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ALGEBRAIC GEOMETRY AND LOCAL DIFFERENTIAL GEOMETRY

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In this paper we shall study the local differential geometry of a manifold in a linear space. We are interested in determining the structure of submanifolds whose position in the linear space fails to be generic in one of five specific ways ([2]). Algebraic geometry enters the problem quite naturally when we observe that the structure of sucessive infinitesimal neighborhoods of a point p in the manifold is described by a sequence of linear systems ([3]) in the tangent spaces $T_p(M)$ constituting the 1st, 2nd, 3rd, 4th, ... fundamental forms. These linear systems vary with p in a prescribed manner. Non-genericity turns out to be described by very strong pointwise implications on these linear systems. For fixed p these implications may be studied by algebro-geometric methods (Bertini's Theorem, Bezout's Theorem, elementary properties of base loci of rational maps, etc.). Then when p varies these algebro-geometric conclusions must fit together i.e. must satisfy integrability conditions, which leads to still further restrictions, and so forth.

In more detail, we are primarily interested in the local differential geometry of a complex-analytic submanifold of projective space, written

$$M \subset \mathbb{P}^N, \tag{0.1}$$

and with the full projective group as group of symmetries ([4]). On a few occasions we shall also be concerned with submanifolds of $\mathbb{C}^N$ having either the affine linear or affine unitary group as symmetry group. In a neighborhood of a generic point on M ([5]), we may attach to each point $p \in M$ a sequence of linear systems

$$|II|, \quad |III|, \quad |IV|, \quad \ldots$$

consisting of quadrics, cubics, quartics, ... in the projectivized tangent spaces $\mathbb{P}T_p(M) \cong \mathbb{P}^{n-1}$ where $\dim M = n$. These linear systems constitute the 2nd, 3rd,

([2]) These are discussed individually in paragraphs 2-6.

([3]) Here we mean linear systems of divisors on projective space. In Appendix A we have collected the relevant definitions and facts from algebraic geometry.

([4]) Aside from the results of paragraph 4 everything we shall do holds for real analytic submanifolds of $\mathbb{RP}^N$. Also, projective space may be replaced by any manifold having a flat projective connection, so that e.g. our results are meaningful for submanifolds of non-Euclidean spaces.

([5]) More precisely, we should be in an open set where a finite number of holomorphically varying matrices have constant maximal rank.

4th, ... fundamental forms of M in $\mathbb{P}^N$. Collectively their properties that are invariant under the projective group acting on $\mathbb{P}^{n-1}$ represent the basic invariants of the position of M in $\mathbb{P}^N$. The sequence of fundamental forms is linked by the remarkable property that the Jacobian system of the $(k+1)$-st is contained in the k-th. Roughly speaking, by assigning to $p \in M$ the linear systems $|II|, |III|, |IV|, \ldots$, we attach to the point p a sequence of image points in algebro-geometric moduli spaces, and as p varies in M these moduli points vary subject to precise conditions (6). In a sense the purpose of this paper is to put this vague philosophy in a form amenable to reasonably straightforward computations, and from these to draw some conclusions.

Following a preliminary discussion of frame manifolds, the fundamental forms are introduced and some examples computed in paragraphs 1 (*b*)-(*d*). The definition of the fundamental forms is by means of the osculating sequences associated to curves lying on M; essentially we invert the Theorem of Meusnier-Euler. In paragraph 1 (*e*) an alternate interpretation in terms of the Gauss mappings, the k-th one of which associates to a generic point $p \in M$ the k-th osculating space $\tilde{T}_p^{(k)}(M)$, is given. For example, the first Gauss mapping

$$\gamma: \quad M \to \mathbb{G}(n, N) \tag{0.2}$$

is the usual tangential one, and the 2nd fundamental form may be interpreted as the differential of γ.

Consideration of the sequence of osculating spaces is of course classical for curves, and it is not surprising that a general formalism should exist. What was unexpected was the extent to which algebro-geometric reasoning could be applied to draw differential-geometric conclusions on submanifolds in special position, and then in turn the local differential geometry can be applied to deduce global algebro-geometric conclusions. The simplest case of this occurs when the Gauss mapping (0.2) is assumed to be degenerate in the sense that $\dim \gamma(M) < \dim M$. It is easily shown that this is equivalent to all quadrics $Q \in |II|$ being singular along a $\mathbb{P}^{k-1}$ $(k \geqq 1)$, and then when this condition is suitable differentiated we are able to completely determine the local structure of submanifolds (0.1) with degenerate Gauss mappings [*cf.* § 2 (*b*)]. For real-analytic surfaces in $\mathbb{R}^3$ it is classical that those with Gaussian curvature zero (7) are pieces of planes, cones, or developable ruled surfaces. Once one has the concept of a multi-developable ruled variety (8) the general description turns out to not be significantly more complicated.

A global consequence is that any smooth projective variety $V \subset \mathbb{P}^N$ with a degenerate Gauss mapping must be $\mathbb{P}^n$. Our proof of this is by using the local description to actually locate the singularities on such a V which is not $\mathbb{P}^n$ (9).

Next we study manifolds (0.1) whose dual is degenerate. Recall that the dual $M^* \subset \mathbb{P}^{N^*}$ is the set of tangent hyperplanes to M, and degeneracy of the dual means that

(6) These conditions are the projective form of the Gauss-Codazzi equations.

(7) The vanishing of the Gaussian curvature (but not its numerical value) is a projectively invariant property.

(8) These occur only in dimensions $\geqq 3$.

(9) For example, for surfaces it is the vertex of the cone or the edge of regression of the developable ruled surface.

dim $M^* \leqq N-2$. In terms of 2nd fundamental forms this is expressed by the property that every quadric $Q \in |II|$ is singular. From Bertini's Theorem we conclude that $|II|$ has a non-empty base locus along which all $Q \in |II|$ are singular, and this then leads to the conclusion that M contains special families of linear spaces.

An easy Corollary is that if $V \subset \mathbb{P}^N$ is any projective variety whose Kodaira number $\varkappa(V) \geqq 0$ (10), then the dual variety is non-degenerate.

We are also able to devise computational methods for deciding when every $Q \in |II|$ is singular, and using these we can list all low-dimensional M with degenerate duals. A Corollary is that if $V \subset \mathbb{P}^N$ is a smooth projective variety with a degenerate dual, then dim $V \geqq 3$. Moreover, if equality holds then $N \geqq 5$ and V is geometrically ruled by $\mathbb{P}^2$'s.

In paragraph 4 we turn our attention to submanifolds $M \subset \mathbb{C}^N$ with the affine unitary group as structure group. It is well known that the Chern forms $c_q(\Omega_M)$ constructed from the curvature matrix Ω_M satisfy a pointwise inequality

$$(-1)^q c_q(\Omega_M) \geqq 0,$$

but it does not seem to have been determined when equality can hold. We observe that the condition $c_q(\Omega_M) \equiv 0$ is a projectively invariant property [*cf.* footnote (7)], and then we show that this happens exactly when every quadric $Q \in |II|$ has rank $\leqq q-1$. For $q=n$, therefore, $c_n(\Omega_M) \equiv 0$ if, and only if, the dual M^* is degenerate, and in this case we may apply the preceding analysis to conclude that M contains a lot of linear spaces.

A global conclusion can be drawn by considering a closed complex-analytic subvariety V of an abelian variety $\mathbb{C}^N/\Lambda$. We find that the conditions

$$\begin{cases} c_n(\Omega_V) \equiv 0, \\ c_1(\Omega_V)^n \equiv 0 \end{cases}$$

are equivalent, and are satisfied if, and only if, V is ruled by abelian subvarieties. When V is smooth this was proved using global techniques by Smythe [12].

Next, in paragraph 5, we discuss manifolds M in $\mathbb{P}^N$ whose tangential variety $\tau(M)$ is degenerate. First we express this condition in terms of the 2nd fundamental form. Using some algebro-geometric reasoning it turns out that $|II|$ must have a base locus with quite unusual properties. Using this and applying suitable differentiation we find what is perhaps our deepest result, namely, that the Gauss mapping on $\tau(M)$ has fibres of dimension $\geqq 2$. Now all our previous local results have a global implication, but we have been unsuccessful in determining any global consequence of this Theorem. Because it is somewhat subtle we suspect that it may have something to do with the beautiful recent result of Fulton and Hansen [5].

Next, in paragraph 5(*c*) and (*d*) we give a structure Theorem for manifolds having a degenerate tangential variety and that are in a certain sense generic among manifolds with this property. In particular, this includes all but one such M with dim $M \leqq 4$, and we are

(10) Here V may be singular. We recall that the Kodaira number is the transcendence degree of the canonical ring; in particular $-1 \leqq \varkappa(V) \leqq n$.

then able to list these. A Corollary is that if $V \subset \mathbb{P}^N$ is a smooth projective variety having a degenerate tangential variety, then either V lies in a $\mathbb{P}^{2n-1}$ or else dim $V \geqq 4$. Moreover, if equality holds then V has the same 2nd fundamental form as the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. We also see that there are varieties with arbitrary Kodaira number having a degenerate tangential variety.

Finally we turn to manifolds (0.1) having a degenerate secant variety $\sigma(M)$. Here we should like to mention that, whereas our information on the first three types of degeneracy is in a sense fairly complete, our results on degenerate tangential varieties are only a part of what one should reasonably be able to find, and this is even more the case for degenerate secant varieties. The degeneracy of $\sigma(M)$ is a condition on $M \times M$, which when expanded in a power series about the diagonal has as leading coefficient an expression involving the 2nd fundamental form $|II|$ and a refinement of the 3rd fundamental form. Examination of this term gives a result of which a curious easy consequence is that if $\sigma(M)$ is degenerate but $\tau(M)$ is not, then M lies in a $\mathbb{P}^{n(n+3)/2}$. Moreover, when dim $M=2$ then either M lies in a $\mathbb{P}^4$ or else it is part of the Veronese surface $\mathbb{P}^2 \subset \mathbb{P}^5$. This last result is proved in paragraph 6 (*c*); its argument requires the most intricate computation of any result in the paper.

It is hardly necessary to mention that the complete story on the secant variety will require examination of more than the first term in the power series, but we have not seriously tried to do this.

In Appendix A we have collected the results from multilinear algebra and algebraic geometry that are required for our study. Since the material concerning linear systems may not be so well known among differential geometers we have attempted to at least explain what we need from this area. What is difficult to capture in a few lines is the use of geometric reasoning with linear systems for computations; undoubtedly this comes best by looking at low-dimensional examples, of which there are a lot scattered throughout [9].

Our technique for differential-geometric computations is to use moving frames. These provide an efficient formalism, so that for instance none of our calculations exceeds a few lines and only one proof requires more than a couple of paragraphs. In fact, computations involving successive prolongations (i. e., going to higher order information) and carried out by moving frames have an algorithmic character (11). In Appendix B, then, we have attempted to explain how to use moving frames and why they should apply to our problem. Specifically, although we have not formulated it precisely it is clear from the discussion in paragraph B (*a*) that *in general* the moduli point described by the sequence of linear systems $|II|, |III|, |IV|, \ldots$ completely determines the position of M in $\mathbb{P}^N$ up to a projective transformation. However, this rigidity statement will fail in certain cases where more subtle higher order invariants are required. For example, if $M \subset \mathbb{P}^{n+1}$ is a hypersurface then the 2nd fundamental form amounts to a single quadric in $\mathbb{P}T_p(M) \cong \mathbb{P}^{n-1}$. Assuming that Q is non-degenerate (12) it has a normal form and so the moduli point is constant. Putting Q in this standard form and using the method of moving

(11) In this regard, *cf.* [6] and [10].

(12) The case when Q is degenerate is completely described in paragraph 3 (*b*).

frames — i.e., differentiating the condition that the moduli point is constant — leads to a cubic V on $\mathbb{P}^{n-1}$ whose residual intersection with Q is an algebro-geometric invariant representing a "refined moduli point" associated to $p \in M$. Aside from degenerate cases this moduli point determines the position of M in $\mathbb{P}^{n+1}$ when $n \geqq 3$.

When $n=2$ we are in the classically much studied topic of the projective differential geometry of a surface S in $\mathbb{P}^3$ (13). Here the intersection $Q \cap V$ is in general empty, but for somewhat subtle reasons the ideal generated by Q and V in the ring of homogeneous forms on $\mathbb{P}T_p(S) \cong \mathbb{P}^1$ determines S. This is a beautiful Theorem of Fubini, and in paragraph B(*b*) we have discussed its proof, together with the rigidity of hypersurfaces when $n \geqq 3$, as a means of illustrating the algorithmic character of the method of moving frames. The paper concludes with some observations and questions on submanifolds of codimension two in $\mathbb{P}^N$. A list of notations and ranges of indices appears just before the bibliography.

It is our pleasure to thank Mark Green for numerous comments and suggestions, and Bill Fulton for encouragement and correspondence concerning his joint work [5] with Hansen. Also we should like to express appreciation to the referee for several suggestions that helped clarify the presentation.

1. Differential geometric preliminaries

(*a*) STRUCTURE EQUATIONS FOR THE FRAME MANIFOLDS. — In order to have a formalism which most simply expresses differential-geometric relationships we shall use the calculus of moving frames (14), and we begin by establishing notation.

In $\mathbb{C}^N$ a frame is denoted by $\{z; e_1, \ldots, e_N\}$; it is given by a position vector z and basis $e_1, \ldots, e_N$ for $\mathbb{C}^N$ (15). The set of all frames forms a complex manifold $\mathscr{F}(\mathbb{C}^N)$, that upon choice of a reference frame may be identified with the affine general linear group $\widetilde{GL}_N$. Using the index range $1 \leqq a, b \leqq N$ each of the vectors z and e_a may be viewed as a vector-valued mapping

$$v: \mathscr{F}(\mathbb{C}^N) \to \mathbb{C}^N.$$

Expressing the exterior derivative dv in terms of the basis e_a gives

$$(1.1) \qquad \begin{cases} dz = \sum_a \theta_a e_a, \\ de_a = \sum_b \theta_{ab} e_b. \end{cases}$$

(13) There is an incredibly vast literature here (*cf.* [4]), perhaps even more extensive than that on theta functions.

(14) The classic source is [2]. More recent expositions appear in [6], [7] and [10]. An informal discussion of the method is given in Appendix B.

(15) Only in paragraph 4 will we be concerned about a metric structure.

The $N+N^2=N(N+1)$ differential 1-forms θ_a, θ_{ab} are the Maurer-Cartan forms on the group $\widetilde{GL}_N$, and by taking the exterior derivatives in (1.1) we obtain the Maurer-Cartan equations

$$(1.2)\qquad \begin{cases} d\theta_a=\sum_b \theta_b \wedge \theta_{ba}, \\ d\theta_{ab}=\sum_c \theta_{ac} \wedge \theta_{cb}. \end{cases}$$

Most of our study will be concerned with the local differential geometry of a submanifold of $\mathbb{P}^N$, and for this we shall use projective frames. A frame $\{A_0, A_1, \ldots, A_N\}$ for $\mathbb{P}^N$ is given by a basis $A_0, A_1, \ldots, A_N$ for $\mathbb{C}^{N+1}$. The manifold of all such frames may be identified with GL_{N+1} ([16]). Using the index range $0 \leqq i, j, k \leqq N$ the structure equations of a moving frame

$$(1.3)\qquad \begin{cases} dA_i=\sum_j \omega_{ij} A_j, \\ d\omega_{ij}=\sum_k \omega_{ik} \wedge \omega_{kj} \end{cases}$$

are valid for the same reasons as (1.1) and (1.2). Geometrically we may think of $\{A_0, A_1, \ldots, A_N\}$ as defining a coordinate simplex in $\mathbb{P}^N$, and then $\{\omega_{ij}\}$ gives the rotation matrix when this coordinate simplex is infinitesimally displaced.

There is a fibering

$$(1.4)\qquad \pi:\quad \mathcal{F}(\mathbb{P}^N) \to \mathbb{P}^N$$

given by

$$\pi\{A_0, A_1, \ldots, A_N\}=A_0 \quad ([17]).$$

The fibre $\pi^{-1}(p)$ over a point $p \in \mathbb{P}^N$ consists of all frames $\{A_0, A_1, \ldots, A_N\}$ whose first vector lies over p [*cf.* (1.5) below]. If we set

$$\omega_i=\omega_{0i}$$

then the 1-forms

$$\omega_1, \ldots, \omega_N=\{\omega_a\}$$

are horizontal for the fibering (1.4); i.e., they vanish on the fibres $\pi^{-1}(p)$. In fact, under a

([16]) In a way it is more natural to use frames satisfying $A_0 \wedge A_1 \wedge \ldots \wedge A_N=1$, so that the corresponding group is SL_{N+1}. For our purposes it is notationally simpler to not make this normalization.

([17]) Here we mean the point in $\mathbb{P}^N$ determined by $A_0 \in \mathbb{C}^{N+1}-\{0\}$.

change of frame in $\pi^{-1}(p)$

$$
(1.5) \qquad \left\{ \begin{array}{l} A_0 = \mu \tilde{A}_0, \\ A_a = \sum_b g_{ab} \tilde{A}_b + \lambda_a \tilde{A}_0, \end{array} \right.
$$

where $\mu \det (g_{ab}) \neq 0$, we have

$$
(1.6) \qquad \left\{ \begin{array}{l} \tilde{\omega}_0 = \omega_0 - d \log \mu + \mu^{-1} \sum_a \lambda_a \omega_a, \\ \tilde{\omega}_b = \mu^{-1} \sum_a \omega_a g_{ab}, \end{array} \right.
$$

which shows clearly the horizontality of the ω_a's. From

$$
d\omega_a = \sum_b \omega_b \wedge \omega_{ba}
$$

we see that the forms $\{ \omega_a \}$ satisfy the Frobenius integrability condition. Thus we may think of the fibration (1.4) as defined by the foliation

$$
\omega_1 = \ldots = \omega_N = 0.
$$

The equation

$$
(1.7) \qquad dA_0 = \sum_i \omega_i A_i \equiv \sum_a \omega_a A_a \bmod A_0
$$

has the following geometric interpretation:

For each choice of frame $\{ A_0, A_1, \ldots, A_N \}$ lying over $p \in \mathbb{P}^N$ the horizontal 1-forms $\omega_1, \ldots, \omega_N$ give a basis for the cotangent space $T_p^*(\mathbb{P}^N)$. The corresponding basis $v_1, \ldots, v_n \in T_p(\mathbb{P}^N)$ for the tangent space has the property that v_a is tangent to the line $\overline{A_0 A_a}$.

We shall also use frames to study the Grassmannian $G(n, N)$ of n-planes through the origin in $\mathbb{C}^N$. Here the frame manifold $\mathscr{F}(G(n, N))$ consists of all bases $\{ e_1, \ldots, e_N \}$ for $\mathbb{C}^N$. Using the ranges of indices

$$
1 \leqq a, b \leqq N; \qquad 1 \leqq \alpha, \beta \leqq n; \qquad n+1 \leqq \mu, \nu \leqq N
$$

the fibering

$$
(1.8) \qquad \pi : \mathscr{F}(G(n, N)) \to G(n, N)
$$

associates to each frame $\{ e_1, \ldots, e_N \}$ the n-plane S spanned by $e_1, \ldots, e_n$. The 1-forms $\{ \theta_{\alpha\mu} \}$ are horizontal for the fibration (1.8), which as before is given by the foliation

$$
\theta_{\alpha\mu} = 0.
$$

The equation

$$(1.9) \qquad de_\alpha \equiv \sum_\mu \theta_{\alpha\mu} e_\mu \bmod S$$

has the following geometric interpretation:

Over the Grassmannian there are the universal sub- and quotient-bundles, which we also denote by S and Q. Then by (1.9) there is an isomorphism

$$(1.10) \qquad T(G(n, N)) \cong \operatorname{Hom}(S, Q).$$

More precisely, the pullbacks to $\mathscr{F}(G(n, N))$ of all three bundles are canonically trivialized, and by (1.9) there is an isomorphism

$$\pi^* T(G(n, N)) \cong \operatorname{Hom}(\pi^* S, \pi^* Q)$$

that is invariant along the fibres in (1.8). In the future we shall use, without comment, equations such as (1.9) up on the frame manifold to describe geometric relations such as (1.10) that hold on the quotient space.

We shall also have occasion to consider the Grassmannian $\mathbb{G}(n, N)$ of all $\mathbb{P}^n$'s in $\mathbb{P}^N$. This manifold may be identified with $G(n+1, N+1)$, and frames for it will be denoted by $\{A_0, A_1, \ldots, A_N\}$ where $A_0, A_1, \ldots, A_n$ span the $\mathbb{P}^n$ in question. We shall also denote by $\{\omega_{ij}\}$ the Maurer-Cartan matrix attached to a moving frame.

Finally we remark that a list of notations and ranges of indices is given at the end of this paper.

(*b*) THE 2nd FUNDAMENTAL FORM. — We assume given a connected n-dimensional complex submanifold

$$M \subset \mathbb{P}^N.$$

At each point $p \in M$ the projective tangent space $\tilde{T}_p(M)$ is defined as the limiting position of all chords $\overline{pq}$ as $q \to p$. We shall view $\tilde{T}_p(M)$ either as a $\mathbb{P}^n \subset \mathbb{P}^N$ or as a $\mathbb{C}^{n+1} \subset \mathbb{C}^{N+1}$. With this latter interpretation there is an isomorphism, defined up to scale factors,

$$T_p(M) \cong \tilde{T}_p(M)/\mathbb{C}.A_0 \quad (^{18}),$$

where $A_0 \in \mathbb{C}^{N+1} - \{0\}$ is any point lying over p.

Equivalently, in any affine open set $\mathbb{C}^N \subset \mathbb{P}^N$ for each point $p \in M \cap \mathbb{C}^N$ we may consider the usual affine tangent n-plane to $M \cap \mathbb{C}^N$ in $\mathbb{C}^N$, and then $\tilde{T}_p(M)$ is the corresponding $\mathbb{P}^n$ in $\mathbb{P}^N$.

[18] We recall here the Euler sequence

$$(1.11) \qquad 0 \to \mathcal{O} \to \tilde{T}(M) \to T(M) \to 0$$

where $\tilde{T}(M) = \bigcup_{p \in M} \tilde{T}_p(M)$ is the corresponding abstract $\mathbb{C}^{n+1}$-bundle.

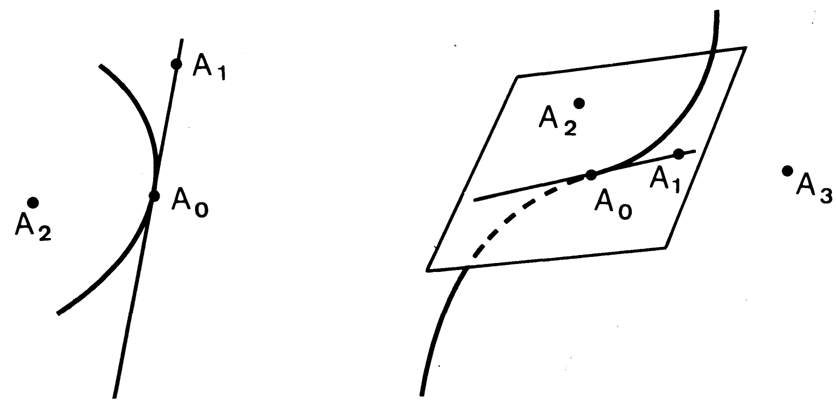
Associated to $M \subset \mathbb{P}^N$ is the submanifold $\mathscr{F}(M) \subset \mathscr{F}(\mathbb{P}^N)$ of Darboux frames

$$\{A_0; A_1, \ldots, A_n; A_{n+1}, \ldots, A_N\} = \{A_0; A_\alpha; A_\mu\}$$

defined by the conditions

$$(1.12) \qquad \left\{ \begin{array}{l} A_0 \text{ lies over } p \in M, \\ \text{and} \\ A_0, A_1, \ldots, A_n \text{ spans } \tilde{T}_p(M). \end{array} \right.$$

For a curve in $\mathbb{P}^2$ or surface in $\mathbb{P}^3$ we have in mind the pictures



We shall use the additional index ranges

$$1 \leqq \alpha, \beta, \gamma \leqq n, \qquad n+1 \leqq \mu, \quad \nu \leqq N,$$

and recall that we have set $\omega_{0i} = \omega_i$. From (1.7) we infer that the condition

$$dA_0 \in \tilde{T}_p(M)$$

on $\mathscr{F}(M)$ is equivalent to

$$(1.13) \qquad \left\{ \begin{array}{l} \text{the } \omega_\alpha \text{ give a basis for } T_p^*(M) \\ \text{and} \\ \omega_\mu = 0. \end{array} \right.$$

From (1.6) we see that it is natural to think of the ω_α as homogeneous coordinates in the projectivized tangent spaces

$$\mathbb{P}\,T_p(M) \cong \mathbb{P}^{n-1}.$$

For example, a quadric in $\mathbb{P}\,T_p(M)$ is defined by an equation

$$\sum_{\alpha, \beta} q_{\alpha\beta}\, \omega_\alpha \omega_\beta = 0, \qquad q_{\alpha\beta} = q_{\beta\alpha}.$$

We shall also use the notation $v_\alpha \in T_p(M)$ for the basis of tangent vectors dual to the 1-forms ω_α. As mentioned above, v_α is tangent to a curve $p(t)$ in M such that the chords $\overline{pp(t)}$ have as limiting position the line $\overline{A_0 A_\alpha}$.

