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ON THE POINT SPECTRUM OF SCHRÖDINGER OPERATORS

BY ANNE BERTHIER

1. Introduction

This paper is an extension of a work [2] on the spectral analysis of partial differential operators of Schrödinger type. The problem was the following: Let A be a compact subset of $\mathbb{R}^n$, Σ a finite interval in $\mathbb{R}$ and H a self-adjoint elliptic differential operator in the complex Hilbert space $\mathscr{H} = L^2(\mathbb{R}^n)$. We define $F(\Sigma)$ to be the spectral projection of H associated with the interval Σ and $E(A)$ the multiplication operator by the characteristic function χ_A of A. Do there exist vectors in $L^2(\mathbb{R}^n)$ which are contained both in the range $E(A)\mathscr{H}$ of $E(A)$ and in $F(\Sigma)\mathscr{H}$?

It turns out that the closed subspace $\mathscr{H}_p(H)$ generated by the set of eigenvectors of H plays a different role from the subspace $\mathscr{H}_c(H) = \mathscr{H}_p(H)^\perp$ associated with the continuous spectrum of H. Notice that it is shown in [2], under regularity and integrability conditions on the coefficients of the differential operator, that there do not exist vectors of $\mathscr{H}_c(H)$ which belong both to $E(A)\mathscr{H}$ and to $F(\Sigma)\mathscr{H}$. On the other hand, to prove the non-existence of vectors in $\mathscr{H}_p(H)$ belonging to $E(A)\mathscr{H} \cap F(\Sigma)\mathscr{H}$, we used an unique continuation theorem for solutions of the differential equation associated with H. Now, if for example $H = -\Delta + V$, where V is the multiplication operator by a real function $v(\vec{x})$, the known results on unique continuation require a condition $L^\infty(\mathbb{R}^n \setminus N)$ on v, where N is a closed set of measure zero such that $\mathbb{R}^n \setminus N$ is connected ([3], [5]).

In the present paper, we propose to show that:

$$(1) \qquad \mathscr{H}_p(H) \cap E(A)\mathscr{H} \cap F(\Sigma)\mathscr{H} = \{0\},$$

by imposing only an integrability condition on the function v. More precisely, we will prove (1) under the hypothesis that $v \in L^s_{\mathrm{Loc}}(\mathbb{R}^n)$ with $s=2$ if $n=1, 2, 3$ and $s>n-2$ if $n \geqq 4$.

This result shows that, under the above conditions on v, the operator $-\Delta+v$ has no eigenvector with compact support. This is essentially the content of our Theorem 1 in paragraph 2. (In the case $n=1$, one obtains *ordinary* differential operators for which results of this type have been known for a long time [9]).

This result is also interesting from the point of view of "non-existence of positive eigenvalues of the operator H". In the literature (for example [2], [12]) the non-existence of positive eigenvalues is obtained in two steps:

(i) under suitable decay conditions at infinity on the function v, it is shown that all eigenfunctions f associated with a strictly positive eigenvalue of H have compact support;

(ii) then one imposes suitable local conditions on v (e. g. $v \in L^{\infty}_{\text{Loc}}(\mathbb{R}^n \setminus N)$ in order to apply the unique continuation theorem, which then leads to $f \equiv 0$. It turns out that the non-existence of positive eigenvalues is also obtained by assuming in (ii) as a local condition that $v \in L^s_{\text{Loc}}(\mathbb{R}^n)$ with $s=2$ if $n=1, 2, 3$ and $s>n-2$ if $n \geqq 4$ (Thm. 2).

Finally our method implies also the spectral continuity of a class of Schrödinger operators with periodic potentials $v(\vec{x})$.

The organization of the paper is a follows: first we give the principal results and deduce Theorems 1 and 2 from Theorem 3 in section 2, and we introduce a direct integral representation of Schrödinger operators in section 3. This representation will be used in section 4 for proving Theorem 3. The principal estimate of the proof is the subject of the last section 5.

2. Statements of the results

Let $v : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. We always suppose that:

$$v \in L^s_{\text{Loc}}(\mathbb{R}^n) \quad \text{with } s=2 \quad \text{if } n=1, 2, 3; \qquad s>n-2 \quad \text{if } n \geqq 4. \tag{2}$$

Notice that $s>n-2$ in all cases.

The function v will be called *periodic* if there exist n linearly independent vectors $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^n$ such that $v(\vec{x}+\vec{a}_i)=v(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$. A periodic function will be called *ortho-periodic* if:

$$\vec{a}_j . \vec{a}_k = L^2 \delta_{jk}, \tag{3}$$

with $L>0$, i. e. if the vectors of the form $\sum_{i=1}^{n} \alpha_i . \vec{a}_i$, $0 \leqq \alpha_i < 1$, define a cube C^n with side L.

We denote by $\hat{H}$ the symmetric operator:

$$\hat{H} = -\Delta + v(\vec{x}), \tag{4}$$

with domain $D(\hat{H}) = C_0^{\infty}(\mathbb{R}^n)$ and by H_0 the unique self-adjoint extension of $\hat{H}_0 = -\Delta$, $D(\hat{H}_0) = C_0^{\infty}(\mathbb{R}^n)$. Let H a self-adjoint extension of $\hat{H}$. We have the following lemma:

LEMMA 1. — *Assume that* (2) *and one of the following conditions are satisfied:*

(i) *v is periodic;*

(ii) $v \in L^{\infty}(\complement B_R)$ *where* $B_R = \{\vec{x} \in \mathbb{R}^n \mid |\vec{x}| \leqq R\}$ *and* $\complement B_R$ *denotes the complement of* B_R.

Then:

(*a*) *v is H_0-bounded with H_0-bound* 0;

(*b*) $\hat{\mathrm{H}}$ *is essentially self-adjoint;*

(*c*) $\mathrm{D(H)=D(H_0)}$, *where* H *is the unique self-adjoint extension of* H.

Proof. — (*b*) and (*c*) follow from (*a*) by using the Kato-Rellich Theorem ([7], Chapt. 5.4.1). Under hypothesis (i), (*a*) follows from Theorem XIII.96 of [11], whereas under the assumption (ii), (*a*) can be proved by the method used in the proof of Lemma 3 in [10]. Both cases are treated in [4].

We now state our principal results. In Theorem 2 we choose as conditions on the potential v at infinity those used in [4].

