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ANALYTICITY OF RELATIVE FUNDAMENTAL SOLUTIONS AND PROJECTIONS FOR LEFT INVARIANT OPERATORS ON THE HEISENBERG GROUP

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1. Introduction

We show that for certain classes of unsolvable, non-hypoelliptic differential operators on the Heisenberg group there exist left (respectively right) inverses modulo the orthogonal projection onto the L^2 nullspace of the operator (resp. the adjoint of the operator). We also show that these relative inverses and the projections preserve analyticity locally.

Let G be the Heisenberg group and let $X_1, X_2, \dots, X_{2n}, T$ be a basis for the Lie algebra $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$ of G with X_1, X_2, \dots, X_{2n} a basis of \mathcal{G}_1 , \mathcal{G}_2 spanned by (T) and $[\mathcal{G}_1, \mathcal{G}_1] = \mathcal{G}_2 =$ the center of \mathcal{G} . A left invariant differential operator L on G is said to be *homogeneous of degree d* if there is a homogeneous non-commutative polynomial p such that $L = p(X_1, X_2, \dots, X_{2n})$. L is *elliptic in the generating directions* if $p(\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_{2n})$ is elliptic on \mathbb{R}^{2n} .

Our main result is the following.

(1.1) THEOREM. — *Let L be a homogeneous, left invariant differential operator on the Heisenberg group G elliptic in the generating directions. Then there are distributions k_1 and k_2 such that:*

$$(1.2) \quad L f * k_1 = f - \Pi_1 f$$

$$(1.3) \quad L(f * k_2) = f - \Pi_2 f$$

for $f \in C_0^\infty(G)$, where Π_1 and Π_2 are orthogonal projections onto the L^2 nullspaces of L and L^ respectively, and $*$ denotes group convolution. Furthermore, the operators $f \rightarrow f * k_i$ and $f \rightarrow \Pi_i f$, $i = 1, 2$, all preserve analyticity locally.*

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COROLLARY. — If $u, f \in C_0^\infty(G)$ and:

$$(1.4) \quad Lu = f \quad \text{in } U,$$

U open, then $u_1 = (I - \Pi_1)u$ is analytic in every open subset of U where f is, and u_1 is again a solution of (1.4).

Proof. — If $Kv = v * k_1$, then:

$$(I - \Pi_1)u = Kf + K(Lu - f).$$

By Theorem 1.1, the right hand side is analytic in U .

Theorem 1.1 was proved by Greiner, Kohn and Stein [4] for the case where $L = \square_b$, the boundary Laplace operator. The analyticity of the projections Π_1 and Π_2 was proved by Geller [2], who also proved the existence of distributions k'_1, k'_2 satisfying (1.2) and (1.3) and preserving local smoothness. The general result was conjectured by Stein [2]. For the general case, Métivier, by the methods in [11], obtained a proof (unpublished). However, our method is a direct reduction to the hypoelliptic case.

A differential operator D is called C^∞ hypoelliptic (resp. analytic hypoelliptic) in U if $Du = f$ in U with f smooth (resp. analytic) in any open subset $V \subset U$ implies u is smooth in V (resp. u is analytic in V if u is smooth in V). Tartakoff [16], [17] and Trèves [18] have shown that for homogeneous left invariant differential operators on the Heisenberg group, analytic hypoellipticity is implied by C^∞ hypoellipticity. For C^∞ hypoellipticity, necessary and sufficient conditions for operators of the above type on groups with dilations have been given by Rockland [14] and Helffer and Nourrigat [6].

2. The self adjoint case

One can easily reduce the proof of Theorem 1.1 to the case where L is self adjoint and of large homogeneous degree d , with $d/2$ even. Indeed, suppose the result is known for $(L^*L)^n = L_1$ and $(LL^*)^n = L_2$. Then there exists k'_2 such that:

$$(2.1) \quad L_2(f * k'_2) = f - \Pi_2 f,$$

since $\ker L_2 = \ker L^*$. Since for any left invariant vector field X we have $X(f * k) = f * Xk$, from (2.1):

$$L(f * L^*(LL^*)^{n-1}k'_2) = f - \Pi_2 f,$$

so that (1.3) follows with $k_2 = L^*(LL^*)^{n-1}k'_2$. Furthermore, if convolution with k'_2 preserves local analyticity, so does convolution with k_2 . (1.2) is obtained similarly from L_1 .

Theorem 1.1 is then a consequence of the following, which is partly based on an idea of Beals and Greiner [1].

(2.2) THEOREM. — Let L be a self adjoint operator satisfying the hypotheses of Theorem 1.1 and of sufficiently high degree divisible by 4. Then there is a closed contour Γ around 0 in \mathbb{C} such that $L_\alpha = L - \alpha(-iT)^{d/2}$ is hypoelliptic for all $\alpha \in \Gamma$. There exist distributions k_α satisfying $L_\alpha k_\alpha = \delta$, with $\alpha \rightarrow \|D^p(f * k_\alpha)\|_{L^\infty}$ bounded on Γ for all multi-indices and any $f \in C_0^\infty(G)$. Hence define:

$$K, S: C_0^\infty(G) \rightarrow C^\infty(G)$$

by:

$$Kf = \frac{1}{2\pi i} \int_\Gamma \alpha^{-1} f * k_\alpha d\alpha,$$

$$Sf = \frac{1}{2\pi i} \int_\Gamma (-iT)^{d/2} f * k_\alpha d\alpha.$$

Then:

$$(2.3) \quad LKf = K * Lf = f - Sf, \quad f \in C_0^\infty(G),$$

and $S = \Pi$, the orthogonal projection onto the L^2 kernel of L . Furthermore, K and Π preserve local analyticity.

The proof of Theorem 2.2 will proceed as follows. First, one must construct the k_α . For this we use the construction given by Métivier [11] for a single operator and check that the k_α vary well with α . The first equality in (2.3) is an immediate consequence of the self adjointness of L and Π , while the second is easily obtained by writing $L = L_\alpha + \alpha(-iT)^{d/2}$. The proof that $S = \Pi$ will be obtained by applying the irreducible unitary representations of G to both operators and then using the Plancherel theorem for G .

Finally, to prove that K and S preserve analyticity, it suffices to obtain local estimates for derivatives of $f * k_\alpha$ independent of α . For this we use the methods of the second author [16], checking that the constants obtained in the L^2 estimates can be chosen independent of α .

3. Unitary representations and the Plancherel formula for G

We summarize some facts about the irreducible unitary representations of G which will be used in the construction of k_α and in the proof that $S = \Pi$. Let $X'_i, X''_i, i = 1, 2, \dots, n$, T be a basis for \mathcal{G} with $[X'_i, X'_j] = \delta_{ij}T$, all other commutators zero. For every $\lambda \in \mathbb{R} - \{0\}$, let π_λ be the irreducible unitary representation of G on $L^2(\mathbb{R}^n)$ defined by:

$$(3.1) \quad \pi_\lambda(x', x'', t) f(u) = e^{i(\text{sgn } \lambda)|\lambda|^{1/2}x'' \cdot u + \lambda t + \lambda x' \cdot x''/2} f(u - |\lambda|^{1/2} x').$$

Here (x', x'', t) are the coordinates given by:

$$(x', x'', t) \leftrightarrow \exp(x' \cdot X' + x'' \cdot X'' + tT),$$

where $x' \cdot X' = \sum_{i=1}^n x'_i X'_i$ and \exp denotes the exponential map.

These induce the following on \mathcal{G} :

$$\begin{aligned}\pi_\lambda(T) &= i\lambda, \\ \pi_\lambda(X_j) &= |\lambda|^{1/2} \frac{\partial}{\partial u_j}, \\ \pi_\lambda(X'_j) &= i \operatorname{sgn} \lambda |\lambda|^{1/2} u_j.\end{aligned}$$

If $\varphi \in C_0^\infty(G)$, let $\pi_\lambda(\varphi)$ be the bounded operator on $L^2(\mathbb{R}^n)$ given by:

$$\pi_\lambda(\varphi) = \int \varphi(g) \pi_\lambda(g^{-1}) du(g),$$

where $du(g) = dx' dx'' dt$ is a Haar measure on G . If $L \in U(\mathcal{G})$, the universal enveloping algebra of \mathcal{G} , then:

$$(3.2) \quad \pi_\lambda(L\varphi) = \pi_\lambda(L) \pi_\lambda(\varphi),$$

where:

$$\pi_\lambda: \mathcal{G} \rightarrow \operatorname{End}(L^2(\mathbb{R}^n))$$

is the corresponding representation of \mathcal{G} .