For later use we want to discuss the second condition in (1.13) in a somewhat broader context. On a manifold $\mathscr{F}$ a *differential system* is given by a finite collection of differential forms φ_λ. An integral manifold is a submanifold $\mathscr{F}' \subset \mathscr{F}$ on which all φ_λ restrict to zero. We may think of $\mathscr{F}'$ as defined by equations

$$\varphi_\lambda = 0 \quad (^{19}). \tag{1.14}$$

A differential ideal is an ideal I in the exterior algebra of differential forms that is closed under exterior differentiation. There is a smallest differential ideal $I(\varphi_\lambda)$ containing any collection of forms φ_λ, and it is clear that any $\varphi \in I(\varphi_\lambda)$ restricts to zero on any integral manifold of the differential system. The simplest differential systems are the completely integrable ones; i.e., they are generated by linearly independent 1-forms $\varphi_1, \ldots, \varphi_k$ satisfying

$$d\varphi_\lambda \equiv 0 \mod \varphi_1, \ldots, \varphi_k. \tag{1.15}$$

By the Frobenius Theorem there is through each point of $\mathscr{F}$ a unique $(n\text{-}k)$-dimensional integral manifold constituting a leaf of the foliation defined by (1.14). In general the failure of (1.15) to hold cuts down the dimension of integral manifolds; the existence and uniqueness results are embodied, at least in the real analytic case, in the Cartan-Kähler Theorem.

On the frame manifold $\mathscr{F}(\mathbb{P}^N)$ we consider the differential system

$$\omega_\mu = 0. \tag{1.16}$$

For each submanifold $M \subset \mathbb{P}^N$ the Darboux frames $\mathscr{F}(M) \subset \mathscr{F}(\mathbb{P}^N)$ constitute an integral manifold; moreover, any maximal integral manifold satisfying a mild general position requirement is given in this way. Consequently the projectively invariant properties of $M \subset \mathbb{P}^N$ are embodied in the equations (1.16) and the consequences obtained by exterior differentiation of them [*cf.* Appendix B, Lemmas (B.2) and (B.3)]. By the second equation in (1.3), on $\mathscr{F}(M)$:

$$0 = d\omega_\mu = \sum_\alpha \omega_\alpha \wedge \omega_{\alpha\mu}.$$

Appealing to the Cartan Lemma (A.2) it follows that

$$\omega_{\alpha\mu} = \sum_\beta q_{\alpha\beta\mu}\, \omega_\beta, \qquad q_{\alpha\beta\mu} = q_{\beta\alpha\mu}. \tag{1.17}$$

We set

$$Q_\mu = \sum_{\alpha,\beta} q_{\alpha\beta\mu}\, \omega_\alpha \omega_\beta, \qquad \mu = n+1, \ldots, N,$$

(19) These are really the equations of the tangent spaces to $\mathscr{F}'$.

and define the *2nd fundamental form* II to be the linear system $|\mathrm{II}|$ of quadrics in $\mathbb{P}\,T_p(M) \cong \mathbb{P}^{n-1}$ obtained in this way ([20]). One may think of $|\mathrm{II}|$ as the initial algebro-geometric invariant attached to $p \in M$. In general this invariant will have moduli and the corresponding mapping $M \to \{\text{moduli space}\}$ will go a fair distance towards determining the position of M in $\mathbb{P}^N$ ([21]). But in certain cases such as low codimension (say 1) or dimension (say 1 or 2), or when $|\mathrm{II}|$ is the full system of quadrics or else is strangely degenerate, it will be necessary to go to higher order invariants — *cf.* Appendix B for further discussion.

It is useful to express the 2nd fundamental form in more intrinsic terms. We define the normal space at $p \in M$ by

$$N_p(M) = \mathbb{C}^{N+1}/\tilde{T}_p(M).$$

Then the 2nd fundamental form may intrinsically be thought of as a map

$$\mathrm{II}: \quad \mathrm{Sym}^{(2)}\, T(M) \to N(M) \tag{1.18}$$

given in coordinates by

$$\mathrm{II}(v) = \sum_{\alpha,\,\beta,\,\mu} q_{\alpha\beta\mu}\, \omega_\alpha(v)\, \omega_\beta(v)\, A_\mu,$$

where $v = \sum_\alpha \omega_\alpha(v)\, v_\alpha \in T_p(M)$. Thus $|\mathrm{II}|$ is a linear system of quadrics parametrized by the dual $N_p^*(M)$ of the normal space. The algebro-geometric properties of this linear system, such as its dimension, base locus, singular locus, fibre dimension, etc. may then be expected to reflect, and be reflected in, the local differential geometry of $M \subset \mathbb{P}^N$.

In addition to (1.18) we shall also on occasion use the notation

$$Q: \quad T(M) \otimes T(M) \to N(M)$$

for the 2nd fundamental form, this to emphasize its quadratic character.

The 2nd fundamental form has been defined analytically, but of course it arose geometrically. For our purposes there will be two geometric interpretations. One is via the Gauss map that will be explained in paragraph 1 (*d*), and the other is via the classical Theorem of Meusnier-Euler that we shall discuss now. Recall that if $\{p(t)\}$ is any holomorphic arc in $\mathbb{P}^N$ described by a vector-valued function $A_0(t)$, then the osculating sequence is the sequence of linear spaces spanned by the following collections of vectors (primes denote derivatives):

$$\{A_0(t)\}, \qquad \{A_0(t),\, A_0'(t)\}, \qquad \{A_0(t),\, A_0'(t),\, A_0''(t)\}, \qquad \ldots$$

We then have the Theorem of Meusnier-Euler:

([20]) In paragraph (*b*) of Appendix A there is a discussion of those aspects of linear systems that we shall use.

([21]) In this regard we mention that there are enormous possibilities for linear systems of quadrics. For example, under a suitable projective embedding any algebraic variety may be given as the base of a linear system of quadrics.

(1.19) *For a tangent vector* $v \in T_p(M)$, *the normal vector*

$$II(v) \in N_p(M)$$

gives the projection in $\mathbb{C}^{N+1}/\tilde{T}_p(M)$ *of the 2nd osculating space to* any *curve* $p(t)$ *with tangent* v *at* $t=0$.

To prove this we choose an arbitrary field of Darboux frames $\{A_i(t)\}$ along $p(t)$ and write

$$\frac{dA_0}{dt} \equiv \sum_{\alpha} \left(\frac{\omega_\alpha}{dt}\right) A_\alpha \bmod A_0,$$

$$\frac{d^2 A_0}{dt^2} \equiv \sum_{\alpha} \left(\frac{\omega_\alpha}{dt}\right)\left(\frac{dA_\alpha}{dt}\right) \bmod A_0, A_1, \ldots, A_n = \sum_{\alpha, \beta, \mu} q_{\alpha\beta\mu} \left(\frac{\omega_\alpha}{dt}\right)\left(\frac{\omega_\beta}{dt}\right) A_\mu$$

by (1.17). Q.E.D.

We may symbolically write the second equation in this proof as [22]:

$$d^2 A_0 \equiv \sum_{\alpha, \beta, \mu} q_{\alpha\beta\mu}\, \omega_\alpha\, \omega_\beta\, A_\mu \bmod \tilde{T}(M), \tag{1.20}$$

and geometrically interpret (1.20) as expressing the relation between the 2nd fundamental form as defined analytically and the 2nd osculating spaces to curves in M.

In particular we recall from Appendix B that the base B(II) of the linear system $|II|$ is the subvariety of $\mathbb{P}\, T_p(M) \cong \mathbb{P}^{n-1}$ defined by

$$\sum_{\alpha, \beta} q_{\alpha\beta\mu}\, \omega_\alpha\, \omega_\beta = 0.$$

Geometrically, a vector $v \in T_p(M)$ is in the base exactly when this is a line in $\mathbb{P}^N$ touching M to 3rd order at p and with tangent direction v.

In paragraph 1 (*d*) we will be able to easily prove that

(1.21) *The* 2*nd fundamental form* II *is identically zero if, and only if,* M *is part of a* $\mathbb{P}^n$ [23].

In general, dim $|II|$ will be related to the lowest dimensional linear space containing M – *cf.* (1.62).

We remark that if

$$M \subset \mathbb{P}^{n+1}$$

is a hypersurface, then the 2nd fundamental form reduces to a single quadric Q. It is well known that Q has a normal form

$$Q = \sum_{\alpha=1}^{r} \omega_\alpha^2,$$

[22] It might have been more consistent to adopt (1.20) as a definition of the 2nd fundamental form.

[23] Of course, we could prove this here but the result will be a consequence of (1.51).

where r is the rank. In this case it is necessary to go to higher order to determine the position of M in $\mathbb{P}^{n+1}$ — *cf.* Appendix B.

If $Q \in |II|$ is any fixed quadric, say $Q = \sum_{\alpha, \beta} q_{\alpha\beta\, n+1}\, \omega_\alpha \omega_\beta$ corresponds to $A_{n+1} \in N_p(M)$, then it is easy to check that Q is the 2nd fundamental form at p of the projection of M into the $\mathbb{P}^{n+1}$ spanned by $A_0, \ldots, A_{n+1}$. The general principle is this:

(1.22) *Linear sub-systems of* $|II|$ *correspond to the 2nd fundamental forms at* p *of projections of* M *into linear sub-spaces of* $\mathbb{P}^N$.

(*c*) EXAMPLES. — It is interesting to explicitly compute the 2nd fundamental forms of some familiar homogeneous spaces.

Segre varieties. — Given vector spaces V and W the Segre variety is the image Σ of the natural inclusion

$$\mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V \otimes W). \tag{1.23}$$

If $\dim V = m+1$ and $\dim W = n+1$ then, by choosing coordinates, (1.23) is the image of

$$\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$$

given by the mapping

$$(\{X_\alpha\}, \{Y_\mu\}) \to \{X_\alpha Y_\mu\}. \tag{1.24}$$

The Segre variety has two rulings by the families of linear spaces

$$v \otimes \mathbb{P}(W), \quad \mathbb{P}(V) \otimes w, \quad v \in V, \quad w \in W.$$

Algebraically, Σ may be thought of as the decomposable tensors in $\mathbb{P}(V \otimes W)$. The tensors of rank r; i.e., those expressed as sums

$$v_1 \otimes w_1 + \ldots + v_r \otimes w_r,$$

may be thought of as the r-fold secant planes to Σ.

The simplest Segre variety is the embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, whose image is the quadric with the familiar double ruling by straight lines.

We denote by $\{A_0, A_1, \ldots, A_m\}$ and $\{B_0, B_1, \ldots, B_n\}$ frames for $\mathbb{P}(V)$ and $\mathbb{P}(W)$. The ranges of indices

$$1 \leqq i, j \leqq m, \qquad 1 \leqq \alpha, \beta \leqq n$$

will be used in this example. If A_0 lies over $v_0 \in \mathbb{P}(V)$ and B_0 over $w_0 \in \mathbb{P}(W)$, then the frame

$$\{A_0 \otimes B_0, A_0 \otimes B_\alpha, A_i \otimes B_0, A_i \otimes B_\alpha\} \tag{1.25}$$

lies over $v_0 \otimes w_0$. In other words, (1.25) gives an embedding

$$\mathscr{F}(\mathbb{P}(V)) \times \mathscr{F}(\mathbb{P}(W)) \to \mathscr{F}(\mathbb{P}(V \otimes W))$$

lying over the inclusion (1.23). If we write

$$dA_0 \equiv \sum_i \varphi_i A_i \bmod A_0,$$

$$dB_0 \equiv \sum_\alpha \psi_\alpha B_\alpha \bmod B_0,$$

then

$$(1.26) \qquad d(A_0 \otimes B_0) \equiv \sum_\alpha \psi_\alpha A_0 \otimes B_\alpha + \sum_i \varphi_i A_i \otimes B_0 \bmod A_0 \otimes B_0$$

and

$$(1.27) \qquad d^2(A_0 \otimes B_0) \equiv \sum_{i,\alpha} \varphi_i \psi_\alpha A_i \otimes B_\alpha \bmod \{A_0 \otimes B_0, A_i \otimes B_0, A_0 \otimes B_\alpha\}.$$

It follows from (1.26) that (1.25) gives a Darboux frame for Σ and

$$(1.28) \qquad \tilde{T}_{v_0 \otimes w_0}(\Sigma) \cong V \otimes w_0 + v_0 \otimes W,$$

so that

$$(1.29) \qquad N_{v_0 \otimes w_0}(\Sigma) \cong (V \otimes W)/(V \otimes w_0 + v_0 \otimes W).$$

From (1.27) we infer that the symmetric bilinear form associated to the 2nd fundamental form of Σ is given by

$$(1.30) \qquad II(v_0 \otimes w + v \otimes w_0, v_0 \otimes \tilde{w} + \tilde{v} \otimes w_0) \equiv v \otimes \tilde{w} + \tilde{v} \otimes w \bmod V \otimes w_0 + v_0 \otimes W.$$

In terms of homogeneous coordinates $\{\varphi_i\}$ for $T_{v_0}(\mathbb{P}(V))$ and $\{\psi_\alpha\}$ for $T_{w_0}(\mathbb{P}(W))$, $|II|$ is the linear system of quadrics

$$\sum_{i,\alpha} q_{i\alpha} \varphi_i \psi_\alpha$$

for any matrix $(q_{i\alpha})$. The two subspaces $\{\varphi_i = 0\}$ and $\{\psi_\alpha = 0\}$ are the tangent spaces to the two rulings, from which it is then clear that these two linear spaces constitute the base of the linear system $|II|$. Passing to coordinates we have:

(1.31) *The projectivized tangent space* $\mathbb{P}T_{v_0 \otimes w_0}(\Sigma) \cong \mathbb{P}^{m+n-1}$, *and* $|II|$ *is the complete linear system of quadrics having as base locus the union* $\mathbb{P}^{m-1} \cup \mathbb{P}^{n-1}$ *of two skew linear subspaces.*

For example, for $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ the base is the union of two skew lines in $\mathbb{P}^3$ [*cf.* (5.61) below].

Veronese varieties. — We begin with a useful alternate description of the 2nd fundamental form of M in $\mathbb{P}^N$. Given $p \in M$ we choose a homogeneous coordinate system $[X_0, X_1, \ldots, X_N]$ for $\mathbb{P}^N$ so that $p = [1, 0, \ldots, 0]$ and $\tilde{T}_p(M)$ is the $\mathbb{P}^n$ spanned by the first $n+1$ coordinate vectors. If $x_1 = X_1/X_0, \ldots, x_N = X_N/X_0$ is the corresponding affine coordinate system and $(z_1, \ldots, z_n)$ is any holomorphic coordinate system for M centered at p, then for $n+1 \leq \mu \leq N$:

$$x_\mu | M = x_\mu(z)$$

vanishes to 2nd order at $z=0$. Therefore

$$x_\mu(z)=\sum_{\alpha,\beta} q_{\alpha\beta\mu} z_\alpha z_\beta + (\text{higher order terms}).$$

The 2nd fundamental form is the linear system of quadrics generated by $\sum_{\alpha,\beta} q_{\alpha\beta\mu} dz_\alpha dz_\beta$.

For example, let V be a vector space and consider the Veronese variety Ξ, defined as the image of the natural inclusion

$$\mathbb{P}(\mathrm{V}) \to \mathbb{P}(\mathrm{Sym}^d \mathrm{V}). \tag{1.32}$$

In terms of homogeneous coordinates $[\mathrm{X}_0, \ldots, \mathrm{X}_n]$ for $\mathbb{P}(\mathrm{V})$ the mapping (1.32) is given by

$$[\ldots, \mathrm{X}_\alpha, \ldots] \to [\ldots, \mathrm{F}_\lambda(\mathrm{X}), \ldots]$$

where $\mathrm{F}_\lambda(\mathrm{X})$ varies over a basis for the homogeneous forms of degree d. Given a point $v_0 \in \mathbb{P}(\mathrm{V})$, e.g., $v_0=[1, 0, \ldots, 0]$, we identify the projectivized tangent space with $\mathbb{P}^{n-1}$ and then, by our previous remark,

(1.33) *The 2nd fundamental form is the linear system of all quadrics in* $\mathbb{P}^{n-1}$.

A simple but interesting case is the Veronese surface $\mathbb{P}^2 \subset \mathbb{P}^5$. The hyperplanes in $\mathbb{P}^5$ cut out on $\mathbb{P}^2$ the system of all conics, and the hyperplanes that contain the tangent plane $\tilde{\mathrm{T}}_p(\mathbb{P}^2)$ correspond to the conics that are singular at p. This is the system of all pairs of lines through p, which then cuts out the complete system of all quadrics in $\mathbb{P}\,\mathrm{T}_p(\mathbb{P}^2) \cong \mathbb{P}^1$. We note in passing that for any pair of points $p, q \in \mathbb{P}^2$ the intersection

$$\tilde{\mathrm{T}}_p(\mathbb{P}^2) \cap \tilde{\mathrm{T}}_q(\mathbb{P}^2) = 2\,\overline{pq} \tag{1.34}$$

is the conic corresponding to the line $\overline{pq}$ counted twice, and consequently any two tangent planes to the Veronese surface in $\mathbb{P}^5$ will meet in a point [*cf.*, (6.2) and (6.18)].

Grassmannians. — Given an N-dimensional vector space E we denote by $\mathrm{G}(n, \mathrm{E})$ the Grassmannian of n-planes $\mathrm{S} \subset \mathrm{E}$. It will be convenient to describe the Plücker embedding

$$\mathrm{G}(n, \mathrm{E}) \to \mathbb{P}(\Lambda^n \mathrm{E}) \tag{1.35}$$

by lifting it to an embedding

$$\mathscr{F}(\mathrm{G}(n, \mathrm{E})) \to \mathscr{F}(\mathbb{P}(\Lambda^n \mathrm{E})) \tag{1.36}$$

of frame manifolds. Recalling out notation in paragraph 1 (a) for the frame manifold $\mathscr{F}(\mathrm{G}(n, \mathrm{E}))$, for each index set $\mathrm{I}=\{a_1, \ldots, a_n\}$ with $1 \leqq a_1 < \ldots < a_n \leqq \mathrm{N}$ we set

$$e_\mathrm{I} = e_{a_1} \wedge \ldots \wedge e_{a_n},$$

and then the embedding (1.36) is given by

$$\{\ldots, e_a, \ldots\} \to \{\ldots, e_\mathrm{I}, \ldots\}.$$

We will use the ranges of indices

$$1 \leqq \alpha,\ \beta \leqq n; \qquad n+1 \leqq \mu,\ \nu \leqq N$$

and the notations

$$A_0 = e_1 \wedge \ldots \wedge e_n,$$
$$A_{\alpha\mu} = (-1)^{n-\alpha+1} e_1 \wedge \ldots \wedge \hat{e}_\alpha \wedge \ldots \wedge e_n \wedge e_\mu,$$
$$A_{\alpha\beta\mu\nu} = (-1)^{n-\alpha-\beta} e_1 \wedge \ldots \wedge \hat{e}_\alpha \wedge \ldots \wedge \hat{e}_\beta \wedge \ldots \wedge e_n \wedge e_\mu \wedge e_\nu,$$

where $\alpha < \beta$ and $\mu < \nu$. Recalling the structure equations (1.9) we have for the 1st two derivatives of (1.36):

(1.37) $$dA_0 \equiv \sum_{\alpha,\ \mu} \theta_{\alpha\mu} A_{\alpha\mu} \bmod A_0$$

and

(1.38) $$d^2 A_0 \equiv \sum_{\alpha,\ \beta,\ \mu,\ \nu} \theta_{\alpha\mu} \theta_{\beta\nu} A_{\alpha\beta\mu\nu} \bmod \{ A_0,\ A_{\alpha\mu} \}.$$

It follows from (1.37) that

$$\{ A_0;\ A_{\alpha\mu};\ A_{\alpha\beta\mu\nu};\ \ldots \} \in \mathscr{F}(\mathbb{P}(\Lambda^n E))$$

gives a Darboux frame for the Grassmannian; here we may compare (1.37) with the identification (1.10) of the tangent bundle of $G(n.\ E)$. To describe the 2nd fundamental form we will intrinsically interpret (1.38). For this recall that each n-plane $S \subset E$ defines a filtration

(1.39) $$F_S^n \subset F_S^{n-1} \subset \ldots \subset F_S^0 = \Lambda^n E,$$

where

$$F_S^k = \text{image of } \Lambda^k S \otimes \Lambda^{n-k} E \to \Lambda^n E.$$

In terms of indices F_S^k is spanned by exterior monomials having at least k α-indices. Setting $Q = E/S$ it follows that

(1.40) $$F_S^k / F_S^{k+1} \cong \Lambda^k S \otimes \Lambda^{n-k} Q.$$

Referring to (1.10) we have

(1.41) $$T_S(G(n,\ E)) \cong S^* \otimes Q \cong F_S^{n-1} / F_S^n.$$

Comparing (1.41) with (1.38) the 2nd fundamental form is described by the map

$$\text{II:} \quad \operatorname{Sym}^2(S^* \otimes Q) \to \Lambda^2 S^* \otimes \Lambda^2 Q$$

defined by the symmetric bilinear form

$$\text{II}(e^*\otimes\tilde{f}, \tilde{e}^*\otimes f)=(e^*\wedge\tilde{e}^*)\otimes(f\wedge\tilde{f}). \tag{1.42}$$

From this description it is easy to verify that:

(1.43) *The base locus* $B(\text{II})\subset\mathbb{P}(S^*\otimes Q)$ *is given by the Segre variety* $\mathbb{P}(S^*)\times\mathbb{P}(Q)\subset\mathbb{P}(S^*\otimes Q)$.

Alternatively, using the identification (1.10) the base is given by all non-zero $\psi\in\text{Hom}(S, Q)$ that have rank one. In terms of coordinates $S\cong\mathbb{C}^n$, $Q\cong\mathbb{C}^{N-n}$ the tangent space to $G(n, N)$ is given by all $n\times(N-n)$ matrices $\psi_{\alpha\mu}$, the linear system $|\text{II}|$ is generated by all 2×2 minors $\psi_{\alpha\mu}\psi_{\beta\nu}-\psi_{\alpha\nu}\psi_{\beta\mu}$, and the base is those transformations of rank one.

(*d*) THE HIGHER FUNDAMENTAL FORMS. — We will define the higher fundamental forms of a submanifold $M\subset\mathbb{P}^N$, give their basic properties, and compute a few examples.

As above we consider Darboux frames $\{A_0; A_\alpha; A_\mu\}$, and recall that $v_1, \ldots, v_n\in T_p(M)$ is the basis dual to the basis $\omega_1, \ldots, \omega_n$ of $T_p^*(M)$ defined by the equation [24]:

$$dA_0\equiv\sum_\alpha \omega_\alpha A_\alpha \bmod A_0.$$

Thus we may write

$$\frac{dA_0}{dv_a}\equiv A_\alpha \bmod A_0,$$

and for the 2nd fundamental form (1.20) we have

$$\frac{d^2 A_0}{dv_\alpha\, dv_\beta}\equiv\sum_\mu q_{\alpha\beta\mu}A_\mu \bmod \tilde{T}_p(M). \tag{1.44}$$

The symmetry of the 2nd fundamental form may be expressed by

$$\frac{dA_\alpha}{dv_\beta}\equiv\frac{dA_\beta}{dv_\alpha} \bmod \tilde{T}_p(M).$$

The 2nd osculating space $\tilde{T}_p^{(2)}(M)$ is defined to be the span of the collection of vectors

$$A_0, \qquad \frac{dA_0}{dv_\alpha}=A_\alpha, \qquad \frac{dA_\alpha}{dv_\beta}=\frac{dA_\beta}{dv_\alpha}.$$

Equivalently, it is the span of the 2nd osculating spaces at p to all curves lying in M. If the 2nd fundamental form viewed as a linear system $|\text{II}|$ of quadrics has projective dimension $r-1$, then $\tilde{T}_p^{(2)}(M)$ is a $\mathbb{P}^{n+r}$ in $\mathbb{P}^N$; i.e.

$$\dim|\text{II}|=r-1 \quad\Rightarrow\quad \dim\tilde{T}_p^{(2)}(M)=n+r. \tag{1.45}$$

[24] In a sense this defines the 1st fundamental form.

