cap \hat{W} \times [\mu_5, \infty) \text{ by } (b)$$
$$\supseteq \hat{X} \times [\mu_5, \infty) \quad \text{by } (g) \text{ and } (h)$$
$$\supseteq n(X) \cap \hat{W} \times [\mu_5, \infty) \text{ by } (a).$$

Therefore $n(X) \subseteq g_2(X)$. Further, if we choose k_1 and k_2 so that they are isotopic to the identity by isotopies with supports in $\hat{X}_\gamma \times (\mu_3, \infty)$ and $\hat{X}_\varepsilon \times (\mu_5, \infty)$ respectively, it is easy to check that $g_2^{-1} g_1$ is isotopic to the identity by an isotopy with support in $\hat{X}_\varepsilon \times (\mu_3, \infty)$.

The final step is to alter g_2 outside X to an element g which is the identity on $(\hat{X}_\varepsilon - \hat{X}_{-\varepsilon}) \times (\mu_7, \infty)$ rather than just on $(X_\alpha - X_{-\varepsilon}) \times (\mu_7, \infty)$. Clearly this may be done so that g satisfies all the conditions required by (a) and (c).

Now consider (b). By arguing as in (a) with X replaced by X_ε, we can find $g \in N_\varepsilon^*$ such that $g = n$ on W^λ, $g^{-1} n \in U$ and:

$$n(X_\varepsilon) \subseteq g(X_\varepsilon) \subseteq n(X_\varepsilon) \cup (\hat{X}_{2\varepsilon} \times (\lambda', \infty)).$$

Here λ' can be arbitrarily large. In fact, by hypothesis on n we can choose $\mu_3 > \lambda$ so that:

$$n(X_\varepsilon) \subseteq W^{\mu_3} \cup (\hat{X} \times (\mu_3, \infty)).$$

Then g may be chosen so that $g = n$ on $X_\varepsilon \cap W^{\mu_3}$ and $g(X_\varepsilon) \subseteq n(X_\varepsilon) \cup (\hat{X}_{2\varepsilon} \times (\mu_3, \infty))$. It now suffices to alter g on $(\hat{X}_{3\varepsilon} - \hat{X}) \times (\mu_3, \infty)$, so as to make $g(X_\varepsilon) \subseteq n(X_\varepsilon) \cup (\hat{X}_\varepsilon \times (\mu_3, \infty))$. $\square$

This completes the proof of Proposition 4.12. The proof of Proposition 4.5 when $S \neq \emptyset$ is almost exactly the same and will be left to the reader. Thus the case $A = \emptyset$ of Theorem 4.1 is finally proved. In order to prove the general case of Theorem 4.1, it suffices to consider the case when A is a top dimensional submanifold of W which is a product both near infinity and near ∂X. The reader may check that the proof goes through in this generality without essential change.

(4.25) PROOF OF (2.8)

Consider the diagram:

$$\overline{B} E_0(W, X, \text{rel } A \cup Y) \rightarrow \overline{B} E_0(W, X, Y, \text{rel } A) \rightarrow \overline{B} E_0(Y, \text{rel } A)$$
$$\downarrow i_2 \qquad\qquad\qquad \downarrow i_1 \qquad\qquad\qquad \|$$
$$\overline{B} E_0(W, \text{rel } A \cup Y) \rightarrow \overline{B} E_0(W, Y, \text{rel } A) \rightarrow \overline{B} E_0(Y, \text{rel } A).$$

Here the rows are simple homology fibration sequences by (2.9). [Because $Y \subseteq \text{Int } X$, the proof of (2.9) goes through for the top row.] Theorem 4.1 implies that i_2 is a **Z**-homology equivalence. Thus i_1 is also. Hence, by composing i_1 with the **Z**-homology equivalence $\overline{B} E_0(W, Y, \text{rel } A) \hookrightarrow \overline{B} E_0(W, \text{rel } A)$, one sees that the inclusion $\overline{B} E_0(W, X, Y, \text{rel } A) \hookrightarrow \overline{B} E_0(W, \text{rel } A)$ is a **Z**-homology equivalence. Now consider the diagram:

$$\overline{B} E_0(W, \text{rel } A \cup X) \rightarrow \overline{B} E_0(W, X, Y, \text{rel } A) \rightarrow \overline{B} E_0(X, Y, \text{rel } A)$$
$$\| \qquad\qquad\qquad \downarrow i_3 \qquad\qquad\qquad \downarrow i_4$$
$$\overline{B} E_0(W, \text{rel } A \cup X) \rightarrow \overline{B} E_0(W, X, \text{rel } A) \rightarrow \overline{B} E_0(X, \text{rel } A)$$

Here, i_3 is a **Z**-homology equivalence because its composite with the inclusion $\overline{B} E_0(W, X, \text{rel } A) \hookrightarrow \overline{B} E_0(W, \text{rel } A)$ is. Therefore, i_4 must also be a **Z**-homology equivalence. $\square$

Remark. — Because $Y \subset \text{Int } X$ one could also show that the inclusion:

$$\overline{B} E_0(W, X, Y, \text{rel } A) \hookrightarrow \overline{B} E_0(W, \text{rel } A),$$

is a **Z**-homology equivalence by repeating the proof of Theorem 4.1.

5. Normalization

In this section we verify the last of the properties in §2, showing that when $n \geq 3$,

$$f_D : \quad \overline{B} \, E_0 \, (D^{n-1} \times \mathbf{R}) \to \overline{B} \, \Gamma^n_{sl},$$

is a homology equivalence. Our starting point is Lemma 2.4 of [10] which says that the homotopy fibre of the map:

$$\overline{\beta} : \quad B \, \mathrm{Emb}^g_{k\omega} \, D^n \to B \, \mathbf{R}^\delta,$$

is weakly (homotopy) equivalent to $B \, \Gamma^n_{sl}$. Here $\mathrm{Emb}^g_{k\omega} \, D^n$ denotes the discrete monoid formed by the germs at D^n of self-embeddings of the pair $(\mathbf{R}^n, D^n)$ which preserve the volume form ω up to a constant factor. Also, $\mathbf{R}^\delta$ is the discrete group of additive reals, and $\overline{\beta}$ is induced by the homomorphism $\beta : \mathrm{Emb}^g_{k\omega} \, D^n \to \mathbf{R}^\delta$ given by $\beta(m) = \log \, (m_* \omega / \omega)$. It follows easily from this that $\overline{B} \, \Gamma^n_{sl}$ is weakly equivalent to the homotopy fibre F of the map $\overline{B} \, \mathscr{E}mb^g_{k\omega, \, 0} \, D^n \to \overline{B} \, \mathbf{R}$. (One can see this by arguing as in the proof of Theorem 1.1 given at the end of §2 in [9].) We will show that there is an isomorphism between $H_*(F)$ and $H_*(\overline{B} \, E_0 \, (D^{n-1} \times \mathbf{R}))$ which is induced by a sequence of inclusion and restriction maps each of which gives rise to an isomorphism on H_*. Since the equivalence $F \to \overline{B} \, \Gamma^n_{sl}$ is induced by the usual map f it will follow that there is a commutative diagram:

$$H_*(F) \xrightarrow{\cong} H_*(\overline{B} \, \Gamma^n_{sl})$$
$$\cong \searrow \quad \nearrow {}_{(f_D)_*}$$
$$H_*(\overline{B} \, E_0 \, (D^{n-1} \times \mathbf{R})).$$

Hence f_D is a $\mathbf{Z}$-homology equivalence. This proves (2.5), since $\overline{B} \, \Gamma^n_{sl}$ is simply connected.

