ANNALES SCIENTIFIQUES DE L'É.N.S.

MARLIES GERBER
ANATOLE KATOK

Smooth models of Thurston's pseudo-Anosov maps

Annales scientifiques de l'É.N.S. 4e série, tome 15, nº 1 (1982), p. 173-204
<http://www.numdam.org/item?id=ASENS_1982_4_15_1_173_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 15, 1982, p. 173 à 204.

SMOOTH MODELS OF THURSTON'S PSEUDO-ANOSOV MAPS

BY MARLIES GERBER (*) AND ANATOLE KATOK (**)

Dedicated to the memory of V. M. Alexeyev (1932-1980)

1. Introduction

Pseudo-Anosov maps were singled out by W. Thurston in connection with the problem of classifying diffeomorphisms of a compact connected C^∞ surface M up to isotopy (*see* [T], [F-L-P]). According to Thurston's classification, every diffeomorphism f of M is isotopic to an f' satisfying one of the following:

(i) f' is of finite order and is an isometry with respect to a Riemannian metric of constant curvature on M;

(ii) f' is a "reducible" diffeomorphism, i. e. it leaves a closed curve (possibly having several components) invariant. (In this case f' can be further analyzed by cutting along that curve);

(iii) f' is a pseudo-Anosov map (*cf.* paragraph 2).

For every diffeomorphism f the possibilities of getting an f' satysfying (i) and (iii) [respectively (ii) and (iii)] are mutually exclusive.

Thurston's pseudo-Anosov maps are homeomorphisms which are C^∞ diffeomorphisms except at finitely many points (singularities). A pseudo-Anosov map preserves a natural absolutely continuous measure whose density is C^∞ and positive except at the singularities, at which it vanishes. It is Bernoulli with respect to this measure [F-L-P, paragraphs 9-10]. If M is a torus, then a pseudo-Anosov map is an Anosov diffeomorphism. As will be observed below (*cf.* paragraph 2.4), for M with genus greater than 1, a pseudo-Anosov map cannot be made a diffeomorphism by a coordinate change which is smooth outside the singularities or even outside a sufficiently small neighborhood of the singularities. Thus, in order to find diffeomorphism models for (iii) with the same dynamical properties as pseudo-

(*) Supported by NSF Post-Doctoral Fellowship No. MCS79-19149.

(**) Partially supported by NSF Grant No. MCS79-030.

Anosov maps it is necessary to apply a non-trivial constructions which is global in nature. In this paper we will construct, for every pseudo-Anosov map f, a C^∞ diffeomorphism g which is topologically conjugate to f through a homeomorphism isotopic to the identity and which is Bernoulli with respect to a smooth measure (i.e., one whose density is C^∞ and positive everywhere). In our paper we will assume as in [F-L-P, paragraphs 9-10] that M is orientable and without boundary. At the end we will indicate modifications which are needed in the general case.

We are grateful to the referee who made several useful observations and comments (*cf.* for example corollary at the end of Section 2 and the discussion of the non-orientable case in Section 8) and also pointed out several errors in the original text.

2. Definitions and preliminaries

2.1. Measured foliations. — For each $r>0$, let $\mathscr{D}_r=\{z : |z|<r\}\subset \mathbb{C}$.

A *measured foliation* of M is a foliation $\mathfrak{F}$ with a finite set of singular points $x_1, \ldots, x_m$ and a transverse measure μ given as follows. There is a collection of C^∞ charts (φ_i, U_i), $i=1, \ldots, L$, $L>m$, with $\bigcup_i U_i = M$ such that for each i, $1\leqq i\leqq m$, there is an integer $p=p(i)\geqq 3$ (equal to the number of prongs of the singularity of $\mathfrak{F}$ at x_i) for which the chart (φ_i, U_i) satisfies:

(i) $\varphi_i(U_i)=\mathscr{D}_{a_i}$, for some $a_i>0$;

(ii) $\varphi_i(x_i)=0$;

(iii) leaves of $\mathfrak{F}$ get mapped to components of the sets $\{\operatorname{Im} z^{p/2}=\text{constant}\}\cap\varphi_i(U_i)$;

(iv) on U_i, μ is given by the image of $|\operatorname{Im} z^{(p-2)/2}\,dz|$ and for $i>m$, the chart (φ_i, U_i) satisfies:

(i) $\varphi_i(U_i)=(0, b_i)\times(0, c_i)\subset\mathbb{R}^2$, for some $b_i, c_i>0$;

(ii) leaves of $\mathfrak{F}$ get mapped to segments $\{y=\text{constant}\}\cap\varphi_i(U_i)$;

(iii) on U_i, μ is given by $|dy|$.

Of course, the transverse measures are required to be consistently defined on chart overlaps.

2.2. Pseudo-Anosov maps. — Thurston's pseudo-Anosov maps have the following form. There are two measured foliations $(\mathfrak{F}^s, \mu^s)$ and $(\mathfrak{F}^u, \mu^u)$ with the same singularities $x_1, \ldots, x_m$ and the same number of prongs $p=p(i)$ at each x_i, $1\leqq i\leqq m$, which are transversal in the usual sense at nonsingular points and which have C^∞ charts (φ_i, U_i), $1\leqq i\leqq m$, satisfying:

$$(2.1)\quad \begin{cases} \text{(i) } \varphi_i(U_i)=\mathscr{D}_{a_i} \text{ for some } a_i>0, \\ \text{(ii) } \varphi_i(x_i)=0, \\ \text{(iii) leaves of } \mathfrak{F}^s \text{ get mapped to components of the sets } \{\operatorname{Re} z^{p/2}=\text{constant}\}\cap\mathscr{D}_{a_i}, \\ \text{(iv) leaves of } \mathfrak{F}^u \text{ get mapped to components of the sets } \{\operatorname{Im} z^{p/2}=\text{constant}\}\cap\mathscr{D}_{a_i} \end{cases}$$

and there exists a constant $\lambda > 1$ such that:

$$f(\mathfrak{F}^s, \mu^s) = \left(\mathfrak{F}^s, \frac{1}{\lambda}\mu^s\right) \quad \text{and} \quad f(\mathfrak{F}^u, \mu^u) = (\mathfrak{F}^u, \lambda\mu^u). \tag{2.2}$$

In fact, the pseudo-Anosov maps in [F-L-P, paragraph 9] are constructed so that:

$$\begin{cases} \text{the transverse measures } \mu^s \text{ and } \mu^u \text{ on } U_i \text{ are given by:} \\ \qquad |\operatorname{Re} z^{(p-2)/2}\, dz| \quad \text{and} \quad |\operatorname{Im} z^{(p-2)/2}\, dz|. \end{cases} \tag{2.3}$$

respectively. The first examples of maps on the surfaces of genus greater than 1 with properties (2.1) and (2.2) were constructed by T. O'Brien and W. Reddy in 1970 [O-R].

At each singular point x_i, consider the stable and unstable prongs,

$$P^s_{i,j} = \varphi_i^{-1}\left\{ z = \rho e^{i\tau}: 0 \leqq \rho \leqq a_i,\ \tau = \frac{2j+1}{p}\pi \right\},$$
$$j = 0, 1, \ldots, p-1,$$

and:

$$P^u_{i,j} = \varphi_i^{-1}\left\{ z = \rho e^{i\tau}: 0 \leqq \rho \leqq a_i,\ \tau = \frac{2j}{p}\pi \right\},$$
$$j = 0, 1, \ldots, p-1,$$

and the stable and unstable sectors:

$$S^s_{i,j} = \varphi_i^{-1}\left\{ z = \rho e^{i\tau}: 0 \leqq \rho \leqq a_i,\ \frac{2j-1}{p}\pi \leqq \tau \leqq \frac{2j+1}{p}\pi \right\},$$
$$j = 0, ., \ldots, p-1$$

and:

$$S^u_{i,j} = \varphi_i^{-1}\left\{ z = \rho e^{i\tau}: 0 \leqq \rho \leqq a_i,\ \frac{2j}{p}\pi \leqq \tau \leqq \frac{2j+2}{p}\pi \right\},$$
$$j = 0, 1, \ldots, p-1.$$

A singular leaf of $\mathfrak{F}^s$ [$\mathfrak{F}^u$] is defined to be a singularity x_i together with the extension along $\mathfrak{F}^s$ [$\mathfrak{F}^u$] (away from x_i) of a stable [unstable] prong at x_i.

Since f is a homeomorphism, $f(x_i) = x_{\sigma(i)}$, $i = 1, \ldots, m$ where σ is a permutation of $\{1, \ldots, m\}$ such that $p(i) = p(\sigma(i))$ and f maps the stable prongs at x_i into the stable prongs at $x_{\sigma(i)}$ (provided the $a_i's$ are chosen so that $a_i/\lambda^{2/p} \leqq a_{\sigma(i)}$). Henceforth (except in paragraph 2.4), we will assume that $\sigma(i) = i$, $i = 1, \ldots, m$, and $f(P^s_{i,j}) \subset P^s_{i,j}$, $j = 0, \ldots, p-1, i = 1, \ldots, m$. (The arguments in the general case are the same, but require more cumbersome notation.) Now consider the mapping $\Phi_{i,j}:\varphi_i S^s_{i,j}\to\{z:\mathrm{Re}\, z\geqq 0\}$ given by $\Phi_{i,j}(z)=(2/p)z^{p/2}$. Let $\Phi_{i,j}(z)=w=s_1+is_2$. Note that:

$$(2.4)\qquad (\Phi^{-1}_{i,j})^*\,|\mathrm{Re}\, z^{(p-2)/2}\,dz|=|ds_1|$$

and:

$$(2.5)\qquad (\Phi^{-1}_{i,j})^*\,|\mathrm{Im}\, z^{(p-2)/2}\,dz|=|ds_2|.$$

Thus, in $S^s_{ij}\cap f^{-1}S^1_{ij}$ is given by:

$$(2.6)\qquad f=\varphi_i^{-1}\,\Phi^{-1}_{i,j}\,F\,\Phi_{i,j}\,\varphi_i,$$

where $f(w)=F((s_1, s_2))=(\lambda s_1, (1/\lambda)s_2)$. It is easy to show that f is expanding linearly in $p(\geqq 3)$ directions and contracting linearly in $p(\geqq 3)$ directions at x_i, and consequently f is not differentiable at x_i. On $M\setminus\{x_1, \ldots, x_m\}$, f can be constructed to be C^∞.

2.3. Markov partitions. — The construction in [F-L-P, paragraph 9] provides a Markov partition for a pseudo-Anosov map f. It consists of "rectangles" $R_1, \ldots, R_N$, which are subsets of M satisfying:

(i) $\bigcup_{i=1}^{N} R_i=M$.

(ii) For each $i=1, \ldots, N$, there are positive numbers l_i, $\overline{l}_i$ and a homeomorphism $\psi_i:[0, l_i]\times[0, \overline{l}_i]\to R_i$. If $x\in R_i$ is a singular point of f then x must be one of the corner points, namely, $\psi_i(0, 0)$, $\psi_i(l_i, 0)$, $\psi_i(0, \overline{l}_i)$ or $\psi_i(l_i, \overline{l}_i)$; ψ_i is a C^∞ embedding outside of the set of singularities. Moreover, for each $t\in[0, l_i]$, $\psi_i(\{t\}\times[0, \overline{l}_i])$ is contained in a leaf of $\mathfrak{F}^s$ and for each $t\in[0, \overline{l}_i]$, $\psi_i([0, l_i]\times\{t\})$ is contained in a leaf of $\mathfrak{F}^u$. [For $x\in R_i$ with $(u, v)=\psi_i^{-1}(x)$, let $\mathfrak{F}^s(x, R_i)=\psi_i(\{u\}\times[0, \overline{l}_i])$ and $\mathfrak{F}^u(x, R_i)=\psi_i([0, l_i]\times\{v\})$.]

(iii) $\mathrm{Int}\, R_i\cap\mathrm{Int}\, R_j=\emptyset$ for $i\neq j$.

(iv) If $x\in\mathrm{Int}\, R_i$ and $f(x)\in\mathrm{Int}\, R_j$ then:

$$f(\mathfrak{F}^s(x, R_i))\subset\mathfrak{F}^s(f(x), R_j)\qquad\text{and}\qquad f^{-1}(\mathfrak{F}^u(f(x), R_j))\subset\mathfrak{F}^u(x, R_i).$$

(v) If $x\in\mathrm{Int}\, R_i$ and $f(x)\in\mathrm{Int}\, R_j$, then:

$$f(\mathfrak{F}^u(x, R_i))\cap R_j=\mathfrak{F}^u(f(x), R_j)\qquad\text{and}\qquad f^{-1}(\mathfrak{F}^s(f(x), R_j))\cap R_i=\mathfrak{F}^s(x, R_i),$$

which means that $f(R_i)$ goes across R_j just one time.

In fact, in the construction in [F-L-P], the ψ_i's can be chosen so that for each $(u, v)\in[0, l_i]\times[0, \overline{l}_i]$,

$$(2.7)\qquad \mu^s(\psi_i([0, u]\times\{v\}))=u$$

and:

$$(2.8)\qquad \mu^u(\psi_i(\{u\}\times[0, v]))=v.$$

Also, if we let $\partial_u R_i = \psi_i([0, l_i] \times \{0, \bar{l}_i\})$ and $\partial_s R_i = \psi_i(\{0, l_i\} \times [0, \bar{l}_i])$, then we can get:

$$(2.9)\quad \left\{ \begin{array}{l} \bigcup_{i=1}^{N} \partial_u R_i = L^u, \\ \text{where:} \\ L^u = \bigcup_{\substack{i=1, \ldots, m \\ j=0, \ldots, p(i)-1}} L^u_{i,j}, \end{array} \right.$$

with each $L^u_{i,j}$ being a finite segment (with x_i as one of its endpoints) of a singular leaf extending $P^u_{i,j}$, and similarly for $\bigcup_{i=1}^{N} \partial_s R_i$, $L^s_{i,j}$ and L^s.

The transition matrix $A^f = (a_{ij})_{1 \leq i, j \leq N}$ for the above Markov partition is given by:

$$(2.10)\qquad a_{ij} = \begin{cases} 1 & \text{if } \operatorname{Int} f(R_i) \cap \operatorname{Int} R_j \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The number λ defined in (2.2) is the maximum absolute value of eigenvalues of A; $\log \lambda$ is equal to the topological entropy of f [F-L-P, paragraph 10].

As is shown in [F-L-P], the partition $\mathscr{R} = \{R_1, \ldots, R_N\}$ generates the Lebesgue sigma field under f and $(f, \mathscr{R})$ is a mixing Markov process with respect to the normalization of the invariant measure on M given locally as a Cartesian product $\mu^u \times \mu^s$. Hence, by [0] f is Bernoulli with respect to this (non-smooth) measure.

