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THE COMPACTIFIED JACOBIAN

By C. J. REGO

Let X be a reduced and irreducible curve over an algebraically closed field k. For X singular the generalized Jacobian variety of X i.e. the group variety parametrising line bundles of degree zero, on X, is an extension of an Abelian variety by a commutative affine group. In particular it is not complete. In [11] Mumford and Mayer proposed a natural compactification of the Jacobian consisting of torsion free O_X modules of rank 1 with Euler characteristic equal to $\chi(O_X)$. The construction of this compact scheme was settled in D'Souza's thesis where more was proved. The main results of [6] are:

(i) For any integer d let P_d be defined as follows. Fix a regular point "y" $\in X$ so for any k-scheme S we get a section defined by $\sigma_S(S)=(y)\times S$.

$$\overline{P}_d(S)=\{\text{isomorphism classes of coherent } O_{X\times S} \text{ modules } F_S, \text{ flat over S, inducing on the geometric fibres of } f_S : X\times S\to S, \text{ torsion free sheaves } F_{s_0} \text{ of rank 1 and } \chi(F_{s_0})=d, \text{ plus isomorphisms } \sigma_S^* F_S \approx O_S\}.$$

Then $\overline{P}_d$ is a representable functor.

(ii) The morphism of functors

$$\Phi_d : \quad \text{Hilb}^{-d}\to\overline{P}_d$$

[obtained by considering an ideal sheaf $I_S\subset O_{X\times S}$ flat on S as an element of $\overline{P}_d(S)$] is smooth at points $F\in\overline{P}_d(k)$, where F is an O_X module of Gorenstein dimension zero, whenever $-d\gg 0$. In particular Φ_d is smooth when X is Gorenstein (and $-d\gg 0$). (Recall that a module M over a local ring A has Gorenstein dimension zero if:

(i) M is reflexive;

(ii) $\text{Ext}^1(M, A)=\text{Ext}^1(M^*, A)=0$.

We say F is of Gorenstein dimension zero if each stalk satisfies the above conditions.)

(iii) If at each point $x \in X$ the δ invariant at x i.e. length [normalization $(O_{X,x})/O_{X,x}$] is less than or equal to one then $\overline{P}_d$ is reduced and irreducible. If the singularities of X have multiplicity at most two then $\overline{P}_d$ is irreducible.

See [2] for related material.

It is observed in [6] that (ii) implies the method of Chow-Matsusaka-Grothendieck for the construction of the Picard scheme extends to represent $\overline{P}_d$ in the Gorenstein case. In general (ii) is false and the equidimensionality of Φ_d, $-d \gg 0$, implies that X is Gorenstein, as is verified in [12].

The main results of this article are:

THEOREM A. — *If the singularities of* X *have embedding dimension two then* $\overline{P}$ *is irreducible. If* X *has a singularity of embedding dimension* $\geqq 3$ *then* $\overline{P}$ *is reducible.*

THEOREM B. — *The boundary* $\overline{P} - \mathrm{Pic}^0(X)$ *of* $\overline{P}$, *when* X *has planar singularities, is a union of m irreducible, codimension one subsets of* $\overline{P}$ *where*

$$m = \sum_{Q \in X} (\textit{multiplicity } O_{X,Q} - 1).$$

The first statement of Theorem A is deduced in [1] from Iarrobino's calculation of the dimension of the Punctual Hilbert scheme of $k[X, Y]$ (*see* [10]). We give a short self contained proof by induction on the multiplicity of a singular point of X. The induction works because the "polar is an adjoint curve of lower multiplicity than the given curve". We find it convenient to work with the scheme E of paragraph 2 rather than $\overline{P}$. Since Iarrobino's estimate appears as a Corollary of our method the treatment may be viewed as an application of curves to punctual Hilbert schemes of smooth surfaces. The proof of Theorem B utilizes Briançon's recent result [4] that the Punctual Hilbert scheme of $k[X, Y]$ is irreducible. It seems likely that Briançon's Theorem may be provable using the method of Theorem A.

The scheme E of paragraph 2 is useful also in describing the boundary of $\overline{P}$ when X has singularities of module type in the sense of [14].

An amusing aspect of the techniques used here is the amount of mileage one can get from the use of the fact that $\mathfrak{a}^{**} = \mathfrak{a}$ when $\mathfrak{a}$ is an ideal in a one dimensional Gorenstein ring.

1. Preliminaries and Notation

We write $\overline{P}$ for $\overline{P}_d$,

$$d = \chi(O_X) = \mathrm{rank}\ H^0(X, O_X) - \mathrm{rank}\ H^1(X, O_X).$$

The functor $\overline{P}$ is identified with the scheme representing it. As $\overline{P}$ can be constructed for a family $X_S \to S$ we sometimes write $\overline{P}(X)$ or $\overline{P}(X_S \mid S)$. Note that the algebraic group $\mathrm{Pic}^0(X)$ is contained as an open subset in $\overline{P}$ but $\overline{\mathrm{Pic}^0(X)} \neq \overline{P}$ in general. The morphism $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(\overline{X})$ obtained by pulling back line bundles to the normalization $\overline{X}$ is surjective with kernel G. One can think of G as O_X submodules L of K = the function field of X, with $L_y = O_{X,y}$, for smooth points y and $L_{x_i} = u_i . O_{X,x_i}$, for x_i singular points and where u_i is a unit in the normalization of O_{X,x_i}. Hence dimension $G = \delta = \mathrm{rank}\ H^0(X, O_{\overline{X}}/O_X)$. Note that $\mathrm{Pic}^0(X)$ and hence G acts on $\overline{P}$ by tensoring. Suppose $F \in \overline{P}(k)$ and $L \in G(k)$, $L_{x_i} = u_i . O_{X,x_i}$ then if $F \otimes L = F'$, $F \neq F'$ if and only if $u_i \in \mathrm{End}(F_{x_i})$ for some i. Hence the dimension of the G orbit through F is equal to rank $H^0(O_{\overline{X}}/\mathrm{End}(F))$. Remembering that if two fractional ideals over a domain are isomorphic then one is a multiple of the other by an element of the quotient field, we see immediately that the two torsion free O_X modules which are locally isomorphic "differ" by a line bundle.