THEOREM 1. — *Let* $v \in \mathrm{L}^s_{\mathrm{Loc}}(\mathbb{R}^n)$ *with s satisfying* (2) *and let* H *be a self-adjoint extension of* H:

(*a*) *suppose that* $f \in \mathrm{L}^2(\mathbb{R}^n)$ *satisfies* $\mathrm{H}f=\lambda f$ *for some* $\lambda \in \mathbb{R}$ *and* $\mathrm{E(A)}f=f$ *for some compact subset* A *of* $\mathbb{R}^n$. (*i. e. f is an eigenvector of* H *with compact support in* $\mathbb{R}^n$). *Then* $f=0$;

(*b*) *for each compact subset* A *of* $\mathbb{R}^n$ *and each bounded interval* Σ, *one has:*

$$\mathscr{H}_p(\mathrm{H}) \cap \mathrm{E(A)}\mathscr{H} \cap \mathrm{F}(\Sigma)\mathscr{H}=\{0\}.$$

THEOREM 2. — *Suppose that:*

(i) $v \in \mathrm{L}^s(\mathrm{B_R})$ *with s satisfying* (2) *for some* $\mathrm{R}<\infty$;

(ii) $v=v_1+v_2$ *such that:*

(α) $v_1, v_2 \in \mathrm{L}^\infty(\complement \mathrm{B_R})$,

(β) $|\vec{x}|\, v_1(\vec{x}) \to 0$ as $|\vec{x}| \to \infty$,

(γ) $v_2(\vec{x}) \to 0$ as $|\vec{x}| \to \infty$,

(δ) $r \mapsto v_2(r,\ .)$

is differentiable as a function from (R, ∞) *to* $\mathrm{L}^\infty(\mathrm{S}^{n-1})$, *and* $\limsup_{r\to\infty} \partial v_2/\partial r \leqq 0$.
(S^{n-1} *denotes the unit sphere in* $\mathbb{R}^n$.)

Then $\mathrm{H=H_0+V}$ *has no eigenvalues in* $(0, \infty)$.

THEOREM 3. — *Let v be ortho-periodic and* $v \in \mathrm{L}^s_{\mathrm{Loc}}(\mathbb{R}^n)$ *with s satisfying* (2). *Then the spectrum of* $\mathrm{H=H_0+V}$ *is purely continuous.*

Remark 1. — By following the proof of Theorem XIII.100 in [11], it is possible to show that the operator H in Theorem 3 is absolutely continuous. Other comments on Theorem 3 will be made at the end of this paper.

Remark 2. — Contrarily to [2], where the operator $\hat{\mathrm{H}}$ was defined by:

$$\hat{\mathrm{H}}=\sum_{j,\,k=1}^{n} a_{jk}\left(-i\frac{\partial}{\partial x_j}+b_j(\vec{x})\right)\left(-i\frac{\partial}{\partial x_k}+b_k(\vec{x})\right)+\mathrm{V}(\vec{x}),$$

we assume here that the vector potential $\vec{b}=\{b_k\}$ is equal to zero. It is possible to generalize Theorem 1 to the case where $\vec{b}\neq 0$.

Theorem 2 follows from results of [11] and [6], and from Theorem 1 as indicated in the introduction. (If $\mathrm{H}f=\lambda f$ with $\lambda>0$, then f has compact support by Theorem XIII.58 of

[11], and consequently $f=0$ by our Theorem 1.) Theorem 1 (a) is deduced from Theorem 3: By the proof of Proposition 4 of [2], the vector f belongs to $D(H_0)\cap D(V)$ and $Hf=H_0 f+VE(A)f$. Let w be an ortho-periodic function such that $w\in L^s_{Loc}(\mathbb{R}^n)$ and $w(\vec{x})=v(\vec{x})$ for $\vec{x}\in A$. If H_1 denotes the periodic Schrödinger operator $H_1=H_0+W$ then $H_1 f=Hf=\lambda f$. Therefore we deduce from Theorem 3 that $f=0$.

To show Theorem 1(b), let $S=E(A)\cap F(\Sigma)$ (the orthogonal projection with range $E(A)\mathscr{H}\cap F(\Sigma)\mathscr{H}$) and suppose that $f\in\mathscr{H}_p(H)$ satisfies $Sf=f$. f is a linear combination of eigenvectors of H, i.e. $f=\sum_k \alpha_k . g_k$, where $Hg_k=\lambda_k g_k$ with $\lambda_k\in\Sigma$. It follows that:

$$Sf=f=\sum_k \alpha_k S g_k.$$

Now, by Proposition 2 of [2], S commutes with H; in particular $HSg_k=SHg_k=\lambda_k S g_k$. This implies that each Sg_k is an eigenvector of H of compact support in A, hence $Sg_k=0$ by the part (a) of Theorem 1. We deduce from this that $f=\sum_k \alpha_k S g_k=0$. The condition "$\Sigma$ bounded" is fundamental: we can choose a potential V such that $\mathscr{H}_p(H)=\mathscr{H}$, i.e. such that the eigenvectors of $\mathscr{H}$ generate $\mathscr{H}$. In this case, we have:

$$\mathscr{H}_p(H)\cap E(A)\mathscr{H}=E(A)\mathscr{H}\neq\{0\}.]$$

3. Reduction of the translation group of the lattice

In this part, let v be an ortho-periodic potential. In a natural way, this implies a decomposition of the Hilbert space $\mathscr{H}=L^2(\mathbb{R}^n)$ and of the operators H and H_0 into direct integrals. This decomposition will be used in the next part for the proof of Theorem 3.

The potential v satisfies $v(\vec{x}+\vec{a}_i)=v(\vec{x})$ where $\vec{a}_1,\ldots,\vec{a}_n$ are as in (3). The points of the form $\vec{z}=\sum_{i=1}^{n} q_i\vec{a}_i$, $\vec{q}=\{q_i\}\in\mathbb{Z}^n$, form a cubic lattice in $\mathbb{R}^n$ which is invariant under the translations:

$$\vec{z}\mapsto\vec{z}+\sum_i q'_i\vec{a}_i,\qquad \vec{q}\,'\in\mathbb{Z}^n.$$

In $L^2(\mathbb{R}^n)$, we consider the unitary representation $U(\vec{q})$ of the additive group $\mathbb{Z}^n$ given by:

$$[U(\vec{q})f](\vec{x})=f(\vec{x}-\sum_i q_i\vec{a}_i)=f(x-L\vec{q}),\tag{5}$$

where we have written $\sum_i q_i\vec{a}_i=L\vec{q}$, assuming that the directions of the $\vec{a}_i$ coïncide with Cartesian coordinate system.