We restrict our attention to an open set on M where dim $|\mathrm{II}|$ is constant, and define the 1st normal space

$$N_p^{(1)}(M)=\tilde{T}_p^{(2)}(M)/\tilde{T}_p(M).$$

Clearly, $N_p^{(1)}(M)$ is the image of the quadratic mapping (1.18) which defines II.

The 3rd fundamental form III is most easily defined by generalizing the description (1.20) of the 2nd fundamental form. Namely, for any given tangent vector $v\in T_p(M)$ we choose a curve $p(t)$ in M with tangent v at $t=0$, and then

$$\frac{d^3 A_0}{dt^3}\in\mathbb{C}^{N+1}/\tilde{T}_p^{(2)}(M)$$

depends only on v. Accordingly we set

$$\mathrm{III}(v)=\frac{d^3 A_0}{dv^3} \bmod T_p^{(2)}(M). \tag{1.45}$$

This defines a mapping $T_p(M)\to\mathbb{C}^{N+1}/\tilde{T}_p^{(2)}(M)$ which is homogeneous of degree three, and is therefore given by a mapping ([25])

$$\mathrm{III}:\quad \mathrm{Sym}^3 T(M)\to\mathbb{C}^{N+1}/\tilde{T}^{(2)}(M). \tag{1.46}$$

This procedure may be repeated to define the higher fundamental forms IV, V, ...

We may also view the 3rd fundamental form as a linear system $|\mathrm{III}|$ of cubics on $\mathbb{P}T_p(M)\cong\mathbb{P}^{n-1}$, and a basic property is this:

(1.47) *The 3rd fundamental form is a linear system of cubics* $|\mathrm{III}|$ *whose Jacobian system is contained in* $|\mathrm{II}|$ ([26]).

Proof. — The idea is that the 3rd fundamental form may also be defined analytically as was the case for II, and when this is done (1.47) will fall out.

It will be convenient to use the ranges of indices

$$1\leqq\alpha,\ \beta\leqq n;\qquad n+1\leqq\mu,\ \nu\leqq n+r;\qquad n+r+1\leqq s,\ t\leqq N.$$

We consider frames

$$\{A_0;\ A_1,\ \ldots,\ A_n;\ A_{n+1},\ \ldots,\ A_{n+r};\ A_{n+r+1},\ \ldots,\ A_N\}=\{A_0;\ A_\alpha;\ A_\mu;\ A_s\}$$

which are adapted to the filtration

$$\mathbb{C}.A_0\subset\tilde{T}_p(M)\subset\tilde{T}_p^{(2)}(M)\subset\mathbb{P}^N$$

([25]) The point is that any function F(X) on $\mathbb{C}^n$ which satisfies $F(\lambda X)=\lambda^d F(X)$ is given by a homogeneous polynomial of degree d.

([26]) The Jacobian system $\mathscr{G}(|\mathrm{III}|)$ is discussed in paragraph (*b*) of Appendix A. This proposition remains true for all the higher fundamental forms.

on $\mathbb{P}^N$. Since by definition

$$dA_\alpha \equiv 0 \bmod \tilde{T}_p^{(2)}(M)$$

we have from (1.3) that

$$\omega_{\alpha s} = 0.$$

From the second equation in (1.3):

$$0 = d\omega_{\alpha s} = \sum_\mu \omega_{\alpha\mu} \wedge \omega_{\mu s}. \tag{1.48}$$

Now the forms $\omega_{\mu s}$ are obviously horizontal for the fibration $\tilde{T}^{(2)}(M) \to M$, and hence are linear combinations of $\omega_1, \ldots, \omega_n$. Thinking of these as homogeneous coordinates in $\mathbb{P}(T_p^*(M)) = \mathbb{P}^{n-1}$, (1.48) may be written

$$\sum_\mu \frac{\partial \omega_{\alpha\mu}}{\partial \omega_\beta} \omega_{\mu s} = \sum_\mu \omega_{\alpha\mu} \frac{\partial \omega_{\mu s}}{\partial \omega_\beta}, \qquad \beta = 1, \ldots, n. \tag{1.49}$$

The 3rd fundamental form is

$$d^3 A_0 \equiv d^2 \left(\sum_\alpha \omega_\alpha A_\alpha \right) \equiv d \left(\sum_{\alpha, \mu} \omega_\alpha \omega_{\alpha\mu} A_\mu \right) \equiv \sum_{\alpha, \mu, s} \omega_\alpha \omega_{\alpha\mu} \omega_{\mu s} A_s \bmod \tilde{T}^{(2)}(M).$$

Thus $|\mathrm{III}|$ is the linear system of cubics generated by

$$V_s = \sum_{\alpha, \mu} \omega_\alpha \omega_{\alpha\mu} \omega_{\mu s}, \qquad s = n+r+1, \ldots, N.$$

Then

$$\frac{\partial V_s}{\partial \omega_\beta} = \sum_\mu \omega_{\beta\mu} \omega_{\mu s} + \sum_{\alpha, \mu} \omega_\alpha \frac{\partial \omega_{\alpha\mu}}{\partial \omega_\beta} \omega_{\mu s} + \sum_{\alpha, \mu} \omega_\alpha \omega_{\alpha\mu} \frac{\partial \omega_{\mu s}}{\partial \omega_\beta} = \sum_{\alpha, \mu} q_{\beta\alpha\mu} \omega_\alpha \omega_{\mu s} + 2 \sum_{\alpha, \mu} \omega_\alpha \omega_{\alpha\mu} \frac{\partial \omega_{\mu s}}{\partial \omega_\beta}$$

by (1.49):

$$= \sum_{\alpha, \mu} \omega_\alpha \frac{\partial \omega_{\alpha\mu}}{\partial \omega_\beta} \omega_{\mu s} + 2 \sum_{\alpha, \mu} \omega_\alpha \omega_{\alpha\mu} \frac{\partial \omega_{\mu s}}{\partial \omega_\beta}$$

since $q_{\beta\alpha\mu} = q_{\alpha\beta\mu}$

$$= 3 \sum_{\alpha, \mu} \omega_\alpha \omega_{\alpha\mu} \frac{\partial \omega_{\mu s}}{\partial \omega_\beta}$$

by (1.49) again.

Q.E.D.

We will now discuss a few examples.

Curves. — We will only make one brief remark here. Suppose that $C \subset \mathbb{P}^N$ is a non-degenerate holomorphic curve given locally by a vector-valued holomorphic function $A(t) \in \mathbb{C}^{N+1} - \{0\}$. Then the Wronskian

$$\Delta(t) = A(t) \wedge A'(t) \wedge \ldots \wedge A^{(N)}(t)$$

is not identically zero. It follows that:

(1.50) *On the open set of regular points where* $\Delta(t) \neq 0$ *the kth fundamental form is just* $(dt)^k$ [27].

Surfaces. — We consider a surface $S \subset \mathbb{P}^N$. The dimension possibilities for the 2nd fundamental form are

$$\dim |II| = -1, 0, 1, 2.$$

To handle the first case we prove the general statement:

(1.51) *For* $M \subset \mathbb{P}^N$ *to be an open set in a* $\mathbb{P}^n$ *it is necessary and sufficient that the 2nd fundamental form be zero.*

Proof. — Clearly $II = 0$ when $M = \mathbb{P}^n$.

For the converse we assume that $II = 0$. Then from (1.47) it follows that $III = 0$. Repeating the argument inductively we conclude that $IV = V = \ldots = 0$. In other words, for any $p \in M$ and any k

$$d^k A_0 \equiv 0 \bmod \tilde{T}_p(M).$$

This means that any curve in M passing through p lies in $\tilde{T}_p(M)$; i.e.,

$$M \subset \tilde{T}_p(M).$$

But then M is an open set in $\tilde{T}_p(M)$

Q.E.D.

The other extreme where $\dim |II| = 2$ is the "general" case when $|II|$ is the complete linear system of quadrics on $\mathbb{P}^1$. In this situation it is necessary to go to higher order information to determine the position of S in $\mathbb{P}^N$; *cf.* Appendix B.

The case $\dim |II| = 0$ means that there is one independent quadric $Q \in |II|$, which will then have rank equal to 1 or 2. By properly choosing frames we may take this quadric to be respectively

$$Q = \omega_1^2,$$
$$Q = \omega_1^2 + \omega_2^2.$$

In the first instance, from (1.47) we see that $|III|$ can at most contain ω_1^3. Similarly, $|IV|$ can at most contain ω_1^4, etc. In fact, comparing with (1.50) we have something resembling

[27] The infinitesimal projective invariants of C in $\mathbb{P}^N$ are given in [6].

the sequence of fundamental forms of a curve, and it will be a consequence of (2.20) that:

(1.52) *If the sequence of fundamental forms at a generic point of a surface* $S \subset \mathbb{P}^N$ *is*

$$\omega_1^2, \quad \omega_1^3, \quad \ldots, \quad \omega_1^k, \quad 0, \quad \ldots, \quad 0,$$

then S *is either a cone over a curve in* $\mathbb{P}^k$, *or is the tangential ruled surface associated to a curve in* $\mathbb{P}^k$ [28].

In the remaining case when $Q = \omega_1^2 + \omega_2^2$ it follows from (1.47) that $III = 0$.

Proof. — If V is a cubic such that

$$\frac{\partial V}{\partial \omega_1} = \lambda_1 Q, \qquad \frac{\partial V}{\partial \omega_2} = \lambda_2 Q,$$

then

$$\frac{\partial^2 V}{\partial \omega_1 \partial \omega_2} = 2 \lambda_1 \omega_2, \qquad \frac{\partial^2 V}{\partial \omega_2 \partial \omega_1} = 2 \lambda_2 \omega_1$$

and so $\lambda_1 = \lambda_2 = V = 0$.

A similar proof as that of (1.51) gives in general:

(1.52) *If for* $M \subset \mathbb{P}^N$ *the 3rd fundamental form* $III = 0$, *then* M *lies in* $\tilde{T}_p^{(2)}(M)$.

In particular, a surface of the type we are considering must lie in a $\mathbb{P}^3$ [*cf.* (1.69) for a more general result].

Finally we consider the case $\dim |II| = 1$. Then the 2nd fundamental form consists of a pencil of quadrics on $\mathbb{P}^1$, and there are two possibilities

$$(1.53) \qquad \begin{cases} |II| \text{ has a base point,} \\ |II| \text{ does not have a base point.} \end{cases}$$

In the first case we may suppose the base point is given by $\omega_1 = 0$. Then we may take the linear system $|II|$ to be generated by

$$\omega_1^2, \quad \omega_1 \omega_2.$$

If $V \in |III|$ then from $\partial V / \partial \omega_1$ and $\partial V / \partial \omega_2 \in |II|$ we conclude that V does not contain the monomials $\omega_1 \omega_2^2$ and ω_2^3. Thus the 3rd fundamental form is contained in the linear system of cubics generated by

$$\omega_1^3, \qquad \omega_1^2 \omega_2.$$

Continuing in this way we arrive at the conclusion:

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[28] Thus, in either case S is a "curve in disguise".

If $S \subset \mathbb{P}^N$ *is any surface whose 2nd fundamental form at a generic point is a pencil having a simple base point, then for the kth osculating space*

$$\dim \tilde{T}_p^{(k)}(S) \leqq 2k,$$

with equality holding exactly when

$$(1.54) \qquad II = \{ \omega_1^2, \omega_1 \omega_2 \}, \qquad III = \{ \omega_1^3, \omega_1^2 \omega_2 \}, \qquad IV = \{ \omega_1^4, \omega_1^3 \omega_2 \}, \qquad \ldots$$

To give a specific example when (1.54) occurs, recall that a ruled surface is constructed from a pair of curves C and C′ in $\mathbb{P}^N$ having a common parameter, and taking as our surface the locus of ∞^1 lines joining points on C and C′ with the same parameter value. It is not difficult to check that a general ruled surface satisfies (1.54) [*cf.* (2.3) for a proof]; the converse seems likely but we are unable to prove it.

The last possibility is when $|II|$ is a base point free pencil. The 2 : 1 mapping

$$i_{|II|}: \quad \mathbb{P}^1 \to \mathbb{P}^1,$$

then has two branch points, which we take to be our basis for $\mathbb{P}^1 \cong \mathbb{P} T_p(S)$. With this choice it follows that $|II|$ is spanned by the quadrics

$$\omega_1^2, \quad \omega_2^2.$$

If $V \in |III|$ then from the Jacobian condition (1.47) we conclude that V does not contain $\omega_1^2 \omega_2$ or $\omega_1 \omega_2^2$; consequently the 3rd fundamental form is contained in the linear system of cubics generated by ω_1^3 and ω_2^3. Continuing in this way leads to the conclusion:

If $S \subset \mathbb{P}^N$ *is any surface whose 2nd fundamental form at a generic point is a base point free pencil, then*

$$\dim \tilde{T}_p^{(k)}(S) \leqq 2k,$$

with equality holding exactly when

$$(1.55) \qquad II = \{ \omega_1^2, \omega_2^2 \}, \qquad III = \{ \omega_1^3, \omega_2^3 \}, \qquad IV = \{ \omega_1^4, \omega_2^4 \}, \qquad \ldots$$

Examples of such surfaces are minimal surfaces in a sphere, or surfaces of translation type. It is interesting that from our point of view these two classes of surfaces are members of the same continuous family.

Grassmannians. — We retain the notation from paragraph 1 (*c*) above. Given $S \in G(n, E)$ we consider the filtration (1.39). By an obvious extension of the computations (1.37) and (1.38) we deduce that the *k*th osculating space is given by

$$(1.56) \qquad \tilde{T}_S^{(k)}(G(n, N)) = F_S^{n-k}.$$

Moreover, the sequence of fundamental forms is given by the standard maps, of which we write only the first two

(1.57)
$$\begin{cases} \text{II:} & \operatorname{Sym}^{(2)}(S^* \otimes Q) \to \Lambda^2 S^* \otimes \Lambda^2 Q, \\ \text{III:} & \operatorname{Sym}^{(3)}(S^* \otimes Q) \to \Lambda^3 S^* \otimes \Lambda^3 Q. \end{cases}$$

In terms of coordinates if we make the identification (1.10) and choose isomorphisms $S \cong \mathbb{C}^n$, $Q \cong \mathbb{C}^{N-n}$ so that

$$T_S(G(n, N)) \cong n \times (N-n) \text{ matrices},$$

then

(1.58)
$$\begin{cases} \text{II} = \text{quadrics generated by } 2 \times 2 \text{ minors}, \\ \text{III} = \text{cubics generated by } 3 \times 3 \text{ minors}, \end{cases}$$

etc. It follows that

$$\begin{cases} \text{base of } |\text{II}| = \text{transformations of rank one}, \\ \text{base of } |\text{III}| = \text{transformations of rank two}, \end{cases}$$

etc.

This illustrates a special case of the following general phenomenon, which is a consequence of (1.47) and (A.15) in Appendix A:

(1.59) *For a manifold* M *in* $\mathbb{P}^N$ *with fundamental forms* II, III, *etc., we have for the base loci*

$$B(\text{II}) \subset B(\text{III}),$$
$$B(\text{III}) \subset B(\text{IV}).$$

(*e*) The Gauss mappings. — Another interpretation of the 2nd fundamental form is via the Gauss mapping

(1.60)
$$\gamma: \quad M \to \mathbb{G}(n, M)$$

defined by

$$\gamma(p) = \tilde{T}_p(M).$$

Using Darboux frames $\{A_0; A_\alpha; A_\mu\}$ associated to M and the Plücker embedding (1.35) of the Grassmannian, we may describe γ by

$$\gamma(p) = A_0 \wedge A_1 \wedge \ldots \wedge A_n.$$

Then, since $dA_0 \equiv 0 \mod \tilde{T}_p(M)$:

(1.61)
$$d\gamma \equiv \sum_{\alpha, \mu} (-1)^{n-\alpha+1} \omega_{\alpha\mu} A_0 \wedge A_1 \wedge \ldots \wedge \hat{A}_\alpha \wedge \ldots \wedge A_n \wedge A_\mu \mod A_0 \wedge A_1 \wedge \ldots \wedge A_n.$$

Comparing with (1.9) we conclude that:

(1.62) *The 2nd fundamental form gives the differential of the Gauss mapping.*

More precisely, according to (1.10) the differential of γ is

$$\gamma_*: \ T_p(M) \to \operatorname{Hom}(\tilde{T}_p(M), N_p(M)).$$

But since $dA_0 \in \tilde{T}_p(M)$ and

$$\tilde{T}_p(M)/\mathbb{C}.A_0 \cong T_p(M),$$

this map factors to induce

$$\gamma_*: \ T_p(M) \to \operatorname{Hom}(T_p(M), N_p(M)). \tag{1.63}$$

By a standard linear algebra identification this is the same as giving a mapping

$$T_p(M) \otimes T_p(M) \to N_p(M),$$

which is then just the 2nd fundamental form (1.18).

Now we would like to compare our projective 2nd fundamental form to the usual one for a complex manifold $M \subset \mathbb{C}^N$. Here one considers the manifold $\mathscr{F}(M) \subset \mathscr{F}(\mathbb{C}^N)$ of Darboux frames $\{z; e_\alpha; e_\mu\}$ defined by the conditions

$$\begin{cases} z \in M, \text{ and } e_1, \ldots, e_n \text{ span the translate} \\ T_z(M) \text{ of the tangent space to the origin.} \end{cases}$$

Just as before, on $\mathscr{F}(M)$ the relations

$$\begin{cases} dz = \sum_\alpha \theta_\alpha e_\alpha, \\ \theta_\mu = 0 \end{cases} \tag{1.64}$$

are valid. Taking the exterior derivative of the second equation and using (1.2) gives

$$\sum_\alpha \theta_\alpha \wedge \theta_{\alpha\mu} = 0.$$

By the Cartan Lemma (A.2):

$$\theta_{\alpha\mu} = \sum_\beta \tilde{q}_{\alpha\beta\mu} \theta_\beta, \qquad \tilde{q}_{\alpha\beta\mu} = \tilde{q}_{\beta\alpha\mu}.$$

Setting

$$\tilde{Q}_\mu = \sum_{\alpha, \beta} \tilde{q}_{\alpha\beta\mu} \theta_\alpha \theta_\beta$$

the quadrics $\tilde{Q}_\mu$ generate a linear system that we may call the Euclidean 2nd fundamental form $\tilde{II}$ of M in $\mathbb{C}^N$.

We want to show that there is a natural identification

$$\text{(1.65)} \qquad \mathrm{II}=\widetilde{\mathrm{II}}.$$

For this we consider the Euclidean Gauss map

$$\tilde{\gamma}:\quad \mathrm{M}\to \mathrm{G}(n,\,\mathrm{N})$$

defined by

$$\tilde{\gamma}(z)=\mathrm{T}_z(\mathrm{M}).$$

From

$$d(e_1\wedge\ldots\wedge e_n)\equiv\sum_{\alpha,\,\mu}(-1)^{n-\alpha+1}\,\theta_{\alpha\mu}\,e_1\wedge\ldots\wedge\hat{e}_\alpha\wedge\ldots\wedge e_n\wedge e_\mu \bmod e_1\wedge\ldots\wedge e_n$$

it follows that the differential of $\tilde{\gamma}$ is given by the matrix of 1-forms $\theta_{\alpha\mu}$.

Suppose now that we have $\mathrm{M}\subset\mathbb{C}^{\mathrm{N}}\subset\mathbb{P}^{\mathrm{N}}$. To each Euclidean Darboux frame $\{z;\,e_\alpha;\,e_\mu\}$ we shall attach a projective Darboux frame $\{\mathrm{A}_0;\,\mathrm{A}_\alpha;\,\mathrm{A}_\mu\}$ and shall then compare Gauss mappings. For $v=(v_1,\,\ldots,\,v_{\mathrm{N}})\in\mathbb{C}^{\mathrm{N}}$ we set $\tilde{v}=(0,\,v_1,\,\ldots,\,v_{\mathrm{N}})\in\mathbb{C}^{\mathrm{N}+1}$ and let $e_0=(1,\,0,\,\ldots,\,0)\in\mathbb{C}^{\mathrm{N}+1}$. Then $e_0+\tilde{v}$ gives the homogeneous coordinates of $v\in\mathbb{C}^{\mathrm{N}}\subset\mathbb{P}^{\mathrm{N}}$. Now define

$$\text{(1.66)} \qquad \begin{cases} \mathrm{A}_0=e_0+\tilde{z},\\ \mathrm{A}_\alpha=e_0+\tilde{z}+\tilde{e}_\alpha,\\ \mathrm{A}_\mu=e_0+\tilde{z}+\tilde{e}_\mu.\end{cases}$$

On the one hand

$$d\mathrm{A}_0=\sum_\alpha\theta_\alpha\,\tilde{e}_\alpha\equiv\sum_\alpha\theta_\alpha\,\mathrm{A}_\alpha \bmod \mathrm{A}_0,$$

and

$$d\mathrm{A}_\alpha=\sum_\beta\theta_\beta\,\tilde{e}_\beta+\sum_\mu\theta_{\alpha\mu}\,\tilde{e}_\mu\equiv\sum_\mu\theta_{\alpha\mu}\,\mathrm{A}_\mu \bmod \mathrm{A}_0,\,\mathrm{A}_1,\,\ldots,\,\mathrm{A}_n.$$

It follows that under the frame mapping (1.66):

$$\omega_\alpha=\theta_\alpha,\qquad \omega_{\alpha\mu}=\theta_{\alpha\mu},$$

which establishes (1.65).

Using the higher osculating spaces $\{\tilde{\mathrm{T}}_p^{(k)}(\mathrm{M})\}$ associated to $\mathrm{M}\subset\mathbb{P}^{\mathrm{N}}$ it is possible to define the higher Gauss mappings. For example, retaining the notations from paragraph 1 (*d*) the 2nd Gauss mapping

$$\text{(1.67)} \qquad \gamma^{(2)}:\quad \mathrm{M}\to\mathbb{G}(n+r,\,\mathrm{N})$$

is defined by

$$\gamma^{(2)}(p)=\tilde{T}_p^{(2)}(M).$$

We shall prove that:

(1.68) *The 2nd Gauss mapping $\gamma^{(2)}$ is constant along the fibres of the 1st Gauss mapping γ (and so forth for the higher Gauss mappings).*

Proof. — We retain the notations from the proof of (1.47). The fibres of the 1st Gauss mapping (1.60) are, according to (1.62), defined by

$$\omega_{\alpha\mu}=0$$

(these forms may not be linearly independent, but we select a linearly independent subset). Similarly, the differential of $\gamma^{(2)}$ is given by the 3rd fundamental form, and the fibres of $\gamma^{(2)}$ are defined by

$$\omega_{\mu s}=0.$$

By (1.48), for any α and s:

$$\sum_{\mu}\omega_{\alpha\mu}\wedge\omega_{\mu s}=0.$$

This is exactly the situation of the second variant (A.4) of the Cartan Lemma, and by (A.5) we have

$$\omega_{\mu s}\equiv 0 \bmod \omega_{\alpha\mu}.$$

This implies (1.68).

Q.E.D.

Another way in which $\gamma^{(2)}$ turns up naturally is in the following:

(1.69) *Suppose that we have $M\subset\mathbb{P}^N$ with $N\geqq 2n$ and $\dim|II|\leqq n-2$. Then the 2nd Gauss mapping $\gamma^{(2)}$ is degenerate* ([29]).