DEFINITION 1.0. — We say $F \in \overline{P}(k)$ is a boundary point if F is not locally free and there is a coherent module $\mathscr{F}$ on $X \times \mathrm{Spec}\, k[t]$ flat over $\mathrm{Spec}\, k[t]$ with $\mathscr{F}/t.\mathscr{F} \approx F$ and $\mathscr{F} \otimes k((t))$ on $X \times \mathrm{Spec}\, k((t))$ a locally free rank one $O_{X \times \mathrm{Spec}\, k((t))}$ module.

Remark 1.1. — For an arbitrary flat deformation of F as above we have $\mathscr{F}$ to be of maximal depth, hence principal, at all smooth points of $X \times \mathrm{Spec}\, k[t]$. Hence the property of being a boundary point is local around the singular points $\{x_i\}$ — and depends only on the O_{X,x_i} modules F_{x_i}. If the modules F_{x_i}, for every i, can be deformed (flatly) on $O_{X,x_i} \otimes_k k[t]$ to a (generically) locally principal module then F is a boundary point. To see this assume for simplicity that X has one singular point (x_0) and write $S = \mathrm{Spec}\, k[t]$. The deformation of F_{x_0} defines a torsion free module $\mathscr{F}_V$ on $V \times S$, for an affine open neighbourhood V of x_0, with the property $\mathscr{F}_V \mid (V \times S) - (x_0) \times$ (closed point of S), is locally free. Extend $\mathscr{F}_V$ as a coherent sheaf to $X \times S$ and double dualize to get $\mathscr{F}'$. Now $\mathscr{F}'$, being reflexive and rank one, $\mathscr{F}'$ is flat over S. Put $\mathscr{F}'/t.\mathscr{F}' = F'$ and note that $F'_{x_0} \approx F_{x_0}$, so $F'_{x_0} = f.F_{x_0}$, where f is a rational function on X. Tensoring by a suitable line bundle L we get $L \otimes F' \approx F$. Then $L \otimes_k k[t] \otimes \mathscr{F}' = \mathscr{F}$ has F for special fibre and exhibits F as a boundary point. The case of several singular points is left to the reader. We will usually speak of boundary points as being modules over the local ring O_{X,x_0}.

The simplest non-trivial example of a boundary point is the maximal ideal. Write $0 = O_{X,x_0}$ and look at the diagonal ideal $I \subset O \otimes_k O$ and consider one O as parameter. The generic fibre of I is supported at smooth points, hence is locally principal, and the special fibre is just the maximal ideal. Since boundary points form a closed subset of $\overline{P}$ the limit of boundary points is a boundary point.

In the study of boundary points it suffices for most purposes to work with the points in the closure of G in $\overline{P}$. This is because of the:

PROPOSITION 1.2. — *If* $F \in \overline{P}$ *is a limit of line bundles then there is a line bundle* L *such that* $F \otimes L$ *is a limit of line bundles belonging to* G *i.e.*: $F \otimes L \in \overline{G}$.

Proof. — Suppose $\mathscr{F}$ is an $O_{X \times \operatorname{Spec} k[t]}$ module expressing F an a boundary point so $\mathscr{F}/t.\mathscr{F} \approx F$ and defines a morphism $h : \operatorname{Spec} k[t] \to \overline{P}$ with generic point of $h(\operatorname{Spec} k[t])$ in $\operatorname{Pic}^0(X)$. By composition with the morphism $\operatorname{Pic}^0(X) \to \operatorname{Pic}^0(\overline{X})$ we have a morphism $p' : \operatorname{Spec}\ k((t)) \to \operatorname{Pic}^0(\overline{X})$ and since $\operatorname{Pic}^0(\overline{X})$ is complete p' can be extended to $p : \operatorname{Spec} k[t] \to \operatorname{Pic}^0(\overline{X})$. By smoothness of $\operatorname{Pic}^0(X) \to \operatorname{Pic}^0(\overline{X})$ we can lift $p(\operatorname{Spec} k[t])$ to a curve T in $\operatorname{Pic}^0(X)$ and we have a morphism $p_0 : \operatorname{Spec} k[t] \to \operatorname{Pic}^0(X)$ with image T. Write $\mathscr{L}^{-1}$ for the line bundle on $X \times \operatorname{Spec} k[t]$ defined by p_0. By construction $\mathscr{L} \otimes \mathscr{F}$ is a family of O_X modules with the generic member a point in $G(k((t)))$ and with limit equal to $L \otimes F$, $L \approx \mathscr{L}/t.\mathscr{L}$. This proves the proposition.