In order to relate F to $E_0 \, (D^{n-1} \times \mathbf{R})$ we introduce the following monoids. Let $H \subseteq \mathbf{R}^n$ be the half space $\{ x \in \mathbf{R}^n : x_1 \geq 0 \}$. Then H is a nice submanifold of $\mathbf{R}^n$, and one can define the monoid:

$$\mathbf{E}(\mathbf{R}^n, \, H^+) \subset \mathbf{Diff}_\omega \, \mathbf{R}^n = \mathbf{E}(\mathbf{R}^n),$$

as in Definition 4.3, where $+$ denotes the single end of $\mathbf{R}^n$. We will write $\mathbf{E}_{k\omega} \, (\mathbf{R}^n, \, H^+)$ for the analogous submonoid of the total singular complex $\mathbf{Diff}_{k\omega} \, \mathbf{R}^n$ of the group of all diffeomorphisms of $\mathbf{R}^n$ which preserve ω up to a constant. Also, $\mathbf{E}_{k\omega, \, 0} \, (\mathbf{R}^n, \, H^+, \, D^n)$ is defined to be the (connected) monoid:

$$\mathbf{E}_{k\omega, \, 0} \, (\mathbf{R}^n, \, H^+) \cap \mathbf{E}_{k\omega} \, (\mathbf{R}^n, \, D^n).$$

Consider the diagram:

(5.1)

$$F \xleftarrow{\ \rho_1\ } F_1 \xrightarrow{\ i_1\ } \overline{B}\, E_0(\mathbf{R}^n, H^+)$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$\overline{B}\,\mathbf{Emb}^g_{k\omega,\,0}\, D^n \xleftarrow{\ \rho\ } \overline{B}\, E_{k\omega,\,0}(\mathbf{R}^n, H^+, D^n) \xhookrightarrow{\ i\ } \overline{B}\, E_{k\omega,\,0}(\mathbf{R}^n, H^+)$$

$$\downarrow{\scriptstyle\overline{\beta}} \qquad\qquad\qquad \downarrow{\scriptstyle\overline{\beta}} \qquad\qquad\qquad \downarrow{\scriptstyle\overline{\beta}}$$

$$\overline{B}\,V \ =\!=\!=\!=\ \overline{B}\,V \ =\!=\!=\!=\ \overline{B}\,V$$

Here V is the total singular complex of the contractible group $\mathbf{R}$. Also, F_1 is defined to be the homotopy fibre of $\overline{\beta}$, and ρ and i are the obvious restriction and inclusion. Observe that the right hand column is a fibration sequence since $E_{k\omega,\,0}(\mathbf{R}^n, H^+)$ is the semi-direct product of $E_0(\mathbf{R}^n, H^+)$ with V. It follows easily from Lemma 4.11 that i is an equivalence. Moreover, ρ is a homology equivalence, because:

$$\overline{B}\, E_0(\mathbf{R}^n, H^+, \mathrm{rel}\ D^n) \to \overline{B}\, E_{k\omega,\,0}(\mathbf{R}^n, H^+, D^n) \to \overline{B}\,\mathbf{Emb}^g_{k\omega,\,0}\, D^n,$$

is a homology fibration sequence whose fibre is acyclic by (4.5) and (4.6). Thus both ρ_1 and i_1 are $\mathbf{Z}$-homology equivalences.

Now consider the monoid $E_0^+(H)$ of germs at H^+ of the elements of $E_0(\mathbf{R}^n, H^+)$. It is easy to check that $E_0^+(H)$ is isomorphic to the monoid $E_0^+(D^{n-1} \times \mathbf{R})$ of germs at the end $D^{n-1} \times \infty$ of the elements of $E_0(D^{n-1} \times \mathbf{R})$. Therefore, there are maps:

$$q_1:\quad \overline{B}\, E_0(\mathbf{R}^n, H^+) \to \overline{B}\, E_0^+(H),$$

and:

$$q_2:\quad \overline{B}\, E_0(D^{n-1} \times \mathbf{R}) \to \overline{B}\, E_0^+(D^{n-1} \times \mathbf{R}) \cong \overline{B}\, E_0^+(H).$$

Clearly it will suffice to show that both q_1 and q_2 are $\mathbf{Z}$-homology equivalences. This is easily seen for q_1 since its homology fibre is the space $\overline{B}\, E_0(\mathbf{R}^n, \mathrm{rel}\ H^+)$, which is acyclic by (2.7). However the proof for q_2 involves some diagram chasing, because the usual fibration lemma does not apply to q_2. (The trouble is that we are not restricting to a submanifold in the *interior* of $D^{n-1} \times \mathbf{R}$.) We therefore consider the following diagrams, where $D = D^{n-1}$ is embedded in $S = S^{n-1}$.

$$\overline{B}\, E_0(S \times \mathbf{R}, \mathrm{rel}\ D \times \mathbf{R}) \longrightarrow \overline{B}\, E_0(S \times \mathbf{R}, D \times \mathbf{R}) \longrightarrow \overline{B}\, E_0(D \times \mathbf{R})$$

$$\cap{\scriptstyle j_1} \qquad\qquad\qquad\qquad \cap{\scriptstyle j_2} \qquad\qquad\qquad\qquad \downarrow{\scriptstyle q_2} \qquad\qquad (\mathrm{A})$$

$$\overline{B}\, E_0(S \times \mathbf{R}, (D \times \mathbf{R})^-, \mathrm{rel}\,(D \times \mathbf{R})^+) \to \overline{B}\, E_0(S \times \mathbf{R}, (D \times \mathbf{R})^\pm) \to \overline{B}\, E_0^+(D \times \mathbf{R})$$

$$\overline{B}\, E_0(S \times \mathbf{R}, \mathrm{rel}\ D \times \mathbf{R}) \xrightarrow{\ j_4\ } \overline{B}\, E_0(S \times \mathbf{R}, D \times \mathbf{R}, \mathrm{rel}\,(D \times \mathbf{R})^+) \to \overline{B}\, E_0(D \times \mathbf{R}, \mathrm{rel}\,(D \times \mathbf{R})^+)$$

$$\qquad\qquad {\scriptstyle j_1}\searrow\qquad\qquad\qquad \downarrow{\scriptstyle j_3} \qquad\qquad\qquad\qquad (\mathrm{B})$$

$$\overline{B}\, E_0(S \times \mathbf{R}, (D \times \mathbf{R})^-, \mathrm{rel}\,(D \times \mathbf{R})^+)$$

$$\overline{B} E_0 (S \times R, \text{ rel } D \times R \cup (S \times R)^+) \to \overline{B} E_0 (S \times R, D \times R, \text{ rel}(S \times R)^+)$$

$$\to \overline{B} E_0 (D \times R, \text{ rel}(D \times R)^+)(C).$$

Using suitable variants of (2.9), one can easily check that all the rows are simple homology fibrations. Moreover j_2 and j_3 are **Z**-homology equivalences by (4.5) and the proof of (4.2). In diagram (C) both:

$$\overline{B} E_0 (S \times R, \text{ rel } D \times R \cup (S \times R)^+) \quad \text{and} \quad \overline{B} E_0 (S \times R, D \times R, \text{ rel}(S \times R)^+)$$

are acyclic by (2.7) and (4.6). Hence $\overline{B} E_0 (D \times R, \text{ rel } (D \times R)^+)$ is acyclic. Therefore, in diagram (B) j_4 is a **Z**-homology equivalence, and so $j_1 = j_3 \circ j_4$ is too. Finally, by looking at (A) one sees that q_2 is also a **Z**-homology equivalence. This completes the proof of (2.5). $\square$

6. Proof of Theorems 1.1 and 1.2 when $n \geq 3$

Theorem 1.1 would follow immediately from Theorem 2.1 if the fibration:

$$\overline{B} \mathcal{D}iff^{\Phi}_{\omega 0} W \to \overline{B} \mathcal{D}iff_{\omega 0} W \to \overline{B} \mathcal{V},$$

were simple. However Rousseau's calculations in [12] show that $\pi_1 \overline{B} \mathcal{V}$ acts non-trivially on $H_1 (\overline{B} \mathcal{D}iff^{c\Phi}_{\omega 0} \mathcal{W})$ when $W = T^2 - pt$, and so there is no reason to suppose that this is so.