We also introduce the *reciprocal lattice* which is the set of points of the following form:

$$\vec{z}=\sum_{i=1}^{n} q_i\vec{e}_i,\qquad \vec{q}\in\mathbb{Z}^n,$$

where the vectors $\vec{e}_1, \ldots, \vec{e}_n$ are defined by:

$$\vec{e}_i . \vec{a}_k = 2\pi\delta_{ik}. \tag{6}$$

We may write $\vec{z} = \mathrm{E}\,\vec{q}$, with $\mathrm{E} = 2\pi\mathrm{L}^{-1}$. Let again:

$$\Gamma^n = \left\{ k \in \mathbb{R}^n \mid k = \sum_{i=1}^{n} \lambda_i e_i,\ 0 \leqq \lambda_i < 1 \right\}.$$

Consider the Hilbert space $\mathscr{G}$ of square-integrable functions $f : \Gamma^n \to l_n^2 \equiv l^2(\mathbb{Z}^n)$:

$$\mathscr{G} = \mathrm{L}^2(\Gamma^n;\ l_n^2).$$

We write $f(\vec{k})_{\vec{q}}$ for the component $\vec{q}\,(\vec{q} \in \mathbb{Z}^n)$ of f at the point $\vec{k} \in \mathbb{Z}^n$. Thus, we have:

$$\| f \|_{\mathscr{G}}^2 = \int_{\Gamma^n} dk \sum_{\vec{q} \in \mathbb{Z}^n} | f(\vec{k})_{\vec{q}} |^2.$$

Now, let $\mathscr{U} : \mathscr{H} \to \mathscr{G}$ be the operator defined by:

$$(\mathscr{U} f)(\vec{k})_{\vec{q}} = \hat{f}(\vec{k} + \mathrm{E}\,\vec{q}), \tag{7}$$

where $\hat{f}$ is the Fourier transform of the function f:

$$\hat{f}(\xi) = (2\pi)^{-n/2}\ \mathrm{Lim} \int_{\mathbb{R}^n} dx\ \exp(-i\vec{x}.\vec{\xi})\ f(\vec{x}).$$

It follows from Plancherel's Theorem that the operator $\mathscr{U}$ is unitary, and its inverse is given by:

$$\mathscr{F}[\mathscr{U}^{-1}\{ f(.) \}](\xi) = f(\vec{k})_{\vec{q}},$$

where $\vec{q} \in \mathbb{Z}^n$ and $\vec{k} \in \Gamma^n$ are determined by $\vec{k} + \mathrm{E}\,\vec{q} = \vec{\xi}$. If $\vec{m} \in \mathbb{Z}^n$, one has:

$$[\mathscr{U}\,\mathrm{U}(\vec{m}) f](\vec{k})_{\vec{q}} = \exp(-i\mathrm{L}\,\vec{k}.\vec{m})(\mathscr{U} f)(\vec{k})_{\vec{q}}, \tag{8}$$

i.e. $\mathscr{U}\,\mathrm{U}(m)\,\mathscr{U}^{-1}$ is diagonalizable in $\mathscr{G}$ (i.e. a multiplication operator by a function of $\vec{k}$). As the functions $\{\exp(i\mathrm{L}\,\vec{k}.\vec{m})\}_{\vec{m} \in \mathbb{Z}^n}$ form a basis of $\mathrm{L}^2(\Gamma^n)$, each bounded diagonalizable operator is a function of $\{\mathscr{U}\,\mathrm{U}(m)\,\mathscr{U}^{-1}\}$. As H_0, V and H commute with every $\mathrm{U}(\vec{m})$, these operators commute with each diagonalizable operator, i.e. $\mathscr{U}\,\mathrm{H}_0\,\mathscr{U}^{-1}$, $\mathscr{U}\,\mathrm{V}\,\mathscr{U}^{-1}$ and $\mathscr{U}\,\mathrm{H}\,\mathscr{U}^{-1}$ are decomposable in $\mathrm{L}^2(\Gamma^n;\ l_n^2)$. Therefore there exist in l_n^2 measurable families of self-adjoint operators $\mathrm{H}_0(\vec{k})$, $\mathrm{V}(\vec{k})$ and $\mathrm{H}(\vec{k})\,(\vec{k} \in \Gamma^n)$ such that, for $f \in \mathrm{D}(\mathrm{H}_0)$:

$$\left\{ \begin{aligned} (\mathscr{U}\,\mathrm{H}_0 f)(\vec{k}) &= \mathrm{H}_0(\vec{k})\, f(\vec{k}), \\ (\mathscr{U}\,\mathrm{V} f)(\vec{k}) &= \mathrm{V}(\vec{k})\, f(\vec{k}), \\ (\mathscr{U}\,\mathrm{H} f)(\vec{k}) &= \mathrm{H}(\vec{k})\, f(\vec{k}). \end{aligned} \right. \tag{9}$$

Now let us give the explicit form and the properties of these three families of operators.

LEMMA 2. — (i) $H_0(\vec{k})$ *is the self-adjoint multiplication operator in* l_n^2 *by* $\varphi_{\vec{k}}(\vec{q})=(\vec{k}+E\,\vec{q})^2$: *If* $g=\{g_{\vec{q}}\}\in l_n^2$, *then:*

$$(H_0(\vec{k})\,g)_{\vec{q}}=(\vec{k}+E\,\vec{q})^2\,g_{\vec{q}};$$

(ii) *the domain of* $D(H_0(\vec{k}))$ *is independent of* $\vec{k}$ *and is given by:*

$$D(H_0(\vec{k}))=D_0=\{g\in l_n^2 \mid \sum_{q\in\mathbb{Z}^n} |\vec{q}^2\,g_{\vec{q}}|^2<\infty\};$$

(iii) *the resolvent* $(H_0(\vec{k})-\mu)^{-1}$ *of* $H_0(\vec{k})$ *is a compact operator for all* $\mu\notin\sigma(H_0(\vec{k}))$, *where* $\sigma(H_0(\vec{k}))$ *is the spectrum of* $H_0(\vec{k})$.

Proof. — (i) and (ii) are obvious, since:

$$(H_0 f)\,(\xi)=\xi^2\,\hat{f}\,(\xi).$$

(iii) The resolvent $(H_0(\vec{k})-\mu)^{-1}$ is the multiplication operator by:

$$\psi(\vec{q})=[(\vec{k}+E\,\vec{q})^2-\mu]^{-1}.$$

Let χ_M be the characteristic function of the set $\{\vec{q}\in\mathbb{Z}^n \mid \vec{q}^2\leqq M\}$ and D_M the multiplication operator by $\psi(\vec{q})\,\chi_M(\vec{q})$. D_M is a compact (even nuclear) operator, and:

$$\text{(10)}\qquad \|(H_0(\vec{k})-\mu)^{-1}-D_M\|=\operatorname*{Sup}_{\vec{q}>M}[(\vec{k}+E\,\vec{q})^2-\mu]^{-1}\to 0,$$

as $M\to\infty$. Thus $(H_0(\vec{k})-\mu)^{-1}$ is compact as the uniform limit of the sequence $\{D_M\}$ of compact operators. ■

Let us denote by $\{\hat{v}_{\vec{q}}\}_{\vec{q}\in\mathbb{Z}^n}$ the Fourier coefficients of the periodic function v:

$$\text{(11)}\qquad \hat{v}_{\vec{q}}=L^{-n/2}\int_{C^n} dx\,\exp(-i\,E.\vec{q}.\vec{x})\,v(\vec{x}).$$