Proof. — Suppose that $\dim|II|=r-1$ where $r\leqq n-1$, and choose Darboux frames

$$\{A_0;\ A_1,\ \ldots,\ A_n;\ A_{n+1},\ \ldots,\ A_{n+r};\ A_{n+r+1},\ \ldots,\ A_N\},$$

where $A_0, A_1, \ldots, A_{n+r}$ spans $\tilde{T}_p^{(2)}(M)$. We consider the rational quadratic map

$$Q:\ \mathbb{P}^{n-1}\dashrightarrow\mathbb{P}^{r-1}$$

([29]) This result will be used in paragraph 5 (*a*). The surface example given below (1.53) shows that the statement is false if we only assume $\dim|II|\leqq n-1$.

defined by $|\mathrm{II}|$ (this map is denoted by i_{II} in Appendix A). If we assume that the generic fibre of Q has dimension k, then since $r \leqq n-1$:

$$1 \leqq n-r \leqq k.$$

For generic $v \in \mathbb{P}^{n-1}$:

$$(1.70) \qquad \begin{array}{ccc} \dfrac{\partial \mathrm{Q}}{\partial w_1}(v) \wedge & \ldots & \wedge \dfrac{\partial \mathrm{Q}}{\partial w_{n-k}}(v) \\ \| & & \| \\ \mathrm{Q}(v, w_1) \wedge & \ldots & \wedge \mathrm{Q}(v, w_{n-k}) \not\equiv 0 \end{array}$$

as a function of $w_1, \ldots, w_{n-k}$, while for any v and $w_1, \ldots, w_{n-k+1}$:

$$\mathrm{Q}(v, w_1) \wedge \ldots \wedge \mathrm{Q}(v, w_{n-k+1}) = 0.$$

Suppose now we have chosen our basis $\{v_\alpha\}$ for $\mathrm{T}_p(\mathrm{M})$ so that (1.70) holds for $v = v_\alpha$. Then we claim that:

$$(1.71) \qquad \omega_{\alpha, n+1}, \ldots, \omega_{\alpha, n+r} \text{ span a } \mathbb{C}_\alpha^{n-k} \text{ in } \mathbb{C}^n \cong \mathrm{T}_p(\mathrm{M})^*.$$

Indeed, for any $w \in \mathrm{T}_p(\mathrm{M})$:

$$\mathrm{Q}(v_\alpha, w) = \sum \langle \omega_{\alpha\mu}, w \rangle \mathrm{A}_\mu \equiv \frac{d^2 \mathrm{A}_0}{dv_\alpha\, dw} \bmod \tilde{\mathrm{T}}_p(\mathrm{M})$$

by (1.20). Consequently, for $v = v_\alpha$ (1.70) is equivalent to saying that the vector-valued 1-form

$$d\left(\frac{d\mathrm{A}_0}{dv_\alpha}\right) \bmod \tilde{\mathrm{T}}_p(\mathrm{M})$$

contains $n-k$ independent 1-forms from among $\omega_{\alpha, n+1}, \ldots, \omega_{\alpha, n+r}$, so that (1.71) is an alternate formulation of Q having generic fibre dimension k. Assuming that $\omega_{\alpha, n+1}, \ldots, \omega_{\alpha, 2n-k}$ are linearly independent, we use the additional ranges of indices

$$n+1 \leqq \mu \leqq n+r; \qquad n+r+1 \leqq s \leqq \mathrm{N},$$
$$1 \leqq \rho \leqq n-k; \qquad n-k+1 \leqq \lambda \leqq r.$$

From (1.48):

$$0 = \sum_\mu \omega_{\alpha\mu} \wedge \omega_{\mu s},$$

so that from (A.3) in Appendix A we have

$$(1.72) \qquad \omega_{n+\rho, s} \equiv 0 \bmod \omega_{n+\lambda, s}, \omega_{\alpha, n+\rho}.$$

Since the number of forms on the right is $(r-n+k)+(n-k)=r$ it follows that $\gamma^{(2)}$ has fibres of dimension $\geqq n-r \geqq 1$.

Q.E.D.

We remark that the proof gives:

$$\text{rank } \gamma_*^{(2)} \leqq \dim |\mathrm{II}| + 1. \tag{1.73}$$

In concluding we should like to make a general observation concerning a holomorphic mapping

$$f: \quad \mathrm{M} \to \mathrm{G}(n, \mathrm{N}),$$

where say $\dim \mathrm{M} = m$. Choosing bases and making the identification (1.10) the differential is given by a linear map

$$\mathbb{C}^m \to \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{\mathrm{N}-n}). \tag{1.74}$$

Except when m, n, or $\mathrm{N}-n$ is small such a mapping does not have a normal form [30], so that the infinitesimal study of maps to Grassmannians involves generally difficult questions in tri-linear algebra.

If we assume that f is an embedding and use the Plücker embedding of the Grassmannian, then we have

$$\mathrm{M} \subset \mathrm{G}(n, \mathrm{N}) \subset \mathbb{P}(\Lambda^n \mathbb{C}^{\mathrm{N}}). \tag{1.75}$$

According to the general Meusnier-Euler Theorem the basic invariant of (1.75) is the intersection of $\mathrm{T}_p(\mathrm{M})$ with base of the 2nd, 3rd, 4th, ... fundamental forms of $\mathrm{G}(n, \mathrm{N})$ in $\mathbb{P}(\Lambda^n \mathbb{C}^{\mathrm{N}})$. By (1.58) and (1.59) this suggests the study of the determinental varieties

$$\Sigma_1 \subset \Sigma_2 \subset \ldots \subset \Sigma_n = \mathbb{P}^{m-1},$$

where Σ_k are the tangent vectors $v \in \mathrm{T}_p(\mathrm{M})$ such that $f_*(v) \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{\mathrm{N}-n})$ has rank $\leqq k$.

When $m = n$ and f is the Gauss mapping the differential (1.74) is equivalent to a map

$$\mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^{\mathrm{N}-n}.$$

As we have seen the basic fact peculiar to Gauss maps is the *symmetry* of its differential. In particular, the tri-linear algebra reduces to linear systems of quadrics, about which algebraic geometry has much more to say. It is this philosophy that we shall exploit.

2. Varieties with degenerate Gauss mappings

(*a*) A NORMAL FORM FOR CURVES IN GRASSMANNIANS. — Much of our discussion will use elementary properties of a holomorphic mapping

$$f: \quad \mathrm{B} \to \mathrm{G}(m, \mathrm{N})$$

[30] About the best general result seems to be Kronecker's Pencil Lemma, which gives a normal form when $m = 2$.

of a complex manifold B into a Grassmannian. We denote points of B by y and set

$$S_y = f(y), \qquad Q_y = \mathbb{C}^N / S_y.$$

As noted in (1.10) the differential of f is

$$f_*: \quad T_y(B) \to \mathrm{Hom}(S_y, Q_y).$$

Explicitly, if $y_1, \ldots, y_l$ are local coordinates on B and $e_1(y), \ldots, e_m(y)$ a holomorphically varying basis for S_y, then by definition

$$(2.1) \qquad f_*\left(\frac{\partial}{\partial y_\lambda}\right) e_\alpha(y) \equiv \frac{\partial e_\alpha}{\partial y_\lambda}(y) \bmod S_y.$$

For each tangent vector $w \in T_y(B)$ the linear subspaces of $\mathbb{C}^N$:

$$\begin{cases} S_y + \dfrac{dS_y}{dw}, \\ S_y \cap \dfrac{dS_y}{dw} \end{cases}$$

are well-defined, where the latter is taken to be $\ker f_*(w)$.

The statements

$$\begin{cases} \dim\left(S_y + \dfrac{dS_y}{dw}\right) = m + r, \\ \operatorname{rank} f_*(w) = r, \\ \dim\left(S_y \cap \dfrac{dS_y}{dw}\right) = m - r \end{cases}$$

are all equivalent. We shall sometimes use the notation $\mathbb{C}^m_y$ for S_y.

Similar considerations apply to holomorphic mappings

$$f: \quad B \to \mathbb{G}(m, N)$$

into the Grassmannian of $\mathbb{P}^m$'s in $\mathbb{P}^N$. Setting $\mathbb{P}^m_y = f(y)$ and taking $w \in \mathbb{P}\,T_y(B)$, the linear subspaces of $\mathbb{P}^N$

$$\begin{cases} \mathbb{P}^m_y + \dfrac{d\mathbb{P}^m_y}{dw}, \\ \mathbb{P}^m_y \cap \dfrac{d\mathbb{P}^m_y}{dw} \end{cases}$$

are defined. Geometrically, they represent the span of $\mathbb{P}^m_y$ with the "infinitely near" space $d\mathbb{P}^m_y/dw$, and the intersection of $\mathbb{P}^m_y$ with this space.

In case $\dim B = 1$ so that $\{\mathbb{C}_y^m\}$ is a curve in $G(n, N)$ with parameter y, the following will be of considerable use:

(2.2) LEMMA. — *Suppose that for generic y:*

$$\dim\left(\mathbb{C}_y^m \cap \frac{d\mathbb{C}_y^m}{dy}\right) = m - r.$$

Then we can find $\mathbb{C}^N$-valued functions $e_1(y), \ldots, e_r(y)$, positive integers $a_1, \ldots, a_r$, and a fixed $\mathbb{C}^{m-\Sigma a_i}$ such that

$$\mathbb{C}_y^m = \{e_1(y), \ldots, e_1^{(a_1-1)}(y); \ldots; e_r(y), \ldots, e_r^{(a_r-1)}(y); \mathbb{C}^{m-\Sigma a_i}\}$$

where

$$e^{(k)}(y) = \frac{d^k e(y)}{dy^k}.$$

Proof. — We return to the notation S_y for $\mathbb{C}_y^m$, and set

$$S_y^{(1)} = \ker \frac{d}{dy} : S_y \to \mathbb{C}^N / S_y;$$

$$S_y^{(2)} = \ker \frac{d}{dy} : S_y^{(1)} \to \mathbb{C}^N / S_y^{(1)} = \ker \frac{d^2}{dy^2} : S_y^{(1)} \to \mathbb{C}^N / S_y;$$

$$S_y^{(3)} = \ker \frac{d}{dy} : S_y^{(2)} \to \mathbb{C}^N / S_y^{(2)} = \ker \frac{d^3}{dy^3} : S_y^{(2)} \to \mathbb{C}^N / S_y;$$

etc. Then $S_y \supset S_y^{(1)} \supset S_y^{(2)} \supset S_y^{(3)} \supset \ldots$, and the subspace $\cap S_y^{(k)}$ is fixed under d/dy and is therefore a constant $\mathbb{C}^l$.

Working modulo this space we consider the largest k such that

$$S_y^{(k-1)} \neq 0, \qquad S_y^{(k)} = 0.$$

If $e(y) \in S_y^{(k-1)}$, then

$$e(y), \quad \frac{de}{dy}, \quad \ldots, \quad \frac{d^{k-1}e}{dy^{k-1}} \in S_y,$$

$$\frac{d^k e}{dy^k} \notin S_y.$$

Choosing a basis $e_1(y), \ldots, e_s(y)$ for $S_y^{(k-1)}$ we have

$$S_y \supset \{e_1(y), \ldots, e_1^{(k-1)}(y); \ldots; e_s(y), \ldots, e_s^{(k-1)}(y); \mathbb{C}^l\}.$$

If we work modulo the subspace on the right and repeat the argument inductively then we arrive at the normal form of the Lemma.

Q.E.D.

This Lemma will be most frequently applied in projective form. Then we imagine the curve $\{\mathbb{P}^m_y\}$ in $\mathbb{G}(m, N)$ as tracing out a ruled $(m+1)$-fold V in $\mathbb{P}^N$, and where $\mathbb{P}^m_y \cap d\mathbb{P}^m_y/dy$ is the intersection of a generator with an infinitely near one.

For example, let us examine the Lemma in the simplest case $m=1$ of a ruled surface S in $\mathbb{P}^N$. Then $\mathbb{P}^1_y \subset \mathbb{P}^N$ corresponds to $\mathbb{C}^2_y \subset \mathbb{C}^{N+1}$, and we consider the possible cases in Lemma 2.2:

$r=2$. Then $\mathbb{C}^2_y = \{e_1(y), e_2(y)\}$ and a generic $\mathbb{P}^1_y$ does not meet the infinitely near one. Each of the vectors $e_1(y)$ and $e_2(y)$ describes a curve in $\mathbb{P}^N$. These curves have a common parameter y and S is the surface obtained by linearly joining corresponding points. A general point on S is

$$A_0 = e_1(y) + te_2(y).$$

Then

$$dA_0 \equiv (e_1' + te_2')\,dy + e_2\,dt \mod A_0,$$
$$d^2 A_0 \equiv (e_1'' + te_2'')\,dy^2 + 2e_2'\,dy\,dt \mod \tilde{T}_p(S).$$

It follows that:

(2.3) *For a general ruled surface the 2nd fundamental form is a pencil of quadrics on $\mathbb{P}^1$ having a simple base point.*

Here, "general" means that the vectors

$$e_1'' + te_2'', \qquad e_2'$$

should be linearly independent at a generic point of S. The referee points out that, for the (algebraic) ruled surface obtained by joining corresponding points on a pair of lines we have $e_1'' + te_2'' \equiv 0$, and the 2nd fundamental form reduces to a single conic.

$r=1$. In this case each generator $\mathbb{P}^1_y$ meets the infinitely near line in a point $p(y)$, and there are two subcases according to whether or not the curve traced out by $p(y)$ is constant, as follows:

$a_1=1$. Then $\mathbb{C}^2_y = \{e_1(y), \mathbb{C}\}$ where $\mathbb{C} \subset \mathbb{C}^{N+1}$ projects onto a fixed point $p_0 \in \mathbb{P}^N$. In this case the ruled surface is a cone with vertex p_0 and base curve described by $e_1(y)$.

$a_1=2$. Then we have $e_1(y)=e(y)$, $e_2(y)=e'(y)$, and

$$\mathbb{C}^2_y = \{e(y), e'(y)\}.$$

Geometrically, $e(y)$ describes a curve C in $\mathbb{P}^N$, and $\mathbb{P}^1_y$ is the tangent line. The locus of these tangent lines is a developable ruled surface with C being in classical terminology the "edge of regression". It is a curve of singularities on the ruled surface S.

In general suppose that $e(y) \in \mathbb{C}^{N+1} - \{0\}$ describes a curve C in $\mathbb{P}^N$ with $e(y) \wedge e'(y) \wedge \ldots \wedge e^{(a-1)}(y) \neq 0$. Denoting the osculating $(a-1)$-plane by

$$\mathbb{P}^{(a-1)}_y(C) = \{e(y), e'(y), \ldots, e^{(a-1)}(y)\},$$

the locus of the $\mathbb{P}_y^{(a-1)}(C)$ traces out the osculating a-fold associated to the curve. Clearly

$$\dim\left(\mathbb{P}_y^{(a-1)}(C)\cap\frac{d\mathbb{P}_y^{(a-1)}}{dy}(C)\right)=a-2.$$

Lemma 2.2 gives us the following picture of a general curve in $\mathbb{G}(m, N)$:

(2.4) *Given in* $\mathbb{G}(m, N)$ *a curve* $\mathbb{P}_y^m$ *with*

$$\dim\left(\mathbb{P}_y^m\cap\frac{d\mathbb{P}_y^m}{dy}\right)=m-r$$

at a generic point, then there are curves $C_1, \ldots, C_r$ *in* $\mathbb{P}^N$ *each having a common parameter* y, *positive integers* $a_1, \ldots, a_r$, *and a fixed* $\mathbb{P}^l$ *where* $l=m-\sum a_i$, *such that* $\mathbb{P}_y^m$ *is the span of* $\mathbb{P}^l$ *together with the osculating* (a_i-1)*-planes to* C_i *at the point* y; *i.e.*

$$\mathbb{P}_y^m=\mathbb{P}_y^{(a_1-1)}(C_1)+\ldots+\mathbb{P}_y^{(a_r-1)}(C_r)+\mathbb{P}^l.$$

(*b*) MANIFOLDS HAVING DEGENERATE GAUSS MAPPINGS. — We will describe those submanifolds $M\subset\mathbb{P}^N$ whose Gauss map

(2.5) $$\gamma:\quad M\to\mathbb{G}(n, N)$$

is degenerate, i.e., has positive dimensional fibres. The final result is (2.27). We begin by proving:

(2.6) *The Gauss mapping* (2.5) *is degenerate with m-dimensional fibres if, and only if, at a generic point of* M *all quadrics* $Q\in|II|$ *are singular along a* $\mathbb{P}^{m-1}\subset\mathbb{P}T_p(M)$.

Proof. — We consider a field of Darboux frames $\{A_0; A_\alpha; A_\mu\}$ and recall the basic structure equations (1.17):

(2.7) $$\begin{cases} dA_0\equiv\sum_\alpha \omega_\alpha A_\alpha \mod A_0,\\ dA_\alpha\equiv\sum_\mu \omega_{\alpha\mu} A_\mu \mod A_0, \ldots, A_n,\\ \omega_{\alpha\mu}=\sum_\beta q_{\alpha\beta\mu}\omega_\beta, \qquad q_{\alpha\beta\mu}=q_{\beta\alpha\mu}.\end{cases}$$

We also recall from (1.61) that $\omega_{\alpha\mu}$ gives the differential of γ, so that if the generic rank of γ_* is n-m then among the forms $\omega_{\alpha\mu}$ there are exactly n-m which are linearly independent. In fact, let us choose our frame field so that $v_1, \ldots, v_m$ spans the subspace $\omega_{\alpha\mu}=0$. Then

(2.8) $$\omega_{\alpha\mu}\equiv 0 \mod \omega_{m+1}, \ldots, \omega_n.$$

Using the additional index range $1\leqq\rho, \sigma\leqq m$ we obtain from (2.8) that

(2.9) $$q_{\rho\alpha\mu}=q_{\alpha\rho\mu}=0$$

for all α, ρ, μ. Using first $q_{\rho\sigma\mu}=0$ we infer that the quadrics

$$Q_{\mu}=\sum_{\alpha,\beta} q_{\alpha\beta\mu}\,\omega_{\alpha}\,\omega_{\beta}$$

all vanish on the $\mathbb{P}^{m-1}$ spanned by $v_1, \ldots, v_m$, and then the conditions $q_{s\rho\mu}=0$ when $m+1\leqq s\leqq n$ say that all Q_{μ} are singular along this $\mathbb{P}^{m-1}$ [31].

Conversely, suppose that all $Q\in|II|$ are singular along the $\mathbb{P}^{m-1}$ spanned by $v_1, \ldots, v_m$, so that (2.8) holds. From

$$\omega_{m+1}, \ldots, \omega_n\equiv 0 \bmod \omega_{\alpha\mu},$$
$$d\omega_{\alpha\mu}=\sum_{\beta}\omega_{\alpha\beta}\wedge\omega_{\beta\mu}+\sum_{\nu}\omega_{\alpha\nu}\wedge\omega_{\nu\mu}\equiv 0 \bmod \omega_{\beta\nu}$$

we infer that the Frobenius integrability condition for $\omega_{m+1}, \ldots, \omega_n$ is satisfied. From the middle equation in (2.7) we see that the tangent space $\tilde{T}_p(M)$ remains constant along the leaves of the folitation defined by

$$\omega_{m+1}=\ldots=\omega_n=0.$$

This completes the proof of (2.6).

Q.E.D.

It is now easy to show:

(2.10) *The fibres of the Gauss mapping are linear spaces* [32].

Proof. — We retain the notations from the proof of (2.6). From (2.9) we have:

$$(2.11)\qquad \omega_{\rho\mu}=0.$$

Using the additional index range $m+1\leqq s,\ t\leqq n$ and taking into account the structure equation (1.3), the exterior derivative of (2.11) gives

$$(2.12)\qquad \sum_{s}\omega_{\rho s}\wedge\omega_{s\mu}=0$$

for all ρ, μ. We want to apply the refined Cartan Lemma A.5 to conclude that

$$(2.13)\qquad \omega_{\rho s}\equiv 0 \bmod \omega_{s\mu}.$$

For this it is necessary to establish the assertion:

$$(2.14)\qquad \sum_{s} C_s\,\omega_{s\mu}=0 \;\Rightarrow\; C_{m+1}=\ldots=C_n=0.$$

---

(31) We have repeated the proof of (A.15) in this special case.

(32) More precisely, they are open sets in linear spaces, but we shall abuse terminology and refer to them simply as linear spaces. This convention will be followed throught the paper.

If $\sum_s C_s\, \omega_{s\mu}=0$ then

$$0=\sum_s C_s\, q_{s\alpha\mu}=\sum_s C_s\, q_{\alpha s\mu}$$

for all α, μ. Consequently, the vector

$$v=\sum_s C_s\, v_s$$

satisfies

$$\langle \omega_{\alpha\mu}, v \rangle=0$$

for all α, μ. If we recall that

(2.15) $$\text{span}\,\{\omega_{\alpha\mu}\}=\text{span}\,\{\omega_{m+1}, \ldots, \omega_n\},$$

then we conclude that $C_{m+1}=\ldots=C_n=0$.

Having established (2.13), if we denote by $\bar{\omega}$ a form ω considered modulo $\omega_{m+1}, \ldots, \omega_n$ and denote by $\bar{d}$ the operator $\eta \to \overline{d\eta}$ ([33]), then by (2.13) and (2.11):

(2.16) $$\begin{cases} \bar{d}\, A_0 \equiv 0 \mod A_0, A_1, \ldots, A_m, \\ \bar{d}\, A_\rho \equiv 0 \mod A_0, A_1, \ldots, A_m, \end{cases}$$

where the second equation results from

(2.17) $$\bar{\omega}_{\rho s}=\bar{\omega}_{\rho\mu}=0.$$

If we denote by F a fibre of the Gauss mapping, then the two equations in (2.16) say exactly that

$$\begin{cases} \tilde{T}_p(F)=\text{span of } A_0, A_1, \ldots, A_m, \text{ and} \\ \tilde{T}_p(F) \text{ remains constant as } p \text{ varies along } F. \end{cases}$$

This implies that F is (an open subset in) a $\mathbb{P}^m$.

Q.E.D.

We note the decisive role played by the symmetry of the 2nd fundamental form in both (2.6) and (2.10).

If we denote by B the image of the Gauss map, then there is a ruled variety $\{\mathbb{P}^m_y\}_{y\in B}$ in which M is an open set. It is convenient here to simply take M to be this ruled variety. We remark that not every ruled variety has a degenerate Gauss mapping (e. g., a quadric surface in $\mathbb{P}^3$); this is reflected by the fact that we have thus far only used the relation $\bar{\omega}_{\rho\mu}=0$ and not the stronger equation $\omega_{\rho\mu}=0$. These enter into the proof of the following:

([33]) That is, we are denoting by "$^-$" the restriction to the fibres $\omega_1=\ldots=\omega_m=0$ of the Gauss map of M.

(2.18) *If* $\mathbb{P}_y^n$ *denotes the tangent space* $\tilde{T}_p(M)$ *for any* $p \in M$ *lying over* $y \in B$, *then for any* $w \in T_y(B)$:

$$\mathbb{P}_y^m + \frac{d\mathbb{P}_y^m}{dw} \subseteqq \mathbb{P}_y^n \quad ({}^{34}).$$

Proof. — Recall that $\mathbb{P}_y^m$ is spanned by $A_0, A_1, \ldots, A_m$, so that (2.18) follows from

$$dA_\rho \equiv \sum_\mu \omega_{\rho\mu} A_\mu \bmod A_0, \ldots, A_n \equiv 0 \bmod A_0, \ldots, A_n$$

since $\omega_{\rho\mu} = 0$.

Q.E.D.

Summarizing, we have established the following result:

(2.19) *The most general n-dimensional manifold whose Gauss map has m-dimensional fibres is a ruled variety* $M = \bigcup_{y \in B} \mathbb{P}_y^m$ *such that*

$$\mathbb{P}_y^m + \frac{d\mathbb{P}_y^m}{dw} \subseteqq \mathbb{P}_y^n,$$

where $w \in T_y(B)$ *is any tangent vector and where*

$$\mathbb{P}_y^n = \tilde{T}_p(M) \quad \textit{for any} \quad p \in \mathbb{P}_y^m.$$

We may now determine the detailed structure of these manifolds. Perhaps the most instructive way to explain this is by examining the extreme cases. For example, suppose that $m = n-1$ so that the base B of the ruling is a curve. From (2.19):

$$\mathbb{P}_y^{n-1} + \frac{d\mathbb{P}_y^{n-1}}{dy} \subseteqq \mathbb{P}_y^n,$$

and consequently

$$\dim\left(\mathbb{P}_y^{n-1} \cap \frac{d\mathbb{P}_y^{n-1}}{dy}\right) = n-2.$$

From (2.4) we obtain:

(2.20) *The case when* $M = \bigcup_{y \in B} \mathbb{P}_y^{n-1}$ *has the rulings* $\mathbb{P}_y^{n-1}$ *as fibres of its Gauss map occurs exactly when there is a curve* $C \subset \mathbb{P}^N$ *having osculating* $(a-1)$*-planes* $\mathbb{P}_y^{(a-1)}(C)$ *and a fixed* $\mathbb{P}^{n-a-1}$ *such that*

$$\mathbb{P}_y^{n-1} = \mathbb{P}_y^{(a-1)}(C) + \mathbb{P}^{n-a-1}.$$

(34) The notation is explained at the beginning of paragraph 2(*a*).