Remark 1.3. — One may try to prove $\overline{P}$ irreducible as follows. Let $I \subset O_{X,x_0}$, length $(O_{X,x_0}/I) = n$. If I can be deformed to an ideal with non-trivial support at smooth points of X so that its colength at x_0 is less than n, then by induction on n, I is a limit of boundary points hence is a boundary point. In general this argument fails because the Punctual Hilbert scheme $H_0^n(X)$ of ideals in O_X supported at x_0 and of colength n, is a component of $\operatorname{Hilb}^n(X)$. Let X be (locally at x_0) embedded in a smooth surface S. Iarrobino has shown that the dimension of $H_0^n(S)$ is equal to $(n-1)$ so $H_0^n(X) \subsetneq H_0^n(S)$ has dimension less than or equal to $(n-1)$. To prove the irreducibility of $\overline{P}$ in this case it thus suffices to show that the components of $\operatorname{Hilb}^n(X)$ have dimension greater than or equal to n. This can be checked as follows. Suppose $f \in O_S$ defines X at x_0 and $f \in I$ with length $(O_S/I) = n$. By [8] $\operatorname{Hilb}^n(S)$ is smooth with a dense open subset defined by n distinct points on S. Let $\mathscr{I} \subset O_S \otimes k[t]$ define a deformation of O_S/I into "n distinct points" and $f \in \mathscr{I}$ map to f in $\mathscr{I}/t.\mathscr{I} = I$. Then, locally, f defines a family of curves over $\operatorname{Spec} k[t]$ and gives a section of

$$\operatorname{Hilb}^n(O_S \otimes k[t]/(f) \,|\, k[t]) \to \operatorname{Spec} k[t].$$

Look at the generic fibre of the relative Hilbert scheme; it has an n-dimensional component defined by the collection of "n-distinct points on the generic curve". By construction the point of $\operatorname{Hilb}^n(X)$ defined by O_X/I is in the limit of these n dimensional components of "nearby fibres". Since I was arbitrary $\operatorname{Hilb}^n(X)$ is of dimension greater than or equal to n at every point. In [1] this fact was verified as follows. The Poincare sheaf $M = O_H \otimes O_S/\mathscr{I}$ is a rank n vector bundle on $H = \operatorname{Hilb}^n(S)$. Then the section of M given by $1 \otimes f \in O_H \otimes O_S$ vanishes exactly on $\operatorname{Hilb}^n(X) \subset \operatorname{Hilb}^n(S)$. By [8] $\dim H = 2n$ so $\dim \operatorname{Hilb}^n(X) \geqq n$ at every point. In paragraph 3 we will prove that any extra component of $\overline{P}$, when X has planar singularities, has smaller dimension that $\operatorname{Pic}^0(X)$. By D'Souza's Theorem this would yield a component of $\operatorname{Hilb}^d(X)$, $d \gg 0$, of dimension less than d which is impossible. As a Corollary we derive Iarrobino's estimate for dimension $H_0^n(S)$.

One final remark: if a Gorenstein curve has irreducible $\overline{P}$ it has irreducible Hilb^n for every n. To see this take $I \subset O_{X,x_0}$, where I is the stalk at x_0 of $\mathscr{I}$, a sheaf of ideals on X, with $H^0(X, O_X/\mathscr{I})$ of dimension d, $d \gg 0$. By D'Souza's Theorem $\overline{P}$ irreducible $\Rightarrow$ $\operatorname{Hilb}^d(X)$ irreducible. So $\mathscr{I}$ can be deformed to a product of maximal ideals. Restricting this deformation to a neighbourhood of x_0 shows that I is in the closure of the open subset of Hilb defined by n distinct points of X. Hence $\operatorname{Hilb}^n(X)$ is irreducible.

2. The Functor E

Let $\mathscr{C}$ be the sheaf of conductors on X and write $U = X - \{x_i\}$ for the open subset of smooth points of X. Denote by $\mathscr{C}_1$ a subsheaf of $\mathscr{C}$ with $\mathscr{C}_1$ an $O_{\overline{X}}$ module. Let A be the semi local ring of functions regular at the $\{x_i\}$ and C, C_1 the ideals in A corresponding to $\mathscr{C}$ and $\mathscr{C}_1$. For $d \leqq \text{rank } H^0(\overline{X}, O_{\overline{X}}/\mathscr{C}_1) = \text{length } (\overline{A}/C_1)$, $\overline{A}$ the normalization of A, we define the functor $E(d, \mathscr{C}_1)$ by

$$E(d, \mathscr{C}_1)(S) = \{ F_S \mid F_S \in \overline{P}_q(S),\quad q = \chi(O_{\overline{X}}) - d,\ \mathscr{C}_1 \otimes_k O_S \subset F_S \subset O_{\overline{X}} \otimes_k O_S \text{ and } O_{\overline{X}} \otimes O_S / F_S \text{ is a locally free } O_S \text{ module of rank } d \}.$$

Since $\mathscr{C}_1 \otimes O_S = O_{\overline{X}} \otimes O_S$ on $U \times S$ the functor $E(d, \mathscr{C}_1)$ may be identified with the functor $E(d, C_1)$:

$$E(d, C_1)(S) = \{ I_S \mid C_1 \otimes_k O_S \subset I_S \subset \overline{A} \otimes_k O_S,\ I_S \text{ an } A \otimes_k O_S \text{ module and } \overline{A} \otimes_k O_S / I_S \text{ a locally free } O_S \text{ module of rank } d \}.$$

PROPOSITION 2.1. — *$E(d, \mathscr{C}_1)$ is representable by a projective scheme.*