Notice that $v\in L^p(C^n)$ for all $p\in[1, s]$. To establish the relation between the Fourier coefficients of v and the operator $V(\vec{k})$ we need the following result:

LEMMA 3. — *Given* $\varphi, \psi : \mathbb{Z}^n\to\mathbb{C}$, *we define an operator* $A_{\varphi\psi} : l_n^2\to l_n^2$ *as follows:*

$$(A_{\varphi\psi}\,g)_{\vec{q}}=\sum_{\vec{m}\in\mathbb{Z}^n}\varphi(\vec{m})\,\psi(\vec{q}-\vec{m})\,g_{\vec{q}-\vec{m}}.$$

Assume that $2\leqq p<\infty$, $\psi\in l^p(\mathbb{Z}^n)$ and let $\{\varphi(\vec{q})\}$ be the Fourier coefficients of a function Φ belonging to $L^p(\mathbb{C}^n)$. Then $A_{\varphi\psi}$ is a compact operator and one has:

$$\text{(12)}\qquad \|A_{\varphi\psi}\|\leqq L^{-(n/2)-(n/p)}\,\|\Phi\|_{L^p(C^n)}\,\|\psi\|_{l^p(\mathbb{Z}^n)}.$$

Proof. — For $g=\{g_{\vec{q}}\}\in l_n^2$, define $\psi\,g=\{\psi(\vec{q})\,g_{\vec{q}}\}$. By the Hölder inequality, $\psi\,g\in l_n^r$ with $r^{-1}=(1/2)+p^{-1}$, i.e. $1\leqq r<2$, and:

$$\|\psi\,g\|_r\leqq\|\psi\|_p\,\|g\|_2.$$

Let:

$$\gamma(x)=L^{-n/2}\sum_{\vec{q}\in\mathbb{Z}^n}\exp(i\,E\,\vec{q}.\vec{x})\,\psi(\vec{q})\,g_{\vec{q}},\qquad x\in C^n.$$

By the Hausdorff-Young inequality [8], $\gamma\in L^{r'}(C^n)$ with $(r')^{-1}=1-r^{-1}=1/2-p^{-1}$ and:

$$\|\gamma\|_{r'}\leqq L^{(n/r')-(n/2)}\|\psi g\|_r\leqq L^{(n/r')-(n/2)}\|\psi\|_p\|g\|_2. \tag{13}$$

Since $1/2=p^{-1}+(r')^{-1}$ and $\Phi\in L^p(C^n)$, the Hölder inequality implies that $\Phi\gamma\in L^2(C^n)$ and:

$$\|\Phi\gamma\|_2\leqq\|\Phi\|_p\|\gamma\|_{r'}\leqq L^{(n/r')-(n/2)}\|\Phi\|_p\|\psi\|_p\|g\|_2. \tag{14}$$

Now:

$$(A_{\varphi\psi}\,g)_{\vec{q}}=\int_{C^n}dx\,\exp(-i\,E.\vec{q}.\vec{x})\,\Phi(\vec{x})\,\gamma(\vec{x}),$$

and by Plancherel's theorem we have:

$$\|A_{\varphi\psi}\,g\|_2=L^{n/2}\|\Phi\gamma\|_2\leqq L^{n/r'}\|\Phi\|_p\|\psi\|_p\|g\|_2. \tag{15}$$

This shows that $A_{\varphi\psi}$ is defined everywhere with the bound (12) :

(*b*) Let D_M be the multiplication operator by $\psi_M(\vec{q})=\psi(\vec{q})\,\chi_M(\vec{q})$ (*see* the proof of Lemma 2). By (*a*), $A_{\varphi\psi_M}$ is bounded, and $A_{\varphi\psi_M}$ is non-zero only on a subspace of finite dimension. Therefore $A_{\varphi\psi_M}$ is nuclear. By using (12) we obtain:

$$\|A_{\varphi\psi}-A_{\varphi\psi_M}\|\leqq L^{(n/2)-(n/p)}\|\Phi\|_p\|(1-\chi_M)\,\psi\|_p. \tag{16}$$

Since $\psi\in l_n^p$, $\|(1-\chi_M)\,\psi\|_p\to 0$ as $M\to\infty$. This proves the compactness of $A_{\varphi\psi}$.

LEMMA 4. — *Let* Y *be the operator in* l_n^2 *defined by:*

$$(Y\,g)_{\vec{q}}=L^{-n/2}\sum_{\vec{m}\in\mathbb{Z}^n}\hat{v}_{\vec{m}}\,g_{\vec{q}-\vec{m}}. \tag{17}$$

Then:

(i) $D_0\subseteqq D(Y)$ *and* Y *is symmetric on* D_0;

(ii) Y *is relatively compact with respect to* $H_0(\vec{k})$;

(iii) $V(\vec{k})=Y$ *on* D_0, *for all* $\vec{k}\in\Gamma_n$ (*in particular* $V(\vec{k})$ *is independent of* $\vec{k}$);

(iv) $H(\vec{k})=H_0(\vec{k})+Y$ *and* $D(H(\vec{k}))=D_0$.

Proof. — (i) If $g\in D_0$, then $g=[H(\vec{0})+1]^{-1}$ for some $h\in l_n^2$. (15) shows that $\|Y\,g\|_2<\infty$, therefore $D_0\subseteqq D(Y)$. By using $\bar{v}_{-\vec{q}}=v_{\vec{q}}$, one obtains easily that $(f, Y\,g)=(Y\,f, g)$ for $f, g\in D_0$;

(ii) $Y\,(H_0(\vec{k})+1)^{-1}$ is of the form $A_{\varphi\psi}$, with $\Phi(x)=L^{-n/2}\,v(\vec{x})$ and $\psi(\vec{q})=[(k+E\,\vec{q})^2+1]^{-1}$. Notice that $\psi\in l_n^p$ for each $p>n/2$. As $v\in L^s(C^n)$ for $s=2$ if $n=2$, 3 and $s>n/2$ if $n\geqq 4$, Lemma 3 implies that $Y\,(H_0(\vec{k})+1)^{-1}$ is compact;

(iii) this can be verified by calculating the Fourier transform of $\mathrm{V}f$;

(iv) by (i) and (ii), $\mathrm{H}_0(\vec{k})$ is self-adjoint. $\mathrm{H}(\vec{k})=\mathrm{H}_0(\vec{k})+\mathrm{Y}$ follows from (iii) and Lemmas 1 and 2.

4. Proof of Theorem 3

Let f be an eigenvector of H, i.e. $\mathrm{H}f=\lambda f$ for some $\lambda\in\mathbb{R}$. By defining $v'(x)=v(x)-\lambda$ and $\mathrm{H}'=\mathrm{H}_0+\mathrm{V}'$, we have $\mathrm{H}'f=0$. Since V' satisfies also the hypothesis (2), it is possible to assume without loss of generality that $\lambda=0$.