The two extreme cases here are when M is a cone with vertex $\mathbb{P}^{n-2}$ over the curve ($a=1$), and when M is the osculating n-fold to a curve ($a=n$). In general M is a composite of these two.

Going to the other extreme, suppose that the fibres of γ are one-dimensional so that $M=\bigcup_{y\in B}\mathbb{P}^1_y$ is ruled by ∞^{n-1} lines. We will prove that

(2.21) M *is the union of* ∞^{n-2} *surfaces having a degenerate Gauss map. In general,* M *is such a union in* $n-1$ *different ways* ([35]).

Proof. — We seek to determine directions $w\in T_y(B)$ such that

$$\mathbb{P}^1_y\cap\frac{d\mathbb{P}^1_y}{dw}\neq\emptyset.$$

Equivalently, if the line is spanned by $A_0(y)$ and $A_1(y)$, then we want a point $p\in\mathbb{P}^1_y$ represented by a vector

$$A(y,\lambda)=\lambda_0 A_0(y)+\lambda_1 A_1(y)$$

together with a direction w such that

$$\frac{dA}{dw}\equiv 0 \bmod A_0, A_1. \qquad (2.22)$$

Letting $v_1, \ldots, v_n\in T_p(M)$ be a basis such that v_1 is tangent to the ruling $\mathbb{P}^1_y$ then the projections of $v_2, \ldots, v_n$ determine a basis for $T_y(B)$. Using the additional ranges of indices $2\leqq a, b\leqq n$ we write $w=\sum_a w_a v_a$. From (2.17):

$$\omega_{1a}=\sum_b h_{ab}\,\omega_b$$

so that

$$\frac{dA}{dw}\equiv\lambda_0\frac{dA_0}{dw}+\lambda_1\frac{dA_1}{dw} \bmod A_0, A_1\equiv\lambda_0\Big(\sum_a w_a A_a\Big)+\lambda_1\Big(\sum_b h_{ab}\,w_a A_b\Big) \bmod A_0, A_1.$$

Thus (2.22) is equivalent to the system of linear equations

$$\lambda_0 w_b+\lambda_1\sum_a w_a h_{ab}=0, \qquad b=2, \ldots, n. \qquad (2.23)$$

In general (2.23) will have $n-1$ distinct solutions corresponding to the eigenvalues of the matrix (h_{ab}). By relabelling we may assume that $v_2, \ldots, v_n$ are the eigenvectors. Then

$$\mathbb{P}^1_y\cap\frac{d\mathbb{P}^1_y}{dv_a}\neq\emptyset, \qquad a=2, \ldots, n. \qquad (2.24)$$

([35]) The meaning of "in general" will be made precise in the proof.

If we take the flow curves $\Gamma_a \subset B$ along the vector field v_a, then from (2.24) it follows that the ruled surface $\{ \mathbb{P}^1_y \}_{y \in \Gamma_a}$ has a degenerate Gauss mapping.

In case the matrix (h_{ab}) is not diagonizable, then we may not have a spanning set of eigenvectors. However it will be a consequence of the discussion following this proof that this situation is a degeneration of the case when (h_{ab}) is diagonalizable, and so a "general" M is the union of ∞^{n-2} ruled surfaces in $n-1$ different ways.

Q.E.D.

We shall say that a manifold is *multi-developable* in case it is the union, in more than one way, of developable ruled surfaces. The general multi-developable threefold will now be described, and this will then show how one may think of the M's in (2.21).

Suppose that $A_0(y_0, y_1)$ and $A_1(y_0, y_1)$ are chosen as in the preceding proof so that

$$(2.25) \qquad \begin{cases} \dfrac{\partial A_0}{\partial y_1} = \alpha A_0 + \beta A_1, \\ \dfrac{\partial A_1}{\partial y_1} = \gamma A_0 + \delta A_1. \end{cases}$$

In general we will have $\beta \neq 0$, so dividing the first equation through by β gives

$$A_1 = \rho A_0 + \sigma \frac{\partial A_0}{\partial y_0}.$$

Taking $\partial/\partial y_1$ of this equation and using the second equation in (2.25) we obtain a relation

$$(2.26) \qquad \frac{\partial^2 A_0}{\partial y_0 \partial y_1} + \mu \frac{\partial A_0}{\partial y_0} + \nu \frac{\partial A_0}{\partial y_1} + \lambda A_0 = 0.$$

Conversely, given functions μ, ν, λ and appropriate Cauchy initial data for the equation (2.26), we may uniquely solve and thereby construct a multi-developable ruled surface.

Summarizing, we may state the following general result:

(2.27) *Any manifold* M *with degenerate Gauss mapping is a ruled variety* $M = \bigcup_{y \in B} \mathbb{P}^m_y$ *where the rulings are the fibres of the Gauss mapping. There are directions* $w \in T_y(B)$ *for which*

$$(2.28) \qquad \mathbb{P}^m_y \cap \frac{d\mathbb{P}^m_y}{dw} \neq \emptyset.$$

By following integral curves of these directions and applying (2.4) *we may describe* M *as being ultimately built up from cones and developable varieties.*

In particular, the points of intersection in (2.28) are necessarily singular points of M. From this we may draw the following global conclusion:

(2.29) *The only smooth projective variety having a degenerate Gauss mapping is* $\mathbb{P}^n$ *itself* (36).

We may use (2.4) to list explicitly low dimensional varieties having a degenerate Gauss map, as follows:

Surfaces. — Then S is either $\mathbb{P}^2$, a cone, or a developable ruled surface.

Threefolds. — We separate into the cases $m=3$, 2, and 1 in (2.27).

$m=3$. Then M is $\mathbb{P}^3$.

$m=2$. Then M is either the osculating threefold $\{\mathbb{P}_y^{(2)}\}_{y\in B}$ to a curve in $\mathbb{P}^N$, or else is a cone over a developable ruled surface $\{\mathbb{P}_y^{(1)}\}_{y\in B}$.

$m=1$. In general, M is the union of ∞^1 developable ruled surfaces in two distinct ways. In degenerate cases, M may be the union of ∞^1 ruled surfaces and also the union of ∞^1 cones, or else it may be the union of ∞^1 cones in two ways.

3. Varieties with degenerate dual varieties

(*a*) THE DUAL VARIETY AND THE 2nd FUNDAMENTAL FORM. — For a vector space E the associated projective space $\mathbb{P}(E)$ is the set of lines through the origin in E, and the dual projective space $\mathbb{P}(E)^*$ is the set of hyperplanes in $\mathbb{P}(E)$. If $\dim E = N+1$, then because of $E^* \otimes \Lambda^{N+1} E \cong \Lambda^N E$ we may view $\mathbb{P}(E^*)$ as $\mathbb{P}(\Lambda^N E)$. The dual of $\mathbb{P}^N$ will be denoted by $\mathbb{P}^{N*}$.

For a manifold $M \subset \mathbb{P}^N$ the *dual hypersurface* $M^* \subset \mathbb{P}^{N*}$ (37) is the set of tangent hyperplanes to M. Equivalently, if $N = \bigcup_{p\in M} N_p(M)$ is the abstract normal bundle with fibres $N_p(M) = \mathbb{C}^{N+1}/\tilde{T}_p(M)$, then each hyperplane in $\mathbb{P}(N_p(M))$ determines a hyperplane in $\mathbb{P}^N$ and M^* is the image of the mapping

$$(3.1) \qquad \delta: \quad \mathbb{P}(N^*) \to \mathbb{P}^{N*}.$$

We note that when $M \subset \mathbb{P}^{n+1}$ is a hypersurface, then $\mathbb{P}^{n+1*} = \mathbb{G}(n, n+1)$ and δ is the Gauss mapping. The manifold M is said to have a *degenerate dual variety* in case $\dim M^* \leqq N-2$; i.e., in case the Jacobian of the mapping (3.1) has everywhere rank $\leqq N-2$. We shall express this condition in terms of the 2nd fundamental form.

We shall do the computation on the frame manifold $\mathscr{F}(N)$ associated to $\mathbb{P}(N^*)$. Recall that a Darboux frame for M is $\{A_0; A_\alpha; A_\mu\}$ where A_0 determines $p \in M$ and $A_0, \ldots, A_n$

(36) Of course, (2.29) may be proved by global considerations, using e. g., a Chern class argument applied to the tangent bundle along a generic fibre of the Gauss mapping. Also, Alan Landman showed us a proof using some results of his on dual varieties.

(37) M^* need not be a hypersurface, and in fact it is exactly this situation we shall study. Nonetheless we shall retain the classical terminology.

spans $\tilde{T}_p(M)$. A hyperplane ξ in $\mathbb{P}(N_p(M))$ is specified by choosing vectors $A_{n+1}, \ldots, A_{N-1}$ whose projection in $\mathbb{C}^{N+1}/\tilde{T}_p(M)$ spans ξ. This defines the Darboux frames lying over the point $(p, \xi) \in \mathbb{P}(N^*)$, and the totality of these gives the frame manifold $\mathscr{F}(N)$. Using the ranges of indices

$$0 \leqq i, j \leqq N; \qquad 1 \leqq \alpha, \beta \leqq n; \qquad n+1 \leqq \rho, \sigma \leqq N-1$$

the group of the principal fibration $\mathscr{F}(N) \to \mathbb{P}(N^*)$ is given by all substitutions

$$\begin{cases} \tilde{A}_0 = a A_0, \\ \tilde{A}_\alpha = \sum_\beta b_{\alpha\beta} A_\beta + c_\alpha A_0, \\ \tilde{A}_\rho = \sum_\sigma d_{\rho\sigma} A_\sigma + \sum_\alpha e_{\rho\alpha} A_\alpha + f_\rho A_0, \\ A_N = \sum_i g_{Ni} A_i \end{cases}$$

with non-singular coefficient matrix.

In terms of frames the mapping (3.1) is expressed by

$$\delta(p, \xi) = A_0 \wedge A_1 \wedge \ldots \wedge A_{N-1}. \tag{3.2}$$

It will be convenient to set

$$A_i^* = (-1)^{N-i} A_0 \wedge \ldots \wedge \hat{A}_i \wedge \ldots \wedge A_N, \tag{3.3}$$

so that (3.2) becomes

$$\delta(p, \xi) = A_N^*.$$

Recalling that

$$dA_0 \equiv \sum_\alpha \omega_\alpha A_\alpha \mod A_0,$$

$$dA_\alpha \equiv \sum_\rho \omega_{\alpha\rho} A_\rho + \omega_{\alpha, N} A_N \mod A_0, \ldots, A_n$$

it follows that

$$dA_N^* \equiv \sum_\alpha \omega_{\alpha, N} A_\alpha^* + \sum_\rho \omega_{\rho, N} A_\rho^* \mod A_N^*.$$

The $N-n-1$ forms $\omega_{\rho, N}$ restrict to a basis for the forms on the fibres $\mathbb{P}(N_p(M)^*) \cong \mathbb{P}^{N-n-1}$, since they describe the variation of the tangent hyperplane ξ when the point p is held fixed. The forms $\omega_{\alpha, N}$ are horizontal for the fibering $\mathbb{P}(N^*) \to M$, and it follows that

$$(\bigwedge_\alpha \omega_{\alpha, N}) \wedge (\bigwedge_\rho \omega_{\rho, N}) = 0 \iff \bigwedge_\alpha \omega_{\alpha, N} = 0.$$

Equivalently, the dual variety is degenerate if, and only if, for any Darboux frame

$$(3.4) \qquad \bigwedge_{\alpha} \omega_{\alpha, N} = 0.$$

Recalling that

$$\omega_{\alpha, N} = \sum_{\beta} q_{\alpha\beta N} \omega_{\beta}, \qquad q_{\alpha\beta N} = q_{\beta\alpha N}$$

we have

$$\bigwedge_{\alpha} \omega_{\alpha, N} = \det(q_{\alpha\beta N}) \omega_1 \wedge \ldots \wedge \omega_n,$$

so that (3.4) is equivalent to

$$\det(q_{\alpha\beta N}) = 0.$$

Finally, since A_N was any normal vector we arrive at the conclusion:

(3.5) *The dual variety* $M^* \subset \mathbb{P}^{N^*}$ *is degenerate if and only if at a generic point every* $Q \in |II|$ *is singular.*

Now at this point algebraic geometry enters. Namely, by Bertini's (A.6) every $Q \in |II|$ can be singular only if the linear system $|II|$ has a non-empty base locus B, and for generic Q the singularities must occur along B.

For example, consider the Grassmannian $G(2, n+2)$ embedded in $\mathbb{P}(\Lambda^2 \mathbb{C}^{n+2})$ by Plücker [*cf.* below (1.35)]. At a point $S \in G(2, n+2)$ with $Q = \mathbb{C}^{n+2}/S$ we recall from (1.42) that the 2nd fundamental form is the mapping

$$(3.6) \qquad \mathrm{Sym}^2(S^* \otimes Q) \to \Lambda^2 S^* \otimes \Lambda^2 Q$$

defined by

$$(e^* \otimes f, \tilde{e}^* \otimes \tilde{f}) \to (e^* \wedge \tilde{e}^*) \otimes (f \wedge \tilde{f}).$$

From (1.43) the base locus B consists of the decomposable vectors $e^* \otimes f$. To determine which quadrics $Q \in |II|$ are singular somewhere along B we argue as follows: since $\dim S = 2$ the right hand side of (3.6) is isomorphic to $\Lambda^2 Q$, and the dual of this space is $\Lambda^2 Q^*$. If $f_1, \ldots, f_n$ is a basis for Q with dual basis $f_1^*, \ldots, f_n^*$ for Q^*, then any $A \in \Lambda^2 Q^*$ is

$$(3.7) \qquad A = \sum_{\alpha, \beta} A_{\alpha\beta} f_{\alpha}^* \wedge f_{\beta}^*.$$

Using an isomorphism $\Lambda^2 S \cong \mathbb{C}$ the quadrics in $|II|$ are parametrized by $\Lambda^2 Q^*$, and the quadric Q_A corresponding to A in (3.7) is

$$(3.8) \qquad Q_A(e^* \otimes f, \tilde{e}^* \otimes \tilde{f}) = (e^* \wedge \tilde{e}^*) . \langle A, f \wedge \tilde{f} \rangle.$$

Given $e^* \otimes f \in B$ we conclude from (3.8) that Q_A is singular at $e^* \otimes f$ if, and only if, the linear function

$$\tilde{f} \to \langle A, f \wedge \tilde{f} \rangle$$

is identically zero, i.e., if, and only if

$$f \perp A = 0$$

where $\perp$ is the contraction $\Lambda^2 Q^* \to Q^*$. Summarizing:

(3.9) *The quadric* Q_A *is singular if, and only if, the 2-form* $A \in \Lambda^2 Q^*$ *is degenerate.*

In particular, in case $n = \dim Q$ is odd any A is degenerate and consequently:

(3.10) *The Grassmannian* $G(2, 2m+1)$ *has a degenerate dual variety. For any other Grassmannian the dual is non-degenerate.*

(*b*) DUALS OF HYPERSURFACES. — We shall discuss in some detail the dual of a hypersurface $M \subset \mathbb{P}^{n+1}$. Recall that in this case we are just considering the Gauss mapping

$$\gamma : \quad M \to \mathbb{P}^{n+1*},$$

whose image is the dual variety M^*. In terms of Darboux frames $\{ A_0; A_\alpha; A_{n+1} \}$ and using the notation (3.3):

$$\gamma(A_0) = A_{n+1}^*.$$

Then the differential is expressed by

$$dA_{n+1}^* \equiv \sum_\alpha \omega_{\alpha, n+1} A_\alpha^* \mod A_{n+1}^*.$$

Writing

$$\omega_{\alpha, n+1} = \sum_\beta q_{\alpha\beta} \omega_\beta, \qquad q_{\alpha\beta} = q_{\beta\alpha}$$

the 2nd fundamental form is in this case the single quadric $Q = \sum_{\alpha, \beta} q_{\alpha\beta} \omega_\alpha \omega_\beta$, which we assume to have rank $n-k$ at a generic point of M. With appropriate choice of Darboux frames we will have

$$Q = \sum_{\alpha=1}^{n-k} \omega_\alpha^2;$$

i.e.,

(3.11) $$\omega_{\alpha, n+1} = \begin{cases} \omega_\alpha, & 1 \leqq \alpha \leqq n-k, \\ 0, & n-k+1 \leqq \alpha \leqq n. \end{cases}$$

We observe that

$$\dim M^* = n-k.$$

In case $k=0$ so that M^* is again a hypersurface, $\{A^*_{n+1}; A^*_n, \ldots, A^*_1; A^*_0\}$ forms a field of Darboux frames for M^* and its Gauss mapping

$$\gamma^*: \quad M^* \to \mathbb{P}^{n+1}$$

is expressed by

$$\gamma^*(A^*_{n+1}) = A^*_{n+1} \wedge A^*_n \wedge \ldots \wedge A^*_1 = A_0;$$

thus:

(3.12) *With the natural identification* $(\mathbb{P}^{n+1^*})^* = \mathbb{P}^{n+1}$ *we have the double duality*

$$(M^*)^* = M.$$

Actually, (3.12) is always true. If $k>0$ then we have proved in (2.10) that the Gauss mapping

$$\gamma: \quad M \to M^* \tag{3.13}$$

has linear spaces $\mathbb{P}^k_{A_0}$ as fibres. Moreover, using the normalization (3.11) the leaves of the fibration (3.13) are defined by

$$\omega_1 = \ldots = \omega_{n-k} = 0.$$

From

$$dA^*_{n+1} = d(A_0 \wedge \ldots \wedge A_n) \equiv \sum_{\alpha=1}^{n-k} \omega_\alpha (-1)^{n-\alpha} A_0 \wedge \ldots \wedge \hat{A}_\alpha \wedge \ldots \wedge A_{n+1} \bmod A^*_{n+1}$$

we see that the tangent plane to M^* at $\gamma(A_0)$ is spanned by $A^*_{n+1}, A^*_1, \ldots, A^*_{n-k}$; i.e.,

$$\tilde{T}_{\gamma(A_0)}(M^*) = (\mathbb{P}^k_{A_0})^\perp.$$

Consequently, the hyperplanes in $\mathbb{P}^{n+1^*}$ that contain $\tilde{T}_{\gamma(A_0)}(M^*)$ are just $\mathbb{P}^k_{A_0}$ which establishes (3.12) in general. Summarizing:

(3.14) *Suppose that* $M \subset \mathbb{P}^{n+1}$ *is a complex analytic hypersurface whose 2nd fundamental form has rank* $n-k$ *at a generic point. Then the Gaussian image of* M *is an* $(n-k)$*-dimensional subvariety* $M^* \subset \mathbb{P}^{n+1^*}$*, and the double duality*

$$(M^*)^* = M$$

is valid.

This then gives us a pretty good general structure theorem for hypersurfaces with degenerate duals. Going to higher codimension, from (1.22) we have:

(3.16) *A manifold* $M \subset \mathbb{P}^N$ *has a degenerate dual if, and only if, a generic projection* $M' \subset \mathbb{P}^{n+1}$ *has a degenerate dual. In this case the structure of* M' *is given by* (3.14).

A Corollary follows by observing that if a generic projection M′ contains linear spaces then the same must be true of M. Consequently:

(3.17) *If* $M \subset \mathbb{P}^{N^*}$ *has a degenerate dual, then for some* $k>0$ M *contains* ∞^{n-k} $\mathbb{P}^k$'s [38].

We also remark that from (2.19) these $\mathbb{P}^k$'s fit together in a restricted manner, but we don't yet see a good way of describing this.

(*c*) SOME EXAMPLES AND ALGEBRO-GEOMETRIC OBSERVATIONS. — We shall begin by listing all manifolds $M \subset \mathbb{P}^N$ of dimensions two and three whose dual is degenerate. For this we recall from (3.5) and Bertini's Theorem that:

(3.18) *In order that the dual* M* *be degenerate it is necessary and sufficient that at a generic point of* M *the second fundamental form* $|II|$ *have a non-empty base locus* B *such that every* $Q \in |II|$ *is singular somewhere along* B. *In particular, if the Gauss mapping is degenerate then so is the dual. This occurs whenever every* $Q \in |II|$ *is singular and the base consists of isolated points.*

Now we can give our list.

Surfaces. — For $S \subset \mathbb{P}^N$ the 2nd fundamental form is a linear system of quadrics on $\mathbb{P}^1$. If S is not $\mathbb{P}^2$ the base locus can only consist of points, and from (3.18) and (2.20) we conclude

(3.19) *For a surface* S *the dual is degenerate if, and only if, the Gauss mapping is also degenerate. In this case* S *is either:* (i) $\mathbb{P}^2$; (ii) *a cone, or* (iii) *a developable ruled surface.*

Threefolds. — For a 3-dimensional manifold $M \subset \mathbb{P}^N$ the 2nd fundamental form is a linear system of conics in $\mathbb{P}^2$. Assuming that the dual M* is degenerate we have seen that the base B is non-empty, and then there are the following possibilities:

(i) $B = \mathbb{P}^2$, in which case M is $\mathbb{P}^3$.

(ii) B *consists of isolated points*, in which case the Gauss mapping is degenerate. In fact, M is ruled by $\mathbb{P}^1$'s over an algebraic surface, and is a multi-developable variety in the sense of paragraph 2(*b*).

(iii) $B = \mathbb{P}^1$. In general a linear system $|E|$ in $\mathbb{P}^{n-1}$ is said to have a *fixed component* in case there is a hypersurface in $\mathbb{P}^{n-1}$ that is common to all of the divisors in $|E|$. Equivalently, all of the homogeneous forms $F(X) \in E$ should have a common factor $F_0(X)$. In case the linear system consists of quadrics the only possibility for a fixed component is a $\mathbb{P}^{n-2}$. Then the linear system is of the form

$$\mathbb{P}^{n-2} + \mathbb{P}^{n-2}_{\lambda}, \tag{3.20}$$

where $|\mathbb{P}^{n-2}_{\lambda}|$ is a linear system of hyperplanes, and as such consists of all hyperplanes through a fixed $\mathbb{P}^k$. When $k \geqq 1$ all the divisors (3.20) are singular along $\mathbb{P}^{n-2} \cap \mathbb{P}^k$, but when $k=-1, 0$ the common singular locus of the divisors (3.20) is generally empty. We shall show that

[38] Here, $n-k$ is the rank of a generic quadric $Q \in |II|$. As usual, what we mean is that there are ∞^{n-k} distinct $\mathbb{P}^k_t$'s and open sets $U_t \subset \mathbb{P}^k_t \cap M$ such that $\bigcup_t U_t$ contains an open set on M.

(3.21) *If $n \geqq 3$ and if at a generic point the 2nd fundamental form of* $M \subset \mathbb{P}^N$ *has a fixed component then* M *is ruled by* $\mathbb{P}^{n-1}$*'s* (39). *The converse is also true.*

Proof. – The argument will be given when $n=3$. We shall also assume that $|II|$ is the linear system of conics of the form

$$\mathbb{P}^1 + \mathbb{P}^1_\lambda$$

where $|\mathbb{P}^1_\lambda|$ is a pencil of lines through a point in $\mathbb{P}^2$ not on $\mathbb{P}^1$. The other possibilities for $|\mathbb{P}^1_\lambda|$ are: (i) all lines in $\mathbb{P}^2$; (ii) a pencil with a base point on the fixed $\mathbb{P}^1$; (iii) one further fixed line. In the last two the Gauss mapping of M is degenerate, and the argument for all three cases is similar to the one we are about to give.