Proof. — It is more convenient to check that $E(d, C_1)$ is representable. Look at the Grassmanian of vector subspaces of $\overline{A}/C_1$ of codimension d. For a subspace V to be an A module it suffices (and is necessary) that V be closed under the action of the group of units of A/C. In fact an S valued point of the Grassmanian is a locally free O_S module $\overline{I}_S$ where $\overline{I}_S$ comes from I_S, $C_1 \otimes O_S \subset I_S \subset \overline{A} \otimes O_S$. For I_S to be an $A \otimes_k O_S$ module, I_S must be invariant by multiplication by sections of $A \otimes O_S$ and as I_S is an O_S module it is enough that I_S is closed under multiplication by units of A. Since

$$C_1 . I_S \subset C_1 . (\overline{A} \otimes O_S) \subset C_1 \otimes O_S,$$

the finite dimensional algebraic group $(A/C_1)^*$ acts on Grass $(\overline{A}/C_1, d)$ and I_S defines a point of $E(C_1, d)$ iff it is a fixed point for the action of $(A/C_1)^*$. We may therefore apply the results of Fogarty [7] to conclude that E is representable by a closed subscheme of Grass $(A/C_1, d)$.

Remark 2.2. — There is an obvious morphism

$$e = e(C_1, d) : \quad E(C_1, d) \to \overline{P}_q, \qquad q = \chi(O_{\overline{X}}) - d,$$

which is proper as E is projective. Note that $E(d, C_1)$ is defined by A/C_1 so we get the same scheme for two curves with analytically isomorphic singularities. In particular, E is not sensitive to the birational character of the curve.

THEOREM 2.3. — *(a) Given* $\mathscr{C}_2 \subset \mathscr{C}_1 \subset \mathscr{C}$ *there is an injective, proper morphism*

$$q(\mathscr{C}_1, \mathscr{C}_2, d): \quad E(\mathscr{C}_1, d) \to E(\mathscr{C}_2, d).$$

(b) The morphism $e(\mathscr{C}_1, \delta): E(\mathscr{C}_1, \delta) \to \overline{P}$ *has image containing*

$$G = \ker(\mathrm{Pic}^0(\overline{X}) \to \mathrm{Pic}^0(X))$$

and is contained in the set of F *with* $F \mid U \approx O_U$. *In particular, putting* $\mathscr{C}_1 = \mathscr{C}$, *every boundary point defines an element of* $E(\mathscr{C}, \delta)$. *For* $\mathscr{C}_1$ *"sufficiently small" every isomorphism class of fractional ideals modulo multiplication by a line bundle is represented in* $E(\mathscr{C}_1, \delta)$.

(c) The morphism $e(\mathscr{C}_1, d)$ *is finite* $\forall d$ *and is injective if* $O_{\overline{X}}/\mathscr{C}$ *is local. In general* $e(\mathscr{C}_1, \delta)$ *restricted to* $e^{-1}(G)$ *is injective.*

(d) X *is Gorenstein* $\Leftrightarrow$ *every isomorphism class of fractional ideals modulo multiplication by a line bundle is represented in* $E(\mathscr{C}, \delta)$. *In particular if* X *is not Gorenstein then* $\overline{P}$ *is reducible.*

Proof. — The proof of *(a)* is immediate. To verify *(b)* let $F \in E(\mathscr{C}_1, \delta)$ so there is an exact sequence

$$0 \to F \to O_{\overline{X}} \to O_{\overline{X}}/F \to 0,$$

with $\chi(O_{\overline{X}}/F) = \mathrm{rank}\, H^0(O_{\overline{X}}/F) = \delta$. Hence $\chi(F) = \chi(O_{\overline{X}}) - \delta = \chi(O_X)$ so image of e is in $\overline{P}_{\chi(O_X)} = \overline{P}$. Let L be a line bundle with $L \otimes_{O_X} O_{\overline{X}}$ trivial on $\overline{X}$ i.e.: L is defined by $u \in \overline{A}$. Then L can be embedded in $O_{\overline{X}}$ so that $L \mid U = O_{\overline{X}} \mid U$ and $L_{x_i} = u.O_{X, x_i}$. Hence $H^0(O_{\overline{X}}/L)$ has rank δ and as $u.O_{X, x_i} \supset u.\mathscr{C}_{1, x_i} = \mathscr{C}_{1, x_i}$ we find L defines an element of $E(\mathscr{C}_1, \delta)$. This shows that $G \subset e(E(\mathscr{C}_1, \delta))$. It remains to prove the last assertion of *(b)*. Let I be an ideal in A. Since $\overline{A}$ is a P.I.D., $I.\overline{A} = (y).\overline{A}$ and it is easy to verify that y can be chosen in I. Then we have $1 \in y^{-1}.I$ so

$$A \subset y^{-1}.I \subset y^{-1}.y.\overline{A} = \overline{A}.$$

Let $z_1, z_2, \ldots, z_r$ generate the maximal ideals of $\overline{A}$. Any x in the quotient field of A can be written $x = u.\prod z_i^{s_i}$, u a unit in $\overline{A}$ and $s_i \in \mathbb{Z}$. Put $v_i(x) = s_i$. If $x.I \subset \overline{A}$ one checks easily that

$$\mathrm{length}\,(\overline{A}/x.I) = \mathrm{length}\,(\overline{A}/I) + \sum_{i=1}^{r} v_i(x).$$

Choose $C_1 = z_1^{\delta}.C$. Given an A module isomorphic to say an ideal I we can get an isomorphic copy $y^{-1}.I$ between A and $\overline{A}$, as above. Then $z_1^p.y^{-1}.I$ with $p = \mathrm{length}(y^{-1}.I/A)$ contains $z_1^p.C$ and is contained in $\overline{A}$ with length $(\overline{A}/z_1^p.y^{-1}.I) = \delta$. Further as $p \leqq \delta$ we have $z_1^p.C \supset C_1$. So with the above choice of C_1 every fractional ideal is represented in $E(C_1, \delta)$. It is now easy to globalize this fact; given an arbitrary O_X module torsion free of rank one we may assume after tensoring with a line bundle that it contains O_X and is contained in $O_{\overline{X}}$. Now the above argument can be applied. This proves *(b)*.