Let $\Gamma_0=\{\vec{k}\in\Gamma \mid (\mathscr{U} f)(\vec{k})\neq 0 \text{ in } l_n^2\}$. Γ_0 is measurable. Since $\mathrm{H}(\vec{k})(\mathscr{U} f)(\vec{k})=0$, $\mathrm{H}(\vec{k})$ must have the eigenvalue 0 for almost all the $\vec{k}\in\Gamma_0$. We will show that, for all $p\in(k_1, \ldots, k_{n-1}, 0)\in\mathbb{R}^{n-1}$ the set $\theta(\vec{p})$ of the points $k_n\in(0, \mathrm{E})$ such that $0\in\sigma(\mathrm{H}(\vec{p}+k_n\mathrm{E}^{-1}\vec{e}_n))$ is a set of measure zero. Thus the measure of Γ_0 is zero, i.e. $(\mathscr{U} f)(\vec{k})=0$ a.e., i.e. $f=0$. Therefore H cannot have any eigenvalues.

Fix $\vec{p}=(\vec{k}_1, \ldots, \vec{k}_{n-1})$. To show that the measure of $\theta(\vec{p})$ is zero, we shall use the Fredholm theory of holomorphic families of operators of type (A), [7]. Let Ω be the following complex domain:

$$\Omega=\{\mathscr{X}+ir \mid \mathscr{X}\in(0,\ 1),\ r\in\mathbb{R}\}. \tag{18}$$

For $z\in\Omega$, we define $\mathrm{H}_0(\vec{p}, z\vec{e}_n)$ to be the multiplication operator in l_n^2 by $(\vec{p}+z\vec{e}_n+\mathrm{E}\,\vec{q})^2$ and:

$$\mathrm{H}(\vec{p}, z\vec{e}_n)=\mathrm{H}_0(\vec{p}, z\vec{e}_n)+\mathrm{Y}. \tag{19}$$

We shall see that:

(I) $\{\mathrm{H}(\vec{p}, z\vec{e}_n)\}$ is a holomorphic family of type (A) with respect to z. (*See* the terminology in [7]);

(II) the resolvent of $\mathrm{H}(\vec{p}, z\vec{e}_n)$ is compact;

(III) the resolvent set of $\mathrm{H}(\vec{p}, z\vec{e}_n)$ is not empty.

Under these conditions, Theorem VII.1.10 of [7] says that we have the following alternative:

– either $0\in\sigma(\mathrm{H}(\vec{p}, z\vec{e}_n))$ for each $z\in\Omega$;

– or every compact Ω_0 in Ω contains only a finite number of points z such that $0\in\sigma(\mathrm{H}(\vec{p}, z\vec{e}_n))$.

We shall show that:

(IV) 0 belongs to the resolvent set of $\mathrm{H}(\vec{p}, z\vec{e}_n)$ for $\mathrm{Im}\, z$ sufficiently large. Hence the first alternative is excluded, so that the measure of $\theta(\vec{p})$ is zero.

The remainder of the paper is devoted to the verification of the properties I to IV of $\mathrm{H}(\vec{p}, z\vec{e}_n)$. To simplify the notations we write $\mathrm{H}(\vec{p}, \vec{z})$ for $\mathrm{H}(\vec{p}, z\vec{e}_n)$.

LEMMA 5. — (i) $\mathrm{H}_0(\vec{p}, z)$ *is a self-adjoint holomorphic family of type* (A) *in* Ω *with domain* $\mathrm{D}(\mathrm{H}_0(\vec{p}, z))=\mathrm{D}_0$;

(ii) $\forall z \in \Omega$, *the resolvent of* $H_0(\vec{p}, z)$ *is compact;*

(iii) 0 *belongs to the resolvent set* $\rho(H_0(\vec{p}, z))$ *of* $H_0(\vec{p}, z)$ *for all z with* $\operatorname{Im} z \neq 0$.

Proof. — (i) Let $P_j (j=1, \ldots, n)$ be the following operator in l_n^2:

$$P_j g_{\vec{q}} = g_j g_{\vec{q}}. \tag{20}$$

One has:

$$H_0(\vec{p}, z) = (\vec{p} + E\vec{P} + z\vec{e}_n)^2 = (\vec{p} + E\vec{P})^2 + E^2 z^2 + 2E^2 z P_n, \tag{21}$$

and the result is immediate:

(ii) the proof is the same as in Lemma 2 (iv).

(iii) for $z = \mathscr{X} + ir$, we have:

$$\operatorname{Im}(\vec{p} + E\vec{q} + z\vec{e}_n)^2 = 2E^2 r(\mathscr{X} + q_n), \tag{22}$$

which is different from zero if $r \neq 0$. Since $q_n \in \mathbb{Z}$ and $\mathscr{X} \in (0, 1)$ it follows that:

$$\|[H_0(\vec{p}, z)]^{-1}\| = \sup_{\vec{q} \in \mathbb{Z}^n} |(\vec{p} + E\vec{q} + z\vec{e}_n)^2|^{-1} < \infty,$$

i.e. $0 \in \rho(H_0(\vec{p}, z))$. ■

LEMMA 6. — (i) $H(\vec{p}, z)$ *is a self-adjoint holomorphic family of type* (A) *in* Ω *with domain* D_0;

(ii) $\forall z \in \Omega$ *the resolvent of* $H(\vec{p}, z)$ *is compact;*

(iii) *for all* $\vec{p} \in \Gamma^{n-1}$ *and* $z \in \Omega$, $\rho(H(\vec{p}, z))$ *is not empty.*

Proof. — (i) this follows from Lemmas 5 (i) and 4 (ii);

(iii) it suffices to show:

$$\lim_{\lambda \to +\infty} \|Y[H_0(\vec{p}, z) - i\lambda]^{-1}\| = 0, \tag{23}$$

since then the Neumann series for $[H(\vec{p}, z) - i\lambda]^{-1}$, i.e.:

$$[H(\vec{p}, z) - i\lambda]^{-1} = [H_0(\vec{p}, z) - i\lambda]^{-1} \sum_{n=0}^{\infty} \{-Y[H_0(\vec{p}, z) - i\lambda]^{-1}\}^n, \tag{24}$$

is convergent if λ is sufficiently large. Now, by (12):

$$\|Y[H_0(\vec{p}, z) - i\lambda]^{-1}\| \leqq L^{-n/s} \|v\|_s \{\sum_{\vec{q} \in \mathbb{Z}^n} |(\vec{p} + E\vec{q} + z\vec{e}_n)^2 - i\lambda|^{-s}\}^{1/s}. \tag{25}$$

We have with the notations $z = \mathscr{X} + ir$, $\vec{k} = (\vec{p}, \mathscr{X}\vec{e}_n) \in \Gamma^n$:

$$\begin{aligned} |(\vec{p} + E\vec{q} + z\vec{e}_n)^2 - i\lambda|^{-2} &\leqq \{[(\vec{k} + E\vec{q})^2 - E^2 r^2]^2 \\ &+ 4E^4 r^2 [\mathscr{X} + q_n - \lambda(2E^2 r)^{-1}]^2\}^{-1} \leqq [(\vec{k} + E\vec{q})^2 - E^2 r^2]^{-2}. \end{aligned} \tag{26}$$

This shows that each term of the sum in (26) converges to zero as $\lambda \to +\infty$, and that the series in (26) is uniformly majorized in λ by a convergent serie (since $s > n/2$). Therefore (23) is proven.