With a suitable choice of Darboux frames we may assume that $|II|$ is spanned by the two quadrics

$$\omega_1 \omega_3 \quad \text{and} \quad \omega_2 \omega_3,$$

where $\omega_3 = 0$ is the fixed line and $\omega_1 - \lambda\omega_2 = 0$ is the variable pencil with base point defined by $\omega_1 = \omega_2 = 0$. With this choice of frames

$$(3.22) \qquad \begin{cases} \omega_{14} = \omega_3, & \omega_{24} = 0, & \omega_{34} = \omega_1, \\ \omega_{15} = 0, & \omega_{25} = \omega_3, & \omega_{35} = \omega_2, \end{cases}$$

and all remaining $\omega_{\alpha\mu} = 0$ for $\mu \geqq 6$. We will show that the distribution

$$\omega_3 = 0$$

is completely integrable, and therefore defines a foliation of M. From (1.3) and (3.22):

$$(3.23) \qquad \begin{cases} d\omega_3 = \omega_0 \wedge \omega_3 + \omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} + \omega_3 \wedge \omega_{33}, \\ d\omega_{14} = \omega_{11} \wedge \omega_3 + \omega_{13} \wedge \omega_1 + \omega_3 \wedge \omega_{44}, \\ d\omega_{25} = \omega_{22} \wedge \omega_3 + \omega_{23} \wedge \omega_2 + \omega_3 \wedge \omega_{55}. \end{cases}$$

By (3.22) the left hand sides are all equal; consequently,

$$\begin{cases} \omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} \equiv \omega_{13} \wedge \omega_1 \bmod \omega_3, \\ \omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} \equiv \omega_{23} \wedge \omega_2 \bmod \omega_3. \end{cases}$$

By an obvious elimination

$$(3.24) \qquad \begin{cases} \omega_1 \wedge \omega_{13} \equiv 0 \bmod \omega_3, \\ \omega_2 \wedge \omega_{23} \equiv 0 \bmod \omega_3, \end{cases}$$

and then by the first equation in (3.23):

$$d\omega_3 \equiv 0 \bmod \omega_3.$$

(39) This is definitely not true for surfaces; in this case a fixed component is the same as a base point, which occurs for any $S \subset \mathbb{P}^3$.

We denote by $\bar{\omega}$ a form ω considered modulo ω_3. By (3.24), along the surfaces $\omega_3=0$:

$$(3.25)\qquad \begin{cases} \bar{\omega}_{13}=\alpha\bar{\omega}_1, \\ \bar{\omega}_{23}=\beta\bar{\omega}_2, \\ \bar{\omega}_{14}=\bar{\omega}_{24}=\bar{\omega}_{15}=\bar{\omega}_{25}=0. \end{cases}$$

Consequently the 2nd fundamental form $\overline{\mathrm{II}}$ of these surfaces $\{\omega_3=0\}$ is

$$\alpha\bar{\omega}_1^2+\beta\bar{\omega}_2^2.$$

We shall compute their 3rd fundamental form.

Denoting by $\bar{d}$ the restriction of d to the leaves $\omega_3=0$ and using (3.22):

$$\begin{aligned}\overline{\mathrm{III}}=\bar{d}^3 A_0 &\equiv \bar{d}^2(\bar{\omega}_1 A_1+\bar{\omega}_2 A_2) \bmod A_0\\ &\equiv \bar{d}(\bar{\omega}_1\bar{\omega}_{13}+\bar{\omega}_2\bar{\omega}_{23}) A_3 \bmod A_0,\ A_1,\ A_2\\ &\equiv (\alpha\bar{\omega}_1^2+\beta\bar{\omega}_2^2)\bar{\omega}_1 A_4+(\alpha\bar{\omega}_1^2+\beta\bar{\omega}_2^2)\bar{\omega}_2 A_5 \bmod A_0,\ A_1,\ A_2,\ A_3.\end{aligned}$$

Thus $|\overline{\mathrm{III}}|$ is generated by the linear system of cubics

$$\alpha\bar{\omega}_1^3+\beta\bar{\omega}_2^2\bar{\omega}_1, \qquad \alpha\bar{\omega}_1^2\bar{\omega}_2+\beta\bar{\omega}_2^3.$$

Using the Jacobian condition (1.47) we infer that $\alpha=\beta=0$. Then from (3.25):

$$\bar{\omega}_{13}=\bar{\omega}_{23}=0.$$

This says that

$$\bar{d}(A_0\wedge A_1\wedge A_2)\equiv 0 \bmod A_0\wedge A_1\wedge A_2;$$

i.e., the leaves of the foliation $\omega_3=0$ are linear spaces.

Q.E.D.

On the basis of (3.19) and (3.20) we may draw the following global conclusion:

(3.26) *Let* $V\subset\mathbb{P}^N$ *be a smooth projective variety whose dual is degenerate. Then* $\dim V\geqq 3$, *and* $\dim V=3$ *if, and only if,* V *is ruled by* $\mathbb{P}^2$*'s over a curve* ([40]).

As an example, suppose we consider the Segre variety $\mathbb{P}^1\times\mathbb{P}^2\subset\mathbb{P}^5$. According to paragraph 1 (*c*) the 2nd fundamental form $|\mathrm{II}|$ is the linear system of quadrics on $\mathbb{P}^2$ whose base is a point v plus a line $\mathbb{P}^1$.

By (1.22) the 2nd fundamental form of a generic projection of $\mathbb{P}^1\times\mathbb{P}^2$ into a $\mathbb{P}^4$ is a single quadric $Q\in|\mathrm{II}|$, so that this projection is a threefold in $\mathbb{P}^4$ whose Gauss mapping has $\mathbb{P}^1$'s a fibres.

([40]) Note that in this case we must have $N\geqq 5$.

From (3.23) we also have

(3.27) *If* $V \subset \mathbb{P}^N$ *is a (possibly singular) projective algebraic variety whose dual is degenerate, then the Kodaira number* $\varkappa(V) = -1$.

4. Varieties with degenerate Chern forms

(*a*) DEGENERACY OF CHERN FORMS OF MANIFOLDS IN $\mathbb{C}^N$. — We begin by considering a holomorphic mapping

$$f: \quad M \to G(r, E)$$

from a complex manifold into a Grassmannian. Denoting by $S \to M$ the rank-r holomorphic vector bundle induced from the universal sub-bundle ([41]), it is well known that any choice of metric on E induces an Hermitian metric on $S \to M$ which then defines a canonical connection and curvature matrix Ω_S. Moreover, if the Chern forms $c_q(\Omega_S)$ are defined as usual by

$$(4.1) \qquad \sum_{q=0}^{r} c_q(\Omega_S)\, t^{r-q} = \det\left(t I + \frac{\sqrt{-1}}{2\pi} \Omega_S \right),$$

then there is an inequality ([42]):

$$(4.2) \qquad (-1)^q c_q(\Omega_S) \geqq 0.$$

We will examine what it means to have equality in (4.2), and will then apply the result to the Gauss mapping of a complex submanifold of $\mathbb{C}^N$. The outcome is stated in (4.11) below.

It is convenient to first isolate the linear algebra construction underlying the formula for $c_q(\Omega_S)$. Suppose we are given complex vector spaces A, B, C and a linear map

$$(4.3) \qquad T: \quad A \otimes B \to C.$$

We define an induced linear map

$$(4.4) \qquad T^{(q)}: \quad \Lambda^q A \otimes \Lambda^q B \to \mathrm{Sym}^q C$$

([41]) Giving f is equivalent to giving a locally free sheaf $\mathcal{E} \to M$ and subspace $E \subset H^0(M, \mathcal{E})$ of global sections which generates every fibre of $\mathcal{E}$. Then $f(p) = \{\text{sections } s \in E: s(p) = 0\}$.

([42]) A (q, q) form ω is non-negative, written $\omega \geqq 0$, if locally there are $(q, 0)$ forms φ_α such that

$$\omega = (\sqrt{-1})^{q^2} \sum_\alpha \varphi_\alpha \wedge \overline{\varphi_\alpha}$$

In [8], pages 246-247, it is proven that (4.2) holds. In our situation where S is to be the tangent bundle of $M \subset \mathbb{C}^N$ we remark that the forms $c_q(\Omega_S)$ are invariant only under affine unitary transformations, but the condition $c_q(\Omega_S) \equiv 0$ is preserved under all affine linear transformations.

as follows: for vectors $a_1, \ldots, a_q \in A$ and $b_1, \ldots, b_q \in B$ set

$$T^{(q)}(a_1, \ldots, a_q; b_1, \ldots, b_q) = \det_{i,j} T(a_i, b_j) = \sum_\pi \operatorname{sgn} \pi\, T(a_1, b_{\pi(1)}) \ldots T(a_q, b_{\pi(q)}).$$

Since the formula on the right hand side of (4.5) is visibly alternating and linear in the a_i and b_j it indices a mapping (4.4).

To see what it means for $T^{(q)}=0$, we observe that for each linear function $\xi \in C^*$ the composition $\xi \circ T$ induces

$$T_\xi: \quad A \otimes B \to \mathbb{C}.$$

In terms of any basis $\{a_\rho\}$ for A and $\{b_\mu\}$ for B we obtain a matrix by the formula

$$(T_\xi)_{\rho\mu} = T_\xi(a_\rho \otimes b_\mu).$$

Using the fact that a homogeneous polynomial of degree q on C^* is zero if, and only if, its restriction to each line is zero, we infer that

(4.6) *The mapping* $T^{(q)}$ *is zero if, and only if, all of the* $q \times q$ *minors of the matrix* $(T_\xi)_{\rho\mu}$ *are zero for any* $\xi \in C^*$.

Returning to our holomorphic mapping $f: M \to G(r, E)$ we denote by Q the pullback of the universal quotient bundle and use the notations

$$S_p = \text{fibre of } S \text{ at } p \in M,$$
$$Q_p = E/S_p.$$

From (1.10) we recall that the differential

$$f_*: \quad T_p(M) \to \operatorname{Hom}(S_p, Q_p);$$

we consider this as

$$f_*: \quad T_p(M) \otimes S_p \to Q_q.$$

Applying the construction (4.5) we obtain

$$f_*^{(q)}: \quad \Lambda^q T_p(M) \otimes \Lambda^q S_p \to \operatorname{Sym}^q Q_p,$$

which may be thought of as giving

(4.7) $$f_*^{(q)} \in \operatorname{Hom}(\Lambda^q S_p, \operatorname{Sym}^q Q_p) \otimes \Lambda^q T_p(M)^*.$$

Using the metrics on S_p and Q_p we may consider

(4.8) $$\| f_*^{(q)} \|^2 = (f_*^{(q)}, f_*^{(q)}),$$

as a form of type (q, q) in the tangent space at $p \in \mathrm{M}$. The basic formula is: *The Chern forms are given by*

$$(4.9) \qquad (-1)^q c_q(\Omega_S) = \text{Const.} \, \| f_*^{(q)} \|^2,$$

where Const. is a constant depending only on q. In particular, equality holds in (4.2) *if, and only if,* $f^{(q)} = 0$ ([43]).

In fact, the computation required to establish (4.9) has already appeared several times — e. g., in paragraph 5 of [8]. All we have done here is intrinsically interpret the existing calculation.

We now consider the Gauss mapping

$$\tilde{\gamma}: \quad \mathrm{M} \to \mathrm{G}(n, \mathrm{N})$$

associated to a complex submanifold $\mathrm{M} \subset \mathbb{C}^{\mathrm{N}}$. From paragraph 1 (*e*) we recall that in terms of Darboux frames $\{ z; e_\alpha; e_\mu \}$ the 2nd fundamental form is

$$\mathrm{II} = \sum_{\alpha, \beta, \mu} q_{\alpha\beta\mu} \theta_\alpha \theta_\beta \otimes e_\mu, \qquad q_{\alpha\beta\mu} = q_{\beta\alpha\mu} \quad ([44]).$$

We also recall from (1.18) that the differential of $\tilde{\gamma}$ may be naturally identified with II considered as a map

$$\mathrm{II}: \quad \mathrm{T(M)} \otimes \mathrm{T(M)} \to \mathrm{N(M)}.$$

From (4.9) we then deduce the Proposition:

(4.10) *For a complex submanifold* $\mathrm{M} \subset \mathbb{C}^{\mathrm{N}}$ *the Chern forms satisfy*

$$(-1)^q c_q(\Omega_{\mathrm{M}}) \geqq 0,$$

with equality holding if, and only if, every quadric $\mathrm{Q} \in |\mathrm{II}|$ *has rank* $\leqq q-1$.

This result is valid in the tangent space at any point of M.

Here we recall that in a suitable coordinate system any quadric Q on $\mathbb{C}^n$ has the normal form

$$\mathrm{Q(X)} = \sum_{\alpha=1}^{r} \mathrm{X}_\alpha^2,$$

where r is its rank. In particular the rank is n exactly when Q defines a smooth quadric hypersurface in $\mathbb{P}^{n-1}$ [*cf* footnote ([44]) in paragraph A (*b*)]. From this observation and (4.10)

([43]) In particular, the condition $c_q(\Omega_S) \equiv 0$ is independent of the metric on E — *cf.* footnote ([42]).

([44]) Using (1.65) we identify the 2nd fundamental forms of M in $\mathbb{C}^{\mathrm{N}}$ and in $\mathbb{P}^{\mathrm{N}}$.

there follows the Corollary:

(4.11) *For an n-dimensional submanifold* $M \subset \mathbb{C}^N$, *at any point we have*

$$(-1)^n c_n(\Omega_M) \geqq 0,$$

with equality holding if, and only if, every $Q \in |II|$ *is singular. If equality holds at a generic point, then for some* $k \geqq 1$ *the manifold* M *contains* $\infty^{n-k}\,\mathbb{C}^k$*'s.*

The second statement follows from (3.5) and (3.23). We refer to footnote ([42]) there for amplification of the last sentence in (4.11).

(*b*) APPLICATION TO ABELIAN VARIETIES. — We shall consider an Abelian variety $A = \mathbb{C}^N/\Lambda$, ([45]) and in A we assume given an analytic subvariety V of dimension n. We will say that V *is ruled by abelian subvarieties* if there is a positive-dimensional abelian subvariety $A' \subset A$ such that translation by A' leaves V invariant. Equivalently, a finite covering $\tilde{A} \to A$ should split into a product

$$\tilde{A} = A' \times A''$$

of Abelian subvarieties, and the inverse image $\tilde{V}$ of V should correspondingly decompose as

$$\tilde{V} = A' \times V'', \tag{4.12}$$

where $V'' \subset A''$ is an analytic subvariety.

The most immediate differential-geometric formulation of V being ruled by abelian subvarieties comes by considering the Gauss mapping. Namely, we may assume that V is irreducible and has multiplicity one and denote by $V^* \subset V$ the open dense set of smooth points. Identifying the tangent space to A at the origin with $\mathbb{C}^N$ we may use translation to define the Gauss mapping

$$\gamma_A : \quad V^* \to G(n, N) \quad ([46]). \tag{4.13}$$

Then we have:

(4.14) V *is ruled by abelian subvarieties if, and only if, the Gauss mapping* γ_A *is degenerate. In fact the orbits of* A' *are just the fibres of* γ_A.

Proof. — In case there is a splitting (4.12) it is clear that the Gauss mapping is constant along the orbits $A' \times \{v''\}$ of A' $(v'' \in V'')$. Since this local description is invariant under the finite covering group of $\tilde{A} \to A$, it follows that the same is true of the image V of $A' \times V''$.

Conversely, suppose that γ_A is degenerate. Then it follows from (2.10) that the fibres of γ_A are the projections in $\mathbb{C}^N/\Lambda$ of affine linear subspaces of $\mathbb{C}^N$. Since these fibres are also

([45]) The notation means that Λ is a lattice in $\mathbb{C}^N$. We recall that if $A' \subset A$ is an abelian subvariety, then there is an abelian variety A'' and finite covering mapping $A' \times A'' \to A$ which is the given inclusion on the A' factor.

([46]) Alternatively, we may consider the usual Gauss mapping γ on the inverse image $\tilde{V}^*$ of V^* in $\mathbb{C}^N$, and observe that $\gamma(z+\lambda) = \gamma(z)$ for any lattice vector $\lambda \in \Lambda$.

closed analytic subvarieties of V, we conclude that the fibres of γ_A are translates of abelian subvarieties of A. On the other hand it is well known that a continuous family of abelian subvarieties of A must be constant. Indeed such a family is given by a continuous family $\{\mathbb{C}^k_y\}_{y\in B}$ of k-planes such that each intersection $\mathbb{C}^k_y \cap \Lambda$ is a lattice, and the sublattices of Λ are clearly a countable discrete set. It follows that the fibres of γ_A are translates of a fixed abelian subvariety $A' \subset A$, and hence translation by A' leaves V invariant.

Q.E.D.

We may rephrase (4.14) in terms of the Chern forms $c_q(\Omega_V)$ [*cf.* (4.1)] defined on the open set V* of smooth points of V. There it states that the following are equivalent:

(4.15)
$$\begin{cases} \text{V } \textit{is ruled by abelian subvarieties}, \\ c_1(\Omega_V)^n \equiv 0. \end{cases}$$

Indeed, if we compose γ_A with the Plücker embedding $G(n, N) \subset \mathbb{P}(\Lambda^n \mathbb{C}^N)$, then $c_1(\Omega_V)$ is the pullback to V* of the standard Kähler form on $\mathbb{P}(\Lambda^n \mathbb{C}^N)$. Consequently $c_1(\Omega_V)^n \equiv 0$ if, and only if, γ_A is degenerate.

We remark that in local holomorphic coordinates $z_1, \ldots, z_n$ on V*:

$$c_1(\Omega_V) = \frac{\sqrt{-1}}{2} \sum_{\alpha, \beta} R_{\alpha\bar{\beta}}\, dz_\alpha \wedge d\bar{z}_\beta$$

is the Ricci curvature form. The fibres of γ_A are the nullspaces of this form, and in this formulation (4.15) was known classically (*cf.* [1]).

Somewhat less obvious than (4.14) is the following:

(4.16) *For an n-dimensional analytic subvariety* V *of an abelian variety, the following are equivalent:*

$$\begin{cases} \text{V } \textit{is ruled by abelian subvarieties}, \\ (-1)^n c_n(\Omega_V) \equiv 0. \end{cases}$$

Proof. — Since

$$c_n(\Omega_V) = \gamma_A^* ((n, n) \text{ form on } G(n, N))$$

we conclude from (4.14) that $c_n(\Omega_V) \equiv 0$ in case V is ruled by abelian varieties.

For the converse we assume that $c_n(\Omega_V) \equiv 0$. By (4.11) the inverse image $\tilde{V}$ of V in $\mathbb{C}^N$ contains an open subset of a family $\{\mathbb{P}^k_y\}_{y\in B}$ of ∞^{n-k} $\mathbb{P}^k$'s. The affine k-plane $\mathbb{C}^k_y = \mathbb{C}^N \cap \mathbb{P}^k_y$ projects on a subset of V that is a translate of a not-necessarily-closed subgroup G_y of A. However, the closure $\bar{G}_y = A'_y$ is an abelian subvariety of which a translate is contained in V. Letting y vary we may argue as in the proof of (4.14) that A'_y is independent of y, and then conclude as before that V is ruled by abelian subvarieties.

Q.E.D.

Remark. — If we view the curvature as a matrix $\Omega_V = (\Omega_{\alpha\bar{\beta}})$ whose entries are (1,1) forms, then $-\sqrt{-1}\,\Omega_V$ is Hermitian in the sense that

$$\overline{(-\sqrt{-1}\,\Omega_V)^t} = -\sqrt{-1}\,\Omega_V.$$

Moreover, it is positive semi-definite, as follows by writing locally

$$-\Omega_V = B \wedge {}^t\bar{B}$$

where B is a matrix of (1,0) forms (*cf.* [8], p. 195). For such an Ω_V there will always be an inequality

$$\det(-\sqrt{-1}\,\Omega_{\alpha\bar{\beta}}) \leqq \text{Const.}\ (\text{trace}(-\sqrt{-1}\,\Omega_{\alpha\bar{\beta}}))^n \quad (^{47}) \tag{4.17}$$

similar to the inequality of arithmetic and geometric means. What (4.16) gives is a converse to (4.17), in the sense that for a global analytic subvariety V of an abelian variety both sides are identically zero exactly when V is ruled by Abelian subvarieties.

In case V is smooth (4.16) has been proved, using global techniques, by Smythe [12]. His work was partly in response to a question raised by Ochiai [11] concerning work by A. Bloch that Ochiai was attempting to complete. Bloch's conjecture follows from (4.16). Quite independently of this result Mark Green found a complete proof of Bloch's conjecture using new methods. The general question of holomorphic curves in algebraic varieties is the subject of a joint paper that will appear shortly.

5. Varieties with degenerate tangential varieties

(*a*) THE TANGENTIAL VARIETY AND THE 2nd FUNDAMENTAL FORM. — Given a submanifold $M \subset \mathbb{P}^N$ we recall that $\tilde{T}_p(M)$ denotes the projective tangent space at $p \in M$. We shall denote by $\tilde{T}(M) = \bigcup_{p \in M} \tilde{T}_p(M)$ the corresponding abstract $\mathbb{P}^n$-bundle over M. Then $\dim \tilde{T}(M) = 2n$, and the tautological map

$$\tau : \quad \tilde{T}(M) \to \mathbb{P}^N \tag{5.1}$$

has as image the *tangential variety* $\tau(M)$; this is the variety swept out in $\mathbb{P}^N$ by the tangent spaces $\tilde{T}_p(M)$ as p varies over M. The expected dimension of $\tau(M)$ is $\min(N, 2n)$, and assuming that $N \geqq 2n$ (^{48}) we shall be interested in the local structure of submanifolds of $\mathbb{P}^N$ whose tangential variety is degenerate. This is equivalent to the differential τ_* of the

(^{47}) The multiplication on both sides is exterior multiplication of (1,1) forms. This inequality is valid locally for any piece of submanifold in $\mathbb{C}^N$.

(^{48}) By modifying slightly our methods the general case may be treated. We should think of $\tau(M)$ as a generalized developable ruled variety, since when $n=1$ we simply have the tangential ruled surface to a curve.

mapping (5.1) being everywhere singular, and it is the local consequences of this condition which we shall investigate.

We may think of $\tilde{T}(M)$ as the subvariety of $\mathbb{P}^N \times \mathbb{P}^N$ defined by the incidence relation

$$\tilde{T}(M) = \{(p, q) : p \in M \text{ and } q \in \tilde{T}_p(M)\}.$$

Over the point (p, q) we consider Darboux frames

$$\{A_0; A_1; A_2, \ldots, A_n; A_{n+1}, \ldots, A_N\}$$

where A_0 projects onto p and A_1 onto q. The set of all such frames forms a manifold $\mathscr{F}(\tilde{T}(M))$, which is a principal bundle over $\tilde{T}(M)$. Using the additional range of indices $2 \leqq \rho, \sigma \leqq n$, the set of all frames lying over a fixed point $(p, q) \in \tilde{T}(M)$ is obtained from a fixed frame by

$$\begin{cases} \tilde{A}_0 = \lambda A_0, \\ \tilde{A}_1 = \mu A_1, \\ \tilde{A}_\rho = \sum_\sigma g_{\rho\sigma} A_\sigma + \xi_\rho A_1 + \eta_\rho A_0, \\ \tilde{A}_\mu = \sum_\nu g_{\mu\nu} A_\nu + \sum_\sigma g_{\mu\sigma} A_\sigma + \gamma_\mu A_1 + \xi_\mu A_0. \end{cases}$$

The mapping (5.1) is expressed by

$$\tau_{(p, q)} = A_1.$$

Then

(5.2) $$dA_1 \equiv \omega_{10} A_0 + \sum_\rho \omega_{1\rho} A_\rho + \sum_\mu \omega_{1\mu} A_\mu \bmod A_1.$$

The 1-forms

$$\omega_{10}, \omega_{12}, \ldots, \omega_{1n},$$

of which there are n, restrict to a basis for the forms along the fibres $\tilde{T}_p(M)$ of $\tilde{T}(M) \to M$. Geometrically, they measure how A_1 is infinitesimally moving in this $\mathbb{P}^n$. The remaining 1-forms

$$\omega_{1\mu} = \sum_\alpha q_{1\alpha\mu} \omega_\alpha$$

are horizontal for the fibering $\tilde{T}(M) \to M$, and from (5.2) we draw the conclusion:

(5.3) *The rank of* τ_* *is* $n + \{$*number of linearly independent forms* $\omega_{1\mu}\}$.

Put somewhat differently, for each $v \in T_p(M)$ we consider the linear map

$$Q_v : \quad T_p(M) \to N_p(M)$$

defined by

(5.4) $$Q_v(w) = Q(v, w),$$

where

$$Q: \quad T_p(M) \otimes T_p(M) \to N_p(M)$$

is the 2nd fundamental form (1.18). From (5.2) and (5.3) we infer:

(5.5) *If* $v \in T_p(M)$ *points in the direction* $\overline{pq}$, *then the kernel of* τ_* *is isomorphic to* $\ker Q_v$ *where* Q_v *is defined by* (5.4).

More precisely, we consider the differential

$$\pi_*: \quad T_{(p,q)}(\tilde{T}(M)) \to T_p(M)$$

of the projection $\pi: \tilde{T}(M) \to M$. Then τ_* is injective on $\ker \pi_*$, and

$$\ker \tau_* \subset T_{(p,q)}(\tilde{T}(M))/\ker \pi_*$$

projects isomorphically onto $\ker Q_v$.