We begin with an elementary proof (not using Theorem 2.1) that f_W does induce a map from $B \mathcal{D}iff^{\Phi}_{\omega 0} W$ to $\tilde{S}_\omega(W)$.

LEMMA 6.1. — *If* (W, A) *is an admissible pair, there is a commutative diagram:*

$$\begin{array}{ccc} \overline{B} \mathcal{D}iff_{\omega 0}(W, \text{ rel } A) & \overset{\Phi}{\to} & \overline{B} \mathcal{V} \\ \downarrow^{f_W} & & \| \\ S_\omega(W, \text{ rel } A) & \overset{\theta}{\to} & \overline{B} \mathcal{V} \end{array}$$

where θ *induces an isomorphism on* π_1 *and* $\mathcal{V} = H^{n-1}(W, A; R) \cong R^p$.

Proof. — Since $\overline{B} \mathcal{V}$ is a K $(\pi, 1)$, the map θ is determined by the induced map θ_* on π_1. We will begin by giving an explicit description of θ_*. Let $u \in H^n (\overline{B} \Gamma^n_{sl}; R)$ be the "universal transverse volume form". Thus, if I is a foliation on M with transverse volume form α, and if $g : M \to B \Gamma^n_{sl}$ classifies I, then $g^* u = [\alpha]$. We showed in [10], Lemma 2 that a map $g : S^n \to \overline{B} \Gamma^n_{sl}$ is null-homotopic if and only if:

$$g^* i^* (u) = 0, \quad \text{where} \quad i : \overline{B} \Gamma^n_{sl} \to B \Gamma^n_{sl}$$

is the inclusion.

Consider the commutative diagram:

$$\xymatrix{ & & \mathbf{B}\,\Gamma^n_{sl} \ar[d]^{v} \\ \mathbf{M} \ar[rr] \ar[urr]^{h} \ar[ur]_{g} & & \mathbf{B}\,\mathscr{S}\mathscr{L}(n,\mathbf{R}) }$$

where v is a Hurevicz fibration. Obstruction theory implies that, if M has the homotopy type of an n-dimensional complex, then h is fibrewise homotopic to g if and only if $h^* u = g^* u$. Because the elements of $S_\omega(W)$ may be considered to be liftings of the classifying map τ of the tangent bundle to W, an element λ of $\pi_1(S_\omega(W, \mathrm{rel}\, A))$ is represented by a commutative diagram:

$$\xymatrix{ & & \mathbf{B}\,\Gamma^n_{sl} \ar[d]^{v} \\ \mathbf{W}\times\mathbf{S}^1 \ar[rr]_{\tau\cdot pr} \ar[urr]^{h_\lambda} \ar[ur]_{s_1} & & \mathbf{B}\,\mathscr{S}\mathscr{L}(n,\mathbf{R}) }$$

where s_1 is the composite $W \times S^1 \overset{pr}{\to} W \overset{s_0}{\to} B\Gamma^n_{sl}$, and where $h_\lambda = s_1$ over $(A \times S^1) \cup (W \times x_0)$. [Recall that s_0 is the base point of $S_\omega(W)$.] The above remarks imply that λ is trivial if and only if:

$$h_\lambda{}^* u = s_1{}^* u \quad \text{in} \quad H^n(W \times S^1, A \times S^1; \mathbf{R}).$$

But $s_1{}^* u = 0$, and $\pi_1(\overline{B\mathscr{V}}) \cong \mathscr{V} \cong H^{n-1}(W, A; \mathbf{R}) \cong H^n(W \times S^1, A \times S^1; \mathbf{R})$. Therefore, up to the choice of isomorphism $V \cong H^n(W \times S^1, A \times S^1; \mathbf{R})$, we have $\theta_*(\lambda) = h_\lambda{}^* u$.

We have to check that the two homomorphisms from $\pi_1(\overline{B}\,\mathscr{D}\!\mathit{iff}_{\omega 0}(W, \mathrm{rel}\,A))$ to V given by Φ and $\theta_* \circ f_W$ differ at most by an isomorphism of V. Now $\pi_1(\overline{B}\,\mathscr{D}\!\mathit{iff}_{\omega 0}(W, \mathrm{rel}\,A))$ is just the universal cover of $\mathscr{D}\!\mathit{iff}_{\omega 0}(W, \mathrm{rel}\,A)$, but considered with the discrete topology. Therefore, its elements are pairs $(g, [g_t])$, where $g \in \mathscr{D}\mathrm{iff}_{\omega 0}(W, \mathrm{rel}\,A)$ and $[g_t]$ is the homotopy class of the path g_t relative to its end points $g_0 = \mathrm{id}$ and $g_1 = g$. It is easy to check that:

$$\Phi((g, [g_t])) = \Phi(g) = [g^* \Omega - \Omega] \quad \text{in} \quad V,$$

where Ω is any $(n-1)$-form on W such that $d\Omega = \omega$.

The element $(g, [g_t])$ also gives rise to a codimension n foliation $\mathscr{I}$ of the bundle $W \times S^1 \to S^1$ in the obvious way: if $W \times S^1$ is identified with $W \times [0, 1]/(x, 0) = (x, 1)$, then the leaf through (x, t) will contain the points $(g_s g_t^{-1}(x), s)$ for $0 \le s \le 1$. (In order to make $\mathscr{I}$ smooth near $t = 0$ one can assume that the path g_t is constant near its endpoints.) Notice that $\mathscr{I}$ has a transverse volume form α which restricts to ω on each fibre $W \times t$. In fact, if $v_t(x)$ is the vector field on W which is tangent to the path $s \mapsto g_s g_t^{-1}(x)$ at t, then one can check that $\alpha = \omega - dt \wedge (v_t \lrcorner \omega)$.

Now consider the image λ of $(g, [g_t])$ under f_W. The definition of f_W implies that the map $h_\lambda : W \times S^1 \to B\Gamma^n_{sl}$ which represents λ as above is exactly the classifying map of the pair $(\mathscr{I}, \alpha)$. In particular, $h_\lambda{}^*(u) = [\alpha]$. By integration along the fibre S^1, one finds that

$[\alpha] \in H^n$ $(W \times S^1,\ A \times S^1 : \mathbf{R})$ corresponds to $[g^* \Omega - \Omega] \in H^{n-1}$ $(W, A; \mathbf{R})$. The result follows. $\square$

It follows that there is a commutative diagram:

$$
\begin{array}{ccccc}
\overline{B}\,\mathbf{D}_0(W, \text{rel } A) & \to & \overline{B}\,\mathbf{E}_0(W, \text{rel } A) & \overset{\Phi}{\to} & \overline{B}\,V \\
\downarrow{\tilde{f}_W} & & \downarrow{f_W} & & \| \\
\tilde{S}_\omega(W, \text{rel } A) & \to & S_\omega(W, \text{rel } A) & \overset{\theta}{\to} & \overline{B}V
\end{array}
$$

(This notation is defined in (3.1) and the beginning of § 4.)