To verify *(c)* suppose J_1, J_2, are A modules contained in $\overline{A}$ representing two points of $E(C_1, d) \equiv E(\mathscr{C}_1, d)$. If $J_1 \approx J_2$ then there is an x in the quotient field with $J_1 = x.J_2$. If $v_i(x)$ is too large or too small for some i then $x.J_2 \not\supset C_1$ or $x.J_2 \not\subset \overline{A}$ so $\forall i$, $v_i(x)$ is bounded above and below. Hence modulo multiplication by elements of $\overline{A}$ there are finitely many x satisfying $x.J_2 = J_1$. But for a unit $u \in \overline{A}$ with $u.J_1 \neq J_1$ we have J_1 and $u.J_1$ mapping to different points in $\overline{P}_q$, $q=\chi(O_{\overline{X}})-d$. On the other hand if $\overline{A}$ has only one maximal ideal the above considerations show that $J_1 \approx J_2$ imply $J_1 = u.J_2$, $u \in \overline{A}^*$, so $e(C_1, d)$ is injective. Finally if $e(C_1, \delta)(J_1) \in G$ i.e.: $J_1 \approx A$ then $x.J_1 \subset \overline{A}$ implies $x.A \subset \overline{A}$ so that x is in $\overline{A}$ and $v_i(x) \geqq 0$, $\forall i$. But as length $(\overline{A}/J_1)=$ length $(\overline{A}/x.J_1)$ we have $\Sigma v_i(x)=0$ so x is a unit. This proves (c).

From the preceeding it follows that to prove (d) we must verify that A is Gorenstein $\Leftrightarrow$ every A submodule of the quotient field is represented by an element of $E(C, \delta)$. So suppose A is Gorenstein and let $C \subset J \subset A$ with $y \in A$ and $J.\overline{A} = y.\overline{A}$, $y = u.\prod . z_i^{s_i}$. We claim $\sum s_i \geqq$ length (A/J). To see this look at the picture

$$\begin{array}{ccc} y.\overline{A} & \subsetneq & \overline{A} \\ \cup & & \cup, \\ y.A \subset J \subset A & & \end{array} \qquad s = \sum s_i,$$

which shows that

length $(A/J) \leqq$ length $(y.\overline{A}/y.A)+$ length $(\overline{A}/y.\overline{A})-$ length $(\overline{A}/A)=\delta+s-\delta=s$. Hence $\exists (l_1, l_2, \ldots, l_r)$, $l_i \leqq s_i$, $\forall i$ and $J_1 = \prod z_i^{-l_i}.J \subset \overline{A}$ with $\Sigma l_i =$ length (A/J). But as length $(\overline{A}/J_1)=\delta$ and $C \subset \prod z_i^{-l_i}.C \subset J_1$, J_1 defines an element of $E(C, \delta)$. We must now show that every isomorphism class is represented by an ideal between C an A. But if J is an arbitrary fractional ideal then by Gorenstein duality we can write $J = N^{-1}$ and embed N in $\overline{A}$ so $A \subset N \subset \overline{A}$. Then $J \approx N^{-1}$ is isomorphic to an ideal of A containing C.

To complete the proof of (d) we will verify that for A not Gorenstein there is a module J with $A \subset J$ and length $(J/A)=1$; but no multiple of J defines an element of $E(C, \delta)$. We may assume that A is local. Let $A \subset J \subset \overline{A}$ with length $(J/A)=1$ and suppose there is a y with $y.J \subset \overline{A}$, length $(\overline{A}/y.J)=\delta$. Since length $(\overline{A}/J)=\delta-1$, $y = u.z_i$ for some i and u a unit in $\overline{A}$. If $C=\prod z_j^{c_j}.\overline{A}$ then $z_i^{-1}.C \supset C$ so if $C \subset z_i.A$ we get $z_i^{-1}.C \subset A$ which contradicts the definition of C as the largest $\overline{A}$ ideal in A. Hence $C \not\subset z_i.A$ and $C + z_i.A \supset z_i.A$ which gives

$$u.z_i.A + u.C = u.z_i.A + C \supsetneq u.z_i.A \subset y.J.$$

Length considerations give $J = A + z_i^{-1}.C$. So any point of $E(C, \delta)$ defined by a J with $J \supset A$ and length $(J/A)=1$ must be of the above type for some i. But if A is non Gorenstein length (End $(\mathfrak{m})/A) > 1$, $\mathfrak{m}$ the maximal ideal of A. Further every one dimensional subspace of End $(\mathfrak{m})/A$ defines an A module of the required type and since k is infinite (algebraically closed) there are infinitely many such. Hence for A non-Gorenstein there is a fractional ideal not represented in $E(C, \delta)$ and we are through.

Remark 2.5. — If J defines an element of $E(C_1, d)$, $d > \delta$ we have length $(\overline{A}/J) > \delta$ so J cannot contain a unit of $\overline{A}$. Hence $J.\overline{A} = \prod z_i^{r_i}.\overline{A}$, $r_i \geqq 0$, some $r_j > 0$. If say $r_1 > 0$ then $C_1 \subset z_1^{-1}.C_1 \subset z_1^{-1} J \subset \overline{A}$ which defines an element of $E(C_1, d-1)$. If $\overline{A}$ is local there is only one z_i and we get a map $E(C_1, d) \rightarrow E(C_1, d-1)$. It is easily checked (using the fact that every A module in $\overline{A}$ is represented by one between A and $\overline{A}$) that the $E(C, d)$, $d < \delta$ "cover" ($E(C_1, \delta)$-G) for C_1 sufficiently small. Here a map is defined by multiplying J by an element of $\overline{A}$ of suitable valuation.