It will be useful to know the distribution kernel $a_{\varphi, \lambda}(u, v)$ of the operator $\pi_\lambda(\varphi)$. By direct calculation, for $f \in L^2(\mathbb{R}^n)$:

$$\pi_\lambda(\varphi) f(u) = \int \varphi(x', x'', t) e^{-i(\nu_\lambda \cdot u + x_\lambda \cdot \nu_\lambda / 2 + \lambda t)} f(u - x_\lambda) dx' dx'' dt$$

where $x_\lambda = |\lambda|^{1/2} x'$ and $\nu_\lambda = (\operatorname{sgn} \lambda) |\lambda|^{1/2} x''$. Since $dx' dx'' = (\operatorname{sgn} \lambda) |\lambda|^{-n} dx_\lambda dy_\lambda$:

$$\pi_\lambda(\varphi) f(u) = |\lambda|^{-n} \int \varphi_\lambda(x_\lambda, y_\lambda, t) e^{-i(\nu_\lambda \cdot u - x_\lambda \cdot \nu_\lambda / 2 + \lambda t)} f(u - x_\lambda) dx_\lambda dy_\lambda dt$$

where:

$$\varphi_\lambda(x_\lambda, y_\lambda, t) = \varphi(x', x'', t).$$

Hence a simple change of variables shows that:

$$(3.3) \quad a_{\varphi, \lambda}(u, v) = |\lambda|^{-n} \varphi_{\lambda} \hat{\ } \left(u - v, \frac{u + v}{2}, \lambda \right),$$

where $\hat{\ }$ denotes the Euclidean Fourier transform of φ_{λ} in the last two sets of variables. The reader is referred to Métivier [11] for a more detailed account of the above calculation for a more general class of groups.

We shall need two versions of the Plancherel theorem for G . The first is the following equality for $\varphi \in C_0^{\infty}(G)$:

$$(3.4) \quad \varphi(0) = \int_{\mathbb{R} - \{0\}} \text{tr}(\pi_{\lambda}(\varphi)) d\mu(\lambda),$$

where $d\mu(\lambda) = c|\lambda|^n d\lambda$, c constant, and tr denotes trace. The second version of the Plancherel theorem states that π_{λ} extends to a Hilbert space isomorphism:

$$\pi_{\lambda}: L^2(G) \rightarrow L^2(\mathbb{R} - \{0\}, H - S),$$

where $L^2(\mathbb{R} - \{0\}, H - S)$ is the space of all functions F from $\mathbb{R} - \{0\}$ to the space $H - S$ of all Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$ satisfying:

$$\int_{\mathbb{R} - \{0\}} \text{tr}(F(\lambda) F(\lambda)^*) d\lambda < \infty.$$

The norm is:

$$\|F(\lambda)\|_{L^2(\mathbb{R} - \{0\}, H - S)}^2 = \int_{\mathbb{R} - \{0\}} \text{tr}(F(\lambda) F(\lambda)^*) d\mu'(\lambda),$$

where $d\mu'(\lambda) = c'|\lambda|^n d\lambda$, where c' is a constant. In particular, for $f, g \in L^2(G)$;

$$(3.5) \quad (f, g)_{L^2} = \int_{\mathbb{R} - \{0\}} \text{tr}(\pi_{\lambda}(f) \pi_{\lambda}(g)^*) d\mu'(\lambda).$$

The reader is referred to Kirillov [8] or Pukanszky [13] for a complete account of the Plancherel theorem for nilpotent groups.

4. Construction of the fundamental solutions k_{α} of L_{α}

(4.1) LEMMA. — Let L be as in Theorem 2.2, and let $L_{\alpha} = L - \alpha(-iT)^{d/2}$. Then if $\varepsilon > 0$ is sufficiently small, L_{α} is hypoelliptic for all α , $\varepsilon \leq |\alpha| \leq 2\varepsilon$.

Proof. — By Rockland [14], L_α is hypoelliptic if and only if $\pi_1(L_\alpha)$ and $\pi_{-1}(L_\alpha)$ are injective on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Now:

$$\pi_{\pm 1}(L_\alpha) = \pi_{\pm 1}(L) \pm \alpha.$$

By Grušin [5], the eigenvalues of $\pi_{\pm 1}(L)$ are discrete, and so if:

$$\mu = \min_{\substack{\sigma \text{ eigenvalue of } \pi_{\pm 1}(L) \\ \sigma \neq 0}} |\sigma|,$$

then any $\varepsilon < \mu/2$ will satisfy the lemma.

A family σ_α of distributions on a manifold M will be called *uniformly bounded* if for every compact set $K \subset M$ there exist C and M independent of α such that:

$$|\sigma_\alpha(\varphi)| \leq C \sup_{|\beta| \leq M} |D^\beta \varphi(x)|$$

for all $\varphi \in C_0^\infty(K)$. Our proof of analyticity requires that the k_α be uniformly bounded.

(4.2) PROPOSITION. — *Let $\varepsilon > 0$ be chosen as in Lemma 4.1. If d is sufficiently large and $d/2$ even, there is a uniformly bounded family of fundamental solutions k_α , $L_\alpha k_\alpha = \delta$, all $\alpha \in \mathbb{C}$, $\varepsilon \leq |\alpha| \leq 2\varepsilon$, such that $S_\alpha: C^\infty(G) \rightarrow C^\infty(G)$ defined by:*

$$S_\alpha \varphi = (-iT)^{d/2} (\varphi \star k_\alpha),$$

extends to a bounded mapping of $L^2(G)$ into itself satisfying the following:

$$(4.3) \quad \|S_\alpha \varphi\|_{L^2} \leq C \|\varphi\|_{L^2}$$

C independent of α , and

$$(4.4) \quad \pi_\lambda(S_\alpha \varphi) = \begin{cases} \pi_1(L_\alpha)^{-1} \pi_\lambda(\varphi), & \lambda > 0 \\ \pi_{-1}(L_\alpha)^{-1} \pi_\lambda(\varphi), & \lambda < 0 \end{cases}$$

for almost all $\lambda \in \mathbb{R} - \{0\}$.

To prove Proposition 4.2 we shall follow a similar construction in Métivier [11] (where one of the ideas is attributed to Lion [9]), keeping track of the dependence on α . We let $B_\varepsilon = \{\alpha \in \mathbb{C}: \varepsilon \leq |\alpha| \leq 2\varepsilon\}$.

(4.5) LEMMA. — *For $\alpha \in B_\varepsilon$, let $I_{\lambda, \alpha}(u, v)$ be the distribution kernel of the operator $\pi_\lambda(L_\alpha)^{-1}: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$. Then if d is sufficiently large $I_{\lambda, \alpha}$ is continuous and satisfies, for $n/2 < k < d - n/2$;*

$$(4.6) \quad |I_{\varepsilon_\lambda, \alpha}(v, u)| \leq C_\lambda (1 + |v|)^{n/2 - d + k} (1 + |u|)^{n/2 - k}$$

for some constant C independent of $\alpha \in B_\varepsilon$, where $\varepsilon_\lambda = (-1)^{\text{sgn } \lambda}$.

Proof. — The estimate (4.6) is given in [11] for α fixed in B_ϵ . Let

$$H^k = \{f \in L^2(\mathbb{R}^n) : u^\beta D_u^\gamma f \in L^2, \quad \text{all } |\beta| + |\gamma| \leq k\}$$

with norm $\|f\|_{H^k}^2 = \sum_{|\beta|+|\gamma| \leq k} \|u^\beta D_u^\gamma f\|_{L^2}^2$, and let H^{-k} be the dual space. By Grusin [5], each $\pi_{\pm 1}(L_\alpha)^{-1}$ is bounded from L^2 to H^d i. e., there exists C' such that:

$$(4.7) \quad \|f\|_{H^d} \leq C' \|\pi_{\pm 1}(L_\alpha) f\|_{L^2}$$

with C' dependent on α . An easy perturbation argument (see [15], for details) shows that one can choose C' independent of α for α varying in B_ϵ . From (4.7) one obtains for each j

$$\|f\|_{H_j} \leq C_0 \|\pi_{\pm 1}(L_\alpha)^{-1} f\|_{H^{d+j}}$$

for all $j \in \mathbb{Z}$, as in [11], Lemma 12.

Now the proof is exactly as given in [11].