2.4. Essential non-smoothness of f at singularities

Proposition. — *Let f be a pseudo-Anosov map on M of the form described above, where the genus of M is greater than 1, and let g be a C^1 diffeomorphism of M. Then f cannot be topologically conjugate to g via a homeomorphism which is a C^1 diffeomorphism except at the singularities of f.*

Proof. — Suppose $f = hgh^{-1}$, where h is a homeomorphism which is a C^1 diffeomorphism except at the singularities of f. Note that for each positive integer k, $f^k = hg^k h^{-1}$ and f^k is also a pseudo-Anosov map. Thus by replacing f by f^k if necessary, we may again assume that the singularities of f are fixed under f and f maps each stable [unstable] prong to itself. Let $(\mathfrak{F}^s, \mu^s)$ and $(\mathfrak{F}^u, \mu^u)$ be the stable and unstable measured foliations for f with expansion constant $\lambda > 1$, as described in paragraph 2.2. Let $\mathscr{R} = \{R_1, \ldots, R_N\}$ be a Markov partition for f with transition matrix $A^f = (a_{ij})_{1 \leq i, j \leq N}$ such that the conditions given in paragraph 2.3 are satisfied.

By the Euler-Poincaré index theorem (generalized to foliations) [F-L-P, paragraph 5], the genus of M being greater than 1 implies that $(\mathfrak{F}^s, \mu^s)$ and $(\mathfrak{F}^u, \mu^u)$ have at least one singularity each. Thus f has at least one singularity.

Let y be a singularity of f. Assume that the elements of $\mathscr{R}$ have been labeled so that y is a corner point of R_1. Since f fixes the stable and unstable prongs at y, it is easy to see that

$a_{11}=1$. Also there exists an allowable sequence $k_1, k_2, \ldots, k_l$, $l \geqq 3$, of elements of $\{1, \ldots, N\}$ (i.e. $a_{k_1 k_2}=1, \ldots, a_{k_{l-1} k_l}=1$) such that $k_1=k_l=1$ and $k_2, \ldots, k_{l-1}$ are not equal to 1. For each integer $q \geqq 1$, let y_q be the point whose name with respect to the Markov partition $\mathscr{R}$ is $\ldots b_{-2} b_{-1} b_0 b_1 b_2 \ldots$ $\left(\text{i.e. } \{y_q\} = \bigcap_{n=0}^{\infty} \operatorname{cl}\left[\operatorname{int}\left(\bigcap_{i=-n}^{n} f^{-i} R_{b_i}\right)\right]\right)$ where:

$$b_i = k_j \quad \text{if} \quad i \equiv j \bmod (q+l), \qquad j=1, \ldots, l$$

and $b_i=1$, otherwise.

Clearly y_q is a periodic point for f, and it is easy to check that it cannot be a singular point for f. Thus, for each y_q, there exists a neighborhood $V=V(q)$ of the set of singular points for f such that $f^n y_q \notin V$ for each integer n. Note that at each nonsingular point f and f^{-1} have characteristic exponents [P] $\log \lambda$ and $-\log \lambda$. Since dh and dh^{-1} are bounded on $M \setminus V$ and $h M \setminus h V$, respectively, it follows that g and g^{-1} also have characteristic exponents $\log \lambda$ and $-\log \lambda$ at each hy_q.

We now examine the possibilities for dg_{hy}. Consider a local trivialization of TM near hy in which $B = dg_{hy}$ is a matrix representation in Jordan canonical form.

Case 1 :

$$B = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \qquad |\alpha| \leqq 1.$$

This is the most complicated case. We will give the argument for this case in full detail and be more sketchy with the other cases.

We have:

$$B^K = \begin{pmatrix} \alpha^K & K\alpha^{K-1} \\ 0 & \alpha^K \end{pmatrix}, \qquad K \geqq 1.$$

Choose $\varepsilon > 0$ and $K \geqq 1$ such that:

$$\frac{\log (K+\varepsilon)}{K} < \log \lambda.$$

Then there exists $\delta > 0$ such that if $\| B_i - B \| < \delta$ for $i=0, \ldots, K-1$, then:

$$\| B_{K-1} B_{K-2} \ldots B_0 - B^K \| < \varepsilon.$$

Choose a neighborhood U of y sufficiently small so that if $\bar{y} \in U$, then with respect to the above-chosen coordinate system,

$$\| dg_{h\bar{y}} - B \| < \delta, \qquad \| dg_{gh\bar{y}} - B \| < \delta, \ \ldots, \ \| dg_{g^{K-1} h\bar{y}} - B \| < \delta.$$

Then:

$$\| dg^K_{h\bar{y}} - B^K \| = \| dg_{g^{K-1} h\bar{y}}\, dg_{g^{K-2} h\bar{y}} \ldots dg_{h\bar{y}} - B^K \| < \varepsilon.$$

Thus:

$$\| dg_{h\bar{y}}^{K} \| < \| B^{K} \| + \varepsilon \leqq K + \varepsilon.$$

Next choose q' such that if $\bar{y} \in \bigcap_{n=0}^{q'} \mathrm{cl}\left[\mathrm{int} \bigcap_{i=-n}^{n} f^{-i} R_1\right]$, then $\bar{y} \in U$. Finally let $q = 2q' + nK$, where n is large. Then:

$$\overline{\lim_{k \to \infty}} \frac{\log \| dg_{hy_q}^{k} \|}{k} \leqq \frac{\log [(K+\varepsilon)^n \| dg \|^{2q'+l}]}{Kn + 2q' + l}.$$

Note that the limit of the right hand side as $n \to \infty$ is $\log(K+\varepsilon)/K < \log \lambda$. Thus, if n is chosen sufficiently large, we have:

$$\overline{\lim_{k \to \infty}} \frac{\log \| dg_{hy_q}^{k} \|}{k} < \log \lambda.$$

This implies that the characteristic exponents of g at hy_q are both less than $\log \lambda$, a contradiction.

Case 2 :

$$B = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \qquad |\alpha| > 1.$$

Then B^{-1} has Jordan form $\begin{pmatrix} 1/\alpha & 1 \\ 0 & 1/\alpha \end{pmatrix}$. Thus by the argument for Case 1, g^{-1} would have both characteristic exponents less than $\log \lambda$ at hy_q for sufficiently large q.

Case 3 : B has complex eigenvalues α, $\bar{\alpha}$. Suppose $|\alpha| \leqq 1$. Then B is conjugate to a rotation composed with a contraction by $|\alpha|$. Consequently, using the construction described in Case 1 and taking q sufficiently large, we could get the characteristic exponents of g at hy_q to be arbitrarily close to $\log |\alpha|$, in particular less than $\log \lambda$, a contradiction. The case $|\alpha| > 1$ is similar.

Case 4 :

$$B = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \qquad |\alpha_1| \leqq |\alpha_2|.$$

If $|\alpha_1| \leqq 1$, $|\alpha_2| \leqq 1$ or $|\alpha_1| \geqq 1$, $|\alpha_2| \geqq 1$, then the same type of argument as in Case 3 gives a contradiction. Finally, if $|\alpha_1| < 1 < |\alpha_2|$, then according to the Hadamard-Perron Theorem [S, paragraph 1.2], the local stable manifold at hy is homeomorphic to an interval, which is topologically inconsistent with the existence of at least three stable prongs at y.

As the referee pointed out, this proposition implies the following corollary.

COROLLARY. — *There exists a neighborhood* V *of the singular set such that if h is a conjugacy of f to* C^1 *map g then h is not* C^1 *on* $M \setminus V$.

Proof. — Since $h^{-1}gh=f$ then for every n, $h=g^n hf^{-n}$. If the neighborhood V is small enough then each point different from a singularity has an iterate under f which is not in $M\setminus V$. Thus h must be C^1 outside of the singular set which contradics the proposition.

3. Construction of smooth pseudo-Anosov maps

Let f be a pseudo-Anosov map with expansion constant $\lambda>1$, as discussed in paragraph 2. We now describe the construction of a C^∞ diffeomorphism g, which is topologically conjugate to f through a homeomorphism isotopic to the identity, and which is Bernoulli with respect to an invariant measure given by a smooth positive density.

Our construction consists of a local perturbation of f in a neighborhood of each singularity. In each stable sector it coincides up to the coordinate change Φ_{ij} with the "slowing-down" (n° 2) part of the construction from [K, paragraph 2]. The coordinate change Φ_{ij} brings the invariant measure for the slowed-down map back to the Lebesgue measure, thus substituting for the "blowing-up" (n° 3) part of the above-mentioned construction. Henceforth we will refer directly to the equations from [K, paragraph 4] and estimates concerning this construction from [B-F-K, Proposition 2.3].

We should also mention that the proof of Corollary 4.3 (and consequently Proposition 2.1) in [K] is incomplete. For, as M Rees pointed out, the statement from [G] about expansive maps in the closure of Anosov diffeomorphisms is not proved. This gap can be filled by the use of J. Franks' Theorem [F, Prop. 2.1] and the fact that $\lim_{n\to\infty} l(g_1^{-n}\gamma)=\infty$ for every stable curve γ (*cf.* paragraph 5). (This fact is a weaker, non-uniform version of our Corollary 6.2, and it is much easier to prove.)

Indeed, there is a principal difference between the toral case and the pseudo-Anosov case for manifolds of genus greater than 1 because the analog of Franks' theorem is false. Thus, Markov partitions provide a more powerful and more universal (but also more difficult) method of proving the topological conjugacy between the "model" and the perturbed map. The main new technical difficulty in this proof is the establishing of uniform contraction for stable curves. This is done in paragraph 5, which is due completely to the first author.

For $i=1, \ldots, m$, let $a'_i=(2/p(i))a_i^{p(i)/2}$, and for $r>0$ such that $r\leqq a'_i$, let $D^i_r=\varphi_i^{-1}(\mathscr{D}_{[(p(i)/2)r]^{2/p(i)}})$. Then, in particular, $D^i_{a'_i}=\varphi_i^{-1}\mathscr{D}_{a_i}$. Assume that the a_i's were chosen so that $D^i_{a'_i}\cap D^j_{a'_j}=\emptyset$ for $i\neq j$, $1\leqq i, j\leqq m$. For $r>0$ such that $r\leqq a'_i$ for $i=1, \ldots, m$, let $D_r=\bigcup_{i=1}^m D^i_r$.

Since:

$$\Phi_{ij}\varphi_i(D^i_r\cap S^s_{i,j})=\mathscr{D}_r\cap\{z:\operatorname{Re} z\geqq 0\},$$

$$(\Phi_{i,j}\varphi_i)^*|ds_1|=\mu^s \quad\text{and}\quad (\Phi_{i,j}\varphi_i)^*|ds_2|=\mu^u,$$

for $j=0, \ldots, p(i)-1$, $i=1, \ldots, m$, $r\leqq a'_i$, this means that D^i_r is the disk of radius r about x_i in the metric $\rho=du^2+dv^2$ [*cf.* (2.7), (2.8)]. Let us note that this metric is defined consistently on the whole manifold M and it is a Riemannian metric with singularities at the points $x_1, \ldots, x_m$. We will denote the distance generated by this metric by d and the length of curves in this metric by l.

Now choose $0<r_6<r_5<r_4<r_3<r_2<r_1<r_0<r'_3<r'_2<r'_1<r'_0$ such that:

(3.1) $$r_2<r_0/4.$$

(3.2) For each $i=1, \ldots, m$, $F(\overline{\mathscr{D}}_{r'_0})\subset\mathscr{D}_{a'_i}$, where F is as in paragraph 2.2.

(3.3) For each $i=1, \ldots, m$, $FL^u\cap D_{r'_0}\subset\bigcup_{j=1}^{p(i)} P^u_{i,j}$ and :

$$F^{-1}L^s\cap D_{r'_0}\subset\bigcup_{j=1}^{p(i)} P^s_{i,j},\text{ where } L^u \text{ and } L^s \text{ are as in (2.9).}$$

(3.4) $$F(\overline{\mathscr{D}}_{r_{i+1}})\subset\mathscr{D}_{r_i},\quad i=0, 1, 2, 3, 4, 5 \qquad\text{and}\qquad F(\overline{\mathscr{D}}_{r'_{i+1}})\subset\mathscr{D}_{r'_i},\quad i=0, 1, 2.$$

(It then follows from the definition of F that the same inclusions will also hold with F replaced by F^{-1}.)

(3.5) For some positive integer q:

$$F^{-q}((\mathscr{D}_{r_0}\setminus\mathscr{D}_{r_3})\cap\{(s_1, s_2): |s_2|\geqq s_1\geqq 0\})\subset\mathscr{D}_{r'_0}\setminus\mathscr{D}_{r'_3}.$$

(See *Fig.* 1.)

(3.6) If P is the maximum number of prongs at a singularity of f, then:

$$\left(\frac{p}{2}r_5\right)^{2(p-2)/p}<1 \qquad\text{for}\quad p=3, 4, \ldots, \text{P}.$$

(3.7) If H_1 and H_2 are branches of hyperbolas given by $s_1 s_2=\delta_1$, $s_1\geqq 0$, and $s_1 s_2=\delta_2$, $s_1\geqq 0$, respectively, which pass through $\mathscr{D}_{r_0}$ and γ is a connected curve lying between H_1 and H_2 which has a point lying in the region $(\mathscr{D}_{r'_1}\setminus\mathscr{D}_{r'_2})\cap\{(s_1, s_2): s_1\geqq|s_2|\}$ and whose tangent vectors all lie within 45° of the vertical direction in the (s_1, s_2) coordinate system, then γ lies in $\mathscr{D}_{r'_0}\setminus\mathscr{D}_{r'_3}$ and:

$$\bar{l}(\gamma)<\bar{d}(H_1\cap\mathscr{D}_{r_0}, H_2\cap\mathscr{D}_{r_0}),$$

where $\bar{l}$ and $\bar{d}$ denote the usual Cartesian length and distance, respectively, in the (s_1, s_2) coordinate system.