(If z is such that $(\vec{k}+\mathrm{E}\,\vec{q})^2-\mathrm{E}^2 r^2=0$ for certain $\vec{q}\in\mathbb{Z}^n$, then there exist $c>0$ and $\lambda_0<\infty$ such that $4\,\mathrm{E}^4 r^2[\mathscr{X}+q_n-\lambda(2\,\mathrm{E}^2 r)^{-1}]^2 \geqq c$ for all these $\vec{q}$ and for each $\lambda \geqq \lambda_0$. For these values of $\vec{q}$ we can take as majorization in (26) the number c^{-1}).

(ii) Now we use the first and the second resolvent equation:

$$[\mathrm{H}(\vec{p}, z)-\xi]^{-1}=[\mathrm{H}(\vec{p}, z)-\mu]^{-1}+(\xi-\mu)[\mathrm{H}(\vec{p}, z)-\xi]^{-1}[\mathrm{H}(\vec{p}, z)-\mu]^{-1}. \tag{27}$$

$$[\mathrm{H}(\vec{p}, z)-\mu]^{-1}=[\mathrm{H}_0(\vec{p}, z)-\mu]^{-1}-[\mathrm{H}(\vec{p}, z)-\mu]^{-1}\,\mathrm{Y}[\mathrm{H}_0(\vec{p}, z)-\mu]^{-1}. \tag{28}$$

(27) shows that if $[\mathrm{H}(\vec{p}, z)-\mu]^{-1}$ is compact for $\mu\in\rho(\mathrm{H}(\vec{p}, z))$ then $[\mathrm{H}(\vec{p}, z)-\xi]^{-1}$ is compact for each $\xi\in\rho(\mathrm{H}(\vec{p}, z))$. Since $[\mathrm{H}_0(\vec{p}, z)-\mu]^{-1}$ and $\mathrm{Y}[\mathrm{H}_0(\vec{p}, z)-\mu]^{-1}$ are compact if $\mu\in\rho(\mathrm{H}_0(\vec{p}, z))$, by (28) it suffices to show that:

$$\rho(\mathrm{H}_0(\vec{p}, z))\cap\rho(\mathrm{H}(\vec{p}, z))\neq\emptyset.$$

We know from (iii) that there exists a point $\mu_0\in\rho(\mathrm{H}(\vec{p}, z))$. If $\mu_0\notin\rho(\mathrm{H}_0(\vec{p}, z))$, there exists a point close to $\mu\in\rho(\mathrm{H}_0(\vec{p}, z))\cap\rho(\mathrm{H}(\vec{p}, z))$, since:

(α) $\rho(\mathrm{H}(\vec{p}, z))$ is open;

(β) $\sigma(\mathrm{H}_0(\vec{p}, z))$ consists of isolated eigenvalues only, because the resolvent of $\mathrm{H}_0(\vec{p}, z)$ is compact ([7], Thm. III 6.29).

By Lemma 6 we have verified the properties (I) to (III) of the family $\{\mathrm{H}(\vec{p}, z)\}$. It now remains to prove (IV) i.e. $0\in\rho(\mathrm{H}(\vec{p}, z))$ for some $z=\mathscr{X}+ir$ in Ω. We have seen that $0\in\rho(\mathrm{H}_0(\vec{p}, z))$ if $r\neq 0$. We shall show that:

$$\lim_{r\to\infty}\|\mathrm{Y}[\mathrm{H}_0(\vec{p}, \mathscr{X}+ir)]^{-1}\|=0. \tag{29}$$

By using the Neumann series (24) with $\lambda=0$ and r sufficiently large, (29) implies $0\in\rho(\mathrm{H}(\vec{p}, z))$ if $r=\operatorname{Im} z$ is sufficiently large.

To obtain (29), we use the inequality (25). By virtue of the first inequality in (26), it suffices to show that:

$$\lim_{r\to\infty}\sum_{\vec{q}\in\mathbb{Z}^n}\left\{\left[\left(\vec{q}+\frac{\vec{k}}{\mathrm{E}}\right)^2-r^2\right]^2+4r^2\,|q_n+\mathscr{X}|^2\right\}^{-s/2}=0, \tag{30}$$

which will be done in the next section.

5. Estimation of the series (30)

We now show that (30) holds if $s=2$ for $n=2, 3$, $s>n-2$ for $n\geqq 4$ and $\mathscr{X}\in(0, 1)$. We use the following notations:

$$a=2r\,|q_n+\mathscr{X}|,\qquad b=(q_n+\mathscr{X})^2-r^2. \tag{31}$$

We set $\vec{p}=\mathrm{E}^{-1}(k_1, \ldots, k_{n-1})\in\Gamma_1^{n-1}$, where $\Gamma_1^{n-1}=\{\vec{p}\in\mathbb{R}^{n-1} \mid 0 \leqq p_j < 1\}$, and:

$$\mathrm{S}(q_n, r)=\sum_{\vec{m}\in\mathbb{Z}^{n-1}} \{[(\vec{m}+\vec{p})^2+b]^2+a^2\}^{-s/2}. \tag{32}$$

(30) is then equivalent to:

$$\operatorname*{Lim}_{r\to\infty} \sum_{q_n\in\mathbb{Z}} \mathrm{S}(q_n, r)=0. \tag{33}$$

To prove (33), we first give a preliminary estimate in Lemma 7.