Proof of Proposition 4.2. — We shall follow the construction given in [11] for a more general class of groups. Put $\check{\varphi}(g) = \varphi(g^{-1})$, $\varphi \in C_0^\infty(G)$. First

$$\text{tr}(\pi_\lambda(L_\alpha)^{-1} \pi_\lambda(\check{\varphi})) = \int I_{\lambda, \alpha}(v, u) \alpha_{\check{\varphi}, \lambda}(u, v) dv du,$$

for $\alpha \in B_\epsilon$, where $\alpha_{\check{\varphi}, \lambda}(u, v)$ is the kernel of $\pi_\lambda(\check{\varphi})$, which by (3.3) is given by

$$(4.8) \quad \alpha_{\check{\varphi}, \lambda}(u, v) = |\lambda|^{-n} (\varphi_\lambda)^\wedge(v - u, -(u + v)/2, -\lambda).$$

Putting

$$J_{\lambda, \alpha}(u, v) = \int e^{-iv \cdot \xi} I_{\lambda, \alpha}\left(\frac{u}{2} - \xi, -\frac{u}{2} - \xi\right) d\xi$$

we obtain

$$\text{tr}(\pi_\lambda(L_\alpha)^{-1} \pi_\lambda(\check{\varphi})) = |\lambda|^{-n} \int J_{\lambda, \alpha}(u, v) \varphi_\lambda^\wedge(u, v, -\lambda) du dv,$$

where φ_λ^\wedge denotes the Fourier transform in the t variable. Finally, let $u_\lambda = |\lambda|^{-1/2} u$, $v_\lambda = (\text{sgn } \lambda) |\lambda|^{-1/2} v$, $u^\lambda = |\lambda|^{1/2} u$, $v^\lambda = (\text{sgn } \lambda) |\lambda|^{1/2} v$ and put

$$K_{\lambda, \alpha}(u_\lambda, v_\lambda) = J_{\lambda, \alpha}(u, v).$$

In view of (3.2) and (3.4) we want to estimate $\text{tr}(\pi_\lambda(L_\alpha)^{-1} \pi_\lambda(\check{\varphi}))$. By the above we have

$$\text{tr}(\pi_\lambda(L_\alpha)^{-1} \pi_\lambda(\check{\varphi})) = \int K_{\lambda, \alpha}(u_\lambda, v_\lambda) \hat{\varphi}(u_\lambda, v_\lambda, -\lambda) du_\lambda dv_\lambda$$

by the definition of φ_λ . It is easy to check that

$$K_{\lambda, \alpha}(u, v) = |\lambda|^{-d/2} K_{\varepsilon_\lambda, \alpha}(u^\lambda, v^\lambda).$$

We shall need to show

$$(4.9) \quad |K_{\varepsilon_\lambda, \alpha}(u^\lambda, v^\lambda)| \leq C,$$

all $\alpha \in B_\varepsilon$, u, v . By definition

$$K_{\varepsilon_\lambda, \alpha}(u^\lambda, v^\lambda) = J_{\varepsilon_\lambda, \alpha}(u^\lambda, \pm v^\lambda) = \int e^{\mp i v^\lambda \xi} I_{\varepsilon_\lambda, \alpha} \left(\frac{u^\lambda}{2} - \xi, \frac{-u^\lambda}{2} - \xi \right) d\xi.$$

Hence by (4.6)

$$(4.10) \quad |K_{\varepsilon_\lambda, \alpha}(u^\lambda, v^\lambda)| \leq C_0 \sup(1 + |\xi|)^{n+1} \left(1 + \left| \frac{u^\lambda}{2} - \xi \right| \right)^{n/2 - d + k} \left(1 + \left| -\frac{u^\lambda}{2} - \xi \right| \right)^{n/2 - k}$$

for $n/2 < k < d - n/2$. Choose k to be the smallest integer larger than $(3n/2) + 1$. Then for $d > 3n + 4$, k is in the range $n/2 < k < d - n/2$. Now for any $a \in \mathbb{R}$

$$(4.11) \quad (1 + |\xi|) \leq \sup((1 + |a - \xi|), (1 + |-a - \xi|)).$$

Then (4.11), together with (4.10), proves (4.9).

Hence, if $\chi(u, v, \lambda) \in \mathcal{S}(\mathbb{R}^{2n+1})$, with $\chi(u, v, \lambda)$ vanishing in λ to order at least $d/2 - n$ at $\lambda = 0$, the integral

$$\int |K_{\lambda, \alpha}(u, v)| |\chi(u, v, \lambda)| du dv d\lambda$$

exists and is bounded, independent of $\alpha \in B_\varepsilon$.

To handle the singularity near $\lambda = 0$ we proceed as in [11]. Let $\psi \in C_0^\infty(\mathbb{R})$ be chosen with

$$\psi(\lambda) = \begin{cases} 1 & \text{for } |\lambda| \leq 1/2, \\ 0 & \text{for } |\lambda| \geq 1. \end{cases}$$

Let $\mathcal{X}: \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$ be defined by

$$\mathcal{X}f(\lambda) = f(\lambda) - \psi(\lambda) \sum_{k \leq (d/2) - n} \frac{\lambda^k}{k!} \left(\frac{\partial}{\partial \lambda} \right)^k f(0).$$

Then

$$\mathcal{X}f(\lambda) = f(\lambda) \quad \text{for } |\lambda| \geq 1,$$

and

$\mathcal{X}f(\lambda)$ vanishes to order $d/2 - n$ at $\lambda = 0$.

Now define $k_{\alpha, 1}$ by:

$$k_{\alpha, 1}(\varphi) = c \int K_{\lambda, \alpha}(u, v, \lambda) \mathcal{X}\varphi(u, v, -\lambda) |\lambda|^n du dv d\lambda,$$

with c as in (3.4). Then $k_{\alpha, 1}$ is a uniformly bounded family of distributions for $\alpha \in B_\epsilon$.

We will now construct a uniformly bounded family $k_{\alpha, 2}$ of distributions such that $k_{\alpha, 1} + k_{\alpha, 2}$ is a fundamental solution for L_α , i. e.,

$$L_\alpha(k_{\alpha, 1} + k_{\alpha, 2}) = \delta.$$

For this, let r_α be the distribution defined by

$$-r_\alpha = L_\alpha k_{\alpha, 1} - \delta.$$

Clearly r_α is a uniformly bounded family, and we must find $k_{\alpha, 2}$ satisfying

$$L_\alpha k_{\alpha, 2} = r_\alpha.$$

As in [11] we note that since $T^s r_\alpha = 0$ for $s \geq d/2 - n$ we may write r_α in the form:

$$r_\alpha = \sum_{|j| \leq d/2 - n} r_{\alpha, j}(x', x'') t^j$$

where $r_{\alpha, j}(x', x'')$ is a uniformly bounded family of distributions on \mathbb{R}^{2n} . Let L^0 be the constant coefficient differential operator elliptic in \mathbb{R}^{2n} , corresponding to the principal symbol at 0 (in the classical sense) of L_α [i. e., $L^0 = p(\partial/\partial x'_1, \dots, \partial/\partial x'_n, \partial/\partial x''_1, \dots, \partial/\partial x''_n)$. See the definition of "elliptic in the generating directions" on the first page. Note that L^0 is independent of α , since the parameter occurs as a coefficient of a term of lower degree.] Now we may seek to find $k_{\alpha, 2}$ in the form

$$k_{\alpha, 2} = \sum_{j \leq d/2 - n} W_{\alpha, j}(x', x'') t^j.$$

The $W_{\alpha, j}$ may be found by downward recursion by writing

$$L = L^0 + \sum_{0 < j \leq d} L_{\alpha, j} \frac{\partial^j}{\partial t^j}$$

and solving recursively for $W_{\alpha, j}$ satisfying

$$(4.12) \quad L^0 W_{\alpha, j}(x', x'') + \sum_{k > j} L_{\alpha, k} \frac{(j+k)!}{j!} W_{\alpha, j+k} = r_{\alpha, j}.$$

with the convention that $W_{\alpha, j+k} = 0$ for $j+k > d/2 - n$. We must still show that (4.12) can be solved with $W_{\alpha, j}$ a uniformly bounded family. For this we use the following modification of [7], Theorem 3.6.4.

(4.13) LEMMA. — *Suppose that $\{f_{\alpha}\}$ is a uniformly bounded family of distributions on an open set $\Omega \subset \mathbb{R}^N$ which is strongly convex for the constant coefficient differential operator $P(D)$. Then there exists a uniformly bounded family u_{α} on Ω such that:*

$$P(D) u_{\alpha} = f_{\alpha}.$$

The proof of Lemma 4.13 is an easy modification of [7], Theorem 3.6.4, where the result is proved for fixed α .