We now define the "slow-down function" Ψ_p corresponding to a p-prong singularity, $3\leqq p\leqq$ P, on the interval $[0, \infty)$ by:

(i) $$\Psi_p(u)=\left(\frac{p}{2}\right)^{(2p-4)/p} u^{(p-2)/p} \qquad\text{for}\quad u\leqq r_5^2;$$

(ii) $$\Psi_p \text{ is } C^\infty \text{ except at } 0;$$

(iii) $$\Psi'_p(u)\geqq 0 \qquad\text{for}\quad u>0;$$

(iv) $$\Psi_p(u)=1 \qquad\text{for}\quad u\geqq r_4^2.$$

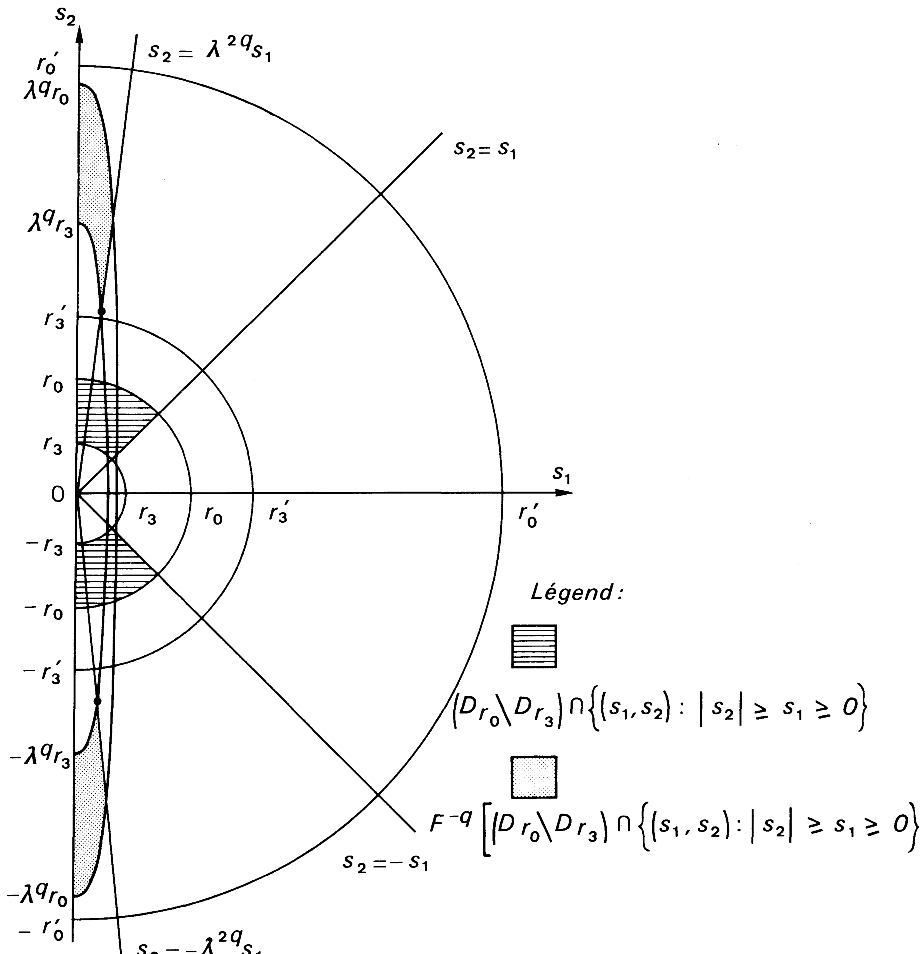

Fig. 1

Note that F is the time-one map for the vector field v given by:

$$\dot{s}_1 = (ln \lambda) s_1,$$
$$\dot{s}_2 = -(ln \lambda) s_2.$$

Now consider the vector field v_{Ψ_p} given by:

$$\dot{s}_1 = (ln \lambda) s_1 \Psi_p(s_1^2 + s_2^2),$$
$$\dot{s}_2 = -(ln \lambda) s_2 \Psi_p(s_1^2 + s_2^2).$$

Let G_p be defined on $\mathscr{D}_{r_0'}$ as the time-one map of v_{Ψ_p}. Since $\overline{\mathscr{D}}_{r_4} \subset F(\mathscr{D}_{r_3})$, G_p coincides with F in $\mathscr{D}_{r_0'} \setminus \mathscr{D}_{r_3}$. Thus we define $g : M \to M$ by:

$$g(x) = \begin{cases} f(x) & \text{if } x \in D^i_{r_2}, \\ \varphi_i^{-1} \Phi_{i,j}^{-1} G_{p(i)} \Phi_{i,j} \varphi_i(x) & \text{if } x \in S^s_{i,j} \cap D^i_{r_0'}. \end{cases}$$

(These two definitions coincide on $\bigcup_{i=1}^{m} D^i_{r'_0} \setminus D^i_{r_1}$, because in $D^i_{r'_0} \cap S^s_{i,j}$, f is given by $\varphi_i^{-1} \Phi_{i,j}^{-1} F \Phi_{i,j} \varphi_i(x)$.)

Obviously, g is a homeomorphism which is C^∞ everywhere except possibly at the singularities $x_1, \ldots, x_m$. In fact, we will see that it is C^∞ everywhere; moreover, in a neighborhood of each singular point, g is real-analytic. In order to prove this, we will show that locally the vector field $\bigcup_{j=0}^{p(i)-1} (\varphi_i^{-1} \Phi_{i,j}^{-1})_* (v_{\Psi_{p(i)}})$, which generates g, is Hamiltonian with respect to the volume element generating the Lebesgue measure and its Hamiltonian function is real-analytic.

4. Smoothness of g

Fix a singular point x_i and let $p = p(i)$. Consider the $z = t_1 + it_2 = \rho e^{i\tau}$ coordinate systems on $D^i_{r_0}$ given by the chart (φ_i, U_i) and the coordinate systems $w = s_1 + is_2 = re^{i\theta}$ on $\mathscr{D}_{r_0}$, where $\Phi_{i,j}^{-1}$, $j = 0, \ldots, p-1$, is defined on $\mathscr{D}_{r_0} \cap \{(s_1, s_2) : s_1 \geqq 0\}$ by:

$$z = \Phi_{i,j}^{-1}(w) = \left(\frac{p}{2} w\right)^{2/p},$$

or, equivalently,

$$\rho = \left(\frac{p}{2} r\right)^{2/p}, \qquad \tau = \frac{2}{p}\theta + \frac{2j\pi}{p} \qquad \text{where} \quad -\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}.$$

In each sector $D^i_{r_0} \cap S^s_{i,j}$, g is given, in the z-coordinate system, as the time-one map of the vector field $(\Phi_{i,j}^{-1})_* v_{\Psi_p}$.

The function H given by $H(s_1, s_2) = (ln\lambda) s_1 s_2$ is Hamiltonian for the vector field v with respect to the volume element $ds_1\, ds_2$. Hence H is also Hamiltonian for v_{Ψ_p} with respect to $ds_1\, ds_2 / \Psi_p(s_1^2 + s_2^2)$. Then $H_1 = H \circ \Phi_{i,j}$ is Hamiltonian for $(\Phi_{i,j}^{-1})_*(v_{\Psi_p})$ with respect to $(\Phi_{i,j})_* (ds_1\, ds_2 / \Psi_p(s_1^2 + s_2^2))$. We will show that in $\mathscr{D}_{r_6}$ $(\Phi_{i,j})^* (ds_1\, ds_2 / \Psi_p(s_1^2 + s_2^2))$ is the Lebesgue measure $\rho\, d\rho\, d\tau = dt_1\, dt_2$ and H_1 is real analytic in the (t_1, t_2) coordinate system. It then follows that g is real analytic in $D^i_{r_6}$.

We first compute $(\Phi_{i,j})^* (ds_1\, ds_2 / \Psi_p(s_1^2 + s_2^2))$. We have:

$$d\rho = \left(\frac{p}{2}\right)^{(2-p)/p} r^{(2-p)/p}\, dr \qquad \text{and} \qquad d\tau = \frac{2}{p} d\theta.$$

Thus:

$$\left(\frac{p}{2}\right)^{(p-2)/p} r^{(2p-2)/p}\, d\rho = r dr \qquad \text{and} \qquad \frac{p}{2} d\tau = d\theta.$$

Hence:

$$(4.1) \quad (\Phi_{i,j})^* \left(\frac{ds_1\, ds_2}{\Psi_p(s_1^2 + s_2^2)}\right) = (\Phi_{i,j})^* \left(\frac{r dr d\theta}{\Psi_p(r^2)}\right)$$
$$= \frac{((p/2)^{(p-2)/p} r^{(2p-2)/p}\, d\rho)((p/2)\, d\tau)}{(p/2)^{(2p-4)/p} r^{(2p-4)/p}} = \left(\frac{p}{2}\right)^{2/p} r^{2/p}\, d\rho\, d\tau = \rho\, d\rho\, d\tau.$$

Next we compute H_1. In polar coordinates H is given by:

$$H(re^{i\theta}) = (\ln\lambda)\, r\cos\theta\, r\sin\theta = (\ln\lambda)\,\frac{r^2}{2}\sin 2\theta$$

and:

$$\Phi_{i,j}(\rho e^{i\tau}) = \left(\frac{2}{p}\right)\rho^{p/2}\, e^{ip\tau/2}.$$

Thus we have:

$$H_1(t_1, t_2) = H_1(\rho e^{i\tau}) = (\ln\lambda)\,\frac{(2\rho^{p/2}/p)^2}{2}\sin 2(p\tau/2)$$
$$= \left(\frac{2\ln\lambda}{p^2}\right)\rho^p \sin p\tau = \left(\frac{2\ln\lambda}{p^2}\right)\operatorname{Im}(t_1 + it_2)^p,$$

which is a polynomial in t_1 and t_2, and hence real analytic.

5. Uniform Contraction Lemma

Let (η_1, η_2) denote the natural coordinates in each tangent space at a point of the (s_1, s_2)-plane. Furthermore, let $\mathscr{K}^+_{(s_1, s_2)}$ $[\mathscr{K}^-_{(s_1, s_2)}]$ be the closed cone in this tangent space around the η_1-$[\eta_2$-$]$ axis bounded by the lines $\eta_1 = \pm\eta_2$.

Define a coordinate system (ξ_1, ξ_2) in each tangent space $T_x M$, $x \in M \setminus \{x_1, \ldots, x_m\}$ as follows. Take a system of C^∞ coordinate charts (φ_i, U_i), $i = 1, \ldots, L$, $L > m$, on M such that $\bigcup_i U_i = M$ and for $i = 1, \ldots, m$, (φ_i, U_i) satisfies (2.1) (i)-(iv) and (2.3) and for $i > m$, (φ_i, U_i) satisfies:

(5.1) $$\varphi_i(U_i) = (0, b_i) \times (0, c_i) \subset (s_1, s_2)\text{-plane},$$

for some $b_i, c_i > 0$;

(5.2) leaves of $\mathfrak{F}^s$ get mapped to segments:

$$\{s_1 = \text{constant}\} \cap \varphi_i(U_i);$$

(5.3) leaves of $\mathfrak{F}^u$ get mapped to segments:

$$\{s_2 = \text{constant}\} \cap \varphi_i(U_i);$$

(5.4) on U_i, μ^s and μ^u are given by $|ds_1|$ and $|ds_2|$, respectively.

Then in each U_i, $i > m$, let $(\xi_1, \xi_2) = d\varphi_i^{-1}(\eta_1, \eta_2)$, and in the j-th sector $S^s_{i,j}$ at x_i, $i = 1, \ldots, m$, $j = 0, \ldots, p(i) - 1$, let $(\xi_1, \xi_2) = (d\varphi_i^{-1}\Phi_{ij}^{-1})(\eta_1, \eta_2)$. Since $(\varphi_i)^*$, $i > m$, and $(\Phi_{ij}\varphi_i)^*$, $i = 1, \ldots, m$, $j = 0, \ldots, p(i) - 1$, each maps $|ds_1|$ and $|ds_2|$ to μ^s and μ^u, respectively, we have $\mu^s(\xi_1, \xi_2)=|\xi_1|$ and $\mu^u(\xi_1, \xi_2)=|\xi_2|$. Thus, on chart overlaps the (ξ_1, ξ_2) coordinate systems are consistently defined, up to changes in sign of each of the coordinates. Also, because:

$$f(\mathfrak{F}^s, \mu^s)=\left(\mathfrak{F}^s, \frac{1}{\lambda}\mu^s\right) \quad \text{and} \quad f(\mathfrak{F}^u, \mu^u)=(\mathfrak{F}^u, \lambda\mu^u),$$

we have $f_*(\xi_1, \xi_2)=(\pm\lambda\xi_1, \pm(1/\lambda)\xi_2)$.

For $x\in M\setminus\{x_1, \ldots, x_m\}$ let $E^u_{x,f}$ and $E^s_{x,f}$ be the subspaces of $T_x M$ which are tangent to $\mathfrak{F}^u$ and $\mathfrak{F}^s$, respectively, at x, and let:

$$K_x^+=\{(\xi_1, \xi_2)\in T_x M : |\xi_2|\leqq|\xi_1|\}$$

and:

$$K_x^-=\{(\xi_1, \xi_2)\in T_x M : |\xi_1|\leqq|\xi_2|\}.$$

PROPOSITION 5.1. — *For $x\in M\setminus\{x_1, \ldots, x_m\}$, the families of cones K_x^+ and K_x^- are semi-invariant, i.e.:*

$$dg(K_x^+)\subset K_{gx}^+ \quad \text{and} \quad dg^{-1}(K_x^-)\subset K_{g^{-1}x}^-.$$

For each $x\in M\setminus\{x_1, \ldots, x_m\}$, the intersections:

$$E^u_{x,g}=\bigcap_{n\geqq 0} dg^n K^+_{g^{-n}x} \quad \text{and} \quad E^s_{x,g}=\bigcap_{n\leqq 0} dg^n K^-_{g^{-n}x} \tag{5.5}$$

are one-dimensional subspaces of $T_x M$.

(For $x\in\{x_1, \ldots, x_m\}$, we simply define $E^s_{x,g}$ [$E^u_{x,g}$] to be the set of all rays tangent to the stable [unstable] prongs.)

Proof. — The differential equations for (ξ_1, ξ_2) under the flow given by the vector field $(\varphi_i^{-1}\Phi_{i,j}^{-1})_* v_{\Psi_{p(i)}}$, which generates g in the sector $S^s_{i,j}$ at x_i, written in the $(s_1, s_2)=\Phi_{i,j}(t_1, t_2)$ coordinate system, are identical with those given in (4.1) of [K] (making the notational substitution of Ψ_p for ψ_τ). Since we also have $dg(\xi_1, \xi_2)=df(\xi_1, \xi_2)=(\pm\lambda\xi_1, \pm(1/\lambda)\xi_2)$ at points outside $\bigcup_{i=1}^{m} D^i_{r_3}$, which is the only other fact needed in the proof of the analogous Proposition 4.1 in [K], our proposition follows. □

As in [K], $E^u_{x,g}=E^u_{x,f}$ for $x\in\bigcup_{i,j} P^u_{i,j}\setminus\{x_1, \ldots, x_m\}$ and $E^s_{x,g}=E^s_{x,f}$ for $x\in\bigcup_{i,j} P^s_{i,j}\setminus\{x_1, \ldots, x_m\}$. Now let $L^u_{i,j}$ be given as in (2.9). We claim that $E^u_{x,g}=E^u_{x,f}$ for $x\in L^u_{i,j}\setminus\{x_1, \ldots, x_m\}$. If $L^u_{i,j}\subseteq P^u_{i,j}$ this is clear; otherwise take the first positive integer n such that $f^n(P^u_{i,j})\supseteq L^u_{i,j}$. Note that it follows from (3.3) that $f^n(P^u_{i,j})=g^n(P^u_{i,j})$, which together with the f_*- and g_*- invariance of the line fields $E^u_{x,f}$ and $E^u_{x,g}$, respectively, establishes the claim, i.e., by (2.9), each vector tangent to a $\partial_u R_j$, $j=1, \ldots, N$, lies in $E^u_{x,g}$. Similarly for $\partial_s R_j$ and $E^s_{x,g}$.