THEOREM 6.2. — *If* (W, A) *is an admissible pair and* dim $W \geq 3$, *then the map* $\tilde{f}_W$ *in the above diagram is a homology equivalence.*

Proof. — Let L be a 1-manifold which represents a basis for H^{n-1} $(W, A : \mathbf{R})$ as in Lemma 4.2. Clearly L can be chosen so that each of its components is non-compact and so that $X = L_\varepsilon$ is a nice submanifold of W which does not meet A. Then H^{n-1} $(W, X \cup A; \mathbf{R}) \cong H^{n-1}$ $(W, A; \mathbf{R}) \cong V$, and there is a fibration sequence:

$$\overline{B}\,\mathbf{D}_0(W, \text{rel } X \cup A) \to \overline{B}\,\mathbf{E}_0(W, \text{rel } X \cup A) \to \overline{B}\,V.$$

Moreover this sequence is simple because, by choice of $X = L_\varepsilon$, every element of V may be lifted to a diffeomorphism of W which has support in X_δ, for some $\delta > 0$. [*See* (3.5) above.] Similarly:

$$\tilde{S}_\omega(W, \text{rel } X \cup A) \to S_\omega(W, \text{rel } X \cup A) \to \overline{B}\,V,$$

is a simple homotopy fibration sequence. Hence (2.1) implies that:

$$\tilde{f}_W : \quad \overline{B}\,\mathbf{D}_0(W, \text{rel } X \cup A) \to \tilde{S}_\omega(W, \text{rel } X \cup A),$$

is a homology equivalence.

Now observe that the elements of $\mathbf{E}_0(X)$ lift to elements of $\mathbf{E}_0(W, X, \text{rel } A)$ which have zero flux and have support in X_δ, for any given $\delta > 0$. Also, obstruction theory shows that the groups:

$$\pi_1 (S_\omega(W, \text{rel } X \cup A)) \quad \text{and} \quad \pi_1 (S_\omega(W, \text{rel } A)),$$

are isomorphic.

It follows that there is a commutative diagram:

$$
\begin{array}{ccccc}
\overline{B}\,\mathbf{D}_0(W, \text{rel } X \cup A) & \to & \overline{B}\,\mathbf{D}_0(W, X, \text{rel } A) & \to & \overline{B}\,\mathbf{E}_0(X) \\
\downarrow{\tilde{f}_1} & & \downarrow{\tilde{f}_2} & & \downarrow{f} \\
\tilde{S}_\omega(W, \text{rel } X \cup A) & \to & \tilde{S}_\omega(W, \text{rel } A) & \to & S_\omega(X)
\end{array}
$$

in which the rows are simple homology fibration sequences. But we proved above that $\tilde{f}_1$ is a homology equivalence, and f is also, by Theorem 2.10. [Observe that $S_\omega(X)$ is simply

connected.] Hence $\tilde{f}_2$ is a homology equivalence, and the result follows by (4.1) and (4.2). $\square$

Finally, we prove the following generalization of Theorem 1.2.

THEOREM 6.3. — *If* (W, A) *is an admissible pair with* $\dim W \geqq 3$, *the map:*

$$f_W^c : \quad \overline{B} \, E_0^c (W, \operatorname{rel} A) \to S_{\omega 0}^c (W, \operatorname{rel} A),$$

is a homology equivalence.

Here $E_0^c (W, \operatorname{rel} A)$ is of course the total singular complex of the group $\mathscr{D}iff_{\omega 0}^c (W, \operatorname{rel} A)$: see the beginning of § 3.

Proof. — We may assume that A is a top dimensional submanifold of W whose non-compact components are products at infinity. Then $E_0 (W, \operatorname{rel} A)$ equals $E_0 (Z, \operatorname{rel} \partial Z)$ where $Z = W - \operatorname{Int} A$. Clearly it suffices to consider the case when Z is connected. Let J be the set of ends of Z. Pick out one end Z^+ and put $S = J - \{ + \}$. Then consider the commutative diagram:

$$\overline{B} \, E_0^c (Z, \operatorname{rel} \partial Z) \to \overline{B} \, E_0 (Z, \operatorname{rel} \partial Z \cup Z^S) \to \overline{B} \, E_0^+ (Z, \operatorname{rel} \partial Z \cup Z^S)$$
$$\downarrow f^c \qquad\qquad\qquad \downarrow f \qquad\qquad\qquad \downarrow f^+$$
$$S_\omega^c (Z, \operatorname{rel} \partial Z) \to S_\omega (Z, \operatorname{rel} \partial Z \cup Z^S) \overset{q}{\to} S_\omega^+ (Z, \operatorname{rel} \partial Z)$$

where $S_\omega^+ (Z, \operatorname{rel} \partial Z)$ is the direct limit of the spaces:

$$S_\omega (Z^+ \times [\lambda, \infty), \operatorname{rel} \partial Z^+ \times [\lambda, \infty)) \text{ as } \lambda \to \infty.$$

The groups:

$$E_0 (Z, \operatorname{rel} \partial Z \cup Z^S) \quad \text{and} \quad E_0^+ (Z, \operatorname{rel} \partial Z \cup Z^S)$$

are as in (4.3). In particular, E_0^+ consists of germs of diffeomorphisms at Z^+. Our first aim is to show that f^c induces a **Z**-homology equivalence from $\overline{B} \, E_0^c (Z, \operatorname{rel} \partial Z)$ to the connected component $S_{\omega 0}^c (Z, \operatorname{rel} \partial Z)$.

First note that the top row is a fibration sequence because it comes from an exact sequence of groups. Moreover, because Z^+ is connected, the elements of $E_0^+ (Z, \operatorname{rel} \partial Z \cup Z^S)$ all have zero flux, and so may be lifted to elements of $E_0 (Z, \operatorname{rel} \partial Z \cup Z^S)$ which have support in $Z^+ \times [\lambda, \infty)$, for any given λ. This implies that the top row is simple.

We next show that both f^+ and q lift to the universal cover $\tilde{S}_\omega^+ (Z, \operatorname{rel} \partial Z)$ of $S_\omega^+ (Z, \operatorname{rel} \partial Z)$. This follows easily for q since the map:

$$\pi_1 (S_\omega (Z, \operatorname{rel} \partial Z \cup Z^S)) \to \pi_1 (S_\omega^+ (Z, \operatorname{rel} \partial Z)),$$

corresponds to the map:

$$H^{n-1} (Z, \partial Z \cup Z^S; \mathbf{R}) \to H^{n-1} (Z^+, \partial Z^+; \mathbf{R}),$$

which is zero. To see it for f^+, note that $\mathbf{E}_0^+(Z, \text{rel } \partial Z \cup Z^S)$ is isomorphic to $\mathbf{D}_0^+(Z^+ \times \mathbf{R}, \text{rel } \partial Z^+ \times \mathbf{R})$. It follows from (2.7) (ii) that the quotient map:

$$\overline{\mathbf{B}}\,\mathbf{D}_0(Z^+ \times \mathbf{R}, \text{rel } \partial Z^+ \times \mathbf{R}) \to \overline{\mathbf{B}}\,\mathbf{D}_0^+(Z^+ \times \mathbf{R}, \text{rel } \partial Z^+ \times \mathbf{R})$$

is a $\mathbf{Z}$-homology equivalence. Therefore, by applying Theorem 6.2 to the pair $(Z^+ \times \mathbf{R}, \partial Z^+ \times \mathbf{R})$, one sees that f^+ lifts to a homology equivalence:

$$f^+ : \overline{\mathbf{B}}\,\mathbf{E}_0^+(Z, \text{rel } \partial Z \cup Z^S) \to \tilde{S}_\omega^+(Z, \text{rel } \partial Z).$$

Thus, we have a commutative diagram:

$$\overline{\mathbf{B}}\,\mathbf{E}_0^c(Z, \text{rel } \partial Z) \to \overline{\mathbf{B}}\,\mathbf{E}_0(Z, \text{rel } \partial Z \cup Z^S) \to \overline{\mathbf{B}}\,\mathbf{E}_0^+(Z, \text{rel } \partial Z \cup Z^S)$$

$$\downarrow f^c \qquad\qquad \downarrow f \qquad\qquad \downarrow \tilde{f}^+$$

$$S_{\omega 0}^c(Z, \text{rel } \partial Z) \to S_\omega(Z, \text{rel } \partial Z \cup Z^S) \xrightarrow{\tilde{q}} \tilde{S}_\omega^+(Z, \text{rel } \partial Z)$$

We have already seen that the top row is a simple homology fibration, and that $\tilde{f}^+$ is a homology equivalence. It is easy to check that the bottom row is also a simple homology fibration. Moreover f is a $\mathbf{Z}$-homology equivalence by Theorem 2.1. Hence f^c is a $\mathbf{Z}$-homology equivalence.