Now we may complete the proof of Proposition 4.2 by verifying (4.3) and (4.4). For this, note that since T is bi-invariant, $(-iT)^{d/2}(\varphi \star k_{\alpha}) = ((-iT)^{d/2} \varphi) \star k_{\alpha}$. One easily sees by the definition of $k_{\alpha, 2}$ that $((-iT)^{d/2} \varphi) \star k_{\alpha, 2} = 0$. Hence

$$S_{\alpha} \varphi = (-iT)^{d/2}(\varphi \star k_{\alpha, 1}).$$

Now

$$\begin{aligned} \pi_{\lambda}(S_{\alpha} \varphi) &= \pi_{\lambda}((-iT)^{d/2} \varphi \star k_{\alpha, 1}) = \pi_{\lambda}(L_{\alpha})^{-1} \pi_{\lambda}(-iT)^{d/2} \pi_{\lambda}(\varphi) \\ &= \lambda^{d/2} |\lambda|^{-d/2} \pi_{\varepsilon_{\lambda}}(L_{\alpha})^{-1} \pi_{\lambda}(\varphi) = \pi_{\varepsilon_{\lambda}}(L_{\alpha})^{-1} \pi_{\lambda}(\varphi), \quad \text{since } d/2 \text{ is even,} \end{aligned}$$

which proves (4.4).

To prove (4.3) it suffices, by (4.4) and the Plancherel Theorem (3.5), to show that

$$\|\pi_{\varepsilon_{\lambda}}(L_{\alpha})^{-1} \pi_{\lambda}(\varphi)\|_{H-S} \leq C' \|\pi_{\lambda}(\varphi)\|_{H-S},$$

where $H-S$ denotes the Hilbert Schmidt norm. This follows immediately since $\pi_{\pm 1}(L_{\alpha})^{-1}$ is bounded on L^2 . This completes the proof of Proposition 4.2.

5. Proof that $S = \Pi$.

(5.1) PROPOSITION. — *Let k_{α} be defined as in Proposition 4.2. Then the operator:*

$$Sf = (2\pi i)^{-1} \int_{\Gamma} (-iT)^{d/2} (f \star k_{\alpha}) d\alpha, \quad f \in C_0^{\infty}(G)$$

extends to a bounded operator on L^2 and $S = \Pi$, the orthogonal projection onto the nullspace of L .

The proof of Proposition 5.1 requires some preliminaries.

(5.2) LEMMA. — *For almost all $\lambda \in \mathbb{R} - \{0\}$, for all $f \in C_0^{\infty}(G)$,*

$$\pi_{\lambda}(\Pi f) = P_{\varepsilon_{\lambda}} \pi_{\lambda}(f)$$

where $P_{\varepsilon_{\lambda}}$ is the orthogonal projection onto the nullspace of $\pi_{\varepsilon_{\lambda}}(L)$.

Proof. — This is very similar to Goodman [3]. First, it is clear that

$$\text{Im } \pi_\lambda(\Pi f) \subset \ker \pi_\lambda(L) = \ker \pi_{\varepsilon_\lambda}(L).$$

Furthermore, if $h \in \ker L$, then $\text{Im } \pi_\lambda(h) \subset \ker \pi_\lambda(L)$ and hence $\pi_\lambda(h) = P_{\varepsilon_\lambda} \pi_\lambda(h)$. Hence it suffices to show that if $g \perp \ker L$, then $\text{Im } \pi_\lambda(g) \subset (\ker \pi_\lambda(L))^\perp$.

Now if $g \perp \ker L$, then by the Plancherel formula (3.5),

$$(5.3) \quad \int \text{tr}(\pi_\lambda(f) \pi_\lambda(g)^*) d\mu(\lambda) = 0$$

for all $f \in \ker L$. Let $\{\varphi_j\}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$ such that $\varphi_1, \varphi_2, \dots, \varphi_N$ is a basis of $\ker \pi_{+1}(L)$ (which is of finite dimension by Grusin [5]), and $\varphi_{N+1}, \varphi_{N+2}, \dots$ is a basis of $(\ker \pi_{+1}(L))^\perp$. Then

$$(5.4) \quad \text{tr}(\pi_\lambda(f) \pi_\lambda(g)^*) = \sum_{i=1}^{\infty} (\pi_\lambda(f) \varphi_i, \pi_\lambda(g) \varphi_i)$$

for any $f \in L^2(G)$. Let the indices i and j be fixed with $1 \leq j \leq N$ and let $c(\lambda) \in L^1(\mathbb{R}^1 - \{0\}, d\mu(\lambda))$ be arbitrary with support in \mathbb{R}^+ . Then by the second version of the Plancherel formula define $h_{ij} \in L^2(G)$ by

$$\pi_\lambda(h_{ij}) \varphi_k = \begin{cases} c(\lambda) \delta_{ik} \varphi_j, & 1 \leq i \leq N, \\ 0, & i > N. \end{cases}$$

Then

$$\text{tr}(\pi_\lambda(h_{ij}) \pi_\lambda(g)^*) = \sum_k (\pi_\lambda(h_{ij}) \varphi_k, \pi_\lambda(g) \varphi_k) = c(\lambda) (\varphi_j, \pi_\lambda(g) \varphi_i).$$

By (5.3), since $h_{ij} \in \ker L$

$$\int c(\lambda) (\varphi_j, \pi_\lambda(g) \varphi_i) d\mu(\lambda) = 0.$$

Since $c(\lambda)$ is arbitrary with support in \mathbb{R}^+ , $(\varphi_j, \pi_\lambda(g) \varphi_i) = 0$ for almost all $\lambda > 0$, all i . The proof for $\lambda < 0$ is the same, obtained by using a basis adapted to the decomposition $\ker \pi_{-1}(L) + (\ker \pi_{-1}(L))^\perp$. Hence $\pi_\lambda(g) \varphi_i \perp \ker \pi_\lambda(L)$ as claimed.

From now on, Γ will denote a fixed simple contour in \mathbb{C} lying in B_ε .

$$(5.5) \text{ LEMMA. — } \pi_\lambda \left(\int_\Gamma S_\alpha f d\alpha \right) = \int_\Gamma \pi_\lambda(S_\alpha f) d\alpha \quad \text{for almost all } \lambda \in \mathbb{R} - \{0\}.$$

Proof. — Suppose $g \in L^2(G)$. Then by the Plancherel Theorem (3.5)

$$\left(\int_\Gamma S_\alpha f d\alpha, g \right) = \int_{\mathbb{R} - \{0\}} \text{tr} \left(\pi_\lambda \left(\int_\Gamma S_\alpha f d\alpha \right) \pi_\lambda(g)^* \right) d\mu(\lambda).$$

Now let $\{\varphi_j\}$ be an orthonormal basis for $L^2(\mathbb{R}^n)$. Then

$$(5.6) \quad \int \operatorname{tr} \left(\int_{\Gamma} \pi_{\lambda}(S_{\alpha} f) d\alpha \pi_{\lambda}(g)^* \right) d\mu(\lambda) = \int \sum_i \left(\int_{\Gamma} \pi_{\lambda}(S_{\alpha} f) d\alpha (\pi_{\lambda}(g))^* \varphi_i, \varphi_i \right) d\mu(\lambda).$$

Now since the infinite sum in the right hand side of (5.6) converges absolutely, by the dominated convergence theorem:

$$(5.7) \quad \begin{aligned} \int \operatorname{tr} \left(\int_{\Gamma} \pi_{\lambda}(S_{\alpha} f) d\alpha \pi_{\lambda}(g)^* \right) d\mu(\lambda) \\ &= \iint_{\Gamma} \operatorname{tr} (\pi_{\lambda}(S_{\alpha} f) \pi_{\lambda}(g)^*) d\alpha d\mu(\lambda) \\ &= \int_{\Gamma} \int_{\mathbb{R} - \{0\}} \operatorname{tr} (\pi_{\lambda}(S_{\alpha} f) \pi_{\lambda}(g)^*) d\mu(\lambda) d\alpha = \int_{\Gamma} (S_{\alpha} f, g) d\alpha \\ &= \left(\int_{\Gamma} S_{\alpha} f d\alpha, g \right) = \int \operatorname{tr} \left(\pi_{\lambda} \left(\int_{\Gamma} S_{\alpha} f d\alpha \right) \pi_{\lambda}(g)^* \right) d\mu(\lambda). \end{aligned}$$

Since g is arbitrary, (3.5) implies

$$\int_{\Gamma} \pi_{\lambda}(S_{\alpha} f) d\alpha = \pi_{\lambda} \left(\int_{\Gamma} S_{\alpha} f d\alpha \right)$$

for almost all λ by the Plancherel Theorem. This proves Lemma 5.5.