Given a divisor $\sum n_P . P$ on a smooth curve $\overline{X}$ with $n_P \geqq 0$ there corresponds a curve X with one singular point and with $\overline{X}$ its normalization [14]. Given an affine open neighbourhood of the P with $n_P > 0$ having coordinate ring R then X is defined by the subring of R equal to $k + \mathfrak{m}_p^{n_P}$, $\mathfrak{m}_P$ the maximal ideal of $O_{X,P}$. These singularities are characterized by property that the maximal ideal is the conductor. For these singularities we have $E(C, \delta) \approx \mathbb{P}^\delta$ and as G is of dimension δ we have $E(C, \delta) = \overline{G}$. Hence in this case E yields exactly the boundary points of $\overline{P}$. We leave it to the reader to verify that there are only finitely many G orbits in this case. For example if X is defined by $\operatorname{Spec} k[x^n, x^{n+1}, \ldots, x^{2n}]$ the points in $E(C, \delta)$, $\delta = n-1$ are defined by $J_m = (x^n, \ldots, x^m, x^{m+2}, \ldots, x^{2n})$. There are therefore δ G orbits in $\overline{G} - G$ and these are of decreasing dimension.

PROPOSITION 2.6. — *For* X *rational with one unibranched singularity* $\overline{P}$ *is simply connected.*

Proof. — By the above $\overline{P}$ is bijective with $E(C_1, \delta)$ for C_1 sufficiently small. Now E is defined as a fixed point subset of a Grassmanian under the action of the group of units of A/C_1, A the singular local ring. As $k^* \subset \text{units}(A/C_1)$ acts trivially we have an action of an additive group on Grass. By [7] :

$$\pi_1(E(C_1, \delta)) \approx \pi_1(\text{Grass}) = (e),$$

which proves the proposition.

For an arbitrary family of curves $\varphi : X_S \to S = \operatorname{Spec} k[t]$ it is not clear how to define a relative E functor. Suppose however that the normalization $\overline{X}_S$ is smooth and the induced mapping $\varphi : \overline{X}_S \to S$ has smooth fibres. Also assume that if C is the conductor of X_S then O_{X_S}/C is S flat and $C/t.C$ is the conductor of $\varphi^{-1}(0)$. Then the relative E functor can be defined in an obvious way and is representable. This is because it can be interpreted as a fixed point set in Grass $(O_{\overline{X}_S}/C, d)$ of the group of units of O_{X_S}/C. Note that as O_{X_S}/C is S flat Fogarty's results [8] apply.

PROPOSITION 2.7. — *Dimension* $\overline{P} \leqq$ *genus* $(\overline{X}) + (\delta/2+1)^2$.

Proof. — Dimension $\overline{P}$ = dimension $(\operatorname{Pic}^0(\overline{X}))$ + dimension $E(C_1, \delta)$, C_1 sufficiently small, so we have to estimate the dimension of E. The constructions of [13] show that given any curve singularity X there is a family

$$\varphi : \quad X_S \to S = \operatorname{Spec} k[t]$$

with

$$X_S \otimes k((t)) \approx X \otimes_k k((t)) \qquad \text{and} \qquad X_0 = X_S \otimes_{k[t]} k$$

a singularity associated to a divisor $\sum n_P$ as described above. Further, the family φ satisfies the conditions given above which enable us to construct a relative E scheme over S which yields the E schemes of the fibres. By upper semi-continuity it suffices to obtain the estimate

$$\dim E \leqq (\delta+1)^2/4,$$

for a singularity associated to a divisor $\sum n_P . P$. But as the maximal ideal is the conductor, all the $E(C, d)$'s are Grassmanians and they cover $E(C_1, \delta)$. As $\dim E(C, d) = d(\delta + 1 - d)$, We get the required estimate.

3. Main Theorems

THEOREM A. — $\overline{P}$ *is irreducible* $\Leftrightarrow$ *the embedding dimension of* X *at every point is less than or equal to two.*

Proof. — Let X have planar singularities. By paragraph 1 the property of an O_X module $\mathscr{F}$ being a boundary point is local around the singular points $x_i \in X$. So let there be one singular point x_0. Then it suffices by Theorem 2.3 to show that $E(C, \delta)$ is irreducible (since X is Gorenstein). Finally, the E scheme depends only on $O_{X,x_0}/C$ so we can as we can as well study the completion $\hat{O}_{X,x_0} \approx k[X, Y]/(f) = A$. Put $v = \operatorname{ord} f$ and suppose the initial form of f is not X^v. Then if the characteristic of k is zero one checks easily (or *see* [3]) that $g = f_Y$ is an adjoint i.e.: g defines an element of the conductor C of $\overline{A}$ in A and $\operatorname{ord} g = v - 1$. More generally we have the:

LEMMA. — *In any characteristic there is a "g" in* C *of order* $(v-1)$.