We will call a curve stable [unstable] if it is connected and its tangent vector at each point x on it belongs to $E^s_{x,g} \setminus \{0\}$ $[E^u_{x,g} \setminus \{0\}]$. For convenience we also assume that stable curves do not contain $x_1, \ldots, x_m$ as interior points. Since the same simple argument as in [K, Corollary 4.1] shows that $E^s_{x,g}$ and $E^u_{x,g}$ depend continuously on x for $x \in M \setminus \{x_1, \ldots, x_m\}$, we can conclude from the existence theorems for differential equations that for each $x \in M \setminus \{x_1, \ldots, x_m\}$, there exists at least one stable [unstable] curve through x.

Uniqueness of such stable [unstable] curves will be shown in paragraph 6. In fact this will follow from the Uniform Contraction Lemma below, which is also crucial for establishing the topological conjugacy between f and g.

We now proceed to establish some properties of stable curves which we need to prove the Lemma.

PROPOSITION 5.2. — *Suppose* γ *is a stable curve lying in the domain* $D^i_{a'_i}$ *of a singular chart* φ_i. *Then the following properties hold:*

(5.6) Cone condition for stable curves. *For* $j=0, \ldots, p(i)-1$ *the tangent vector to* $\Phi_{i,j}\varphi_i(\gamma)$ *at any point* $(s_1, s_2) \in \Phi_{i,j}\varphi_i(\gamma)$ *lies in* $\mathcal{K}^-_{(s_1, s_2)}$.

(5.7) *If* γ *is not equal to a segment of a stable prong, then* γ *is contained in the interior of some* $S^s_{i,j}$. *(In any case,* γ *lies in a single* $S^s_{i,j}$, *which is in the domain of* $\Phi_{i,j}\varphi_i$. *Thus we will often work with* $\Phi_{i,j}\varphi_i(\gamma)$ *instead of* γ.)

(5.8) *If* $s \in \Phi_{i,j}\varphi_i(\gamma)$ *where* $s=(s_1, s_2)$, $s_1>0$, $s_2>0$, $s_1 s_2=\delta$, *then the tangent vector to* $\Phi_{i,j}\varphi_i(\gamma)$ *at* x *lies in* [(*top half of* $\mathcal{K}^-_s$) $\cap$ (*vectors pointing into the region* $s_1 s_2>\delta$)] $\cup$ [(*bottom half of* $\mathcal{K}^-_s$) $\cap$ (*vectors pointing into the region* $s_1 s_2<\delta$)]. *Analogous statements hold for* s *in other quadrants.*

Proof. — Property (5.6) is obvious from the definition of a stable curve.

Suppose (5.7) is false. Then there exists a stable sector $S^s_{i,j}$ such that γ intersects one of the stable prongs bounding $S^s_{i,j}$, but also contains points in int $S^s_{i,j}$. Then applying large positive powers of $G_{p(i)}$ to $\Phi_{i,j}\varphi_i(\gamma)$ gives curves which have a point on the s_2-axis arbitrarily close to (0, 0) and a point arbitrarily close to $(r_0, 0)$. But such curves cannot satisfy the cone condition, a contradiction. Hence (5.7) holds.

Finally let us establish (5.8). Suppose the tangent vector η to $\Phi_{i,j}\varphi_i(\gamma)$ at s lies in (top half of $\mathcal{K}^-_s$) $\cap$ (vectors pointing into the region $s_1 s_2<\delta$ or tangent to $s_1 s_2=\delta$). Then under the flow of $v_{\Psi_{p(i)}}$, η is transformed into a vector in (top half of $\mathcal{K}^-_y$) $\cap$ (vectors pointing into the region $s_1 s_2<\delta$ or tangent to $s_1 s_2=\delta$), where y is a point on $s_1 s_2=\delta$, $s_1>s_2>0$. But for such a point y this intersection is empty. Hence η cannot lie in (to half of $\mathcal{K}^-_s$) $\cap$ (vectors pointing into the region $s_1 s_2<\delta$ or tangent to $s_1 s_2=\delta$). By reversing the time parameter of γ, it follows that η cannot lie in (bottom half of $\mathcal{K}^-_s$) $\cap$ (vectors pointing into the region $s_1 s_2>\delta$ or tangent to $s_1 s_2=\delta$). Thus (5.8) holds. □

For each singular chart $(\varphi_i, D^i_{a'_i})$ satisfying (2.1) (i)-(iv) and (2.3), we have $l(\gamma \cap D^i_{a'_i} \cap S^s_{i,j}) = \bar{l}(\Phi_{i,j}\varphi_i(\gamma \cap D^i_{a'_i} \cap S^s_{i,j}))$ and for each chart (φ_i, U_i) satisfying (5.1)-(5.4), we have $l(\gamma \cap U_i) = \bar{l}(\varphi_i(\gamma \cap U_i))$. (Recall that l is the length generated by the metric $\rho = du^2 + dv^2$ (*cf.* paragraph 3) and $\bar{l}$ is the usual Cartesian length.)

UNIFORM CONTRACTION LEMMA. — *There exists* $\tau>0$ *with the following property: for every* $\varepsilon>0$ *there exists* $\mathcal{N}=\mathcal{N}(\varepsilon)$ *such that if* γ *is a stable curve with* $l(\gamma)\leqq\tau$ *and* $n\geqq\mathcal{N}$, *then* $l(g^n\gamma)<\varepsilon$.

Remark. — Clearly if there is some $\tau>0$ for which the lemma is true, then it is true for every $\tau>0$. We will simply choose a τ which is convenient for the proof.

Set up for proof of Uniform Contraction Lemma. — Let:

$$J=\max(\sup_{x\in M}\|dg_x\|,\ \sup_{x\in M}\|dg_x^{-1}\|).$$

By proposition 2.3 in [B-F-K], there exist constants $C_0>1$, $0<K_0<1$ such that:

$$\|(dg^n_{D^c_{r_0}})e_s\|\leqq C_0 K_0^n\|e_s\|,\qquad n\geqq 0 \tag{5.9}$$

for $e_s\in E^s_{x,g}$, $x\in D^c_{r_0}$ such that $g^n_{D^c_{r_0}}x$ is defined. Here we take dg_A, $A\subset M$, to be the map:

$$dg_A:\quad TA\to TA\quad \text{by}\quad (dg_A)_x=(dg^{n_x})_x,$$

where n_x is the first return time of x to A. As in paragraph 3, let P be the largest number of prongs of any singularity of the $(\mathfrak{F}^u, \mu^u)$ foliation.

(5.10) Let a be an integer such that for each p; $3\leqq p\leqq \underline{P}$, and for $x\in\mathcal{D}_{r_2}\diagdown G^{-1}\mathcal{D}_{r_2}$, there exists an integer a', $0<a'\leqq a$, such that:

$$G_p^{a'}(x)\in(\mathcal{D}_{r'_1}\diagdown\mathcal{D}_{r'_2})\cap\{(s_1, s_2): s_1>|s_2|\}.$$

Let $\mathcal{B}_0$ be the region bounded by $\{(s_1, s_2): s_1=0\}$,

$$\partial\mathcal{D}_{r_0}\cap\{(s_1, s_2): |s_2|\geqq s_1\geqq 0\},$$
$$\partial\mathcal{D}_{r'_0}\cap\{(s_1, s_2): s_1\geqq|s_2|\},$$

and the branches of hyperbolas $s_1s_2=r_0^2/2$, $s_1\geqq 0$ and:

$$s_1s_2=\frac{-r_0^2}{2},\qquad s_1\geqq 0.$$

(5.11) Choose τ' such that:

$$0<\tau'<\min\left(1,\ r'_0-r_0,\ r_0-r_1,\ r_1-r_2,\ r_2-r_3,\ \frac{\sqrt{2}}{4}r_2\right).$$

(5.12) Then if $\delta_3>0$ is choosen sufficiently small and $\mathcal{B}_3$ is the region bounded by $\{(s_1, s_2): s_1=0\}$, $\partial\mathcal{D}_{r_0}$, and the branches of the hyperbolas $s_1s_2=\delta_3$, $s_1\geqq 0$ and $s_1s_2=-\delta_3$, $s_1\geqq 0$, we have:

$$\left\{\begin{array}{l}\tau'<\bar{d}(\mathcal{B}_3\cap(\mathcal{D}_{r_1}\diagdown\mathcal{D}_{r_2})\cap\{(s_1, s_2): |s_2|\geqq s_1\},\\ \qquad\qquad\{(s_1, s_2): s_1\geqq|s_2|\}).\end{array}\right. \tag{5.13}$$

Let τ be such that:

(5.14) $$0<\tau<\frac{\tau'}{C_0 J^{\max(a,\, q+2)}},$$

where q is as in (3.5).

(5.15) Choose ε' such that $0<\varepsilon'<\min(\varepsilon, \tau')/10$.

(5.16) Choose a positive integer n_0 such that $G_p^{n_0}(0, r'_0)$ has s_2 coordinate less than ε' for $3\leqq p\leqq P$. Let $\mathscr{B}_1$ be the region bounded by $\{(s_1, s_2); s_1=0\}$,

$$\partial\mathscr{D}_{r'_0}\cap\{(s_1, s_2): |s_2|\geqq s_1\geqq 0\},$$
$$\partial\mathscr{D}_{r_0}\cap\{(s_1, s_2): s_1\geqq |s_2|\},$$

and the branches of hyperbolas $s_1 s_2=\delta_1$, $s_1\geqq 0$ and $s_1 s_2=-\delta_1$, $s_1\geqq 0$, where δ_1 is chosen so that:

$$0<\delta_1<\min\left(\frac{r_2^2}{2}, (\varepsilon')^2\right)$$

and for any point $(s_1, s_2)\in\mathscr{B}_1$ with $|s_2|\geqq s_1$, we have:

(5.17) $$\bar{d}(G_p^n(s_1, s_2), G_p^n(0, s_2))<\varepsilon' \quad \text{for} \quad 3\leqq p\leqq P, \qquad 0\leqq n\leqq n_0.$$

(5.18) Let b be a positive integer such that for each $(s_1, s_2)\in\mathscr{D}_{r_0}\setminus\mathscr{B}_1$ and each p, $3\leqq p\leqq P$, there exists an integer n, $0<n\leqq b$ such that $G_p^{-n}(s_1, s_2)\in\mathscr{D}_{r_0}^c$.

Choose ξ' such that:

(5.19) $$\begin{cases} 0<\xi'<\bar{d}(\mathscr{B}_1\cap(\mathscr{D}_{r_1}\setminus\mathscr{D}_{r_2})\cap\{(s_1, s_2): |s_2|\geqq s_1\}, \\ \qquad\qquad \{(s_1, s_2): s_1\geqq |s_2|\}) \end{cases}$$

and if δ_3 is chosen sufficiently small and $\mathscr{B}_3$ is as in (5.12), then:

(5.20) $$\xi'<\bar{d}(\mathscr{B}_3\cap\mathscr{D}_{r_1}, \mathscr{B}_1^c).$$

Let $\mathscr{B}=\mathscr{B}_3\cap\mathscr{D}_{r_2}$.

Choose ε'' such that:

(5.21) $$0<\varepsilon''<\frac{\min(\varepsilon', \xi', \tau)}{27\, C_0\, J^{\max(a+b,\, q+1)}}.$$

(Since $C_0\geqq 1$ and $J\geqq 1$ we have $\varepsilon''<\varepsilon'$ and $\tau<\tau'$.)

(5.22) Choose a positive integer n_1 so that $G_p^{n_1}(0, r_0)$ has s_2 coordinate less than ε'' for $3\leqq p\leqq P$. Let $\mathscr{B}_2$ be the region bounded by $\{(s_1, s_2): s_1=0\}$, $\partial\mathscr{D}_{r_0}$, and the branches of hyperbolas $s_1 s_2=\delta_2$, $s_1 \geqq 0$ and $s_1 s_2=-\delta_2$, $s_1 \geqq 0$, where δ_2 is chosen so that $0<\delta_2<\min(\delta_1, (\varepsilon'')^2)$ and for any point $(s_1, s_2)\in \mathscr{B}_2$ with $|s_2|\geqq s_1$, we have:

$$\bar{d}(G_p^n(s_1, s_2), G_p^n(0, s_2))<\varepsilon'' \quad \text{for} \quad 3\leqq p \leqq P, \qquad 0\leqq n \leqq n_1. \tag{5.23}$$

Now fix δ_3 such that $0<\delta_3<\delta_2$ and (5.13) and (5.20) hold.

Note that the region $\mathscr{A}\cap\{(s_1, s_2): s_2\geqq 0\}$ is symmetric with respect to the line $s_1=s_2$ for $\mathscr{A}=\mathscr{B}_2, \mathscr{B}_3, \mathscr{B}$ and is not for $\mathscr{A}=\mathscr{B}_0, \mathscr{B}_1$ (see *Fig.* 2). Let $B\subset M$ be the union over the singular points and stable sectors of the inverse images of $\mathscr{B}$ under the corresponding charts $\Phi_{ij}\varphi_i$. Define B_0, B_1, B_2, B_3 similarly. Again, by Proposition 2.3 in [B-F-K], there exist constants $C>1$, $0<K<1$ such that:

$$\|(Dg_{B^c})^n e_s\| \leqq CK^n \|e_s\|, \qquad n\geqq 0, \tag{5.24}$$

for $e_s\in E^s_{x,g}$, $x\in B^c$ such that $g^n_{B^c} x$ is defined.

(5.25) Let $\xi=\bar{d}(\mathscr{B}_3\cap\mathscr{D}_{r_1}, \mathscr{B}_2^c)$.

Choose a positive integer n_2 such that:

$$CK^{n_2}<\min(\xi, \varepsilon''). \tag{5.26}$$

(5.27) Finally choose $\mathscr{N}>(n_2+1)\max(n_1+1, n_2+1)$.

Keeping the notation in the above set-up, we now formulate and prove Lemmas 5.3-5.7, which will be used in the proof of the Uniform Contraction Lemma.