We may now prove Proposition 5.1. By Lemma 5.2, it suffices to show that

$$(5.8) \quad \pi_{\lambda}(Sf) = P_{\varepsilon_{\lambda}} \pi_{\lambda}(f) \quad \text{for } \varphi \in C_0^{\infty}(G).$$

By Lemma 5.5 and (4.4)

$$(5.9) \quad \pi_{\lambda}(Sf) = \int_{\Gamma} \pi_{\lambda}(S_{\alpha} f) d\alpha = \int_{\Gamma} \pi_{\varepsilon_{\lambda}}(L_{\alpha})^{-1} \pi_{\lambda}(f) d\alpha.$$

Suppose $\lambda > 0$. Then from (5.9)

$$\pi_{\lambda}(Sf) = (2\pi i)^{-1} \int_{\Gamma} \pi_1(L_{\alpha})^{-1} \pi_{\lambda}(f) d\alpha = (2\pi i)^{-1} \int_{\Gamma} (\pi_1(L) - \alpha)^{-1} \pi_{\lambda}(f) d\alpha.$$

Since zero is an isolated point of the spectrum of $\pi_1(L)$ by [5],

$$(2\pi i)^{-1} \int_{\Gamma} (\pi_1(L) - \alpha)^{-1} d\alpha = P_1.$$

A similar argument holds for $\lambda < 0$. Hence (5.8) is proved.

6. Analyticity

In this part we complete the proof of Theorem 1.1 by proving.

(6.1) THEOREM. — *The operators K and S constructed above preserve local analyticity.*

(6.2) LEMMA. — *Let $u_\alpha = K_\alpha f$, $f \in C_0^\infty(G)$. Suppose that for any bounded open set U_0 in which f is real analytic and any V_0 with compact closure in U_0 there exists a constant C such that*

$$(6.3) \quad \sup_{x \in V_0} |D^\gamma u_\alpha(x)| \leq C^{|\gamma|+1} |\gamma|!$$

for all multi-indices γ and all $|\alpha| = \varepsilon$. Then K and Π preserve local analyticity.

Proof. — Suppose f is analytic in U_0 . Then since $L_\alpha K_\alpha f = L_\alpha(f * k_\alpha) = f$,

$$\sup_{x \in V_0} |D^\gamma (K_\alpha f)(x)| \leq C^{|\gamma|+1} |\gamma|!$$

and hence

$$\sup_{x \in V_0} |D^\gamma \int_\Gamma \alpha^{-1} K_\alpha f \, d\alpha| \leq \int_\Gamma |\alpha|^{-1} \sup_{x \in V_0} |D^\gamma (K_\alpha f)(x)| \, d\alpha \leq C' C^{|\gamma|+1} |\gamma|!$$

Hence Kf is analytic in V_0 . The proof for Π is the same.

We shall now prove (6.3). For this we shall need the maximal estimate

$$(6.4) \quad \|X_{i_1} X_{i_2} \dots X_{i_d} v\|_{L^2} \leq C (\|L_\alpha v\|_{L^2} + \|v\|_{L^2})$$

for all $v \in C_0^\infty(U_0)$, some constant C, which may be chosen independent of α , $|\alpha| = \varepsilon$. For each fixed α , $|\alpha|$ small but nonzero, the estimate (6.4) follows from the hypoellipticity [6] of L_α and is clearly preserved under sufficiently small changes in α on the circle $|\alpha| = \varepsilon$. Hence (6.4) follows by compactness for some C independent of α .

7. Proof of the uniform estimates on high derivatives

To demonstrate local bounds of the form:

$$|D^\beta u_\alpha(x)| \leq C^{|\beta|+1} |\beta|! \quad \forall \beta, x \in V_0$$

it is sufficient to obtain analogous L^2 bounds:

$$\|D^\beta u_\alpha\|_{L^2(V_1)} \leq C_1^{|\beta|+1} |\beta|! \quad \text{with } V_0 \subset\subset V_1$$

and, as Nelson has shown, we may use the vector fields X_i and T instead of ordinary partial derivatives. Thus we write $X_I = X_{i_1} X_{i_2} \dots X_{i_{|I|}}$ (or $X^{|I|}$, abusively, for short) and shall show the bounds

$$\|X_I T^b u_\alpha\|_{L^2(V_1)} \leq C_2^{|I|+b+1} (|I| + b)!$$

for all I and b , uniformly in α for $|\alpha| = \varepsilon$. Equivalently, we show:

$$(7.1) \quad \|X_I T^b u_\alpha\|_{L^2(V_1)} \leq C_3^{|I|+b+1} N^{|I|+b}$$

uniformly in α , $|\alpha| = \varepsilon$, N , I and b subject to $|I| + b \leq N$, since Stirling's formula yields

$$N^N \leq C_4^N N!$$

What follows is an extension of [16], but we feel much easier to read, to $d \geq 2$ with attention given to the dependence of all estimates on α .

Clearly [see the *a priori* estimate (6.4)], estimating T derivatives is harder than estimating X derivatives, though one cannot, it appears, do one without the other. To use (6.4) effectively, we should at each stage try to retain at least d X 's in our expressions, and yet this is no limitation, since high, pure T derivatives can yield the required X 's by use of the commutation relations between the X 's ($d/2$ times) and it is easy to see that if one has the desired bounds for $|I| \geq d$, one also has them (with a different constant) for all smaller I .

To localize high T derivatives is not simple, for $[X_j, \varphi T^p]$ exhibits insufficient gain in p (at most a gain of $1/2$ power, while a whole derivative lands on the localizing function). One could repeatedly replace X derivatives consumed in this fashion, but to do so would eventually transfer the p T -derivatives into derivatives of order $2p$ on φ , and this will not yield analyticity.

To overcome this obstacle, we introduce a rather complicated (looking) localization of T^p , i. e., a differential operator of order p , equal to T^p in any open set where $\varphi = 1$ and zero outside the support of φ . First, however, we must pick a new basis for \mathcal{G}_1 . An analytic change of coordinates allows us to pick the basis:

$$\begin{aligned} X_j &= X'_j = \partial/\partial x_j, & j \leq n, \\ X_{j+n} &= X'_j = \partial/\partial y_j + x_j \partial/\partial t, & j \leq n, \\ T &= \partial/\partial t, \end{aligned}$$

where the X_j still generate \mathcal{G}_1 , and T generates \mathcal{G}_2 .

(7.2) DEFINITION. — Let the X_j , T be defined as above. Then let:

$$(T^p)_\varphi = T_\varphi^p = \sum_{r=|\beta+\gamma| \leq p} \frac{(-1)^{|\beta|}}{\beta! \gamma!} (X'^\beta X''^\gamma \varphi) X'^\gamma X''^\beta T^{p-|\beta+\gamma|}.$$

(7.3) LEMMA. — With T_ϕ^p defined as above, then modulo C^p terms of the form $\varphi^{(p+1)} X^p / \beta! \gamma!$ where $|\beta + \gamma| = p$,

$$\begin{aligned} [X'_j, T_\phi^p] &\equiv 0, \\ [X''_j, T_\phi^p] &\equiv (T^{p-1})_{T_\phi} X''_j. \end{aligned}$$

Proof. — From (7.2) and the obvious commutation relations $[X'_j, X''^\beta] = \beta_j$ terms, each $X''^{\beta-e_j} T$, where e_j is the multi-index of length one whose only non-zero entry is a 1 in the j th position.

$$(7.4) \quad \begin{aligned} [X'_j, T_\phi^p] &= \sum_{r=|\beta+\gamma|\leq p} \frac{(-1)^{|\beta|}}{\beta! \gamma!} (X'_j X'^\beta X''^\gamma \varphi) X'^\gamma X''^\beta T^{p-|\beta+\gamma|} \\ &\quad + \sum_{r=|\beta+\gamma|\leq p} \frac{(-1)^{|\beta|}}{\beta! \gamma!} \beta_j (X'^\beta X''^\gamma \varphi) X'^\gamma X''^{\beta-e_j} T^{p-|\beta-e_j+\gamma|}. \end{aligned}$$

Note that in the second sum, $r \geq 1$, since for $r=0$, all $\beta_j=0$. But each term in the first sum, except those with $r=p$, is cancelled by a term in the second; a term in the first with $\beta=\beta_0$, $\gamma=\gamma_0$ is cancelled by a term in the second when $\beta=\beta_0+e_j$, $\gamma=\gamma_0$ unless $|\beta_0+\gamma_0|=p$. Only terms from the first sum with $r=p$ remain, and there are fewer than $(2n)^p$ of them.