Before proceeding with the proof of Theorem 6.3 we will give the proof of Corollary 1.3 since this result will be needed.

PROOF THAT $\pi_{n+1}(\overline{\mathbf{B}}\,\Gamma_{sl}^n) = 0$ when $n \geq 3$.

Observe that:

$$\pi_{n+1}(\overline{\mathbf{B}}\,\Gamma_{sl}^n) \cong H_1((\Omega^n \mathbf{B}\,\Gamma_{sl}^n)_0) \cong H_1(\overline{\mathbf{B}}\,\mathscr{D}\!\mathit{iff}_\omega^c\,\mathbf{R}^n) = H_1(\mathbf{B}\tilde{\mathbf{D}}) = \tilde{\mathbf{D}}/[\tilde{\mathbf{D}}, \tilde{\mathbf{D}}],$$

where $\tilde{\mathbf{D}}$ is the universal cover of the group $\mathscr{D}\!\mathit{iff}_{\omega 0}^c\,\mathbf{R}^n$, but considered with the discrete topology. Therefore $\pi_{n+1}(\overline{\mathbf{B}}\,\Gamma_{sl}^n) = 0$ because $\tilde{\mathbf{D}}$ is perfect. [Unfortunately Thurston's proof [16] that $\tilde{\mathbf{D}}$ is perfect has remained unpublished. However his results were later generalized by Banyaga to the symplectic case, and one can (with some difficulty) reconstruct Thurston's argument from [1].] □

The proof that f^c induces an isomorphism on homology for all (twisted) coefficients has two further steps. First consider the subgroup $\mathbf{D}_0(Z, \text{rel } \partial Z \cup Z^S)$ of $\mathbf{E}_0(Z, \text{rel } \partial Z \cup Z^S)$, which consists of all elements of zero flux, and let $\mathbf{D}_0'$ be the intersection $\mathbf{E}_0^c(Z, \text{rel } \partial Z) \cap \mathbf{D}_0(Z, \text{rel } \partial Z \cup Z^S)$. Then there is a commutative diagram of fibrations:

$$\overline{\mathbf{B}}\,\mathbf{D}_0' \to \overline{\mathbf{B}}\,\mathbf{D}_0(Z, \text{rel } \partial Z \cup Z^S) \to \overline{\mathbf{B}}\,\mathbf{E}_0^+(Z, \text{rel } \partial Z \cup Z^S)$$

$$\downarrow f' \qquad\qquad \downarrow \tilde{f} \qquad\qquad \downarrow \tilde{f}^+$$

$$S' \to \tilde{S}_\omega(Z, \text{rel } \partial Z \cup Z^S) \to \tilde{S}_\omega^+(Z, \text{rel } \partial Z),$$

where S' is the pull-back to $S_{\omega 0}^c(Z, \text{rel } \partial Z)$ of the covering $\tilde{S}_\omega(Z, \text{rel } \partial Z \cup Z^S)$ of $S_\omega(Z, \text{rel } \partial Z \cup Z^S)$. It follows as before that f' is a $\mathbf{Z}$-homology equivalence.

Note secondly that, because $\pi_{n+1} \overline{B} \Gamma^n_{sl} = 0$, obstruction theory implies that $\pi_1 S^c_\omega (Z, \text{rel } \partial Z)$ is isomorphic to $H^{n-1}_c (Z, \partial Z; \mathbf{R})$ and so is abelian. Therefore $\pi_1 S'$ is also abelian. Hence, because $H_1 \overline{B} \mathbf{D}^c_0 (Z, \text{rel } \partial Z) = 0$ [16], there is a commutative diagram:

$$
\begin{array}{ccccc}
\overline{B} \mathbf{D}^c_0 (Z, \text{rel } \partial Z) & \to & \overline{B} \mathbf{D}'_0 & \overset{\Phi}{\to} & \overline{B} \mathbf{V}' \\
\downarrow \tilde{f} & & \downarrow f' & & \| \\
\tilde{S} & \longrightarrow & S' & \longrightarrow & \overline{B} \mathbf{V}'
\end{array}
$$

where $\tilde{S}$ is the universal cover of S' and $\mathbf{V}' = \pi_1 S'$. Both rows here are fibrations. Clearly it will suffice to show that they are both simple.

This is true for the top row because $\mathbf{V}'$ is the flux in $\mathbf{E}^c_0 (Z, \text{rel } \partial Z)$ which comes from the end Z^+. More precisely, let $B = Z^+ \times (\lambda, \infty)$. Then one can check that $\mathbf{E}^c_0 (B, \text{rel } \partial B)$ is is contained in $\mathbf{D}'_0 (Z, \text{rel } \partial Z)$ and maps onto $\mathbf{V}'$. It follows easily that the top row is simple. A similar argument works for the bottom row. $\square$

7. The case $n = 2$

The above proof of Theorem 1.1 no longer works when $n = 2$ since the results of [10] are not valid. In particular, it is not known whether $\overline{B} \mathscr{D}iff_\omega(\mathbf{R}^2, \text{rel } D^2)$ is acyclic. However, because the monoids behave much as before once they have been given a preliminary localization at infinity, our previous results on groups of compactly supported diffeomorphisms still hold.

We will begin with the easiest case, proving Theorem 1.2 for $\mathbf{R}^2$. This proof is almost elementary, in that it uses the hard localization lemma (4.5) only once: see the proof of Proposition 7.1 (i) below. The notation we will use is explained in Definitions 3.1 and 4.3 and in § 5. Further, we will denote the end $X \times \{ \infty \}$ of $X \times \mathbf{R}$ by « + ». Thus, $S^+_\omega (I \times \mathbf{R})$, for example, is the direct limit $\varprojlim_\lambda S_\omega(I \times [\lambda, \infty))$. Also, $\tilde{S}^+_\omega$ is the universal cover of S^+_ω.

PROPOSITION 7.1. — *If* I *is the interval* [0, 1], *then:*

(i) $f^+ : \overline{B} \mathbf{E}^+_0 (I \times \mathbf{R}) \to S^+_\omega (I \times \mathbf{R}) \simeq \overline{B} \Gamma^2_{sl}$,

(ii) $\tilde{f}^+ : \overline{B} \mathbf{D}^+_0 (I \times \mathbf{R}, \text{rel } \partial I \times \mathbf{R}) \to \tilde{S}^+_\omega (I \times \mathbf{R}, \text{rel } \partial I \times \mathbf{R})$, *and*

(iii) $f : \overline{B} \mathbf{E}_0 (I \times \mathbf{R}, \text{rel } \partial I \times \mathbf{R}) \to S_\omega (I \times \mathbf{R}, \text{rel } \partial I \times \mathbf{R})$,

are $\mathbf{Z}$*-homology equivalences.*

Proof. — Part (i) is proved in § 5, since the whole discussion of diagram (5.1) and the proof that q_1 is a $\mathbf{Z}$-homology equivalence go through when $n = 2$.