LEMMA 5.3. — *Any curve in the (s_1, s_2)-plane all of whose tangent vectors are within 45° of being vertical cannot intersect both $\mathscr{D}_{r_2}$ and $\mathscr{D}^c_{r_0}\cap\{(s_1, s_2): s_1\geqq|s_2|, |s_1 s_2|\leqq\delta_1\}$.*

Proof. — Suppose $(c_1, c_2)\in\mathscr{D}_{r_2}$ and $(d_1, d_2)\in\mathscr{D}^c_{r_0}\cap\{(s_1, s_2): s_1\geqq|s_2|, |s_1 s_2|\leqq\delta_1\}$. Since $\delta_1<r_2^2/2$, $|d_2|<r_2/\sqrt{2}$. Then since $d_1^2+d_2^2\geqq r_0^2$ and by (3.1) $r_0>4r_2$, we have:

$$d_1>\sqrt{r_0^2-\frac{r_2^2}{2}}>\sqrt{16r_2^2-\frac{r_2^2}{2}}>3r_2.$$

Also, $|c_1|\leqq r_2$ and $|c_2|\leqq r_2$. Thus, $|d_2-c_2|<2r_2$ and $|c_1-d_1|>2r_2$. Hence:

$$\left|\frac{d_2-c_2}{d_1-c_1}\right|<1.$$

The lemma now follows from the mean-value theorem. □

LEMMA 5.4. — *Let $l=0, 1, 2, 3$. Suppose γ is a stable curve contained in B_l and $g\gamma$, $g^2\gamma, \ldots, g^k\gamma$ intersect B. Then $\gamma, g\gamma, \ldots, g^k\gamma$ all lie in the same component of B_l.*

Proof. — We give the argument for B_1. That for B_2 and B_3 is identical, while that for B_0 requires one small change which we mention below. Since γ is connected, it lies in a single component of B_1. Consequently, it is contained in some $D^i_{r_0}\subset D^i_{a_i'}$, the domain of the

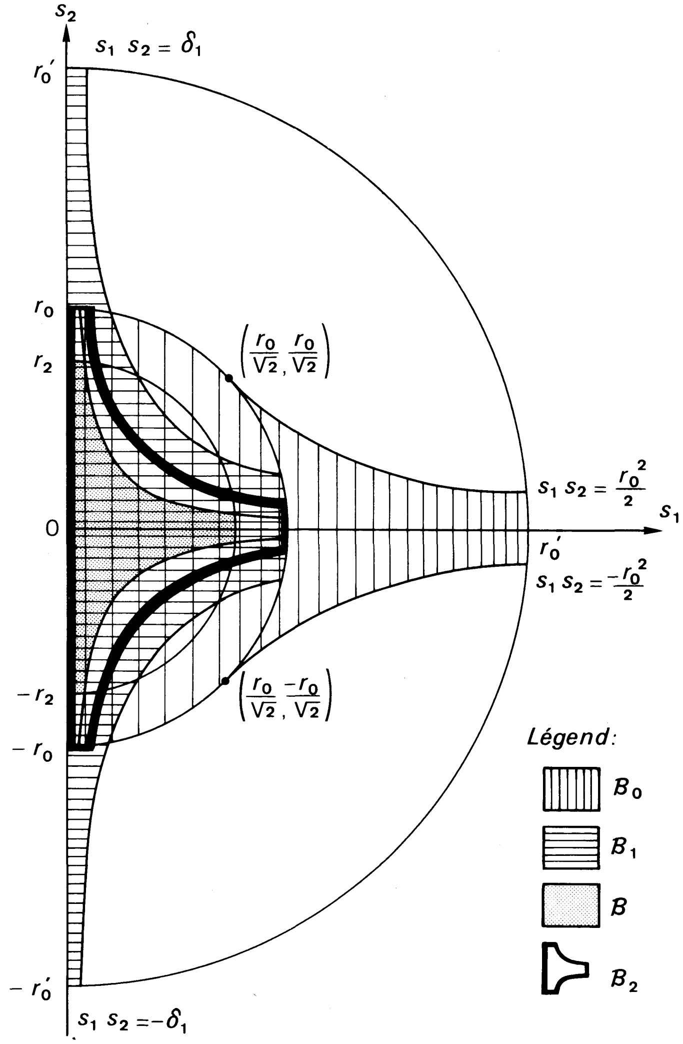

Fig. 2.

chart φ_i. Moreover, by (5.7), γ is contained in a single stable sector S^s_{ij}, the domain of $\Phi_{ij}\varphi_i$. We fix $\Phi\varphi=\Phi_{ij}\varphi_i$ and $G=G_{p(i)}$ throughout the argument. Suppose that for some c, $0\leqq c<k$, $g^c\gamma$ lies in the domain of $\Phi\varphi$ and $g^c\gamma\subset B_1$, but $g^{c+1}\gamma\not\subset B_1$. Since $G(B_1)\subset\mathscr{D}_{a'_i}$ $g^{c+1}\gamma$ is still in the domain of $\Phi\,\varphi$. Then we have $\Phi\,\varphi(g^c\gamma)\subset\mathscr{B}_1$, but $\Phi\varphi(g^{c+1}\gamma)=G(\Phi\varphi(g^c\gamma))\not\subset\mathscr{B}_1$. Thus $\Phi\varphi(g^{c+1}\gamma)$ has a point in $\mathscr{D}^c_{r_0}\cap\{(s_1, s_2): s_1\geqq|s_2|, |s_1 s_2|\leqq\delta_1\}$ and because $g^{c+1}\gamma\cap B\neq\emptyset$, $\Phi\varphi(g^{c+1}\gamma)$ also has a point in $\mathscr{B}\subset\mathscr{D}_{r_2}$. But this contradicts Lemma 5.3. Therefore, for $0\leqq c<k$, if $g^c\gamma$ lies in the domain of $\Phi\varphi$ and $g^c\gamma\subset B_1$, then $g^{c+1}\gamma\subset B_1$. Since $g^{c+1}\gamma$ is in the domain of $\Phi\varphi$, $g^{c+1}\gamma$ lies in the same component of B_1 as $g^c\gamma$. The lemma for the case $l=1$ follows by induction.

Now substitute B_0 for B_1 in the above argument. Then $\Phi\varphi(g^{c+1}\gamma)$ would be a curve lying in $\{(s_1, s_2): |s_1 s_2| \leqq r_0^2/2,\ s_1 \geqq 0\}$ having a point in $\mathscr{D}_{r_2}$ and a point in $\mathscr{D}_{r_0}^c \cap \{(s_1, s_2): s_1 \geqq |s_2|\}$. But such a curve would also have a point in $(\mathscr{D}_{r_1'} \setminus \mathscr{D}_{r_2'}) \cap \{(s_1, s_2): s_1 \geqq |s_2|\}$, contradicting (3.7). □

LEMMA 5.5. — (*Contraction of stable curves near singular stable leaves.*)

(i) *Suppose* γ *is a stable curve and* $\gamma, g\gamma, \ldots, g^k\gamma$ *are contained in* B_1. *Then:*

$$l(g^k\gamma) < 2\,l(\gamma) + 8\,\varepsilon',$$

and if $k \geqq n_0$, $l(g^k\gamma) < 8\,\varepsilon'$.

(ii) *If* γ *is a stable curve with* $l(\gamma) < \xi'$ *and such that* $\gamma \cap B = \emptyset$, *while* $g\gamma, \ldots, g^k\gamma$ *intersect* B, *then:*

$$l(g^k\gamma) < 2\,l(\gamma) + 8\,\varepsilon',$$

and if $k \geqq n_0$, $l(g^k\gamma) < 8\,\varepsilon'$.

(iii) *Suppose* γ *is a stable curve and* $\gamma, g y, \ldots, g^k\gamma$ *are contained in* B_2. *Then:*

$$l(g^k\gamma) < 2\,l(\gamma) + 8\,\varepsilon'',$$

and if $k \geqq n_1$, $(g^k\gamma) < 8\,\varepsilon''$.

(iv) *If the hypothesis of* (ii) *holds with* ξ' *replaced by* ξ, *then:*

$$l(g^k\gamma) < 2\,l(\gamma) + 8\,\varepsilon'',$$

and if $k \geqq n_1$, $l(g^k\gamma) < 8\,\varepsilon''$.

Proof of (i). — Again, γ is contained in a single stable sector S_{ij}^s, which is the domain of $\Phi_{ij}\varphi_i$. Fix $\Phi\varphi = \Phi_{ij}\varphi_i$ and $G = G_{p(i)}$. Recall that the range of $\Phi\varphi$ is $\mathscr{D}_{a_i'} \cap \{(s_1, s_2): s_1 \geqq 0\}$. Also $G(B_1) \subset \mathscr{D}_{r_0'} \cap \{(s_1, s_2): s_1 \geqq 0\}$ and the map g is given by G (with respect to the singular chart $\Phi\varphi$) in S_{ij}^s. Since $\gamma, g\gamma, \ldots, g^k\gamma$ all lie in B_1, it now follows that $\gamma, g\gamma, \ldots, g^k\gamma$ in fact lie in the same component of B_1, and therefore in the domain of $\Phi\varphi$.

Let $c = (c_1, c_2)$, $d = (d_1, d_2)$ be the endpoints of $\Phi\varphi\gamma$.

Case 1. — Assume $k \geqq n_0$. If $c \in \mathscr{B}_1 \cap \{(s_1, s_2): |s_2| \geqq s_1\}$, then by (5.17):

$$\bar{d}(G^{n_0}(c_1, c_2), G^{n_0}(0, c_2)) < \varepsilon',$$

and since by (5.16) $|s_2$-coordinate of $G^{n_0}(0, c_2)| \leqq s_2$-coordinate of $G^{n_0}(0, r_0') < \varepsilon'$, it follows that $|s_2$-coordinate of $G^{n_0}(c)| < 2\,\varepsilon'$.

Upon iteration by G, the absolute value of the s_2-coordinate of any point decreases. Therefore:

$$|s_2\text{-coordinate of } G^k(c)| < 2\varepsilon'.$$

If $c \in \mathscr{B}_1 \cap \{(s_1, s_2): |s_2| \leqq s_1\}$, then we already have $|c_2| < \varepsilon'$, because $\delta_1 < (\varepsilon')^2$, and consequently:

$$|s_2\text{-coordinate of } G^k(c)| < \varepsilon'.$$

Therefore in all subcases we have:

$$|s_2\text{-coordinate of } G^k(c)| < 2\varepsilon'$$

and:

$$|s_2\text{-coordinate of } G^k(d)| < 2\varepsilon'.$$

Hence the s_2-coordinates of the endpoints of $G^k(\Phi\varphi\gamma)$ differ by less than $4\varepsilon'$. Thus we have:

$$l(g^k\gamma) = \bar{l}(G^k(\Phi\varphi\gamma)) < 8\varepsilon',$$

the second inequality following from the cone condition for stable curves (5.6).

Case 2. — Assume $k < n_0$ and $c, d \in \mathscr{B}_1 \cap \{(s_1, s_2): |s_2| \geqq s_1\}$. Again by (5.17) we have:

$$|(s_2\text{-coordinate of } G^k(c)) - (s_2\text{-coordinate of } G^k(0, c_2))| < \varepsilon'$$

and:

$$|(s_2\text{-coordinate of } G^k(d)) - (s_2\text{-coordinate of } G^k(0, d_2))| < \varepsilon'.$$

Also:

$$|(s_2\text{-coordinate of } G^k(0, c_2)) - (s_2\text{-coordinate of } G^k(0, d_2))| \leqq |c_2 - d_2| \leqq \bar{l}(\Phi\varphi\gamma) = l(\gamma).$$

Thus:

$$|(s_2\text{-coordinate of } G^k(c)) - (s_2\text{-coordinate of } G^k(d))| < 2\varepsilon' + l(\gamma).$$

Then applying the cone condition again, we see that:

$$l(g^k\gamma) = \bar{l}(G^k(\Phi\varphi\gamma)) < 4\varepsilon' + 2l(\gamma).$$

Case 3. — Assume $k < n_0$ and $c, d \in \perp_1 \quad \{(s_1, s_2): |s_2| < s_1\}$. Then:

$$|s_2\text{-coordinate of } G^k(c)| < \varepsilon' \text{ and } |s_2\text{-coordinate of } G^k(d))| < \varepsilon'.$$

Consequently:

$$l(g^k\gamma) < 4\varepsilon'.$$

Case 4. — Assume $k \leqq n_0$ and:

$$c \in \mathscr{B}_1 \cap \{(s_1, s_2): |s_2| \geqq s_1\}, \qquad d \in \mathscr{B}_1 \cap \{(s_1, s_2): |s_2| < s_1\}.$$

Then:

$$|(s_2\text{-coordinate of } G^k(c)) - (s_2\text{-coordinate of } G^k(0, c_2))| < \varepsilon', \; |d_2| < \varepsilon',$$

and.

$$|s_2\text{-coordinate of } G^k(d)| < \varepsilon'.$$

Also:

$$|s_2\text{-coordinate of } G^k(0, c_2)| \leqq |c_2| < |c_2 - d_2| + \varepsilon' \leqq \bar{l}(\Phi\varphi\gamma) + \varepsilon' = l(\gamma) + \varepsilon'.$$

Thus:

$$|s_2\text{-coordinate of } G^k(c)| < l(\gamma) + 2\varepsilon'.$$

Therefore:

$$l(g^k\gamma) \leqq 2[l(\gamma) + 2\varepsilon' + \varepsilon'] = 2l(\gamma) + 6\varepsilon'.$$

This completes the proof of (i).

Proof of (ii). — Since $\gamma \cap B = \emptyset$ and $g\gamma \cap B \neq \emptyset$, there is a point $x \in \gamma$ and a $\Phi\varphi = \Phi_{ij}\varphi_i$ such that $\Phi\varphi x \in \mathscr{B}_3 \cap (\mathscr{D}_{r_1} \setminus \mathscr{D}_{r_2})$. Since $l(\gamma) < \xi' < \bar{d}(\mathscr{B}_3 \cap \mathscr{D}_{r_1}, \mathscr{B}_1^c)$, this implies that all of γ lies in the domain of $\Phi\varphi$ and $\Phi\varphi(\gamma) \subset \mathscr{B}_1$. Thus $\gamma \subset B_1$. Now (ii) follows from (i) and Lemma 5.4.

Proof of (iii). — The argument is the same as for (i), because $\mathscr{B}_2$ is defined in terms of ε'', n_1 and r_0 analogously to the way $\mathscr{B}_1$ is defined in terms of ε', n_0 and r_0'.