For the second part of the Lemma, a similar cancellation takes place (with a shift of the γ index this time), the change of sign coming not from the power of -1 , as it did with a shift of β , but from the observation that $[X''_j, X'^\gamma X''^\beta]$ consists of γ_j terms each $X'^{\gamma-e_j} X''^\beta [X''_j, X'_j]$ and $[X''_j, X'_j] = -T$. The more significant difference, however, is that in the first term in (7.4) the extra X'_j sits beside the other X' derivatives on φ , with X''_j it will be on the extreme left, while the others will sit beside φ . Thus what was literal cancellation for the first part of the Lemma will be a commutator here. To be precise:

$$\begin{aligned} [X''_j, T_\phi^p] &= \sum_{|\beta+\gamma|=p} \frac{(-1)^{|\beta|}}{\beta! \gamma!} (X''_j X'^\beta X''^\gamma \varphi) X'^\gamma X''^\beta T^0 \\ &\quad + \sum_{r=|\beta+\gamma|\leq p-1} \frac{(-1)^{|\beta|}}{\beta! \gamma!} (X''_j X'^\beta X''^\gamma \varphi) X'^\gamma X''^\beta T^{p-|\beta+\gamma|} \\ &\quad - \sum_{r=|\beta+\gamma|\leq p} \gamma_j \frac{(-1)^{|\beta|}}{\beta! \gamma!} (X'^\beta X''^\gamma \varphi) X'^{\gamma-e_j} X''^\beta T^{p-|\beta+\gamma-e_j|} \end{aligned}$$

The last term may be rewritten, replacing $\gamma-e_j$ by γ , noting that this term is missing when $r=0$ (since then all γ_j are zero):

$$- \sum_{r=|\beta+\gamma|\leq p-1} \frac{(-1)^{|\beta|}}{\beta! \gamma!} (X'^\beta X''^\gamma \varphi) X'^\gamma X''^\beta T^{p-|\beta+\gamma|}$$

so that we have:

$$\begin{aligned}
 [X'_j, T_\Phi] &= \sum_{|\beta+\gamma|=p} \frac{(-1)^{|\beta|}}{\beta! \gamma!} (X'_j X'^\beta X''^\gamma \varphi) X'^\gamma X''^\beta \\
 &\quad + \sum_{r=|\beta+\gamma| \leq p-1} \frac{(-1)^{|\beta|}}{\beta! \gamma!} ([X'_j, X'^\beta] X''^\gamma \varphi) X'^\gamma X''^\beta T^{p-|\beta+\gamma|}
 \end{aligned}$$

The first term above is the same type of error term as was discussed in proving the first part of the Lemma. The second term above may be written as:

$$\begin{aligned}
 - \sum_{r=|\beta+\gamma|=p-1} \frac{(-1)^{|\beta|}}{\beta! \gamma!} \beta_j (X'^{\beta-e_j} X''^\gamma T \varphi) X'^\gamma X''^{\beta-e_j} T^{p-|\gamma+\beta-e_j|-1} \circ X'_j \\
 = \sum_{r=|\beta+\gamma| \leq p-1} \frac{(-1)^{|\beta|}}{\beta! \gamma!} (X'^\beta X''^\gamma T \varphi) X'^\gamma X''^\beta T^{p-1-|\beta+\gamma|} \circ X'_j
 \end{aligned}$$

(replacing $\beta - e_j$ by β). But this last is nothing but $(T^{p-1})_{T_\Phi} \circ X'_j$.

Let N be fixed for now. We nest $[[\log_2 N]]$ open sets:

$$V_1 = W_0 \subset \subset W_1 \subset \subset \dots \subset \subset W_{[[\log_2 N]]} = U_0$$

(where $[[\log_2 N]]$ denotes the integral part of $\log_2 N$), and choose functions ψ_j , φ_j , and χ_j in $C_0^\infty(W_{j+1})$ with $\psi_j = 1$ near \bar{W}_j , $\varphi_j = 1$ near $\text{supp } \psi_j$, and $\chi_j = 1$ near $\text{supp } \varphi_j$ with specified bounds on their derivatives up to order $2N_j$ where

$$N_j = N/2^j.$$

Namely, we choose the W_j in such a way that if $d = \text{dist}(V_1, U_0^{\text{comp}})$, then $d_j = \text{dist}(W_j, W_{j+1}^{\text{comp}}) = d/2^{j+1}$. Then the ψ_j , φ_j and χ_j may be chosen (cf. [16]) so that

$$|D^\gamma(\psi_j, \varphi_j, \text{ or } \chi_j)| \leq (K d_j^{-1})^{|\gamma|} N_j^{|\gamma|} \quad \text{if } |\gamma| \leq 2N_j$$

with K independent of N (but depending on V_1 and U_0). These families of cut-off functions are just dilations of those introduced by Ehrenpreis and used by Hörmander, Andersson, and others.

Since $\psi_0 = 1$ in $V_1 (= W_0)$, for $a + b$ less than N_0 but $a > d$, we estimate

$$\|X^a T^b u_\alpha\|_{L^2(V_1)} \quad \text{by} \quad \|X^d \psi_0 X^{a-d} T^b u_\alpha\|_{L^2}$$

and use the *a priori* estimate (6.4) on this. On the right there will be fewer X 's:

(7.5) PROPOSITION. — *There exists a constant \tilde{C} depending on f but independent of α , $|\alpha| = \varepsilon$ and N so that if $a + b \leq N_0$,*

$$\|X^a T^b u_\alpha\|_{L^2(W_0)} / N_0^{a+b} \leq \tilde{C} N_0 (d_0^{-1})^{N_0} (1 + \sup_{\substack{d+b' \leq N_0 \\ d' \leq d \\ b'-b \leq (a-d)/2}} \|X^{d'} T^{b'} u_\alpha\|_{L^2(\text{supp } \psi_0)} / N_0^{d'+b'}).$$

Proof. — Using (6.4) we have

$$(7.6) \quad \begin{aligned} \|X^d \psi_0 X^{a-d} T^b u_\alpha\|_{L^2} &\leq C(\|L_\alpha \psi_0 X^{a-d} T^b u_\alpha\|_{L^2} + \|\psi_0 X^{a-d} T^b u_\alpha\|_{L^2}) \\ &\leq C(\|\psi_0 X^{a-d} T^b L_\alpha u_\alpha\|_{L^2} + \|\psi_0 X^{a-d} T^b u_\alpha\|_{L^2} \\ &\quad + \sum_{|I|=d} (c'_I + c''_I |\alpha|) \|[X_I, \psi_0 X^{a-d}] T^b u_\alpha\|_{L^2}). \end{aligned}$$

Here we have written (non-uniquely)

$$(7.7) \quad L_\alpha = \sum_{|I|=d} (c'_I X_I + c''_I \alpha X_I)$$

with constants c'_I and c''_I . Next

$$[X^d, \psi_0 X^{a-d}] = [X^d, \psi_0] X^{a-d} + \psi_0 [X^d, X^{a-d}]$$

consists of terms of the form $X^i \psi'_0 X^{a-i-1}$ ($i < d$, one for each i) arising from the first term on the right above and at most d times $a-d$ terms $\psi_0 X^{a-2} T$ from the second. To avoid constantly commuting X 's to the left, we note that for $i < d$:

$$\|X^i \psi_0 X^{a-i} T^b u_\alpha\|_{L^2} \leq \|X^d \psi_0 X^{a-d} T^b u_\alpha\|_{L^2} + \sum_{j=0}^{d-1} \|X^j \psi'_0 X^{a-j-1} T^b u_\alpha\|_{L^2}.$$

Thus we may generalize (7.6) to:

$$(7.8) \quad \begin{aligned} \sum_{i \leq d} \|X^i \psi_0 X^{a-i} T^b u_\alpha\|_{L^2} &\leq C(\|\psi_0 X^{a-d} T^b f\|_{L^2} \\ &\quad + \sum_{i < d} \|X^i \psi'_0 X^{a-i-1} T^b u_\alpha\|_{L^2} + C(a-d) \sum_{i \leq d} \|X^i \psi_0 X^{a-i-2} T^{b+1} u_\alpha\|_{L^2}) \end{aligned}$$

with a new constant C (depending on d , but uniform in $|\alpha| = \varepsilon$), and now X^e may denote any X_I with $|I| \leq e$.