LEMMA 7. — *Let $\delta > 0$, $c > 0$ and $\mathrm{R} > 0$. Then:*

$$\varepsilon \equiv \inf_{r\geqq \mathrm{R}} \inf_{\substack{a\geqq\delta r\\ b\geqq -r^2\\ t,\, z\geqq 0\\ |t-z|\leqq c}} \frac{(z^2+b)^2+a^2}{(t^2+b)^2+a^2} > 0. \tag{34}$$

Proof. — Setting $\alpha=a/r$, $\beta=br^{-2}$, $\sigma=z/r$, $\tau=t/r$ and $\Omega_r=\{(\alpha, \beta, \sigma, \tau) \mid \alpha \geqq \delta, \beta \geqq -1, \sigma \geqq 0, \tau \geqq 0, |\sigma-\tau| \leqq cr^{-1}\}$, we see that (34) is equivalent to:

$$\varepsilon=\inf_{r\geqq\mathrm{R}} \inf_{\Omega_r} \frac{(\sigma^2+\beta)^2+(\alpha/r)^2}{(\tau^2+\beta)^2+(\alpha/r)^2} > 0. \tag{35}$$

The quotient on the r.h.s. of (35) is $\geqq 1$ if $|\tau^2+\beta| \leqq |\sigma^2+\beta|$. Hence the infimum is obtained by taking $|\tau^2+\beta| \geqq |\sigma^2+\beta|$. Under this restriction we have:

$$\frac{(\sigma^2+\beta)^2+(\alpha/r)^2}{(\tau^2+\beta)^2+(\alpha/r)^2} \geqq \max\left[\frac{(\sigma^2+\beta)^2}{(\tau^2+\beta)^2}, \frac{(\sigma^2+\beta)^2+(\alpha/r)^2}{(\tau^2+\beta)^2+2(\alpha/r)^2}\right]. \tag{36}$$

Also notice the following inequalities, valid on each Ω_r with $r \geqq \mathrm{R}$:

$$\begin{aligned}\tau^2+\beta&=[(\tau-\sigma)+\sigma)]^2+\beta \leqq 2(\tau-\sigma)^2+2\sigma^2+\beta\\ &=2(\sigma^2+\beta)-\beta+2(\tau-\sigma)^2 \leqq 2(\sigma^2+\beta)+1+2c^2\mathrm{R}^{-2}.\end{aligned} \tag{37}$$

$$|(\sigma^2+\beta)-(\tau^2+\beta)| \leqq (\sigma+\tau)\,|\sigma-\tau| \leqq (\sigma+\tau)\,cr^{-1}. \tag{38}$$

(38) implies that:

$$(\tau^2+\beta)^2 \leqq 2(\sigma^2+\beta)^2+2(\sigma+\tau)^2c^2r^{-2}. \tag{39}$$

We denote by ε_+ and ε_- the infimum in (35) under the restriction $\sigma^2+\beta \geqq 1$ and $\sigma^2+\beta\in[-1, +1]$ respectively. It suffices to show that $\varepsilon_+ > 0$ and $\varepsilon_- > 0$. In the first case (i.e. for $\sigma^2+\beta \geqq 1$), we use the first expression on the r.h.s. of (36) and the inequality (37). Setting $x=\sigma^2+\beta$, we see that:

$$\varepsilon_+ = \inf_{x\geqq 1} \frac{x^2}{(2x+1+2c^2\mathrm{R}^{-2})^2} > 0. \tag{40}$$

In the second case (i.e. for $\sigma^2+\beta\in[-1, +1]$), we have $\sigma^2 \leqq 2$, hence $\sigma+\tau \leqq 2\sqrt{2}+c\mathrm{R}^{-2} \equiv \eta$. After inserting this into (39) and using the second expression on the r.h.s. of (36), one obtains by setting $y=(\sigma^2+\beta)^2$:

$$(41) \quad \varepsilon_- = \inf_{r \geqq \mathrm{R}} \inf_{\substack{0 \leqq y \leqq 1 \\ \alpha \geqq \delta}} \frac{y+(\alpha/r)^2}{2y+2\eta^2 c^2 r^{-2}+2(\alpha/r)^2}$$

$$= \inf_{r \geqq \mathrm{R}} \inf_{\alpha \geqq \delta} \frac{(\alpha/r)^2}{2\eta^2 c^2 r^{-2}+2(\alpha/r)^2} = \frac{\delta^2}{2\eta^2 c^2+2\delta^2} > 0. \quad \blacksquare$$

Proof of (33). — Let $\vec{m}\in\mathbb{Z}^{n-1}$ and $\Gamma(\vec{m})$ be the cube:

$$\Gamma(\vec{m})=\{\vec{x}\in\mathbb{R}^{n-1} \mid \vec{x}=\vec{p}+\vec{m}+\vec{y},\ \vec{y}\in\Gamma_1^{n-1}\}.$$

We have $\Gamma(\vec{m})\cap\Gamma(\vec{m}')=\emptyset$ if $\vec{m}\neq\vec{m}'$ and:

$$\mathbb{R}^{n-1}=\bigcup_{\vec{m}\in\mathbb{Z}^{n-1}} \Gamma(\vec{m}).$$

Let $c=\sqrt{n-1}$. Then for each $\vec{x}\in\Gamma(\vec{m})$ and each $\vec{m}\in\mathbb{Z}^{n-1}$:

$$\big|\,|\vec{m}+\vec{p}|-|\vec{x}|\,\big| \leqq c.$$

Let $\delta=1/2 \min(\mathscr{X}, 1-\mathscr{X})$. By assumption $\delta > 0$; since $a \geqq \delta r$ and $b \geqq -r^2$, Lemma 7 implies the existence of a number $\varepsilon > 0$ such that, for each $\vec{m}\in\mathbb{Z}^{n-1}$, each $x\in\Gamma(\vec{m})$, each $a \geqq \delta r$ and $b \geqq -r^2$ and all $r \geqq \mathrm{R}$:

$$(42) \quad [(\vec{m}+\vec{p})^2+b^2]+a^2 \geqq \varepsilon[(\vec{x}^2+b)^2+a^2].$$

Thus:

$$(43) \quad \mathrm{S}(q_n, r)=\sum_{\vec{m}\in\mathbb{Z}^{n-1}} \{[(\vec{m}+\vec{p})^2+b^2]+a^2]^{-s/2}$$

$$= \sum_{\vec{m}\in\mathbb{Z}^{n-1}} \int_{\Gamma(\vec{m})} dx\, \{[(\vec{m}+\vec{p})^2+b]^2+a^2]^{-s/2}$$

$$\leqq \varepsilon^{-1}\sum_{\vec{m}} \int_{\Gamma(\vec{m})} dx\, \{(\vec{x}^2+b)^2+a^2\}^{-s/2}$$

$$= \varepsilon^{-1} \int_{\mathbb{R}^{n-1}} dx\, \{(\vec{x}^2+b)^2+a^2\}^{-s/2}$$

$$= \frac{1}{2}\varepsilon^{-1} w_{n-1} \int_0^\infty y^{(n-3)/2} \{(y+b)^2+a^2\}^{-s/2}\, dy,$$

where we have introduced spherical polar coordinates, $y=|\vec{x}|^2$ and w_{n-1} denotes the area of the unit sphere in $\mathbb{R}^{n-1}$.