Proof of (iv). — The version of Lemma 5.4 with $n=2$ and the fact that $l(\gamma) < \xi < \bar{d}(\mathscr{B}_3 \cap \mathscr{D}_{r_1}, \mathscr{B}_2^c)$ make (iv) follow from (iii) as (ii) follows from (i). □

LEMMA 5.6. — (*Controlling lengths of stable curves during one passage through* B.) *Suppose γ is a stable curve contained in $D_{r_0}^i$ for some i, $\gamma, g\gamma, \ldots, g^{k-1}\gamma$ intersect* B *and $g^k\gamma$ does not intersect* B. *Then $g^{k+a'-1}\gamma \subset D_{r_0}^c$ for some integer a', $0 < a' \leqq a$, where a is defined as in* (5.10), *and:*

$$l(g^k\gamma) < J^{a'-1} l(\gamma) \leqq J^{a-1} l(\gamma).$$

Proof. — By Lemma 5.4, $\gamma, g\gamma, \ldots, g^{k-1}\gamma$ all lie in the same component of B_0, and hence in the domain of a single $\Phi\varphi = \Phi_{ij}\varphi_i$ such that $\Phi\varphi(\gamma), \Phi\varphi(g\gamma), \ldots, \Phi\varphi(g^{k-1}\gamma)$ all lie in $\mathscr{B}_0$. Moreover, since $G(\mathscr{B}_0) \subset G(\mathscr{D}_{r_0'}) \subset \mathscr{D}_{a_1'}$, $g^k\gamma$ also lies in the domain of $\Phi\varphi$. Since $g^{k-1}\gamma \cap B \neq \emptyset$ and $g^k\gamma \cap B = \emptyset$, $\Phi\varphi(g^{k-1}\gamma)$ has a point in $(\mathscr{D}_{r_2} \setminus G^{-1}\mathscr{D}_{r_2})$. Thus by (5.10) there exists an integer a', $0 < a' \leqq a$, such that $G^{a'}(\Phi\varphi(g^{k-1}\gamma))$ has a point in $(\mathscr{D}_{r_1'} \setminus \mathscr{D}_{r_2'}) \cap \{(s_1, s_2): s_1 \geqq |s_2|\}$. Let H_1 and H_2 be branches of hyperbolas of the form $s_1 s_2 = c_1$, $s_1 \geqq 0$, $s_1 s_2 = c_2$, $s_1 \geqq 0$ such that H_1 and H_2 intersect $\Phi\varphi(\gamma)$, and $\Phi\varphi(\gamma)$ lies in the closed region bounded by H_1, H_2, $\partial\mathscr{D}_{r_0}$ and $\{(s_1, s_2): s_1 = 0\}$. (By property (5.8) of stable curves, these hyperbolas pass through the endpoints of $\Phi\varphi(\gamma)$, but we do not need this fact here.) Then $G^{a'}(\Phi\varphi(g^{k-1}\gamma))$ also lies between H_1 and H_2. From (3.7) it follows that $G^{a'}(\Phi\varphi(g^{k-1}\gamma))$ lies in $\mathscr{D}_{r'_0}\setminus\mathscr{D}_{r'_r}$ and thus $g^{k+a'-1}\gamma\subset D^c_{r'_0}$. Also by (3.7), we have:

$$l(g^{k+a'-1}\gamma)=\bar{l}(G^{a'}(\Phi\varphi(g^{k-1}\gamma)))<\bar{d}(H_1\cap\mathscr{D}_{r_0}, H_2\cap\mathscr{D}_{r_0})\leq l(\gamma).$$

Thus:

$$l(g^k\gamma)\leqq J^{a'-1}l(g^{k+a'-1}\gamma)<J^{a'-1}l(\gamma)\leqq J^{a-1}l(\gamma).\quad\square$$

LEMMA 5.7. — (*Controlling length of a stable curve at the beginning of a passage through* B.) *If* γ *is a stable curve with* $l(\gamma)\leqq\tau$ *and* $k>0$ *is such that* $g^{k-1}\gamma\cap B=\emptyset$ *while* $g^k\gamma\cap B\neq\emptyset$, *then:*

$$l(g^k\gamma)<C_0 J^{q+1}l(\gamma)<\tau'.$$

Proof. — Since $g^{k-1}\gamma\cap B=\emptyset$ and $g^k\gamma\cap B\neq\emptyset$, there is some point $x\in g^{k-1}\gamma$ lying in the domain of some $\Phi\varphi=\Phi_{ij}\varphi_i$ such that:

$$\Phi\varphi(x)\in\mathscr{B}_3\cap(\mathscr{D}_{r_1}\setminus\mathscr{D}_{r_2})\cap\{(s_1, s_2):|s_2|\geqq s_1\geqq 0\}.$$

Suppose that $l(g^{k-1}\gamma)\leqq\tau'$. (This assumption will be justified below.) Then by the choice of τ' given in (5.11) and by (5.13), $g^{k-1}\gamma$ lies in the domain of $\Phi\varphi$ and:

$$\Phi\varphi(g^{k-1}\gamma)\subset(\mathscr{D}_{r_0}\setminus\mathscr{D}_{r_3})\cap\{(s_1, s_2):|s_2|\geqq s_1\geqq 0\}.$$

Then by (3.5), we have:

$$\begin{aligned}\Phi\varphi(g^{k-1-q}\gamma)&=G^{-q}(\Phi\varphi(g^{k-1}\gamma))=F^{-q}(\Phi\varphi(g^{k-1}\gamma))\\&\subset(\mathscr{D}_{r'_0}\setminus\mathscr{D}_{r'_3})\cap\{(s_1, s_2):|s_2|\geqq s_1\geqq 0\}\\&\subset(\mathscr{D}_{r'_0}\setminus\mathscr{D}_{r_0})\cap\{(s_1, s_2):|s_2|\geqq s_1\geqq 0\}.\end{aligned}$$

Now divide γ into disjoint subsets $\gamma_0, \gamma_1, \gamma_2, \ldots,$ such that $\gamma_0\subset D^c_{r_0}$ and for $m>0$ there exists a $\overline{\Phi\varphi}=\Phi_{i'j'}\varphi_{i'}$ such that $\overline{\Phi\varphi}(\gamma_m)\subset\mathscr{D}_{r_0}$, $\overline{\Phi\varphi}(g\gamma_m)\subset\mathscr{D}_{r'_2}$, $\overline{\Phi\varphi}(g^2\gamma_m)\subset\mathscr{D}_{r'_2}, \ldots, \overline{\Phi\varphi}(g^{m-1}\gamma_m)\subset\mathscr{D}_{r'_2}$, $\overline{\Phi\varphi}(g^m\gamma_m)\subset(\mathscr{D}_{r'_1}\setminus\mathscr{D}_{r'_2})\cap\{(s_1, s_2):s_1\geqq|s_2|\}$. (Note that since $l(\gamma)<\tau<\tau'<r'_0-r_0$, γ can intersect at most one component of D_{r_0} and hence only one $\Phi_{i'j'}\varphi_{i'}$ is needed here.) By (3.7), we have:

$$l(g^m\gamma_m)=\bar{l}(\overline{\Phi\varphi}(g^m\gamma_m))<\bar{l}(\overline{\Phi\varphi}(\gamma_m))=l(\gamma_m).$$

For $m>0$ such that $\gamma_m\neq\emptyset$, we have $k-1-q>m$. Then since $g^{k-1-q}\gamma_m$ and $g^m\gamma_m$ both lie in $D^c_{r_0}$, it follows from (5.9) that:

$$l(g^{k-1-q}\gamma_m)\leqq C_0 l(g^m\gamma_m)<C_0 l(\gamma_m).$$

Also, since γ_0 and $g^{k-1-q}\gamma_0$ both lie in $D^c_{r_0}$, it follows again from (5.9) that:

$$l(g^{k-1-q}\gamma_0)\leqq C_0 l(\gamma_0).$$

Thus $l(g^{k-1-q}\gamma)\leqq C_0\, l(\gamma)$ and consequently:

$$l(g^k\gamma)\leqq C_0\, J^{q+1}\, l(\gamma)\leqq C_0\, J^{q+1}\, \tau<\frac{\tau'}{J}<\tau'.$$

To see that $l(g^{k-1}\gamma)<\tau'$, we argue by contradiction and suppose that $l(g^{k-1}\gamma)\geqq\tau'$. Then we truncate γ to $\bar{\gamma}$ so that $l(g^{k-1}\bar{\gamma})=\tau'$ and $g^k\bar{\gamma}$ still intersects B. Then $l(g^k\bar{\gamma})<\tau'/J'$ which implies that $g^{k-1}(\bar{\gamma})<\tau'$, a contradiction. □

Proof of Uniform Contraction Lemma. — Fix a stable curve $\gamma\subset M$ with $l(\gamma)\leqq\tau$. Fix $n\geqq\mathcal{N}$. By (5.27), one of the following must hold:

Case 1. — There are at least n_2+1 iterates among $\gamma, g\gamma, \ldots, g^n\gamma$ which lie completely in B^c.

Case 2. — There are at least $\max(n_1+1, n_2+1)$ consecutive iterates among γ, $g\gamma, \ldots, g^n\gamma$ which lie partly in B.

Proof for Case 1. — Let:

$$k_0=\min\{\, j\geqq 0 : g^j\gamma\subset B^c\}$$

and:

$$k_1=\max\{\, j\leqq n : g^j\gamma\subset B^c\}.$$

The argument for Case 1 involves three steps: estimating, in order, $l(g^{k_0}\gamma)$, $l(g^{k_1}\gamma)$ and $l(g^n\gamma)$.

Step 1. — If $k_0=0$, then $l(g^{k_0}\gamma)=l(\gamma)\leqq\tau<\tau'$. Suppose $k_0>0$. Then $\gamma, g\gamma, \ldots, g^{k_0-1}\gamma$ intersect B, while $g^{k_0}\gamma$ does not intersect B. Also, since $\gamma\cap B\neq\emptyset$ and $l(\gamma)<\tau<\tau'<r_0-r_2$, γ is contained in some $D^i_{r_0}$. Hence by Lemma 5.6,

$$l(g^{k_0}\gamma)<J^a\, l(\gamma)\leqq J^a\,\tau<\tau'.$$

Thus we have $l(g^{k_0}\gamma)<\tau'$ in both subcases.

Step 2. — We proceed with the estimate for $l(g^{k_1}\gamma)$. By the assumption of Case 1, if $x\in\gamma$, then $g^{k_1}x=g^{j_x}_{B^c}\, g^{k_0}x$, where $j_x\geqq n_2$. Thus, it follows from (5.24) and (5.26) that:

$$l(g^{k_1}\gamma)\leqq CK^{n_2}\, l(g^{k_0}\gamma)\leqq CK^{n_2}\,\tau'<CK^{n_2}<\varepsilon''.$$

Step 3. — Since $\varepsilon''<\varepsilon'<\varepsilon$, if $k_1=n$, we are done with Case 1. Suppose $k_1<n$. Then $g^{k_1}\gamma\cap B=\emptyset$, while $g^{k_1+1}\gamma, \ldots, g^n\gamma$ intersect B. Since $l(g^{k_1}\gamma)<\varepsilon''<\xi'$, by Lemma 5.5 (ii),

$$l(g^n\gamma)<2\, l(g^{k_1}\gamma)+8\,\varepsilon'<2\,\varepsilon''+8\,\varepsilon'<10\,\varepsilon'<\varepsilon.$$

Proof for Case 2. — Let $k'_0=\min\{\, j\geqq 0 : g^j\gamma, g^{j+1}\gamma, \ldots, g^{j+\max(n_1, n_2)}\gamma$ all intersect B$\}$ and $k'_1=\max\{\, k'_0\leqq j\leqq n : g^{k'_0}\gamma, g^{k'_0+1}\gamma, \ldots, g^j\gamma$ all intersect B$\}$ and let k_1 be as in Case 1, if defined.

We estimate, in order, $l(g^{k'_0}\gamma)$, $l(g^{k'_1}\gamma)$, $l(g^{k_1}\gamma)$ and $l(g^n\gamma)$.

Step 1. — If $k'_0=0$, $l(g^{k'_0}\gamma)=l(\gamma)\leqq\tau<\tau'$. Suppose $k'_0>0$. Then $g^{k'_0-1}\gamma\subset B^c$, while $g^{k'_0}\gamma$ intersects B. Thus, by Lemma 5.7,

$$l(g^{k'_0}\gamma)<\tau'.$$

Step 2. — We now estimate $l(g^{k'_1}\gamma)$. Since $g^{k'_0}\gamma$ intersects $B\subset D_{r_2}$ and $l(g^{k'_0}\gamma)<\tau'<r_1-r_2$, $g^{k'_0}\gamma$ lies in a component of D_{r_1} and hence in the domain of some $\Phi\varphi=\varphi_{ij}\varphi_i$. Divide γ into two pieces γ' and γ'' so that $\Phi\varphi(g^{k'_0}\gamma')\subset\mathscr{B}_3$ and $\Phi\varphi(g^{k'_0}\gamma'')\subset\mathscr{D}_{r_1}\setminus\mathscr{B}_3$. By the properties of stable curves given in Proposition 5.2, γ' is connected and γ'' consists of at most two connected components. Note that by Lemma 5.4, $g^{k'_0}\gamma, g^{k'_0+1}\gamma, \ldots, g^{k'_1}\gamma$ all lie in the same component of B_0. Thus $g^{k'_0}\gamma'', \ldots, g^{k'_1}\gamma''$ all lie in B^c. Hence by (5.24) and (5.26), we have:

$$l(g^{k'_1}\gamma'')\leqq CK^{k'_1-k'_0}\,l(g^{k'_0}\gamma'')\leqq CK^{n_2}\,l(g^{k'_0}\gamma)<CK^{n_2}\tau'<CK^{n_2}<\varepsilon''.$$

Next we estimate $l(g^{k'_1}\gamma')$. Since $g^{k'_0}\gamma'$ is a stable curve contained in $B_3\subset B_2$, and $g^{k'_0+1}\gamma', \ldots, g^{k'_1}\gamma'$ intersect B, it follows from Lemma 5.4 that $g^{k'_0}\gamma', \ldots, g^{k'_1}\gamma'$ all lie in B_2. Then since $k'_1-k'_0\geqq n_1$, Lemma 5.5 (iii) implies that:

$$l(g^{k'_1}\gamma')<8\,\varepsilon''.$$

Thus $l(g^{k'_1}\gamma)<\varepsilon''+8\,\varepsilon''=9\,\varepsilon''$.

Step 3. — If $k'_1=n$, we are done because $9\,\varepsilon''<9\,\varepsilon'<\varepsilon$. Thus we may assume $k'_1<n$. Then k_1 is defined and $k'_1<k_1\leqq n$.

Case 2.1. — $k_1<n$. Then $g^{k_1}\gamma\cap B=\emptyset$ and $g^{k_1+1}\gamma\cap B\neq\emptyset$. Consequently, by Lemma 5.7, since $l(g^{k'_1}\gamma)<9\,\varepsilon''<\tau$,

$$l(g^{k_1}\gamma)<C_0\,J^{q+1}\,l(g^{k'_1}\gamma)<9\,\varepsilon''\,C_0\,J^{q+1}<\min(\varepsilon',\xi').$$

Since $g^{k_1}\gamma\subset B^c$ and $g^{k_1+1}\gamma, \ldots, g^n\gamma$ all intersect B, it follows from Lemma 5.5 (ii) that:

$$l(g^n\gamma)<2\,l(g^{k_1}\gamma)+8\,\varepsilon'<2\,\varepsilon'+8\,\varepsilon'=10\,\varepsilon'<\varepsilon.$$

Case 2.2. — $k_1=n$ and $g^{k_1+1}\gamma\cap B\neq\emptyset$. Then the first estimate in the argument for Case 2.1 gives $l(g^n\gamma)<\varepsilon'<\varepsilon$.