We iterate this process (with a replaced by $a-1$, ψ_0 by ψ'_0 , or with a by $a-2$ and b by $b+1$) on each term which still has at least $d+1$ X 's, [except the first term, of course, since once a term contains $f(x)$, there is no need to iterate further]. One type of term, after a iterations, will be (bounded by)

$$C^a (a-d)^k \sum_{i \leq d} \|X^i \psi_0^{(r)} X^{a-r-2k-i} T^{b+k} u_\alpha\|_{L^2}$$

for some k, r with $a-r-2k \leq d$, and there will be at most $(2d+1)^a$ such terms.

The other terms will all contain f . These, again at most $(2d)^a$ of them, will be of the form

$$C^a (a-d)^k \|\psi_0^{(r)} X^{a-r-2k-d} T^{b+k} f\|_{L^2}$$

In view of the bounds on derivatives of ψ_0 , and the real analyticity of f in U_0 , then, (7.8) yields:

$$(7.9) \quad \sum_{i \leq d} \|X^i \psi_0 X^{a-i} T^b u_\alpha\|_{L^2} \leq C^a \sup_{0 \leq a-2k-r=d' \leq d} a^k (K d_0^{-1})^r N_0^r \|X^{d'} T^{b+k} u_\alpha\|_{L^2(\text{supp } \psi_0)} + C^a \sup_{2k+r \leq a-d} a^k (K d_0^{-1})^r N_0^r K_f^{a-r-k-d+b+1} (a-r-k-d+b)!$$

(The value of C , it should be clear by now, will change from estimate to estimate, but remain uniform in α , $|\alpha|=\varepsilon$ and independent of a , b and N as well as f .) This leads quickly to (7.5) since in the first term on the right in (7.9) we may observe that $a^k N_0^r N_0^{-(a+b)} \leq N_0^{r+k-a-b} \leq N_0^{-d'-b-k}$ if $d'=a-r-2k$ and for the second term on the right in (7.9) we use $a^k N_0^r (a-r-k-d+b)! N_0^{-a-b} \leq N_0^{k+r+a-r-d+b-a-b} \leq 1$. The strings of constants that build up, $C^a K^r$ for the first term in (7.9) and $C^a K^r K_f^{a-r-k-d+b+1}$ for the second, are both bounded by C^{N_0} for a suitable new constant uniform in α , $|\alpha|=\varepsilon$, a , b , and N .

For $d' < d$, further iterations of this type will be useless in proving analyticity, since effective use of (6.4) requires essentially the presence of at least d X 's. Using (7.2), however, we may continue profitably. For we have:

$$(7.10) \quad \|X^{d'} T^{b'} u_\alpha\|_{L^2(\text{supp } \psi_0)} \leq \|X^{d'} (T^{b'})_{\varphi_0} u\|_{L^2}$$

since $\varphi_0=1$ near $\text{supp } \psi_0$, so $(T^{b'})_{\varphi_0} = T^{b'}$ in $\text{supp } \psi_0$.

(7.11) PROPOSITION. — *There exists a constant \tilde{C} depending on f but not on α , $|\alpha|=\varepsilon$ or N so that if $a+b \leq N_0$:*

$$\sup_{\substack{d' \leq d \\ b'+d' \leq N_0 \\ 0 \leq b'-b \leq (a-d)/2}} \|X^{d'} T^{b'} u_\alpha\|_{L^2(\text{supp } \psi_0)} / N_0^{b'+d'} \leq \tilde{C}^{N_0} (d_0^{-1})^{N_0} (1 + \sup_{a'' \leq b+d+(a-d)/2} \|X^{a''} u_\alpha\|_{L^2(\text{supp } \varphi_0)}) / N_1^{a''}$$

Proof. — Since $X_k^* = -X_k$, integration by parts allows us to improve (6.4) by including terms with fewer X derivations on the left:

$$(6.4)' \quad \sum_{d' \leq d} \|X^{d'} v\|_{L^2} \leq C(\|L_\alpha v\|_{L^2} + \|v\|_{L^2}), \quad v \in C_0^\infty(U_0).$$

If we apply this to $v=(T^{b'})_{\varphi_0} u_\alpha$, we obtain, uniformly in α ,

$$(7.12) \quad \sum_{d' \leq d} \|X^{d'} (T^{b'})_{\varphi_0} u_\alpha\|_{L^2} \leq C(\|L_\alpha (T^{b'})_{\varphi_0} u_\alpha\|_{L^2} + \|(T^{b'})_{\varphi_0} u_\alpha\|_{L^2}) \leq C(\|(T^{b'})_{\varphi_0} f\|_{L^2} + \|[L_\alpha, (T^{b'})_{\varphi_0}] u_\alpha\|_{L^2} + \|(T^{b'})_{\varphi_0} u_\alpha\|_{L^2}).$$

The commutator may be expanded using Lemma (7.3) and (7.7):

$$[L_\alpha, (T^{b'})_{\varphi_0}] = \sum_{|I|=d} (c'_I + c''_I \alpha) [X_I, (T^{b'})_{\varphi_0}]$$

and an application of Lemma (7.3) gives:

$$\begin{aligned} (7.13) \quad [X_I, (T^{b'})_{\varphi_0}] &= [X_{i_1} X_{i_2} \dots X_{i_d}, (T^{b'})_{\varphi_0}] \\ &= \sum_{j=0}^{d-1} X_{i_1} \dots X_{i_j} [X_{i_{j+1}}, (T^{b'})_{\varphi_0}] X_{i_{j+2}} \dots X_{i_d} \\ &= -A \sum_{j=0}^{d-1} X_{i_1} \dots X_{i_j} (T^{b'-1})_{T\varphi_0} X_{i_{j+1}} \dots X_{i_d} + C^{b'} \text{ terms } \sum_{j=0}^{d-1} X^j \varphi_0^{(b'+1)} X^{d-j-1+b'}/b'! \end{aligned}$$

where A may be 0 or 1, depending on whether $X_{i_{j+1}}$ is an X' or an X'' . We shall assume that $A=1$ below; when it is zero, that term just doesn't appear. Now to continue to use (6.4) or (6.4)' on the right hand side above we would have to commute all X 's to the left of $(T^{b'-1})_{T\varphi_0}$. This would introduce more terms of the same type, with X 's on both sides. So we choose to estimate generally all divisions of the X 's; i.e., we choose to estimate $\sum_{i \leq d' \leq d} \|X^i (T^{b'})_{\varphi_0} X^{d'-i} u_\alpha\|_{L^2}$. In doing so, we first attempt to bring all X 's to

the left [and then use (7.12)]—the above expansion of the bracket will yield an error which can be estimated by such a *sum* (over $i \leq d'$) but with smaller b' , together with terms free of T altogether and then of course the right hand side of (7.12) followed by another use of (7.13). This gives:

$$\begin{aligned} (7.14) \quad \sum_{i \leq d' \leq d} \|X^i (T^{b'})_{\varphi_0} X^{d'-i} u_\alpha\|_{L^2} &\leq C \left(\sum_{i \leq d' \leq d} \|X^i (T^{b'-1})_{T\varphi_0} X^{d'-i} u_\alpha\|_{L^2} \right. \\ &\quad \left. + \|(T^{b'})_{\varphi_0} f\|_{L^2} + \|(T^{b'})_{\varphi_0} u_\alpha\|_{L^2} + C^{b'} \sum_{j \leq d} \|X^j \varphi_0^{(b'+1)} X^{d-j-1+b'} u_\alpha\|_{L^2} / b'! \right) \end{aligned}$$

We want to iterate this to reduce b' still further. But first we must handle the third term on the right. By the definition;

$$\begin{aligned} (T^{b'})_{\varphi_0} &= (T^{b'-1})_{\varphi_0} T + C^{b'} \text{ terms } (\varphi_0^{(b'+1)} X^{b'}/b'!) \\ &= 2 \text{ terms } (T^{b'-1})_{\varphi_0} X^2 + C^{b'} \text{ terms } (\varphi_0^{(b'+1)} X^{b'}/b'!). \end{aligned}$$

If we (abusively) now write φ'_0 for $T\varphi_0$, $X\varphi_0$, or φ_0 itself, this expansion of $(T^{b'})_{\varphi_0}$ allows the third term in (7.14) to be absorbed by the first and fourth terms (with a new constant):

$$\begin{aligned} (7.15) \quad \sum_{i \leq d' \leq d} \|X^i (T^{b'})_{\varphi_0} X^{d'-i} u_\alpha\|_{L^2} &\leq C \left(\sum_{i \leq d' \leq d} \|X^i (T^{b'-1})_{T\varphi_0} X^{d'-i} u_\alpha\|_{L^2} \right. \\ &\quad \left. + \|(T^{b'})_{\varphi_0} f\|_{L^2} + C^{b'} \sum_{j \leq d} \|X^j \varphi_0^{(b'+1)} X^{d-j-1+b'} u_\alpha\|_{L^2} / b'! \right) \end{aligned}$$

Now we may iterate (7.15) by subjecting the first term on the right to (7.15) again, with b' reduced by one and φ_0 replaced by φ_0' . After at most b' iterations, we obtain $C^{b'}$ terms, each either

$$(7.16) \quad C^k \|(T^{b'-k})_{\varphi_0^{(k)}} f\|_{L^2} \text{ or } C^k C^{b'} \sum_{j \leq d} \|X^j \varphi_0^{(b'+1)} X^{d-j-1+b'-k} u_\alpha\|_{L^2} / (b'-k)!$$

for some $k \leq b'$.