Now consider (ii). Let K be the interval [1/2, 1]. We first claim that the inclusion:

$$\overline{B} \mathbf{E}^+_0 (I \times \mathbf{R}, K \times \mathbf{R}, \text{rel } 0 \times \mathbf{R}) \subsetneq \overline{B} \mathbf{E}^+_0 (I \times \mathbf{R}, \text{rel } 0 \times \mathbf{R}),$$

is a (homotopy) equivalence. To see this, note first that the inclusion on the level BE_0^+ is an equivalence by the easy localization lemma 4.11. For one can take s to be the restriction to $(I \times \mathbf{R})^+$ of a diffeomorphism of $I \times \mathbf{R}$ which takes $[\varepsilon, 1] \times \mathbf{R}$ onto $K \times \mathbf{R}$ for some $\varepsilon > 0$, where ε is chosen so that the elements of T all equal the identity on $([0, \varepsilon] \times \mathbf{R})^+$. A similar argument shows that the inclusion of the simplicial monoids E_0^+ is also an equivalence. Thus $\overline{B} E_0^+ (I \times \mathbf{R}, K \times \mathbf{R}, \text{rel } 0 \times \mathbf{R})$ is equivalent to $\overline{B} E_0^+ (I \times \mathbf{R}, \text{rel } 0 \times \mathbf{R})$, as claimed. Since the latter space is contractible as in (3.3), the former is also.

Now consider the diagram:

$$\overline{B} E_0^+ (I \times \mathbf{R}, \text{rel } (0 \cup K) \times \mathbf{R}) \to \overline{B} E_0^+ (I \times \mathbf{R}, K \times \mathbf{R}, \text{rel } 0 \times \mathbf{R}) \to \overline{B} E_0^+ (K \times \mathbf{R})$$
$$\downarrow{\scriptstyle f_1} \qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle f_2} \qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle f^+}$$
$$S_\omega^+ (I \times \mathbf{R}, \text{rel}(0 \cup K) \times \mathbf{R}) \to S_\omega^+ (I \times \mathbf{R}, \text{rel } 0 \times \mathbf{R}) \to S_\omega^+ (K \times \mathbf{R}).$$

Here both rows are simple homology fibration sequences. *See* (2.3) and (3.5). The map f^+ is a **Z**-homology equivalence by (7.1)(i). Moreover, f_2 is a homotopy equivalence since it is a map between contractible spaces. Therefore f_1 is a **Z**-homology equivalence. Now observe that the fibration sequence:

$$\overline{B} D_0^+ (I \times \mathbf{R}, \text{rel } \partial I \times \mathbf{R}) \to \overline{B} E_0^+ (I \times \mathbf{R}, \text{rel } \partial I \times \mathbf{R}) \to \overline{B} V,$$

which is induced by the flux homomorphism, is simple. [compare (6.2).] Since the corresponding result for S_ω also holds, (ii) follows. Finally note that the homology fibre of the quotient map:

$$\overline{B} E_0 (I \times \mathbf{R}, \text{rel } \partial I \times \mathbf{R}) \to \overline{B} E_0^+ (I \times \mathbf{R}, \text{rel } \partial I \times \mathbf{R})$$

is $\overline{B} E_0 (I \times \mathbf{R}, \text{rel } \partial I \times \mathbf{R} \cup (I \times \mathbf{R})^+)$, which is acyclic by Mather's trick [6]. [*See* also (3.3).] Hence (iii) holds. $\square$

THEOREM 7.2. — *The map:*

$$f^c : \quad \overline{B} \mathscr{D}iff^c_{\omega 0} \mathbf{R}^2 \to S^c_{\omega 0} (\mathbf{R}^2) \simeq (\Omega^2 \overline{B} \Gamma^2_{sl})_0,$$

is a homology equivalence.

Proof. — We first show that f^c is a **Z**-homology equivalence. Consider the commutative diagram:

$$\overline{B} E^c (\mathbf{R}^2) \to \overline{B} E_0 (\mathbf{R}^2, \text{rel } H^+) \to \overline{B} E_0^+ (\mathbf{R}^2, \text{rel } H^+)$$
$$\downarrow{\scriptstyle f^c} \qquad\qquad \downarrow{\scriptstyle f} \qquad\qquad \downarrow{\scriptstyle f^+}$$
$$S^c_\omega (\mathbf{R}^2) \to S_\omega (\mathbf{R}^2, \text{rel } H^+) \to S_\omega^+ (\mathbf{R}^2, \text{rel } H^+)$$

where H is the half-plane $\{ x \in \mathbf{R}^2 : x_1 \geq 0 \}$ as in § 5 and $E^c (\mathbf{R}^2) = E_0^c (\mathbf{R}^2)$ is the total singular complex of $\mathscr{D}iff^c_{\omega 0} \mathbf{R}^2$. Since $E_0^+ (\mathbf{R}^2, \text{rel } H^+) \cong D_0^+ (I \times \mathbf{R}, \text{rel } \partial I \times \mathbf{R})$, the map f^+ lifts to the

universal cover $\tilde{S}_\omega^+ (\mathbf{R}^2, \text{ rel } H^+)$ of $S_\omega^+ (\mathbf{R}^2, \text{ rel } H^+)$. Further, $S_\omega (\mathbf{R}^2, \text{rel } H^+)$ is contractible. Therefore, there is an induced diagram:

$$
\begin{array}{ccccc}
\overline{B}\, E_0^c\,(\mathbf{R}^2) & \to & \overline{B}\, E_0\,(\mathbf{R}^2, \text{ rel } H^+) & \to & \overline{B}\, E_0^+\,(\mathbf{R}^2, \text{ rel } H^+) \\
\downarrow f^c & & \downarrow f & & \downarrow \tilde{f}^+ \\
S_{\omega 0}^c\,(\mathbf{R}^2) & \to & S_\omega\,(\mathbf{R}^2, \text{ rel } H^+) & \to & \tilde{S}_\omega^+\,(\mathbf{R}^2, \text{ rel } H^+).
\end{array}
$$

Since the top row comes from an exact sequence of groups, it is a fibration sequence. Moreover, because the elements of $E_0^+ (\mathbf{R}^2, \text{ rel } H^+)$ may be lifted to elements of $E_0 (\mathbf{R}^2, \text{ rel } H^+)$ which are the identity on any given compact set, one can easily check that this sequence is simple. Now observe that the spaces $S_\omega (\mathbf{R}^2, \text{ rel } H^+)$ and $|E_0 (\mathbf{R}^2, \text{ rel } H^+)|$ are contractible. Further, $BE_0 (\mathbf{R}^2, \text{ rel } H^+)$ is acyclic by Mather's trick [6]. Therefore f is a $\mathbf{Z}$-homology equivalence. But $\tilde{f}^+$ is a $\mathbf{Z}$-homology equivalence by Proposition 7.1 (ii). Therefore f^c is one too.