Case 2.3. — $k_1=n$ and $g^{k_1+1}\gamma\subset B^c$. Assume that $l(g^n\gamma)\leqq\tau'<r'_0-r_0$. (This assumption will be justified later.) Then $g^n\gamma$ intersects at most one component of D_{r_0}. Now divide γ into pieces $\gamma_1, \gamma_2, \gamma_3$ such that $g^n\gamma_1\subset(B_1\cap D_{r_0})\setminus B$, $g^n\gamma_2\subset D_{r_0}\setminus B_1$, and $g^n\gamma_3\subset D^c_{r_0}$. Then by property (5.8) of stable curves, γ_1 is connected, but γ_2 and γ_3 need not be. Note that by Lemma 5.6, since $g^{k'_0}\gamma\subset D_{r_0}$ and $g^{k'_0}\gamma, \ldots, g^{k'_1}\gamma$ intersect B, while $g^{k'_1+1}\gamma\subset B^c$,

$$g^{k'_1+a'}\gamma\subset D^c_{r_0},\ \text{for some } a',\ 0<a'\leqq a. \tag{5.28}$$

Thus, by (5.9), since $g^n\gamma_3\subset D^c_{r_0}$, we have:

$$l(g^n\gamma_3)\leqq J^{a'}\,C_0\,l(g^{k'_1}\gamma_3)\leqq J^a\,C_0\,l(g^{k'_1}\gamma_3).$$

Also, since by (5.18) it is possible to move any point on $g^n\gamma_2$ out of D_{r_0} in at most b iterates of g^{-1}, (5.9) and (5.28) imply that:

$$l(g^n\gamma_2)\leqq J^{a+b}C_0\, l(g^{k'_1}\gamma_2).$$

Finally we estimate $l(g^n\gamma_1)$. Let i be the first positive integer such that for some point $c\in\Phi\varphi(g^n\gamma_1)$, $G^{-i}c\in\mathscr{D}_{r'_1}\setminus\mathscr{D}_{r'_2}$. Then $g^{n-i}\gamma_1, \ldots, g^n\gamma_1$ all lie in B_1. Now if $l(g^{n-i}\gamma_1)\leqq\min(r'_0-r'_1, r'_2-r_0)$, we have:

$$\Phi\varphi(g^{n-i}\gamma_1)\subset(\mathscr{D}_{r'_0}\setminus\mathscr{D}_{r_0})\cap\mathscr{B}_1.$$

Since $g^{n-i}\gamma_1\subset D^c_{r_0}$ and $g^{k'_1+a'}\gamma_1\subset D^c_{r_0}$, where $0<a'\leqq a$, and $k'_1<n-i$,

$$l(g^{n-i}\gamma_1)\leqq J^a C_0\, l(g^{k'_1}\gamma_1)\quad<J^a C_0(9\varepsilon'')<\tau'<\min(r'_0-r'_1, r'_2-r'_0).$$

Thus we can use a truncation argument, as in the proof of Lemma 5.7, to see that, in fact, we must have $l(g^{n-i}\gamma_1)\leqq\min(r'_0-r'_1, r'_2-r_0)$ to begin with and, consequently, $l(g^{n-i}\gamma_1)\leqq 9J^a C_0\varepsilon''$. Since $g^{n-i}\gamma_1, \ldots, g^n\gamma_1$ all lie in B_1. It follows from Lemma 5.5 (i),

$$l(g^n\gamma_1)<2\, l(g^{n-i}\gamma_1)+8\varepsilon'<18J^a C_0\varepsilon''+8\varepsilon'.$$

Hence:

$$\begin{aligned} l(g^n\gamma)&=l(g^n\gamma_1)+l(g^n\gamma_2)+l(g^n\gamma_3)\\ &<18J^aC_0\varepsilon''+8\varepsilon'+J^{a+b}C_0\, l(g^{k'_1}\gamma_2)+J^{a+1}C_0\, l(g^{k'_1}\gamma_3)\\ &<18J^aC_0\varepsilon''+8\varepsilon'+J^{a+b}C_0(9\varepsilon'')\\ &<27C_0J^{a+b}\varepsilon''+8\varepsilon'<9\varepsilon'<\min(\varepsilon, \tau'). \end{aligned}$$

Now another truncation argument can be used to justify the initial assumption that $l(g^n\gamma)\leqq\tau'$. Therefore $l(g^n\gamma)<\varepsilon$. □

6. Corollaries of Uniform Contraction Lemma and Uniqueness of Stable Curves

COROLLARY 6.1 (Global contraction). — *For every stable curve γ, the length $l(g^n\gamma)\to 0$ as $n\to\infty$ and this convergence is uniform for all curves of uniformly bounded length.*

Proof. — Let $K(\gamma)=[l(\gamma)/\tau]+1$. Let us divide γ into pieces $\gamma_1, \ldots, \gamma_{K(\gamma)}$ such that $l(\gamma_i)<\tau$, $i=1, \ldots, K(\gamma)$. By the Uniform Contraction Lemma, we have $l(g^n\gamma_i)\to 0$ uniformly for all γ and all $i=1, \ldots, K(\gamma)$. □

COROLLARY 6.2 (Uniform expansion). — *For any $\varepsilon>0$, $R>0$, there exists a natural number $\mathscr{N}(\varepsilon, R)$ such that for every stable curve γ with $l(\gamma)\geqq\varepsilon$ and for every $n\geqq\mathscr{N}(\varepsilon, R)$, we have $l(g^{-n}\gamma)>R$.*

Proof. — Take $\mathscr{N}(\varepsilon, R)=\mathscr{N}(\varepsilon/([R/\tau]+1))$, where $\mathscr{N}(\delta)$ is the function from the Uniform Contraction Lemma, and divide γ into $[R/\tau]+1$ pieces of equal length. The length of every

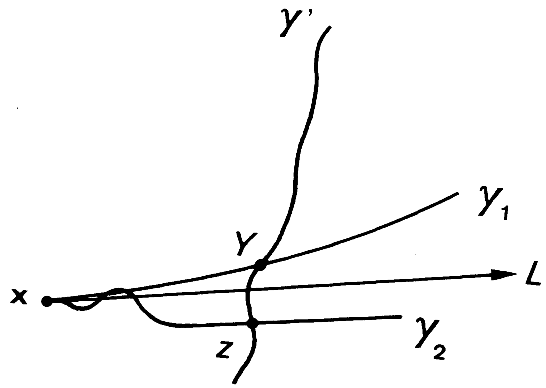

Fig. 3

piece is $\geqq \varepsilon/[R/\tau]+1$. Consequently, by the Uniform Contraction Lemma, the length of the n-th pre-image of every piece is $\geqq \tau$ and $l(g^{-n}\gamma) \geqq \tau([R/\tau]+1) > R$. □

Corollary 6.3 (Expansion and contraction for unstable curves). — *For every unstable curve* γ, $l(g^{-n}\gamma) \to 0$ *as* $n \to \infty$, *the convergence being uniform for curves of uniformly bounded length. For every* $\varepsilon > 0$, $R > 0$, *there exists a natural number* $\mathcal{N}_1(\varepsilon, R)$ *such that for an unstable curve* γ *with* $l(\gamma) \geqq \varepsilon$ *and for* $n \geqq \mathcal{N}_1(\varepsilon, R)$, *we have* $l(g^n\gamma) > R$.

Proof. — Apply Corollaries 6.1 and 6.2 to the map g^{-1}. Unstable curves for g are stable curves for g^{-1}. □

Proposition 6.1 (Uniqueness of stable curves). — *Let* $x \in M \setminus \{x_1, \ldots, x_m\}$, *and let* L *be one of two rays of the space* $E^s_{x, g}$. *Let* γ_1, γ_2 *be two stable curves which begin at* x *and are tangent to* L. *Then one of the curves is a part of the other.*

Proof. — Suppose that the statement were not correct. Then, replacing, if necessary, x by another point on $\gamma_1 \cap \gamma_2$, we can keep all the assumptions and also assume that arbitrarily close to x there are both points belonging to $\gamma_1 \setminus \gamma_2$ and to $\gamma_2 \setminus \gamma_1$.

Let us choose a point $y \in \gamma_1 \setminus \gamma_2$. Since the angle between $E^s_{x, g}$ and $E^u_{x, g}$ is bounded away from 0 outside any fixed neighborhood of $\{x_1, \ldots, x_m\}$, if y is chosen sufficiently close to x, then there is a short unstable curve γ' which passes through y and intersects γ_2. (Recall that close to x the tangents to γ_1 and γ_2 are almost parallel.) Let us denote the point of intersection of γ_2 and γ' by z (cf. *Fig.* 3). Obviously,

$$d(g^n(y), g^n(z)) \leqq d(g^n(y), g^n(x)) + d(g^n(z), g^n(x)) \leqq l(g^n\gamma_1) + l(g^n\gamma_2). \tag{6.1}$$

(Recall that d is the distance generated by our metric ρ (*cf.* paragraph 3).

Thus, by Corollary 6.1, $d(g^n(y), g^n(z))$ goes to 0 as $n \to \infty$ uniformly for all y sufficiently close to x.

On the other hand, by Corollary 6.3, the length of the images of the segment of γ' which connects y and z goes to ∞ as $n \to \infty$. Furthermore, there exist constants $\varepsilon_0 > 0$, $C > 0$, such that if an unstable curve Γ of length $\leqq \varepsilon_0$ connects the points y_1, y_2 then:

(6.2) $$d(y_1, y_2) > C\, l(\Gamma).$$

Choosing the point y sufficiently close to x, one can assure that for some $n > \mathcal{N}(C\,\varepsilon_0/2\,J)$,

(6.3) $$J^{-1}\,\varepsilon_0 \leqq l(g^n \gamma') \leqq \varepsilon_0,$$

where $J = \max_x \| dg_x \|$.

It follows from (6.2) and (6.3) that:

$$d(g^n y, g^n z) > J^{-1}\, C\, \varepsilon_0,$$

which by (6.1) contradicts the Uniform Contraction Lemma. □

7. Topological Conjugacy of the Pseudo-Anosov Map f and the Diffeomorphism g

Let us now consider one of the Markov rectangles R_i provided with a coordinate system (u, v) determined by the invariant foliations and invariant measures of the map f (*cf.* paragraph 2); for the sake of brevity, we will omit the index showing the dependence of the coordinate system on i. The "vertical" sides of the rectangle $u=0$ and $u=l_i$ are pieces of stable leaves of some of the singular points; they will remain stable curves for g as well (*cf.* paragraph 5). Furthermore, a connected component of the intersection of a (sufficiently large) stable curve with R_i has the form:

(7.1) $$\text{graph}\ \varphi = \{(\varphi(v), v), \text{ where } |\varphi(v_1) - \varphi(v_2)| < |v_1 - v_2|\}.$$

Obviously, every connected stable curve in R_i can be extended to a curve of the form (7.1). Thus we can call every stable curve of the form (7.1) a *maximal stable curve* in R_i.

Let us recall that we can associate with f the following $N \times N$ $(0-1)$ matrix:

$$A^f = (a_{ij}), \qquad a_{ij} = \begin{cases} 1 & \text{if } \operatorname{Int}(f R_i \cap R_j) = \varphi, \\ 0 & \text{otherwise.} \end{cases}$$

By (3.3), $g R_i = f R_i$, for every $i = 1, \ldots, N$. Thus the corresponding matrix A^g coincides with A^f. Henceforth, we will denote this matrix simply by A.

As in paragraph 2.4, a finite sequence $\sigma^{(n)} = (\sigma_0, \ldots, \sigma_{n-1})$, $\sigma_i \in \{1, \ldots, N\}$, is called *admissible* if $a_{\sigma_{i-1}\sigma_i} = 1$ for $i = 1, \ldots, n-1$. Accordingly an infinite sequence $\sigma = (\sigma_0, \sigma_1, \ldots)$ is *admissible* if $a_{\sigma_{i-1}\sigma_i} = 1$ for $i = 1, 2, \ldots$ Let us denote for any admissible sequence $\sigma^{(n)} = (\sigma_0, \ldots, \sigma_{n-1})$, the following set:

$$R^g_{\sigma^{(n)}} = \bigcap_{i=0}^{n-1} g^{-i} R_{\sigma_i}.$$

Since the pre-image of a stable curve is a stable curve and the partition $\{R_1, \ldots, R_N\}$ has the Markov property with respect to g, one can see that in the Markov coordinates (u, v) in R_{σ_0}, the set $R^g_{\sigma^{(n)}}$ has the form:

$$\{(u, v) : \varphi_1(v) \leqq u \leqq \varphi_2(v)\}, \tag{7.2}$$

where graph φ_1 and graph φ_2 are two stables curves, and consequently $\varphi_1(v) < \varphi_2(v)$ and:

$$\begin{cases} |\varphi_1(v_1) - \varphi_1(v_2)| \leqq |v_1 - v_2|, \\ |\varphi_2(v_1) - \varphi_2(v_2)| \leqq |v_1 - v_2|. \end{cases} \tag{7.3}$$

PROPOSITION 7.1. — *For any admissible infinite sequence* $\sigma = (\sigma_0, \sigma_1, \ldots)$ *the set:*

$$R^g_\sigma = \bigcap_{i=0}^{\infty} g^{-i} R_{\sigma_i},$$

is a maximal stable curve in R_{σ_0}.

Proof. — Obviously, $R^g_\sigma = \bigcap_{i=0}^{\infty} R^g_{\sigma^{(n)}}$, where $\sigma^{(n)} = (\sigma_0, \ldots, \sigma_{n-1})$. Consequently, R^g_σ has the form (7.2), where (7.3) is satisfied. If $\varphi_1 \equiv \varphi_2$, then the maximal stable curve in R_{σ_0} which begins at the point $(\varphi_1(0), 0)$ must coincide with R^g_σ. For, Proposition 6.1 assures that this maximal stable curve must lie inside every set $R^g_{\sigma^{(n)}}$ and R^g_σ is the only set of the form (7.2) which contains the point $(\varphi_1(0), 0)$ and satisfies that condition. So it remains to prove that $\varphi_1 \equiv \varphi_2$.

Let $R^g_{\sigma^{(n)}} = \{(u, v) : \varphi_1^{\sigma(n)}(v) \leqq u \leqq \varphi_2^{\sigma(n)}(v)\}$. Then the sequence of functions $\varphi_1^{\sigma(n)}$ is non-decreasing, and converges pointwise to φ_1 and similarly for $\varphi_2^{\sigma(n)}$ and φ_2. The Lipschitz condition (7.3) is satisfied for all $\varphi_1^{\sigma(n)}$ and $\varphi_2^{\sigma(n)}$. Consequently, the convergence of $\varphi_1^{\sigma(n)}$ to φ_1 and of $\varphi_2^{\sigma(n)}$ to φ_2 is uniform.

On the other hand, let us consider the set $g^{n-1} R^g_{\sigma^{(n)}} \subset R_{\sigma_{n-1}}$. The images of the curves graph $\varphi_1^{\sigma(n)}$ and graph $\varphi_2^{\sigma(n)}$ are pieces of two components of the stable boundary of the rectangle $R_{\sigma_{n-1}}$. Every maximal unstable curve in $R_{\sigma_{n-1}}$ which begins at a point $y \in g^{n-1}$ (graph $\varphi_1^{\sigma(n)}$) stays within the set $g^{n-1} R^g_{\sigma^{(n)}}$ (uniqueness of unstable curves). Hence, we can connect every such point y to a point at g^{n-1} (graph $\varphi_2^{\sigma(n)}$) by an unstable curve. Applying Corollary 6.3, we can see that given $\varepsilon > 0$, there exists $\mathcal{N}_1(\varepsilon)$ such that for $n > \mathcal{N}_1(\varepsilon)$, the distance from every point $x \in$ graph $\varphi_1^{\sigma(n)}$ to graph $\varphi_2^{\sigma(n)}$ is less than ε. Letting $n \to \infty$ and using the uniform convergence of $\varphi_i^{\sigma(n)}$ to φ_i, $i = 1, 2$, we conclude that graph φ_1 belongs to the closure of graph φ_2, i. e. $\varphi_1 \equiv \varphi_2$. □

The maximal stable curve in R_i which begins at the point $(u, 0)$ can be represented in the form graph φ_u.