For the first type [in (7.16)] we have [see Definition (7.2) where we have $|\beta + \gamma| = r \leq p = b' - k$]:

$$(T^{b'-k})_{\varphi_0^{(k)}} f = C^{b'-k} \text{ terms, each } \varphi_0^{(k+r)} X^r T^{b'-k-r} f / \rho!$$

for some multi-index ρ with $|\rho| = r$, $r \leq b' - k$. Since X derivatives of f have the same type of bounds as ordinary partial derivations,

$$|X_I T^{b_0} f| \leq K_f^{b_0 + |I| + 1} (b_0 + |I|)!$$

in a compact set (despite the coefficients in the X 's) we have:

$$C^k \|(T^{b'-k})_{\varphi_0^{(k)}} f\|_{L^2} \leq C^{b'} \sup_{k+r \leq b'} (K d_0^{-1})^{k+r+1} N_0^{k+r} K_f^{b'-k} (b'-k)! / r! \leq (CKK_f d_0^{-1})^{b'+1} N_0^b$$

[recall that $r+k \leq b'$ so that $(b'-k)! / r! \leq C^b N_0^{b'-k-r}$ for $b' \leq N_0$].

Thus we obtain, from (7.16) and the above,

$$(7.17) \quad \sum_{d \leq d} \|X^d (T^{b'})_{\varphi_0} u_\alpha\|_{L^2} \leq (CKK_f d_0^{-1})^{b'+1} N_0^b + C^{b'} \sup_{k \leq b'} \sum_{j \leq d} \|X^j \varphi_0^{(b'+1)} X^{d-j-1+b'-k} u_\alpha\|_{L^2} / (b'-k)!$$

To bring this last term into a clearer form we commute $\varphi_0^{(b'+1)}$ to the left and bring it out of the norm. Since $[X^j, \varphi_0^{(b'+1)}] = C^j$ terms, each $\varphi_0^{(b'+1+j')} X^{j-j'}$, $j' \leq j$ we have:

$$\begin{aligned} C^{b'} \|X^j \varphi_0^{(b'+1)} X^{d-j-1+b'-k} u_\alpha\|_{L^2} / (b'-k)! \\ \leq C^{d+b'} \sup_{j' \leq j} (K d_0^{-1})^{b'+1+j'} N_0^{b'+1+j'} \|X^{d-j'-1+b'-k} u_\alpha\|_{L^2(\text{supp } \varphi_0)} / (b'-k)! \\ \leq C^{N_0} (K d_0^{-1})^{N_0} N_0^{b'+d} \|X^{d-j'-1+b'-k} u_\alpha\|_{L^2(\text{supp } \varphi_0)} / N_0^{d-j'-1+b'-k} \end{aligned}$$

since $N_0^{b'+1+j'} N_0^{d-j'-1+b'-k} / N_0^{b'+d} (b'-k)! \leq e^{N_0}$ if $b'+d \leq N_0$.

Together with (7.10) and (7.17), this proves Proposition (7.11), since $b' - b \leq (a - d')/2$ implies $d + b'(-j' - 1 - k) \leq d + b + (a - d')/2$.

Next, we once more reduce X derivatives. Actually, this could all have been done at once, as in [16], but breaking it down into three stages should render the proof more readable; this third stage is needed to reduce the total order by half. An application of (7.5) to the right hand side of (7.11) gives:

(7.18) PROPOSITION. — *There exists a constant \tilde{C} depending on f but not on α , $|\alpha| = \varepsilon$ or N so that if $a + b \leq N_0$:*

$$\sup_{\substack{d \leq d \\ b' + d \leq N_0 \\ b' - b \leq (a - d)/2}} \|X^d T^{b'} u_\alpha\|_{L^2(\text{supp } \psi_0)} / N_0^{b'+d} \leq \tilde{C}^{N_0} (d_0^{-1})^{2N_0} \left(1 + \sup_{\substack{d' \leq d \\ b'' = (b + d + (a - 3d')/2)/2}} \|X^{d'} T^{b''} u_\alpha\|_{L^2(\text{supp } \chi_0)} / N_1^{b''+d'}\right)$$

Combining (7.18) with (7.5) gives, for $a_0 + b_0 \leq 2N_0$

$$(7.19) \quad \|X^{a_0} T^{b_0} u_\alpha\|_{L^2(W_0)} / N_0^{a_0+b_0} \leq \tilde{C}^{N_0} (d_0^{-1})^{3N_0} \left(1 + \sup_{a_1 + b_1 \leq d + (a_0 + b_0)/2} \|X^{a_1} T^{b_1} u_\alpha\|_{L^2(W_1)} / N_1^{a_1+b_1}\right)$$

Actually, one calculates $a_1 + b_1 \leq (2b_0 + a_0 + 3d)/4$, but $d + (a_0 + b_0)/2$ will suffice.

(7.20) PROPOSITION. — *There exists a constant \tilde{C} , depending on f but not on α , $|\alpha| = \varepsilon$ or N such that for $j \leq \lceil \log_2 N \rceil$ and for $a_j + b_j \leq 2N_j$:*

$$\|X^{a_j} T^{b_j} u_\alpha\|_{L^2(W_j)} / N_j^{a_j+b_j} \leq \tilde{C}^{N_j} (d_j^{-1})^{3N_j} \left(1 + \sup_{a_{j+1} + b_{j+1} \leq d + (a_j + b_j)/2} \|X^{a_{j+1}} T^{b_{j+1}} u_\alpha\|_{L^2(W_{j+1})} / N_{j+1}^{a_{j+1}+b_{j+1}}\right).$$

Proof. — Exactly the same proof as the proof of (7.19) applies, everything starting with a_j, b_j, W_j, N_j, d_j , etc. instead of a_0, b_0, W_0, N_0, d_0 , etc.

If we now start with $a_0 + b_0 \leq N$ and apply (7.20) repeatedly, we obtain:

$$\begin{aligned} \sup_{a_0 + b_0 \leq N_0} \|X^{a_0} T^{b_0} u_\alpha\|_{L^2(W_0)} / N_0^{a_0+b_0} &\leq \tilde{C}^{N_0} (d_0^{-1})^{3N_0} \left(1 + \tilde{C}^{N_1} (d_1^{-1})^{3N_1} \left(1 + \tilde{C}^{N_2} (d_2^{-1})^{3N_2} \left(1 + \dots \right.\right.\right. \\ &\quad \left.\left.\left. + (1 + \tilde{C}^{2d} (d_{\lceil \log_2 N \rceil}^{-1})^{3\lceil \log_2 N \rceil}) \sup_{a+b < 2d+1} \|X^a T^b u_\alpha\|_{L^2(U_0)}\right) \dots\right) \\ &\leq (2C)^{\Sigma N_j} \Pi (d_j^{-1})^{3N_j} \left(1 + \sup_{\substack{a+b \leq 2d+1 \\ |\alpha| = \varepsilon}} \|X^a T^b u_\alpha\|_{L^2(U_0)}\right), \end{aligned}$$

since $(\dots (((a_0 + b_0)/2 + d)/2 + d)/2 + d) \dots /2 + d \leq 2d + 1$ after $\lceil \log_2 N_0 \rceil$ iterations.

Only in this last line does a supremum over $\alpha, |\alpha| = \varepsilon$ enter. Now $\Sigma N_j \leq 2N$ and we also have the bound:

$$\Pi (d_j^{-1})^{3N_j} \leq C^N$$

since $\Pi (2^j)^{1/2^j} \leq C$. The supremum over $|\alpha| = \varepsilon$ and $a + b \leq 2d$ of $\|X^a T^b u_\alpha\|_{L^2(U_0)}$ is easily seen to be finite in view of the uniform boundedness of the k_α (see the definition following Lemma 4.1), and this finishes the proof of (7.1).

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