Recall from [1], II.4.3 that there is a surjective homomorphism:

$$\rho : \quad \text{Diff}^c_{\omega 0}\, \mathbf{R}^2 \to \mathbf{R}^\delta,$$

with kernel $\text{Diff}^{c\rho}_{\omega 0}\, \mathbf{R}^2$, say. Consider the associated fibration sequence:

$$\overline{B}\, \mathscr{D}\!iff^{c\rho}_{\omega 0}\, \mathbf{R}^2 \to \overline{B}\, \mathscr{D}\!iff^c_{\omega 0}\, \mathbf{R}^2 \to \overline{B}\, \mathscr{V},$$

where $\mathscr{V} = \mathbf{R}$. It is simple, because the elements of $\mathscr{V}$ may be lifted to diffeomorphisms which are the identity on any given compact set. Moreover $H_1 (\overline{B}\, \mathscr{D}\!iff^{c\rho}_{\omega 0}\, \mathbf{R}^2) = 0$ because $\text{Diff}^{c\rho}_{\omega 0}\, \mathbf{R}^2$ is perfect by [1] and $\mathscr{D}\!iff^{c\rho}_{\omega 0}\, \mathbf{R}^2$ is contractible. (In fact, $\mathscr{D}\!iff^{c\rho}_\omega\, \mathbf{R}^2$ has the same homotopy type as $\mathscr{D}\!iff^c\, \mathbf{R}^2$ by [5], and this is contractible by [14].) It follows that there is a commutative diagram:

$$
\begin{array}{ccccc}
\overline{B}\, \mathscr{D}\!iff^{c\rho}_{\omega 0}\, \mathbf{R}^2 & \to & \overline{B}\, \mathscr{D}\!iff^c_{\omega 0}\, \mathbf{R}^2 & \to & \overline{B}\, \mathscr{V} \\
\downarrow \tilde{f}^c & & \downarrow f^c & & \| \\
\widetilde{(\Omega^2\, \overline{B}\, \Gamma^2_{sl})_0} & \to & (\Omega^2\, \overline{B}\, \Gamma^2_{sl})_0 & \to & \overline{B}\, \mathscr{V}.
\end{array}
$$

Since the bottom row is a simple fibration sequence, $\tilde{f}^c$ is a homology equivalence. This completes the proof of Theorem 7.2 and of the case $n = 2$ of Corollary 1.3. $\square$

Our next aim is to prove the following version of Theorem 2.1.

THEOREM 7.3. — *Let* (W, A) *be an admissible pair of dimension 2, and let* X *be a nice submanifold of* W *which is disjoint from* A *and is such that every end of* W *is intersected either by* A *or by* X. *Then the map:*

$$f : \quad \overline{B}\, E_0 (W, X^J, \text{ rel } A) \to S_\omega (W, \text{ rel } A),$$

is a homology equivalence.

Here J is the set of ends of W as in § 4. The proof will use the following localization lemma.

PROPOSITION 7.4. — *Let* W, X *and* A *be as in Theorem* 7.3 *and suppose that* Y *is a nice submanifold of* W *which is disjoint from* X ∪ A. *Then the inclusion:*

$$i_Y : \quad \overline{B}\, E_0\,(W, X^J, Y, \text{rel } A) \hookrightarrow \overline{B}\, E_0\,(W, X^J, \text{rel } A),$$

is a **Z**-*homology equivalence.*

Proof. — First observe that it suffices to prove that the inclusion:

$$\overline{B}\, D_0\,(W, X^J, Y, \text{rel } A) \hookrightarrow \overline{B}\, D_0\,(W, X^J, \text{rel } A),$$

is a **Z**-homology equivalence. To see this one modifies the proof of (4.2). Here one cannot now assume that the L_i are disjoint. However, one can ensure that they do not meet $A \cup \partial X \cup \partial Y$, and this is enough to show that $(\star)$ is a homotopy fibration sequence.

Next, by arguing as in (4.7), one sees that it suffices to show that the inclusion at infinity:

$$i_Y^J : \quad \overline{B}\, D_0^J\,(W, X, Y, \text{rel } A) \hookrightarrow \overline{B}\, D_0^J\,(W, X, \text{rel } A),$$

is a **Z**-homology equivalence. Without loss of generality, we may assume that Y has one end. Consider the commutative diagram:

$$
\begin{array}{ccccc}
\overline{B}\, D_0^J\,(W, Y, \text{rel } A \cup X) & \to & \overline{B}\, D_0^J\,(W, X, Y, \text{rel } A) & \to & \overline{B}\, D_0^J\,(X, \text{rel } A) \\
\downarrow {\scriptstyle i_1} & & \downarrow {\scriptstyle i_Y^J} & & \| \\
\overline{B}\, D_0^J\,(W, \text{rel } A \cup X) & \to & \overline{B}\, D_0^J\,(W, X, \text{rel } A) & \to & \overline{B}\, D_0^J\,(X, \text{rel } A).
\end{array}
$$

Here both rows are simple homology fibration sequences in the usual way. Therefore i_Y^J will be a **Z**-homology equivalence provided that i_1 is. But, because A ∪ X intersects each end of W and because Y has only one end, this amounts to proving that the inclusion:

$$\overline{B}\, D_0^+\,(I \times \mathbf{R}, K \times \mathbf{R}, \text{rel } \partial I \times \mathbf{R}) \to \overline{B}\, D_0^+\,(I \times \mathbf{R}, \text{rel } \partial I \times \mathbf{R}),$$

is a **Z**-homology equivalence, where K is an interval in Int I. As in Proposition 7.1, this may be done using Lemma 4.11. □

(7.5) PROOF OF THEOREM 7.3

First, we use the method of § 2 to show that f is a **Z**-homology equivalence. The following lemma replaces Lemma 2.11 (b). It is an immediate consequence of Proposition 7.4.

LEMMA 7.6. — *Let* (W, Y) *and* (Y, A) *be admissible pairs, and let* X *be a nice submanifold of* W *which is disjoint from* A ∪ Y *and is such that* X ∪ (A ∩ Y) *intersects every end of* W. *Then if the map* f *of* (7.3) *is a* **Z**-*homology equivalence for* $\overline{B}\, E_0\,(W, X^J, \text{rel } B)$ *when* B *is* Y, A *and either* Y ∪ A *or* Y ∩ A, *then it is one for all four choices of* B.

Note that this lemma is identical to (2.11) (b) if A ∩ Y intersects every end of W since then we may take X to be empty. One can now check that the proof of Lemma 2.14 goes through for pairs (W, A) which have a handle decomposition:

$$A \cap W^0 = X_0^0 \subset X_1^0 \subset \dots \subset X_p^0 = W^0,$$

with the property that each component of the ∂X_i^0 intersects ∂W^0. This condition on the pair (W, A) is stronger than requiring A to intersect each end of W. However, one can always satisfy it by adding spikes to A. Notice that the procedure for altering this handle decomposition into the decomposition:

$$A = X_0 \subset X_1 \subset \ldots \subset X_p = Z,$$

works just as before. *See* figure. Thus:

$$\mathbf{E}_0(W, \text{rel } Y) \cong \mathbf{E}_0(I \times \mathbf{R}, \text{rel } \partial I \times \mathbf{R}), \quad \text{and} \quad \mathbf{E}_0(W, \text{rel } Y \cup B),$$

is the product of two copies of $\mathbf{E}_0(I \times [0, \infty), \text{rel } I \times 0 \cup \partial I \times [0, \infty))$. Therefore, f is a **Z**-homology equivalence for the pairs (W, Y) and (W, Y $\cup$ B) by Proposition 7.1 (iii) and by (2.7)(ii), and so the inductive argument of (2.14) goes through. One now shows that f is a **Z**-homology equivalence for arbitrary (W, A) and appropriate X as in (2.15).