Proof. — Let A_1 be the blow up of the maximal ideal $\mathfrak{m}$ of A and C_1 the conductor of A_1 in $\overline{A}$. Recall that $\mathfrak{m}^{v-1}$ is the conductor of A in A_1 and $C = C_1 . \mathfrak{m}^{v-1}$. Also by the definition of blowing up there is a Z in $\mathfrak{m}$ satisfying $Z.A_1 = \mathfrak{m}.A_1$ so that $\mathfrak{m}^{v-1}.A_1 = Z^{v-1}.A_1$.

As $C = C_1 . \mathfrak{m}^{v-1}$, $C \subset \mathfrak{m}^{v-1}$ and we have to show that $C \not\subset \mathfrak{m}^v$. Suppose not, then

$$(3.1.0) \qquad C_1 . \mathfrak{m}^{v-1} \subset \mathfrak{m}^v$$

implies

$$(3.1.1) \quad C_1 \subset \operatorname{Hom}(\mathfrak{m}^{v-1}, \mathfrak{m}^v) = \operatorname{Hom}(Z^{v-1}.A_1, Z^{v-1}.Z.A_1) = \operatorname{Hom}(Z^{-1}.A_1, A_1) = Z.A_1.$$

This says that $Z^{-1}.C_1 \subset A_1$, Z a non-unit in $\overline{A}$ and contradicts the definition of C_1 as the largest $\overline{A}$ ideal in A_1. The Lemma is thereby proved.

Remark. — We refer to any such "g" as a polar of "f".

To continue with the proof assume $\overline{P}$ is irreducible for plane curves of multiplicity less than ν. By the final remark of paragraph 1 this means that the punctual Hilbert scheme $\operatorname{Hilb}_0^n(k[X, Y]/(g))$ has dimension less than or equal to $(n-1)$. As $\operatorname{Hilb}_0^n(A/C) \subsetneq \operatorname{Hilb}_0^n(k[X, Y]/(g))$ we have $\dim \operatorname{Hilb}_0^n(A/C) \leqq n-1$. For $d > \delta$ write $E'(d)$ for the closure of the subscheme of $E(C, d)$ generated by $\operatorname{Hilb}_0^{d-\delta}(A/C) \subsetneq E(C, d)$ *via* translation by elements of $G = \overline{A}^*/A^*$. As noted in Remark 2.5 we do not have morphisms $E(C, d) \to E(C, \delta)$ when $\overline{A}$ is not local and $d > \delta$. However working with $e(E(C, d))$ we see easily that if Z is a closed G-stable subset of $e(E(C, d)) \subset \overline{P}_q$ then "tensoring by a line bundle" of suitable degree defines a bijection $Z \to Z_0 \subset \overline{P}_{\chi(O_X)} = \overline{P}$. In this sense we note that as A is Gorenstein and every fractional ideal lies between C and A, we can cover $e(E(C, \delta) - G)$ by $e(E'(d))$, $\delta < d \leqq 2\delta$. Hence $\overline{P}$-$\mathrm{Pic}^0(X)$ is covered by $\bigcup_d \mathrm{Pic}^0(X).e(E'(d))$. As

$$\dim \mathrm{Pic}^0(X).e(E'(d)) = \dim E'(d) + \dim \mathrm{Pic}^0(X)$$

and by paragraph 1 the dimension of every component of $\overline{P}$ is greater or equal to $\dim \mathrm{Pic}^0(X)$ it suffices to prove:

(3.1.2) $$\dim E'(d) < \delta \quad \text{for} \quad \delta < d \leqq 2\delta.$$

Let $W_d \subset E'(d)$ be an irreducible open subset satisfying the property that the G orbits in W_d are of the same dimension s, where automatically, s is the maximal dimension of the G orbits in the closure of $W_d \equiv \overline{W}_d \subset E'(d)$. Then taking a generic quotient by G we have:

(3.1.2) $$\dim \{\text{isomorphism classes of modules in } W_d\} = \dim W_d - s.$$

Let J define a point in W_d so $C \subset J \subset A$ and length $(A/J) = d - \delta$. The intersection of the G orbit through J with $\mathrm{Hilb}_0^{d-\delta}(A/C)$ is identified with $\{u.J \mid u \in G, u.J \subset A\}$ so we have:

(3.1.3) $$\dim ((G.J) \cap \mathrm{Hilb}_0^{d-\delta}(A/C)) = \text{length } (J^{-1}/\mathrm{End}\,(J)).$$

Further for J in W_d,

(3.1.4) $$\text{length } (\mathrm{End}\,(J)/A) = \text{length } (\overline{A}/A) - \text{length } (\overline{A}/\mathrm{End}\,(J)) = \delta - s.$$

Hence we get,

(3.1.5) $$\begin{aligned}\dim ((G.J) \cap \mathrm{Hilb}_0^{d-\delta}(A/C)) &= \text{length } (\overline{A}/A) - \text{length } (\overline{A}/J^{-1}) - \text{length } (\mathrm{End}(J)/A) \\ (\text{by duality}) &= \delta - \text{length } (J/C) - \delta + s = d + s - 2\delta.\end{aligned}$$

Outside a proper closed subset of W_d every J has $(G.J) \cap \mathrm{Hilb}_0^{d-\delta}(A/C) \neq \emptyset$ and hence we get

(3.1.6) $$\begin{aligned}\dim \mathrm{Hilb}^{d-\delta}(A/C) &= \dim (\text{generic moduli of isomorphism classes in } W_d) \\ &\quad + \dim (G.J) \cap \mathrm{Hilb}_0^{d-\delta}(A/C),\end{aligned}$$

which by the above yields,

$$(3.1.7)\quad (d-\delta)-1 \geqq \dim \mathrm{Hilb}_0^{d-\delta}(k[X, Y]/(g)) \quad (g \text{ a polar})$$
$$\geqq \dim \mathrm{Hilb}_0^{d-\delta}(A/C) \geqq \dim W_d - s + (d+s-2\delta).$$

Since $\overline{W}_d$ is an arbitrary irreducible component of $E'(d)$ we get $\dim E'(d) < \delta$ and so (3.1.2) is proved. Hence $\overline{P}$ is irreducible. For the other implication note that if A is not Gorenstein the result is contained in Theorem 2.3; so let A be Gorenstein.