To estimate the integral in (43), we distinguish the two cases $b \geqq 0$ and $b < 0$. For $b \geqq 0$, we have $\{(y+b)^2+a^2\}^{-s/2} \leqq \{y^2+a^2+b^2\}^{-s/2}$, and (43) leads to:

$$S(q_n, r) \leqq \frac{1}{2} \varepsilon^{-1} w_{n-1} (a^2+b^2)^{-s/2+(n-1)/4} \int_0^\infty z^{(n-3)/2} (z^2+1)^{-s/2} \, dz.$$

Notice that the integral in this expression is convergent since $s > n/2$. By observing that:

$$a^2+b^2 = [(q+\mathscr{X})^2+r^2]^2. \tag{44}$$

we obtain:

$$\sum_{|q_n+\mathscr{X}| \geqq r} S(q_n, r) \leqq \text{Cte} \sum_{|q_n+\mathscr{X}| \geqq r} |q_n+\mathscr{X}|^{-2s+n-1}. \tag{45}$$

The hypothesis $s > n/2$ implies that the last series is convergent so that this term tends to zero as $r \to \infty$.

We now turn to the case $b<0$. We set $z=(y+b)/a$. (43) then gives:

$$S(q_n, r) \leqq \frac{1}{2} \varepsilon^{-1} w_{n-1} a^{-s+1} \int_{b/a}^{+\infty} (az-b)^{(n-3)/2} \{1+z^2\}^{-s/2} \, dz. \tag{46}$$

If $n \geqq 3$, this leads to:

$$\begin{aligned} S(q_n, r) &\leqq c_1 a^{-s+1} \int_{-\infty}^{+\infty} [|az|^{(n-3)/2} + |b|^{(n-3)/2}] \{1+z^2\}^{-s/2} \, dz \\ &\leqq c_2 a^{-s+1} [|a|^{(n-3)/2} + |b|^{(n-3)/2}] \leqq c_3 a^{-s+1} (a^2+b^2)^{(n-3)/4}. \end{aligned} \tag{47}$$

Using (47), (44) and (31), we obtain in this case that:

$$\sum_{|q_n+\mathscr{X}| < r} S(q_n, r) \leqq c_4 r^{-s+1} r^{n-3} \sum_{|q_n+\mathscr{X}| < r} |q_n+\mathscr{X}|^{-s+1} = \mathcal{O}(r^{-s+n-2} \log r),$$

since $s \geqq 2$. Under the hypothesis $s > n-2$, this converges to zero as $r \to \infty$.

Finally, if $n=2$, one may bound the integral in (46) by a constant which is independent of a and b on the set $\{a \geqq a_0 > 0,\ b < 0\}$; this is easily achieved by splitting the domain of integration into $\{z \mid az-b \leqq 1\} \cup \{z \mid az-b > 1\}$. Thus:

$$S(q_n, r) \leqq c_5 r^{-s+1} |q_n+\mathscr{X}|^{-s+1}, \qquad \forall q_n, \quad \forall r \geqq r_0.$$

For any $s > 3/2$, this implies that:

$$\lim_{r \to \infty} \sum_{|q_n+\mathscr{X}| < r} S(q_n, r) = 0. \quad \blacksquare$$

Remark 3. — One sees from the preceding proof that, for $n=3$, the limit in (33) is zero under the weaker hypothesis that $s > 3/2$. By using a modified resolvent equation, one obtains the result of Theorem 1 for $s > 3/2$. The case $s=2$, $n=3$ was first treated by Thomas in [12]. Similarly, for $n=2$, a more careful estimate of the integral in (46) shows that it suffices to require $s > 1$.

Remark 4. — Theorem 3 remains true if the condition of ortho-periodicity of v is replaced by the weaker condition of periodicity. Indeed, the estimation of the series given in section 5, may be applied if, instead of $\vec{a}_i . \vec{a}_j = \delta_{ij}$, one requires only that $\vec{a}_i . \vec{a}_n = \delta_{in}$ (i.e. the vector $\vec{a}_n$ is orthogonal to the hyperplane spanned by $\vec{a}_1, \ldots, \vec{a}_{n-1}$). Clearly the direction $\vec{a}_n$ is distinguished in our estimation. A similar result for an arbitrary periodic lattice is given in Theorem XIII.100 of [11], under a more restrictive assumption on the local behaviour of the function $v(\vec{x})$ than that of Theorem 3.

Remark 5. — We also have the following result which generalizes Theorem 1:

THEOREM 1′. — *Let $v \in L^s_{loc}(\mathbb{R}^n \setminus N)$, where s satisfies $s=2$ if $n=1, 2, 3$ and $s > n-2$ if $n \geqq 4$, and where N is a closed set of measure zero. Let H be a self-adjoint extension of $\hat{H}$, $D(\hat{H}) = C_0^\infty(\mathbb{R}^n \setminus N)$. Suppose that $f \in L^2(\mathbb{R}^n)$ satisfies $Hf = \lambda f$ for some $\lambda \in \mathbb{R}$ and $E(A) f = f$ for some compact subset of $\mathbb{R}^n \setminus N$ (i.e. f is an eigenvector of H having compact support in $\mathbb{R}^n \setminus N$). Then $f=0$.*

Proof. — One has $\chi_A(.)\, v(.) \in L^s(\mathbb{R}^n)$. Let C be a cube in $\mathbb{R}^n$ such that $A \subseteq C$. Define w by:

$$w(\vec{x} + \sum q_i \vec{a}_i) = \chi_A(\vec{x})\, v(\vec{x}), \qquad \vec{x} \in C,$$

w is ortho-periodic and in $L^s_{loc}(\mathbb{R}^n)$. Since $(H_0 + w) f = \lambda f$, one has $f=0$ by Theorem 3. ■

Remark 6. — The hypothesis "Σ bounded" in Theorem 1 (*b*) is essential. Assume for example that v is such that $H_0 + v$ has pure point spectrum (e.g. $v(\vec{x}) \to +\infty$ as $|\vec{x}| \to \infty$). Take $\Sigma = \mathbb{R}$. Then:

$$F(\Sigma)\, \mathscr{H} = \mathscr{H} \qquad \text{and} \qquad E(A)\, \mathscr{H} \cap F(\Sigma)\, \mathscr{H} \cap \mathscr{H}_p(H) = E(A)\, \mathscr{H}.$$

Since $E(A)\mathscr{H} \neq \{0\}$ if A has positive measure, it is clear that one cannot have $E(A)\mathscr{H} \cap F(\Sigma)\mathscr{H} \cap \mathscr{H}_p(H) = \{0\}$ in this case.

Remark 7. — By combining our Theorem 1 with Proposition 4 of [2], one may also prove that $E(A)\, \mathscr{H} \cap F(\Sigma)\, \mathscr{H} = \{0\}$ under assumptions of Theorem 1 (*b*).

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