The proof that f is a homology equivalence may be completed by the argument of (6.2). $\square$
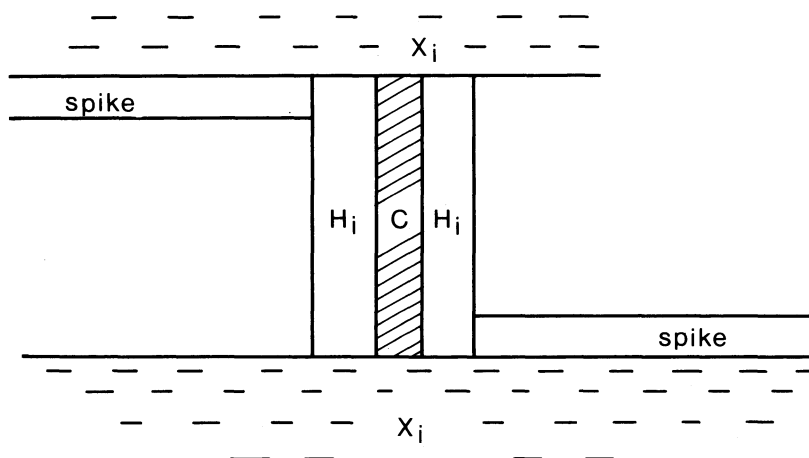
Finally, we prove the following generalization of Theorem 1.2.

THEOREM 7.7. — *If* (W, A) *is an admissible pair of dimension* 2, *then the map*:

$$f^c : \quad \overline{\mathbf{B}} \, \mathbf{E}_0^c(W, \text{rel } A) \to S_{\omega 0}^c(W, \text{rel } A),$$

is a homology equivalence. Further $\pi_1(S_{\omega 0}^c(W, \text{rel } A))$ *is nilpotent.*

Proof. — We use the method of (6.3). Observe that we may assume that A intersects every end of W. For, if A' is a half infinite arc in W disjoint from A, the groups:

$$\mathbf{E}_0^c(W, \text{rel } A \cup A') \quad \text{and} \quad \mathbf{E}_0^c(W, \text{rel } A)$$

are isomorphic because there is an ω-preserving diffeomorphism of $W - (A \cup A')$ onto

$W - A$. Therefore, in (6.3), we may assume that every end of Z has non-empty boundary. It follows that:

$$f^+ : \overline{B} \, E_0^+ (Z, \text{rel } \partial Z \cup Z^S) \to \tilde{S}_\omega^+ (Z, \text{rel } \partial Z)$$

is a homology equivalence by Proposition 7.2 (ii). This assumption also implies that the map f is a $\mathbf{Z}$-homology equivalence by Theorem 7.3. Thus the proof that f^c is a $\mathbf{Z}$-homology equivalence goes through without change.

To proceed further, we must analyse the fundamental groups of $\overline{B} \, E_0^c(\mathring{Z})$ and $S_{\omega 0}^c(\mathring{Z})$. (Here $\mathring{Z}$ denotes the interior $Z - \partial Z$ of Z.) Now the group $E_0^c(\mathring{Z})$ is contractible (*see* 7.2). Therefore $\pi_1(\overline{B} \, E_0^c(\mathring{Z})) \cong E_0^c(\mathring{Z})$. Let:

$$\Phi : \quad E_0^c(\mathring{Z}) \to H_c^1(\mathring{Z}; \mathbf{R})$$

be the flux homomorphism. Its kernel $D_0^c(\mathring{Z})$ surjects onto $\mathbf{R}$ by a homomorphism ρ, whose kernel $D_0^{cp}(\mathring{Z})$ is simple by [1]. Rousseau shows in [12] that the quotient $E_0^c(\mathring{Z})/D_0^{cp}(\mathring{Z})$ is a central extension of $H_c^1(\mathring{Z}; \mathbf{R})$ by $\mathbf{R}$. Moreover this extension is trivial if and only if all products $a.b$ of a, $b \in H_c^1(\mathring{Z}; \mathbf{R})$ vanish in $H_c^2(\mathring{Z}; \mathbf{R})$.

As in (6.1), one can show that there is a commutative diagram:

$$\begin{array}{ccc} \overline{B} \, E_0^c(\mathring{Z}) & \xrightarrow{\Phi} & \overline{B} \, V \\ \downarrow{\scriptstyle f^c} & & \| \\ S_{\omega 0}^c(\mathring{Z}) & \xrightarrow{\theta} & \overline{B} \, V \end{array}$$

where V is the simplicial group corresponding to $H_c^1(\mathring{Z}; \mathbf{R})$. The map θ need not induce an isomorphism on π_1. To see this, let $\overline{Z}$ be a compact manifold with boundary which is a deformation retract of Z, and choose a cell decomposition of $\overline{Z}$ with one vertex and one 2-cell D. Then $H_1(\overline{Z}_1, \partial \overline{Z}) \cong H_1(Z, \partial Z)$, where $\overline{Z}_1$ is the 1-skeleton of $\overline{Z}$. There is a fibration sequence:

$$S_{\omega 0}(D, \text{rel } \partial D) \hookrightarrow S_{\omega 0}(\overline{Z}, \text{rel } \partial \overline{Z}) \xrightarrow{r} S_{\omega 0} | \overline{Z}_1,$$

where r restricts the sections of $S_{\omega 0}(\overline{Z}, \text{rel } \partial \overline{Z})$ to $\overline{Z}_1$. Since $S_{\omega 0}(\overline{Z}, \text{rel } \partial \overline{Z}) \cong S_{\omega 0}^c(\mathring{Z})$, this gives rise to an exact sequence:

$$\pi_1(S_{\omega 0}(D, \text{rel } D)) \to \pi_1 S_{\omega 0}^c(\mathring{Z}) \xrightarrow{r_*} \pi_1(S_{\omega 0} | \overline{Z}_1).$$

But $\pi_1(S_{\omega 0} | \overline{Z}_1) \cong H^1(\overline{Z}_1, \partial \overline{Z}; \mathbf{R}) \cong H_c^1(\mathring{Z}; \mathbf{R})$ by obstruction theory. Also it is easy to see that $r_* = \theta_*$. It follows that $\ker \theta_*$ is an image of:

$$\pi_1(S_{\omega 0}(D, \text{rel } \partial D)) \cong \pi_3 \overline{B} \, \Gamma_{sl}^2 \cong \mathbf{R}$$

and so is abelian.

Let S'' be the covering of $S_{\omega 0}^c(\mathring{Z})$ corresponding to $\ker \theta_*$. Then f^c induces a map $f'' : \overline{B} \, D_0^c(\mathring{Z}) \to S''$. As in (6.3) this is a $\mathbf{Z}$-homology equivalence. Therefore, because

$\pi_1 S'' = \ker \theta_*$ is abelian, f'' induces an isomorphism from $H_1 (\overline{B} \, \mathbf{D}_0^c(\mathring{Z}); \mathbf{Z}) \cong D_0^c(\mathring{Z})/D_0^{cp}(\mathring{Z})$ to $\pi_1 S''$. It follows that $\pi_1 (S_{\omega 0}^c(\mathring{Z}))$ is isomorphic to the nilpotent group $E_0^c(\mathring{Z})/D_0^{cp}(\mathring{Z})$.

To complete the proof, we must show that f'' is a homology equivalence. Consider the commutative diagram:

$$
\begin{array}{ccccc}
\overline{B} \, \mathbf{D}_0^{cp}(\mathring{Z}) & \to & \overline{B} \, \mathbf{D}_0^c(\mathring{Z}) & \overset{\rho}{\to} & \overline{B} \, V \\
\downarrow \tilde{f}^c & & \downarrow f'' & & \| \\
\tilde{S} & \longrightarrow & S'' & \longrightarrow & \overline{B} \, V
\end{array}
$$

where V is the simplicial group of $\mathbf{R}$. The rows are fibration sequences, and it suffices to check that they are simple. But this holds because the loops in $\pi_1 \overline{B} \, V$ may be lifted to elements whose support is disjoint from any given compact subset of Z. $\square$

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