We must show that A has embedding dimension two. If not the vector space $\mathfrak{m}/\mathfrak{m}^2$ with $\mathfrak{m}$ the maximal ideal of A, is of rank greater than or equal to 3. Note that every subspace of $\mathfrak{m}/\mathfrak{m}^2$ yields an ideal so that the projective space of codimension 1 subspaces yields a closed subscheme of $\mathrm{Hilb}_0^2(A)$ of dimension greater than or equal to 2. But for X Gorenstein we have noted in the final remark of paragraph 1 that for $\overline{P}(X)$ to be irreducible every $\mathrm{Hilb}^n(X)$ must be irreducible. In order that $\mathrm{Hilb}^2(X)$ be irreducible it must have the same dimension as the second symmetric product of X i.e. : equal to two. Now $\mathrm{Hilb}_0^2(A)$ is a closed subscheme of $\mathrm{Hilb}^2(X)$ not equal to the whole of it so $\dim \mathrm{Hilb}_0^2(A) \geqq 2$ implies $\dim \mathrm{Hilb}^2(X) \geqq 3$ which proves $\overline{P}(X)$ is reducible.

Remark 3.2. — Essential use is made of D'Souza's smoothness theorem in the last paragraph of the above proof *via* the remark "for X Gorenstein, $\overline{P}$ irreducible $\Leftrightarrow$ $\mathrm{Hilb}^n(X)$ is irreducible $\forall n$".

COROLLARY. — *Dimension* $\mathrm{Hilb}_0^n(k[X, Y]) = n-1$.

Proof. — Let $f \in k[X, Y]$ define a reduced and irreducible curve through (0, 0) with multiplicity n at the origin and Y its projective closure. Now $\mathrm{Hilb}_0^n(k[X, Y]/f)$ being a proper closed subscheme of $\mathrm{Hilb}^n(X)$ (which by the Theorem and paragraph 1 is of dimension n) has dimension less than or equal to $(n-1)$. But

$$\mathrm{Hilb}_0^n(k[X, Y]) = \mathrm{Hilb}_0^n(k[X, Y]/f)$$

as $f \in (X, Y)^n$ and every ideal of length n contains $(X, Y)^n$. It remains only to exhibit a component of dimension $(n-1)$. This is given by the family of ideals

$$g \in (X^n,\ Y + a_1 X + a_2 X^2 + \ldots + a_{n-1} X^{n-1})$$

Recently Briançon [4] has proved that $\mathrm{Hilb}_0^n(k[X, Y])$ is irreducible so the above family is dense open. The above discussion quickly yields.

THEOREM B. — *The boundary of* $\dot{\overline{P}}$ *for a curve with planar singularities has m irreducible components each of codimension one in* $\overline{P}$, *where*

$$(3.3)\qquad m = \sum_{Q \in X} [\textit{multiplicity}\ (Q) - 1].$$

Proof. — It is easily seen and left to the reader to check that the irreducible components of the boundary are "generated" by $\mathrm{Pic}^0(X)$ action by the corresponding subsets of $\overline{G} - G$. It therefore suffices to work with the E scheme of $A=\hat{O}_{X, x_0}$ where x_0 is a typical singular point of X. Let

$$A=k[X, Y]/f, \qquad v=\text{ord } f=\text{mult } (x_0),$$

C the conductor of A. As recalled earlier $\mathfrak{m}_A^{v-1} \supset C$ and in fact $\mathfrak{m}_A^{v-1}$ is the conductor of A in its first blow up. On the one hand, the polar is an adjoint curve of multiplicity $v-1$ and is contained in C. We have

$$\text{Hilb}_0^n(A)=\text{Hilb}_0^n(A/C) \qquad \text{for} \quad n<v-1.$$

On the other hand,

$$\text{Hilb}_0^n(A) \supsetneqq \text{Hilb}_0^n(A/C) \qquad \text{for} \quad n \geqq v.$$

This is because if g is the polar of f then

$$g \in (X^n, Y+a_1 X+a_2 X^2+\ldots+a_{n-1} X^{n-1})$$

for generic choice of a_i since $(g, Y+a_1 X+\ldots)$ will have length $v-1$ for almost all a_i. By Briançon's Theorem dim Hilb_0^n $(A/C)<n-1, n \geqq v$. The calculation of Theorem A shows that $E'(d)$ is irreducible of dimension $\delta-1$ for $d \leqq \delta+v-1$ and dimension $E'(d)<\delta-1$ for $d>\delta+v-1$. Now the $e(E'(d))$ cover $e(E(C, \delta))-G$ in the sense outlined in the proof of Theorem A. Further, since $\overline{P}$ is irreducible (i.e.: every fractional ideal is a boundary point) we have $e(E(C, \delta))=\overline{G}$ by Theorem 2.3. As G is affine $\overline{G}-G$ is a union of codimension one subsets. These are defined by the $E'(d)$ for $\delta<d \leqq \delta+v-1$. This proves the Theorem.

Remark 3.4. – It is likely that Briançon's Theorem is provable by the methods introduced here.

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Addendum

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