Geometrically, the fibre of τ passing through (p, q) projects onto a subvariety of M whose tangent space is defined by

$$\omega_{1\mu} = 0.$$

To interpret (5.5) algebro-geometrically we consider the linear system of quadrics $|II|$ as defining a rational mapping

$$(5.6) \qquad Q: \quad \mathbb{P}^{n-1} \to \mathbb{P}^{N-n-1}$$

at each $p \in M$ ([49]). At a point $v \in \mathbb{P}^{n-1}$ where $Q(v) \neq 0$, i. e., at a point outside the base locus $B = B(II)$, the differential

$$Q_*: \quad T_v(\mathbb{P}^{n-1}) \to T_{Q(v)}(\mathbb{P}^{N-n-1})$$

of the quadratic mapping Q has kernel isomorphic to $\ker Q_v$. From (5.5) we conclude:

(5.7) *For a submanifold* $M \subset \mathbb{P}^N$ *the tangential variety has dimension* $2n-k$ *where* k *is the dimension of a generic fibre of the rational mapping* (5.6). *When* $N \geqq 2n$ *the tangential variety is degenerate exactly when the mapping* (5.6) *has positive dimensional fibres at a generic point of* M.

Referring to (A.10) and (A.11) we arrive at the following conclusion:

(5.8) *If the tangential variety* $\tau(M)$ *has dimension* $2n-k$, *then the base locus* $B = B(II)$ *has dimension* $\geqq k-1$. *Moreover, any fibre* F *of the rational mapping* ι_{II} *meets* B.

([49]) *cf.* (A.7) in Appendix A. We are here using Q to denote ι_{II}, and have chosen bases to have $\mathbb{P}T_p(M) \cong \mathbb{P}^{n-1}$ and $\mathbb{P}N_p(M) \cong \mathbb{P}^{N-n-1}$.

Of course, (5.8) doesn't say much of anything if $\dim |II| \leqq n-2$, and in this regard we recall from (1.69) that:

(5.9) *If we have* $M \subset \mathbb{P}^N$ *with* $N \geqq 2n$ *and* $\dim |II| \leqq n-2$, *at a generic point, then the second Gauss mapping* $\gamma^{(2)}$ *is degenerate.*

In low dimensions, say for $n \leqq 4$, we shall be able to refine the proof of (1.69) to have a much stronger conclusion. On the other hand it is an extremely strong restriction on a linear system $|II|$ of quadrics on $\mathbb{P}^{n-1}$ with $\dim |II| \geqq n-1$ to have positive dimensional fibres, and we shall be able to list all such when $n \leqq 4$. Combining, we will obtain a list of those $M \subset \mathbb{P}^N$ with $\dim M \leqq 4$ that have degenerate tangential varieties. This will be done in paragraphs 5 (*d*) and (*e*) below; for the moment we will give a few examples.

Degenerate Gauss mappings. — If $M \subset \mathbb{P}^N$ has a degenerate Gauss mapping (2.5), then M has a degenerate tangential variety. Indeed, referring to (2.6) the singular set $S(II) \subset \mathbb{P}^{n-1}$ along which all quadrics $Q \in |II|$ are singular is a $\mathbb{P}^{m-1}$ with $m \geqq 1$. If $p \in \mathbb{P}^{n-1}$ is a general point, then on the linear span $\overline{p\,\mathbb{P}^{m-1}} \cong \mathbb{P}^m$ any $Q \in |II|$ reduces to a multiple of the quadric $2\,\mathbb{P}^{m-1}$. Consequently, $\overline{p\,\mathbb{P}^{m-1}}$ is contained in a fibre of the rational mapping ι_{II}.

We recall that varieties having a degenerate Gauss mapping have been classified in paragraph 2 (*b*).

Our other two examples are based on the following observation:

(5.10) *If* $|Q|$ *is a linear system of quadrics on* $\mathbb{P}^{n-1}$ *with base* B *such that the chordal variety of* B *fills up* $\mathbb{P}^{n-1}$, *then the rational mapping* ι_Q *has positive-dimensional fibres* [50].

Proof. — If $v, w \in B$ are any two points, then on the line $L = \overline{vw}$ any quadric vanishing on B is a multiple of the quadric $v+w$ on $L \cong \mathbb{P}^1$. Thus all chords to B are fibres of ι_Q.

Q.E.D.

Segre varieties. — We recall from paragraph 1 (*c*) that the Segre variety is the image of [*cf.* (1.23)]:

$$\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}. \tag{5.11}$$

Moreover, from (1.31) we have:

(5.12) *The base* B *of the 2nd fundamental form consists of two skew linear subspaces* $\mathbb{P}^{m-1}$ *and* $\mathbb{P}^{n-1}$ *in* $\mathbb{P}^{m+n-1}$.

Since these subspaces span $\mathbb{P}^{m+n-1}$ we conclude from (5.7) and (5.10) that:

(4.13) *Whenever* $mn \geqq m+n$, *the Segre variety* (5.11) *has a degenerate tangential variety.*

The first interesting case here is $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$; in this regard, *cf.* (5.61).

(50) The chordal or secant variety is the union of all lines $\overline{vw}$ where $v, w \in B$.

Grassmannians. — At a point $S \in G(n, E)$ we set $Q = E/S$ and recall the identification (1.10):

$$T_S(G(n, E)) \cong S^* \otimes Q.$$

With this identification we have from (1.43):

(5.14) *The base* $B \subset \mathbb{P}(S^* \otimes Q)$ *of the 2nd fundamental form of* $G(n, E) \subset \mathbb{P}(\Lambda^n E)$ *is the Segre variety* $\mathbb{P}(S^*) \times \mathbb{P}(Q) \subset \mathbb{P}(S^* \otimes Q)$.

Choosing bases so that $S^* \cong \mathbb{C}^n$ and $Q \cong \mathbb{C}^m$ $(m = N - n)$, the chordal variety of the image

$$\mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \subset \mathbb{P}^{mn-1}$$

fills up $\mathbb{P}^{mn-1}$ exactly when $n=2$ or $m=2$. This proves one-half of the assertion:

(5.15) *The Grassmannian* $G(n, N) \subset \mathbb{P}^{\binom{N}{n}-1}$ *has a degenerate tangential variety exactly when* $n=2$ *or* $N-n=2$.

The other half may be established by examining the Plücker relations to show that ι_{II} is birational onto its image whenever $n \geqq 3$ and $N-n \geqq 3$.

The first case when $\tau(G(n, N)) \neq \mathbb{P}^{\binom{N}{n}-1}$ occurs for

$$G(2,6) \subset \mathbb{P}^{14};$$

then $\dim G(2,6) = 8$ and $\dim \tau(G(2,6)) = 12$.

(*b*) THE GENERAL STRUCTURE THEOREM. — We now give our main general result concerning varieties having degenerate tangential varieties. In the tangential mapping (5.1) we assume that

$$\dim \tau(M) = 2n - k, \qquad k > 0,$$

and $N \geqq 2n - k + 1$. The Gauss mapping

$$(5.16) \qquad \gamma_\tau : \ \tau(M) \to \mathbb{G}(2n-k, N)$$

is then defined, and it will turn out that γ_τ is constant along lines $\overline{pq}$ where $p \in M$ and $q \in \tilde{T}_p(M)$—in fact, this happens with no assumption on $\tau(M)$ [51].

Our result is this:

(5.17) *In case the tangential variety is degenerate, the Gauss mapping* γ_τ *has fibres of dimension* $\geqq 2$.

Proof. — Over $(p, q) \in \tilde{T}(M)$ we consider Darboux frames

$$\{A_0; A_1; A_2, \ldots, A_n; A_{n+1}, \ldots, A_{2n-k}; A_{2n-k+1}, \ldots, A_N\},$$

[51] The point is that $\tau(M)$ is a developable variety of the type encountered in paragraph 2(*b*).

where

$$\begin{cases} A_0 \text{ projects onto } p \in M, \\ A_1 \text{ projects onto } q \in \tilde{T}_p(M), \\ dA_1 \equiv 0 \bmod A_0, \ldots, A_{2n-k}. \end{cases}$$

Using the additional ranges of indices

$$2 \leqq \rho, \sigma \leqq n, \qquad n+1 \leqq \mu, \nu \leqq 2n-k, \qquad 2n-k+1 \leqq s \leqq N$$

we have

$$dA_1 \equiv \omega_{10} A_0 + \sum_\rho \omega_{1\rho} A_\rho + \sum_\mu \omega_{1\mu} A_\mu \bmod A_1,$$

from which we infer that

$$\tilde{T}_{\tau(p,q)}(\tau(M)) = \text{span of } A_0, \ldots, A_{2n-k}$$

and that the 1-forms, of which there are $2n-k$,

$$\omega_{10}; \quad \omega_{1\rho}; \quad \omega_{1\mu}$$

are the pullbacks to $\tilde{T}(M)$ of linearly independent forms on $\tau(M)$. Next, from $\omega_{1s}=0$ we have

$$0 = d\omega_{1s} = \sum_\rho \omega_{1\rho} \wedge \omega_{\rho s} + \sum_\mu \omega_{1\mu} \wedge \omega_{\mu s},$$

which by the Cartan Lemma implies

$$(5.18) \qquad \begin{cases} \omega_{\rho s} = \sum_\sigma k_{\rho\sigma s}\, \omega_{1\sigma} + \sum_\mu k_{\rho\mu s}\, \omega_{1\mu}, \\ \omega_{\mu s} = \sum_\rho k_{\mu\rho s}\, \omega_{1\rho} + \sum_\nu k_{\mu\nu s}\, \omega_{1\nu}, \end{cases}$$

where $k_{\rho\sigma s} = k_{\sigma\rho s}$, $k_{\rho\mu s} = k_{\sigma\rho s}$, and $k_{\mu\nu s} = k_{\nu\mu s}$. In fact it is clear that these symmetric matrices represent the 2nd fundamental form $\mathrm{II}(\tau(M))$ of $\tau(M)$ in $\mathbb{P}^N$. We note that ω_{10} does not appear in $\mathrm{II}(\tau(M))$, reflecting the fact that the tangent space to $\tau(M)$ is constant along the line $\overline{A_0 A_1}$. This confirms our previous remark about $\tau(M)$ being developable.

It follows that maximum possible rank for the Gauss mapping (5.16) is $2n-k-1$, and if this is achieved then

$$(5.19) \qquad \omega_{1\rho}; \ \omega_{1\mu} \equiv 0 \bmod \{\omega_{\rho s}; \ \omega_{\mu s}\}.$$

Now we consider the composite

$$\tilde{\gamma}: \quad \tilde{T}(M) \to \mathbb{G}(2n-k, N)$$

where $\tilde{\gamma}=\gamma_\tau \circ \tau$. The fibres of $\tilde{\gamma}$ have dimension $\geqq k+1$, and according to (5.19) are defined by

$$\omega_{1\sigma}=0; \qquad \omega_{1\mu}=0. \tag{5.20}$$

Since $k+1 \geqq 2$ the Frobenius integrability condition for the distribution (5.20) is not automatically satisfied, and we shall exploit this fact. Namely, recalling our notation $\omega_\sigma=\omega_{0\sigma}$ from (1.3):

$$d\omega_{1\sigma}=\omega_{10}\wedge\omega_\sigma+\sum_\rho \omega_{1\rho}\wedge\omega_{\rho\sigma}+\sum_\mu \omega_{1\mu}\wedge\omega_{\mu\sigma} \equiv \omega_{10}\wedge\omega_\sigma \bmod \{\omega_{1\rho},\omega_{1\mu}\}.$$

By Frobenius we infer that

$$\omega_{10}\wedge\omega_\sigma \equiv 0 \bmod \{\omega_{1\rho};\omega_{1\mu}\}.$$

This implies

$$\omega_\sigma \equiv 0 \bmod \{\omega_{1\rho};\omega_{1\mu}\}.$$

Now the forms ω_σ are horizontal for the fibration $\tilde{T}(M)\to M$, and consequently

$$\omega_\sigma \equiv 0 \bmod \omega_{1\mu}.$$

Recalling that here $\mu=n+1, \ldots, 2n-k$, since the ω_σ where $\sigma=2, \ldots, n$ are linearly independent this can only happen if $k=1$ and

$$\omega_{1\mu}\equiv 0 \bmod \omega_2, \ldots, \omega_n.$$

But then for any $A_1 \in \tilde{T}_p(M)$:

$$Q(A_1, A_1)=\sum_\mu q_{11\mu}A_\mu=0;$$

i.e., $II=0$. This contradiction arose from the assumption that the Gauss mapping (5.16) had rank $2n-k-1$, and so we have established (5.17).

Q.E.D.

(*c*) ON THE EXPLICIT STRUCTURE OF CERTAIN VARIETIES HAVING DEGENERATE TANGENTIAL VARIETIES. — We shall give heuristic reasoning which suggests breaking up n-dimensional manifolds $M \subset \mathbb{P}^N$ having degenerate tangential varieties into n different classes. Within each class if we make a certain general position assumption then it is possible to describe completely the varieties of that class. In low dimensions, say for $n \leqq 3$, this general position assumption will easily be satisfied, and so we obtain a complete list of surfaces and threefolds having degenerate tangential varieties. In the next section this list will be given, and will be extended to fourfolds in paragraph 5(*e*).

The heuristic reasoning is this: given $M \subset \mathbb{P}^N$, as before we consider the incidence correspondence

$$\tilde{T}(M) \subset M \times \mathbb{P}^N \tag{5.21}$$

defined by

$$\tilde{T}(M) = \{(p, q) : q \in \tilde{T}_p(M)\}.$$

Clearly, $\tilde{T}(M)$ is a complex manifold of dimension $2n$ constituting the bundle of abstract tangent projective spaces to M. The tangential map (5.1) is the projection π_2 onto the second factor in (5.21). Assuming that the tangential variety is degenerate the tangential mapping (5.1) will have generic fibre dimension $k \geqq 1$. It follows that for $p \in M$ a generic point

$$\dim \pi_2^{-1}(\tilde{T}_p(M)) = n + k \quad ([52]). \tag{5.22}$$

We note that

$$\pi_2^{-1}(\tilde{T}_p(M)) = \{(q, r) : r \in \tilde{T}_q(M) \cap \tilde{T}_p(M)\},$$

which implies

$$\pi_1^{-1}(q) \cap \pi_2^{-1}(\tilde{T}_p(M)) = \tilde{T}_q(M) \cap \tilde{T}_p(M), \tag{5.23}$$

where π_1 is the projection on the first factor. By (5.22) the image $\pi_1(\pi_2^{-1}(\tilde{T}_p(M)))$ will have some dimension $d \geqq k$. We may then divide the varieties having degenerate tangential varieties into classes having this d as an invariant:

(5.24) *If* $\dim \tau(M) = 2n - k$, *then for generic* $p \in M$:

$$\dim \{ q \in M : \tilde{T}_q(M) \cap \tilde{T}_p(M) \neq \emptyset \} = d \geqq k,$$

and we say that M *is of class* d. *We note that, in this case for generic* q *with* $\tilde{T}_q(M) \cap \tilde{T}_p(M) \neq \emptyset$:

$$\dim (\tilde{T}_q(M) \cap \tilde{T}_p(M)) = n + k - d. \tag{5.25}$$

For example, when $d = k$ the image of π_1 is a k-fold F_p such that for a generic point $q \in F_p$:

$$\dim (\tilde{T}_q(M) \cap \tilde{T}_p(M)) = n.$$

This means that $\tilde{T}_q(M) = \tilde{T}_p(M)$, so that M has class $d = k$ exactly when Gauss mapping (1.60) is degenerate with k-dimensional fibres. Conversely, we have seen in paragraph 5 (a) that these manifolds have degenerate tangential varieties.

([52]) Actually, all we can say is that some component of $\pi_2^{-1}(\tilde{T}_p(M))$ will have dimension $n+k$, but for the purposes of heuristic reasoning leading to the construction of examples we will not insist on this precision.

At the other extreme, when $d=n$ the image of π_1 is all of M and by (5.25) in this case

$$(5.26) \qquad \dim(\tilde{T}_q(M)\cap\tilde{T}_p(M))=k$$

for generic points $p, q\in M$ (if we put $\geqq$ then the statement holds for any $p, q\in M$). We will see in paragraph 6 below [*cf.* the proof of (6.3)] that (5.26) is equivalent to

$$\dim\sigma(M)=2n+1-k,$$

where $\sigma(M)$ is the secant variety of M. Thus:

(5.27) *If* $M\subset\mathbb{P}^N$ *has degenerate tangential variety and is of class n, then the secant variety is also degenerate. The converse is also true.*

We remark that if M has degenerate tangential variety and is of class $d<n$ then the secant variety may be non-degenerate. This happens, e.g., by taking M to be a general developable ruled surface [*cf.* (6.2) below]. On the other hand, there are varieties with degenerate secant varieties but non-degenerate tangential varieties; e.g., the Veronese surface $\mathbb{P}^2\subset\mathbb{P}^5$ [*cf.* paragraph 1(*d*)]. In this connection we should like to call attention to the beautiful global Theorem of Fulton and Hansen [5], a special case of which states:

(5.28) *If* $V\subset\mathbb{P}^N$ *is a smooth projective variety having a degenerate tangential variety, then the secant variety is also degenerate and the two coincide.*

As mentioned in the introduction, it is our opinion that (5.17) should have a global implication, and the only thing we can reasonably think of is that it should pertain to (5.28).

We will now formulate an infinitesimal analogue of (5.24). For this we fix a generic point $p\in M$, choose isomorphisms

$$\begin{cases} T_p(M)\cong\mathbb{C}^n \\ \text{and} \\ N_p(M)\cong\mathbb{C}^r, \qquad r=N-n \end{cases}$$

(then we also have $\mathbb{P}T_p(M)\cong\mathbb{P}^{n-1}$), and consider the 2nd fundamental form as a symmetric mapping [*cf.* (1.18)]:

$$(5.29) \qquad II:\ \mathbb{C}^n\otimes\mathbb{C}^n\to\mathbb{C}^r \quad (^{53}).$$

We shall also denote by Q a general quadric in the linear system $|II|$. As we saw in (5.7) the condition that $\tau(M)$ be degenerate is that for each $v\in\mathbb{C}^n$ the linear map

$$II_v:\ \mathbb{C}^n\to\mathbb{C}^r$$

[53] This mapping was denoted by Q in paragraph 5(*a*). Also the mapping II_v below was denoted by Q_v in paragraph 5(*a*).

given by

$$\mathrm{II}_v(w)=\mathrm{II}(v, w)$$

should have rank $\leqq n-1$.

(5.30) Définition. — *We shall say that we are in case d if there is a $\mathbb{C}^d \subset \mathbb{C}^n$ such that for each $v \in \mathbb{C}^d$ the mapping II_v has rank $\leqq d-1$.*

For example, when $d=1$ there will be a non-zero $v \in \mathbb{C}^n$ such that $Q(v, w)=0$ for all $w \in \mathbb{C}^n$ and all $Q \in |\mathrm{II}|$, i.e., the Gauss mapping is degenerate.

It will be convenient to have an equivalent algebro-geometric formulation of (5.30). If we are in case d then for each $v \in \mathbb{C}^d$ we may choose a basis $Q_1, \ldots, Q_r$ for the quadrics spanning $|\mathrm{II}|$ such that

$$Q_d(v, w)=\ldots=Q_r(v, w)=0 \tag{5.31}$$

for all $w \in \mathbb{C}^n$. The converse is also clearly true, and thus:

(5.32) *We are in case d if, and only if, there is a $\mathbb{P}^{d-1} \subset \mathbb{P}^{n-1}$ such that for each point $v \in \mathbb{P}^{d-1}$ there are ∞^{r-d} quadrics $Q \in |\mathrm{II}|$ that are singular at v.*

Now it is not automatically the case that, if we are in case d with $2 \leqq d \leqq n$, then the tangential variety is degenerate. However, suppose we make the following

(5.33) *General position assumption* [54]*: there are quadrics $Q_1, \ldots, Q_r$ spanning $|\mathrm{II}|$ such that* (5.31) *holds for all $v \in \mathbb{C}^d$ and all $w \in \mathbb{C}^n$. Equivalently, there is a $\mathbb{P}^{d-1} \subset \mathbb{P}^{n-1}$ along which ∞^{r-d} quadrics $Q \in |\mathrm{II}|$ are everywhere singular* [55].

Then we claim any manifold $M \subset \mathbb{P}^N$ whose 2nd fundamental form satisfies (5.33) will have a degenerate tangential variety. Indeed, given any general point $u \in \mathbb{P}^{n-1}$ the linear span $\overline{u\mathbb{P}^{d-1}}$ will be a $\mathbb{P}^d \subset \mathbb{P}^{n-1}$ such that the restriction to this $\mathbb{P}^d$ of the quadrics $Q_d, \ldots, Q_r$ will all be proportional. Consequently, on this $\mathbb{P}^d$ the quadrics $Q \in |\mathrm{II}|$ cut out a linear system of dimension $\leqq d-1$, and hence

$$\iota_{\mathrm{II}}: \quad \mathbb{P}^d \rightarrow \mathbb{P}^{d-1} \subset \mathbb{P}^{r-1}$$

will have a positive dimensional fibre passing through u.

Our structure Theorem is this:

(5.34) *Suppose that $M \subset \mathbb{P}^N$ has a degenerate tangential variety, that we are in case* (d) *of* (5.30), *and that the general position assumption* (5.33) *is satisfied. Then there is a fibration*

$$M \xrightarrow{\pi} B, \qquad \dim B=n-d,$$

---

[54] As mentioned before this general position assumption will always be satisfied in low dimensions. We note also that in case $d=1$ the general position assumption is always verified.

[55] This means that we may choose $Q_d, \ldots, Q_r$ in (5.31) to be independent of $v \in \mathbb{C}^d$.

and a family of linear spaces $\{\mathbb{P}_y^{n+d-1}\}_{y\in B}$ *such that:*

(i) *if we set* $F_y=\pi^{-1}(y)$, *then for* $p\in F_y$:

$$\tilde{T}_p(M)\subset\mathbb{P}_y^{n+d-1};$$

(ii) *for any* $p\in F_y$ *and tangent vector* $w\in T_y(B)$:

$$\frac{d\tilde{T}_p(F_y)}{dw}\subset\mathbb{P}_y^{n+d-1}\quad (^{56});$$

and

(iii) *for some e with* $0\leqq e\leqq n-1$ *there are linear spaces:*

$$\mathbb{P}_y^{n+d-1-e}\subset\mathbb{P}_y^{n+d-1}\subset\mathbb{P}_y^{n+d-1+e}$$

such that for any $p\in F_y$:

$$\begin{cases}\tilde{T}_p^{(2)}(F_y)\subset\mathbb{P}_y^{n+d-1-e},\\ \tilde{T}_p^{(2)}(M)\subset\mathbb{P}_y^{n+d-1+e}.\end{cases}$$

When $d=1$ we exactly recover our description (2.19) of manifolds having degenerate Gauss mapping. At the other extreme, when $d=n$ it follows from (iii) that M lies in a fixed $\mathbb{P}^{2n-1}$. In general, although for $n\geqq 4$ the general position assumption (5.33) may not always be satisfied, using (5.34) we can at least construct lots of manifolds having a degenerate tangential variety.

We will not give the proof of (5.34) here, but will present the argument for the crucial case $d=2$ in paragraph 5(*e*). The general proof is only notationally more complicated.

(*d*) CLASSIFICATION IN LOW DIMENSIONS. — We shall now completely analyze manifolds $M\subset\mathbb{P}^N$ having a degenerate tangential variety when dim $M=2$ or 3, and in the next section we shall discuss the case dim $M=4$ where, for reasons of space, only the sketch of proofs will be given (57). We assume that

$$\dim|II|=r-1,$$

so that we may view the 2nd fundamental form as a rational mapping

(5.35) $$\iota_{II}:\ \mathbb{P}^{n-1}\rightarrow\mathbb{P}^{r-1},$$

whose image W is an algebraic variety with

(5.36) $$\dim W=k\leqq n-2.$$

(56) Note that this makes sense, since by (i) for $p\in F_y$ all $\tilde{T}_p(F_y)\subset\mathbb{P}_y^{n+d-1}$.

(57) We will, however, completely describe *n*-dimensional manifolds of class $d=2$ in the classification scheme (5.30).

Surfaces. — For a surface $S \subset \mathbb{P}^N$ with degenerate tangential variety, (5.35) becomes

$$\iota_{II}: \quad \mathbb{P}^1 \to \mathbb{P}^{r-1}$$

and by (5.36) we must have

$$k=0, \qquad r=1.$$

The 2nd fundamental form is then a single quadric Q on $\mathbb{C}^2$ of rank $\rho=0$, 1, or 2.