PROPOSITION 7.2. — *The function* $\varphi(u, v)$ *defined by* $\varphi(u, v) = \varphi_u(v)$ *is continuous.*

Proof. — Let us fix u_0 and represent the maximal stable curve graph φ_{u_0} as R^g_σ for some admissible infinite sequence $\sigma = (i, \sigma_1, \sigma_2, \ldots)$. If this representation is unique, then for every n, $\varphi_1^{\sigma(n)} < \varphi_{u_0} < \varphi_2^{\sigma(n)}$, the functions $\varphi_1^{\sigma(n)}$ converge uniformly to φ_{u_0} from below, and the

$\varphi_2^{\sigma(n)}$ converge uniformly to φ_{u_0} from above. If $\varphi_1^{\sigma(n)}(0) < u < \varphi_2^{\sigma(n)}(0)$, then $\varphi_1^{\sigma(n)}(v) < \varphi_u(v) < \varphi_2^{\sigma(n)}(v)$. Thus, the family φ_u is continuous at u_0.

If the representation is not unique, then for some n_0 either $\varphi_{u_0} = \varphi_1^{\sigma(n_0)}(v)$ or $\varphi_{u_0} = \varphi_2^{\sigma(n_0)}(v)$. In the first case there is exactly one more admissible sequence $\tilde{\sigma} = (i, \tilde{\sigma}_1, \ldots)$ such that $\varphi_{u_0} = \varphi_2^{\sigma(n_0)}$. Thus, for all n, $\varphi_1^{\tilde{\sigma}(n)} < \varphi_{u_0} < \varphi_2^{\sigma(n)}$ and both sequences $\varphi_1^{\tilde{\sigma}(n)}$ and $\varphi_2^{\sigma(n)}$ converge to φ_{u_0}. Now the same arguments as above work. The second case is similar. □

Remarks 1. — Since the line field E_g^s is continuous outside the singular points, it follows from Proposition 7.2 that for $u \neq 0$, the family of functions φ_u is continuous in C^1 topology.

2. Since the Markov coordinate systems in adjacent rectangles agree, Proposition 7.2 implies that stable curves form a continuous foliation with singularities at the points $x_1, \ldots, x_m$. The same arguments show that unstable curves form another continuous foliation transversal to the first one outside the singularities. Let us denote these stable and unstable foliations by $\mathfrak{F}_g^s$ and $\mathfrak{F}_g^u$, respectively.

Now we are prepared to carry out the final step in the proof of topological conjugacy of the pseudo-Anosov map f and the diffeomorphism g. Let us recall several notations. The eigenvalue of the matrix A with maximal absolute eigenvalue is denoted by λ. Actually λ is positive, and $\log \lambda$ is equal to the topological entropy of the map f. A positive eigenvector of A corresponds to the eigenvalue λ. We denote this vector $l = (l_1, \ldots, l_N)$. We assume that the vector l is normalized, i. e., $\Sigma_{i=1}^N l_i = 1$. By adjusting the transverse measure $\mu^s = \mu_f^s$ by a constant multiple if necessary, we may also assume that l_i is equal to the transverse measure $\mu_f^s(\gamma_u^i)$, where γ_u^i is an unstable side of the rectangle R_i. Similarly, if we let $\bar{l} = (\bar{l}_1, \ldots, \bar{l}_N)$ be the normalized positive eigenvector of the adjoint matrix A^*, then $\bar{l}_i$ may be assume to be equal to $\mu_f^u(\gamma_s^i)$.

PROPOSITION 7.3. — *There exist non-atomic transversal invariant measures μ_g^s and μ_g^u for the foliations $\mathfrak{F}_g^s$ and $\mathfrak{F}_g^u$ such that:*

(i) $$g_* \mu_g^s = \lambda^{-1} \mu_g^s \quad \text{and} \quad g_* \mu_g^u = \lambda \mu_g^u,$$

(ii) $$\mu_g^s(\gamma_u^i) = l_i \quad \text{and} \quad \mu_g^u(\gamma_s^i) = \bar{l}_i, \qquad i = 1, \ldots, N,$$

(iii) *μ_g^s and μ_g^u are positive on every open transversal interval.*

Proof. — We will give the proof only for the stable foliation. The unstable foliation is considered similarly.

Since the two foliations are transversal, in order to define the measure μ_g^s, it suffices to define it for unstable curves. Moreover, we can do it separately in each rectangle R_i and only for arcs on the unstable boundary of the rectangle, and then check that the measures agree on the intersection of any two rectangles.

We begin with the definition of μ_g^s on γ_u^i by (ii). Since:

$$\sum_{j=1}^{N} a_{ij} l_j = \lambda l_i, \tag{7.4}$$

condition (i) is satisfied for maximal unstable curves in R_i.

For every $n \geqq 1$, we have:

$$R_i = \bigcup_{\sigma^{(n)}} R_{\sigma^{(n)}},$$

where the union is taken over the set of all admissible sequences $\sigma^{(n)} = (i, \sigma_1, \ldots, \sigma_{n-1})$. Let $\gamma_{\sigma^{(n)}}$ be an intersection of a component of the unstable boundary R_i with $R_{\sigma(n)}$. We define:

$$\mu_g^s(\gamma_{\sigma^{(n)}}) = \lambda^{-n} l_{\sigma_{n-1}}. \tag{7.5}$$

It follows easily from (7.4) that this definition is consistent, i. e., for every admissible sequence $\sigma^{(m)} = (i, \sigma_1, \ldots, \sigma_{m-1})$ and every $n > m$, we have:

$$\mu_g^s(\gamma_{\sigma^{(m)}}) = \sum_{\sigma^{(n)} = (\sigma^{(m)}, \sigma^{(n-m)})} \mu_g^s(\gamma_{\sigma^{(n)}}),$$

where the summation is taken over all admissible sequences $\sigma^{(n)}$ which begin with $\sigma^{(m)}$.

It follows from Corollary 6.3 that the maximal length of $\gamma_{\sigma^{(n)}}$ over all admissible $\sigma^{(n)}$ goes to 0 as $n \to \infty$. Thus, applying a standard approximation procedure, we can define $\mu_g^s(\alpha)$ for every arc α of the unstable boundary of R_i.

However, the same arc α has an intersection with a boundary component of at least one more rectangle. In fact, cutting α into pieces, we can consider each piece separately and assume that from the beginning α belongs to exactly one more tectangle. Thus $\alpha \subset \gamma \cap \gamma'$, where γ and γ' are components of the unstable boundaries of the rectangles R_i and R_j, respectively. Let us temporarily denote the measure μ_g^s defined in R_i and R_j by μ_i and μ_j, respectively. So we have to show that:

$$\mu_i(\alpha) = \mu_j(\alpha). \tag{7.6}$$

DEFINITION. — We will call the curve $\gamma_{\sigma^{(n)}}$ a *regular component of* γ if $g^n \gamma_{\sigma^{(n)}} \subset \operatorname{Int} R_{\sigma_n}$.

If $\gamma_{\sigma^{(n)}}$ is a regular component of γ, then it is also a regular component of γ', and by (7.5), obviously:

$$\mu_i(\gamma_{\sigma^{(n)}}) = \mu_j(\gamma_{\sigma^{(n)}}). \tag{7.7}$$

For $\alpha \subset \gamma$, let us denote by $\underline{\alpha}_n$ the union of all $\gamma_{\sigma^{(n)}}$ which lie inside α and by $\bar{\alpha}_n$ the union of all $\gamma_{\sigma^{(n)}}$ which have a non-empty intersection with α. Obviously,

$$\mu_i(\alpha) = \lim_n \mu_i(\underline{\alpha}_n) = \lim_n \mu_i(\bar{\alpha}_n).$$

Equality (7.6) follows immediately from (7.7) and the following lemma.

LEMMA. — *Let $\alpha \subset \gamma$ and suppose both ends of α are different from singular points $x_1, \ldots, x_m$. Then for any sufficiently large n all $\gamma_{\sigma^{(n)}}$ which belong to $\bar{\alpha}_n$ are regular components of γ.*

Proof of the Lemma. — The unstable boundary of the partition $\{R_1, \ldots, R_N\}$ is a union of arcs $\Gamma_1, \ldots, \Gamma_l$ of unstable manifolds of singular points $x_1, \ldots, x_m$. (This is true for both maps f and g.) In particular, let $x \in \Gamma_r$ be a point which lies between the singularity and α. Let us denote the arc of Γ_r from the singularity to x by Γ and the rest of Γ_r by Γ'. By Corollary 6.3, there exists N which depends only on the length of Γ such that for $n \geqq N$, the length $l(g^n \Gamma) > l(\Gamma_r)$. This means that $g^n \Gamma$ contains Γ_r and consequently $g^n \Gamma'$ is disjoint from the unstable boundary of the partition $\{R_1, \ldots, R_N\}$. If n is large enough $\gamma_{\sigma^{(n)}} \subset \bar{\alpha}_n$ implies that $\gamma_{\sigma^{(n)}} \subset \Gamma'$. Thus $\gamma_{\sigma^{(n)}}$ is a regular component of γ.

To finish the proof of Proposition 7.3, it is enough to notice that every unstable curve can be divided into connected pieces belonging to rectangles R_i, and within any rectangle, for every unstable curve α, one can find a curve $\bar{\alpha}$ on the unstable boundary whose ends belong to the same stable curves. We let by definition $\mu_s^g(\alpha) = \mu_i(\bar{\alpha})$. Equalities (7.5) and (7.6) show that the definition is consistent and (i) holds; (iii) follows from Corollary 6.1. □

The measures μ_s^g and μ_u^g provide a local coordinate system in every rectangle R_i. Namely, let us fix a corner z_i in every rectangle. Then for $x \in R_i$ we can draw maximal stable and unstable curves in R_i which interesct the unstable and stable components of ∂R_i containing z_i at the points x' and x'', respectively. Then we let:

$$(6.8) \qquad \begin{cases} u(x) = \mu_u^g((x'', x)), \\ v(x) = \mu_s^g((x', x)). \end{cases}$$

These coordinates change by a translation when we pass from R_i to R_j along a boundary component.

THEOREM. — *The diffeomorphism g is topologically conjugate with the pseudo-Anosov map f, and the conjugating homeomorphism is isotopic to the identity.*

Proof. — The homeomorphism h conjugating g and f is defined in the following way:

(i) h is identical at all vertices of all R_i's;

(ii) $h(R_i) = R_i$;

(iii) within R_i, $h(x)$ is defined by the points whose Markov coordinates are equal to coordinates of x defined by (6.8). It follows immediately from Proposition 7.2 that h is a homeomorphism of M and $g = h^{-1} fh$. Also (i) and (ii) imply that h is isotopic to the identity. □

The fact that g is Bernoulli with respect to a smooth measure can be derived from results in [P], in our case exactly as in [K, paragraph 4].

8. Generalizations

The results of this paper can be generalized in several directions:

1. Diffeomorphisms with stable and unstable measured foliations having one-prong singularities. A one-prong singularity corresponds to a local chart satisfying (2.1) with $p = 1$. Bernoulli diffeomorphisms on the sphere constructed in [K] have four such

singularities (and no multi-prong singularities). In [F-L-P], maps having this type of singularity as well as those previously allowed are called generalized pseudo-Anosov maps. Any generalized pseudo-Anosov map can be modified by a local perturbation consisting of a "slowing-down" and "blowing-up" procedure described in [K, paragraph 2] to produce a diffeomorphism preserving a smooth measure which is topologically conjugate to the original map *via* a homeomorphism isotopic to the identity. Here the "blowing-up" part of the construction cannot be omitted was possible in the previous case, but with this additional step the arguments given above still work.

2. Pseudo-Anosov maps on surfaces with boundary. Singularities on the boundary correspond to local charts satisfying (2.1) with D_{a_i} replaced by $D_{a_i} \cap \{ z : \operatorname{Re} z \geqq 0 \}$ or by $D_{a_i} \cap \{ z : \operatorname{Re} z \leqq 0 \}$. The theory of such maps is outlined in [F-L-P, 11].

The referee pointed out that their description contains a mistake noticed by Jiang. Namely, a Pseudo-Anosov map should not be required to be the identity of the boundary. For exemple, a boundary component may consist of two hyperbolic fixed points x and y together with two branches of the stable manifold of x which at the same time is the unstable manifold of y. Our smoothing construction can be applied to this more general case.

3. Similar comments can be made for pseudo-Anosov maps on non-orientable surfaces with or without boundary. As soon as the generalization of the topological theory mentioned in paragraph 1 of [F-L-P] is developed, the smoothing procedure is completely parallel to that in the orientable case.

Of course, the complications and generalizations given above may be combined and dealt with in the same manner.

REFERENCES

[B-F-K] M. I. Brin, J. Feldman and A. Katok, *Bernoulli Diffeomorphisms and Group Extensions of Dynamical Systems with Non-Zero Characteristic Exponents* (*Annals of Math.*, Vol. 113, 1981, pp. 159-179).

[F-L-P] A. Fathi, F. Laudenbach and V. Poénaru, *Asterisque*, Vol. 66-67, *Travaux de Thurston sur les surfaces* (Séminaire Orsay).

[F] J. Franks, *Anosov Diffeomorphisms* (*Proceedings of a Symposium in Pure Mathematics*, Vol. 14, *Global Analysis*, *A.M.S.*, 1970, pp. 61-93.

[G] A. A. Gura, *On Expansive Diffeomorphisms of the Torus* (*Math. notes*, Vol. 18, No. 1, 1975, pp. 605-610. Translation from Russian).

[K] A. Katok, *Bernoulli Diffeomorphisms on Surfaces* (*Annals of Math.*, Vol. 110, 1979, pp. 529-547).

[O-R] T. O'Brien and W. Reddy, *Each Compact Orientable Surface of Positive Genus Admits an Expansive Homeomorphism* (*Pacific J. of Math.*, Vol. 35, No. 3, 1970, pp. 737-741).

[O] D. Ornstein, *Ergodic Theory, Randomness and Dynamical Systems*, Yale University Press, 1974.

[P] Ya. B. Pesin, *Characteristic Lyapunov Exponents and Smooth Ergodic Theory* (*Russian Math. Surveys*, Vol. 32, No. 4, 1977, pp. 55-114).

[S] S. Smale, *Differentiable Dynamical Systems* (*Bull. A.M.S.*, Vol. 73, 1967, pp. 747-817).

[T] W. Thurston, *On the Geometry and Dynamics of Diffeomorphisms of Surfaces*, preprint.

(Manuscrit reçu le 17 juillet 1981,
révisé le 12 octobre 1981.)

Marlies Gerber and Anatole Katok
Department of Mathematics,
University of Maryland,
College Park, Md 20742
U.S.A.