If $\rho \leqq 1$, then the Gauss mapping of S is degenerate and by (3.13) S is either $\mathbb{P}^2$, a cone, or a developable ruled surface.

If $\rho=2$, then by (1.69) the second Gauss mapping $\gamma^{(2)}$ is degenerate. We shall prove the stronger assertion that $\gamma^{(2)}$ is constant, so that S lies in a $\mathbb{P}^3$. In a suitable Darboux frame field we will have

$$Q=\omega_1^2+\omega_2^2.$$

If $V \in |III|$ is any cubic, then by the Jacobian condition (1.47):

$$\frac{\partial V}{\partial \omega_1}=\alpha Q, \qquad \frac{\partial V}{\partial \omega_2}=\beta Q.$$

Using equality of mixed partials we immediately find that $\alpha=\beta=0$; i.e., $V=0$ ([58]). Summarizing

(5.37) *If* $S \subset \mathbb{P}^N$ *is a surface with degenerate tangential variety, then either* S *lies in a* $\mathbb{P}^3$, *or else is a cone or a developable ruled surface.*

Threefolds. — We suppose that $M \subset \mathbb{P}^N$ is a threefold having a degenerate tangential variety. Then (5.35) becomes

$$\iota_{II}: \quad \mathbb{P}^2 \to \mathbb{P}^{r-1}, \qquad \text{with image}$$
$$\iota_{II}(\mathbb{P}^2)=W \qquad \text{where} \quad \dim W \leqq 1.$$

The possibilities are:

$$(5.38) \qquad \begin{cases} \dim W=0, & r=1, \\ \dim W=1, & r=2 \text{ or } 3. \end{cases}$$

The second follows by observing that W is the image of a generic $\mathbb{P}^1 \subset \mathbb{P}^2$ under a quadratic map, and hence must be a line or a plane conic.

We shall examine separately the various possibilities in (5.38).

$\dim W=0$, $r=1$. Then the 2nd fundamental form consists of a single quadric Q on $\mathbb{C}^3$ having rank ρ where $0 \leqq \rho \leqq 3$. If $\rho \leqq 2$ then the Gauss mapping of M is degenerate and

([58]) This proof shows in general that if $|II|$ contains a single quadric Q of rank $\geqq 2$, then M lies in a $\mathbb{P}^{n+1}$.

by (2.19) we know its local structure. If $\rho=3$ then by the argument given just above [*cf.* footnote (58)] we conclude that M lies in a $\mathbb{P}^4$.

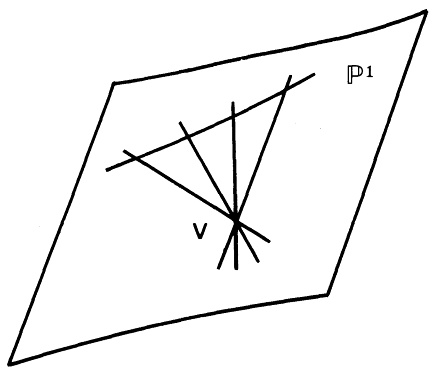
dim $W=1$, $r=2$. In this case W is a line and the restriction of ι_{II} to a generic $\mathbb{P}^1 \subset \mathbb{P}^2$ is either 1 : 1 or 2 : 1. In the first case this restriction must be a linear pencil, so that $|II|$ has a fixed component. Accordingly we shall prove that

(5.39) *If the base* B *of* $|II|$ *has a fixed component, then either the Gauss mapping of* M *is degenerate or else* M *lies in a* $\mathbb{P}^5$ (59).

Proof. — We have that

$$|II| = \mathbb{P}^1 + |\mathbb{P}^1_\lambda|,$$

where $\mathbb{P}^1$ is the fixed component and $|\mathbb{P}^1_\lambda|$ is a linear system of $\mathbb{P}^1$'s, which must then be a pencil since W is itself a line. The picture is



where v is the base point of the pencil $|\mathbb{P}^1_\lambda|$. If $v \in \mathbb{P}^1$, then all quadrics $Q \in |II|$ are singular at v and by (2.6) the Gauss mapping of M is degenerate.

We assume therefore that $v \notin \mathbb{P}^1$. Then in a suitable Darboux frame we will have the equations $\{\omega_3=0\}$ for $\mathbb{P}^1$ and $\{\omega_1=\omega_2=0\}$ for v. It follows that the 2nd fundamental form is spanned by the quadrics

$$\begin{cases} Q_1 = \omega_1 \omega_3, \\ Q_2 = \omega_2 \omega_3. \end{cases}$$

If $V \in |III|$ is any cubic, then according to the Jacobian condition (1.47):

$$\frac{\partial V}{\partial \omega_\alpha} \in |II|, \qquad \alpha = 1, 2, 3.$$

Taking $\alpha=3$ we see that V does not contain the monomials $\omega_1^2\omega_3$, $\omega_1\omega_2\omega_3$, $\omega_2^2\omega_3$, or ω_3^3. Taking $\alpha=1, 2$ we find that V does not contain ω_1^3, $\omega_1^2\omega_2$, ω_2^3, $\omega_2^2\omega_1$, $\omega_1\omega_3^2$, or $\omega_2\omega_3^2$. Thus $V=0$ and $|III|$ is empty. It follows from (1.52) that M lies in a $\mathbb{P}^5$.

Q.E.D.

(59) This is a sharpening of the general result (3.15).

To complete our analysis of the case dim $W=1$, $r=2$ we may assume that $|II|$ is a pencil of conics in $\mathbb{P}^2$ of which a general curve is smooth. There will be three singular members of this pencil, and we begin by showing that:

(5.40) *If no* $Q\in|II|$ *is a double line then* M *lies in a* $\mathbb{P}^5$.

Proof. – Since a generic $Q\in|II|$ is smooth we may assume that this pencil is spanned by (60):

$$(5.41)\qquad \begin{cases} Q_1=\omega_1^2+\omega_2^2+\omega_3^2,\\ Q_2=\lambda_1\,\omega_1^2+\lambda_2\,\omega_2^2+\lambda_3\,\omega_3^2.\end{cases}$$

If $V\in|III|$ is any cubic, then by (1.47):

$$(5.42)\qquad \begin{cases} \dfrac{\partial V}{\partial\omega_1}=\alpha_1\,Q_1+\beta_1\,Q_2,\\[2mm] \dfrac{\partial V}{\partial\omega_2}=\alpha_2\,Q_1+\beta_2\,Q_2,\\[2mm] \dfrac{\partial V}{\partial\omega_3}=\alpha_3\,Q_1+\beta_3\,Q_2.\end{cases}$$

If $\alpha_1=\alpha_2=\alpha_3=0$ then we may assume that $\beta_1\neq 0$. From (5.41) we have

$$\frac{\partial^2 V}{\partial\omega_2\,\partial\omega_1}=2\,\beta_1\,\lambda_2\,\omega_2,$$

$$\frac{\partial^2 V}{\partial\omega_1\,\partial\omega_2}=2\,\beta_2\,\lambda_1\,\omega_1,$$

which implies that $\lambda_2=0=\beta_2\,\lambda_1$. Similarly $\lambda_3=0=\beta_3\,\lambda_1$. But then $\lambda_1\neq 0$ and $\beta_2=\beta_3=0$, so $Q_2=\lambda_3\,\omega_3^2$ and $|II|$ contains the double line $\omega_3=0$.

If we assume that say $\alpha_1\neq 0$, then using (5.42) and equating mixed partials gives

$$\alpha_1\,\omega_2+\beta_1\,\lambda_2\,\omega_2=\alpha_2\,\omega_1+\beta_2\,\lambda_1\,\omega_1,$$

$$\alpha_1\,\omega_3+\beta_1\,\lambda_3\,\omega_3=\alpha_3\,\omega_1+\beta_3\,\lambda_1\,\omega_1.$$

This implies that

$$\alpha_1+\beta_1\,\lambda_2=0=\alpha_2+\beta_2\,\lambda_1,$$

$$\alpha_1+\beta_1\,\lambda_3=0=\alpha_3+\beta_3\,\lambda_1.$$

(60) This is just the simultaneous diagonalization of two quadratic forms Q_1 and Q_2 on $\mathbb{P}^{n-1}$, where Q_1 is assumed smooth. To prove this we consider the n roots $\{t_\alpha\}$ of $\det(Q_1+t\,Q_2)$. In general these roots are distinct and there will be a unique point $v_\alpha\in\mathbb{P}^{n-1}$ satisfying $(Q_1+t_\alpha\,Q_2)(v_\alpha, v_\alpha)=0$. In fact, the referee points out that v_α is the unique singular point of $Q_1+t_\alpha\,Q_2$, so that $(Q_1+t_\alpha\,Q_2)(v_\alpha, x)=0$ for all $x\in\mathbb{P}^{n-1}$. Then the $\{v_\alpha\}$ give a basis relative to which Q_1 and Q_2 are diagonalized. The case when some of the t_α are repeated is obtained by specialization of the generic situation. If, however, $\det(Q_1+t\,Q_2)\equiv 0$ then it may not be possible to simultaneously diagonalize the forms.

Thus $\beta_1 \lambda_2 = \beta_1 \lambda_3 \neq 0$ since $\alpha_1 \neq 0$, so $\lambda_2 = \lambda_3 = \lambda$. Replacing Q_2 by $Q_2 - \lambda Q_1$ gives

$$Q_1 = \omega_1^2 + \omega_2^2 + \omega_3^2,$$
$$Q_2 = \mu \omega_1^2,$$

so that again $|II|$ contains a double line.

In conclusion, if no $Q \in |II|$ is a double line then $|III|$ is empty, which is just what we wanted to show.

Q.E.D.

It remains to analyze the case where $|II|$ contains a double line. Here we will find the following special case of the structure Theorem (5.34) when $d=2$.

(5.43) *In case* $M \subset \mathbb{P}^N$ *is a threefold such that at a generic point the 2nd fundamental form* $|II|$ *is a pencil of conics of which a general curve is smooth but where one member is a double line, then there is a curve* B *in* $\mathbb{P}^N$ *with parameter* y *and in each osculating* 4-*plane* $\mathbb{P}_4^{(y)}$ *a surface* S_y *such that* $M = \bigcup_{y \in B} S_y$.

Proof. — We shall use the ranges of indices

$$0 \leqq j \leqq N; \qquad 1 \leqq \alpha,\ \beta \leqq 3; \qquad 4 \leqq \mu,\ \nu \leqq 5; \qquad 6 \leqq s \leqq N,$$

and consider frames $\{ A_0; A_\alpha; A_\mu; A_s \}$ adapted to the filtration

$$\mathbb{C}.p \subset \tilde{T}_p(M) \subset \tilde{T}_p^{(2)}(M) \subset \mathbb{C}^{N+1}.$$

We may assume that $|II|$ is spanned by the quadrics [*cf.* (5.41)]:

$$\begin{cases} Q_1 = \omega_1^2 + \omega_2^2, \\ Q_2 = \omega_3^2. \end{cases}$$

Letting A_4 correspond to Q_1 and A_5 to Q_2 this means that

$$(5.44) \qquad \begin{cases} \omega_{14} = \omega_1, & \omega_{24} = \omega_2, & \omega_{34} = 0, \\ \omega_{15} = 0, & \omega_{25} = 0, & \omega_{35} = \omega_3. \end{cases}$$

We shall show that the distribution $\{ \omega_3 = 0 \}$ is completely integrable and gives the required fibering $\{ S_y \}$ of M.

By exterior differentiation of $\omega_{3s} = 0$ and using (5.44) we obtain

$$0 = \sum_j \omega_{3j} \wedge \omega_{js} = \omega_{35} \wedge \omega_{5s},$$

which implies that

$$(5.45) \qquad \omega_{5s} = \rho_s \omega_3, \qquad 6 \leqq s \leqq N.$$

From (5.44) we also have

$$d\omega_3 = d\omega_{35} = \omega_{33} \wedge \omega_{35} + \omega_{35} \wedge \omega_{55} \equiv 0 \bmod \omega_3,$$

which proves the complete integrability of the distribution $\{\omega_3 = 0\}$. Finally, from exterior differentiation of $\omega_{1s} = 0 = \omega_{2s}$ and (5.44):

$$0 = \sum_j \omega_{1j} \wedge \omega_{js} = \omega_1 \wedge \omega_{4s},$$

$$0 = \sum_j \omega_{2j} \wedge \omega_{js} = \omega_2 \wedge \omega_{4s},$$

which implies that

(5.46) $$\omega_{4s} = 0.$$

By (5.45) and (5.46) we may set

$$\{A_0, A_1, \ldots, A_4\} = \mathbb{P}_y^4,$$

$$\{A_0, A_1, \ldots, A_5\} = \mathbb{P}_y^5,$$

meaning that the linear spaces on the left are constant along the fibres S_y of the foliation $\omega_3 = 0$. Moreover, by (5.46):

$$\frac{d}{dy}(\mathbb{P}_y^4) \subseteq \mathbb{P}_y^5,$$

so that applying our structure Lemma (2.2) for curves in a Grassmannian we arrive at (5.43).

Q.E.D.

Our analysis of threefolds having degenerate tangential varieties will be completed by showing that:

(5.47) *The case* $\dim W = 1$, $r = 3$ *cannot occur.*

Proof. — If it did, then by a generic projection π of the plane conic onto a line we arrive at a commutative diagram

$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\ {}^t\mathrm{II}\ } & W \\ & {\scriptstyle\lambda}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \mathbb{P}^1 & \end{array}$$

where λ is given by a generic pencil from $|\mathrm{II}|$. Since $|\mathrm{II}|$ contains smooth conics, a generic fibre $\lambda^{-1}(v)$ is irreducible. But this contradicts the fact that π is 2-to-1.

Q.E.D.

Summarizing:

(5.48) *If* $M \subset \mathbb{P}^N$ *is a threefold having a degenerate tangential variety, then either* (i) M *lies in a* $\mathbb{P}^5$; (ii) M *has a degenerate Gauss mapping, or* (iii) M *is a threefold of the type* $d=2$, $e=1$ *described by* (5.43).

We note that these are the cases $d=3$, $d=1$ and $d=2$ in the classification scheme (5.30).

(*e*) HIGHER DIMENSIONAL VARIETIES. — We shall discuss general $M \subset \mathbb{P}^N$ having degenerate tangential varieties with special emphasis on fourfolds. We begin showing that:

(5.49) *In the classification scheme* (5.30), *when* $d=2$ *and the* $\mathbb{C}^2$ *is not contained in the base locus of* $|II|$ *then the general position assumption* (5.33) *is automatically satisfied.*

Proof. — We consider the 2nd fundamental form as a symmetric mapping (5.29) and shall identify a $\mathbb{C}^k \subset \mathbb{C}^n$ with the corresponding $\mathbb{P}^{k-1} \subset \mathbb{P}^{n-1}$. We assume that the $\mathbb{P}^1$ along which II_v has rank $\leqq 1$ is spanned by v_1 and v_2. Since, by assumption, $II(v, v) \not\equiv 0$ for $v \in \mathbb{P}^1$ we may assume that

$$II(v_1, v_1) \neq 0, \qquad II(v_1, v_2) \neq 0, \qquad II(v_2, v_2) \neq 0.$$

If we set $w_1 = II(v_1, v_1)$, then from

$$II(v_1, v_1) \wedge II(v_1, v) = 0,$$
$$II(v_2, v_2) \wedge II(v_2, v) = 0,$$

for all $v \in \mathbb{P}^{n-1}$ it follows that all the vectors $II(v_1, v)$ and $II(v_2, v)$ are multiples of w_1. This is equivalent to (5.33).

Q.E.D.

Remark. — This also explains why we referred to (5.33) as a general position assumption. If we have any of the cases d with $2 \leqq d \leqq n$, and if the general position assumption (5.33) is *not* satisfied, then there must be a further degeneracy.

We will now given the proof of (5.34) in the case $d=2$. The additional ranges of indices

$$1 \leqq a, b \leqq 2; \qquad 3 \leqq \rho, \sigma \leqq n; \qquad n+2 \leqq s \leqq N$$

will be used. With v_1 and v_2 as in the proof of (5.49) and, with A_{n+1} projecting onto $w_1 \in N_p(M) = \mathbb{C}^{N+1} / \tilde{T}_p(M)$, we have

$$II(v_1, v) \equiv 0 \bmod w_1,$$
$$II(v_2, v) \equiv 0 \bmod w_1,$$

for all v, and hence

$$(5.50) \qquad \begin{cases} dA_1 \equiv 0 \bmod A_0, \ldots, A_{n+1}, \\ dA_2 \equiv 0 \bmod A_0, \ldots, A_{n+1}. \end{cases}$$

We will prove that the distribution

(5.51) $$\omega_3 = \ldots = \omega_n = 0$$

is completely integrable; this will give the fibration in (5.34). With the usual notation

$$\omega_{\alpha s} = \sum_{\beta} q_{\alpha\beta s}\,\omega_\beta, \qquad q_{\alpha\beta s} = q_{\beta\alpha s},$$

we have from (5.50) that

$$q_{a\beta s} = 0,$$
$$q_{\rho b s} = q_{b\rho s} = 0.$$

Denoting by $\bar{\omega}$ a form considered modulo $\omega_3, \ldots, \omega_n$ it follows that

(5.52) $$\begin{cases} \omega_{as} = 0, \\ \bar{\omega}_{\rho s} = 0. \end{cases}$$

Taking the exterior derivative of the first equation gives

$$0 = \sum_{\rho} \omega_{a\rho} \wedge \omega_{\rho s} + \omega_{a,\,n+1} \wedge \omega_{n+1,\,s}.$$

By (5.52) we obtain

$$\bar{\omega}_{a,\,n+1} \wedge \bar{\omega}_{n+1,\,s} = 0.$$

It is easy to see that $\bar{\omega}_{1,\,n+1}$ and $\bar{\omega}_{2,\,n+1}$ are linearly independent [this follows from the proof of (5.49)]. Thus

(5.53) $$\bar{\omega}_{n+1,\,s} = 0.$$

We now define a mapping

$$\mu: \quad \mathrm{M} \to \mathbb{G}(n+1, \mathrm{N})$$

by sending $p \in \mathrm{M}$ to the linear space

(5.54) $$\mathbb{P}_p^{n+1} = \overline{\mathrm{A}_0, \ldots, \mathrm{A}_{n+1}}.$$

From (5.52) and (5.53) we infer that the differential of μ is zero on v_1 and v_2. On the other hand, since were are not in the case $d=1$, the Gauss mapping of M is non-degenerate and hence the forms $\omega_{\rho s}$ must span $\omega_3, \ldots, \omega_n$. Consequently the fibres of μ are just the leaves of the foliation defined by (5.51), which is then completely integrable.

We denote this fibering by $\{\mathrm{S}_y\}_{y \in \mathrm{B}}$ where $\dim \mathrm{B} = n-2$, and we write $\mathbb{P}_y^{n+1}$ for the linear space (5.54). At each point $p \in \mathrm{S}_y$ we set $\mathbb{P}_p^2 = \{\mathrm{A}_0, \mathrm{A}_1, \mathrm{A}_2\} = \tilde{\mathrm{T}}_p(\mathrm{S}_y)$. From (5.52) we have for any $w \in \mathrm{T}_p(\mathrm{M})$:

(5.55) $$\frac{d\mathbb{P}_p^2}{dw} \subset \mathbb{P}_y^{n+1}.$$

At this juncture we have proved (i) and (ii) in (5.34).

We shall now establish (iii). The additional index range $3 \leqq \lambda \leqq n+1$ will be used. Then the exterior derivative of the first equation in (5.52) gives

(5.56) $$\sum_{\lambda} \omega_{a\lambda} \wedge \omega_{\lambda s} = 0,$$

while from the second equation and (5.53) we have

(5.57) $$\omega_{\lambda s} = \sum_{\rho} b_{\lambda\rho s}\, \omega_{\rho}.$$

Plugging (5.57) in (5.56) yields

$$\sum_{\rho} \Big(\sum_{\lambda} \omega_{a\lambda}\, b_{\lambda\rho s}\Big) \wedge \omega_{\rho} = 0.$$

By the Cartan Lemma

$$\sum_{\lambda} \omega_{a\lambda}\, b_{\lambda\rho s} \equiv 0 \bmod \omega_{\rho}.$$

Suppose that among these equations there are e which are linearly independent. Using the index range

(5.58) $$3 \leqq \xi \leqq n+1-e; \qquad n+2-e \leqq \tau \leqq n+1,$$

we may make a linear change of the A_{λ}'s to have

(5.59) $$\bar{\omega}_{a\tau} = 0.$$

Taking the exterior derivatives of these equations gives

$$\sum_{\xi} \bar{\omega}_{a\xi} \wedge \bar{\omega}_{\xi\tau} = 0, \qquad a = 1, 2.$$

By the usual argument these equations imply that

(5.60) $$\bar{\omega}_{\xi\tau} = 0.$$

It follows from (5.59) and (5.60) that the span of $\{A_0, \ldots, A_{n+1-e}\}$ is constant along the fibres S_y, and hence gives a $\mathbb{P}_y^{n+1-e}$ such that

$$\bar{d}\, \mathbb{P}_y^2 \subset \mathbb{P}_y^{n+1-e}$$

where $\bar{d} = d \,|\, S_y$. Finally, from (5.57) we see that $\dim(\mathbb{P}_y^{n+1} + d\,\mathbb{P}_y^{n+1}) \leqq n+e+1$, which completes the proof of (iii) in (5.34).

Q.E.D.

By similar arguments it is possible to extend (5.49) to the case $d=3$ and $n=4$, the result being that all possible four-folds with degenerate tangential varieties and falling in the cases $d=1$, 2, or 3 of (5.30) appear in the classification provided by (5.34).

For example, let us describe a typical M when $d=3$ and $e=1$. For this we should be given a general curve C in $\mathbb{P}^N$ $(N \geqq 8)$ with osculating k-planes $\mathbb{P}_y^{(k)}$. In each $\mathbb{P}_y^{(5)}$ we should be given a threefold

$$F_y \subset \mathbb{P}_y^{(5)},$$

and then

$$M = \bigcup_{y \in C} F_y.$$

We observe that for any $p \in F_y$ and tangent vector $v \in T_p(M)$:

$$\begin{cases} \dfrac{d\tilde{T}_p(F_y)}{dv} \subset \mathbb{P}_y^{(6)}, \\ \dfrac{d\tilde{T}_p(M)}{dv} \subset \mathbb{P}_y^{(7)}. \end{cases}$$

It follows that $\tau(M) \subset \bigcup_{y \in C} \mathbb{P}_y^{(6)}$, and consequently M has a degenerate tangential variety.

As a special case we let $V = \bigcup_{y \in C} \mathbb{P}_y^{(5)}$ be the ruled variety traced out by the osculating 5-planes to the curve C. In $\mathbb{P}^N$ we generically choose two algebraic hypersurfaces H_1 and H_2, and then

$$M = V \cap H_1 \cap H_2$$

will have a degenerate tangential variety. If C is an algebraic curve and H_1, H_2 are of sufficiently high degree, then by adjunction it is easy to see that the Kodaira number $\varkappa(M)=4$ is maximal. Thus there are no restrictions, such as (3.21), imposed on varieties having a degenerate tangential variety.

When $d=n=4$ the argument used to prove (5.34) breaks down, and then it becomes necessary to directly examine the possible linear systems of quadrics satisfying (5.32). What we are interested in, then, are linear systems of quadrics $|Q|$ on $\mathbb{P}^3$ that satisfy these conditions:

(i) $\dim |Q| = r-1 \geqq 3$ [61];

(ii) the rational mapping

$$i_Q: \quad \mathbb{P}^3 \to \mathbb{P}^{r-1}$$

has positive dimensional fibres; and

(iii) if for each $v \in \mathbb{P}^3$ we denote by $S(v)$ the linear subsystem of quadrics singular at v, then

$$\dim S(v) = r-3.$$

[61] The case when $r \leqq 3$ may be treated by considerations similar to those in paragraph 5(*a*) and in the proof of (1.69).

It follows from (A.10) that $|Q|$ has a non-empty base B, and we will list the possible base loci such that $|Q|$ is a subsystem of the complete linear system of all quadrics with base locus B. It is necessary here to be somewhat more precise and give the base by an ideal I, and with this understood here is the list:

$\dim\|\mathcal{O}_{\mathbb{P}^3}(2)\otimes I\|$	I
5	$2p$
3	$2p+q_1+q_2$, q_1, q_2 distinct
3	$2p+q_1+q_2$, q_2 infinitely near q_1
3	l_1+l_2, l_1, l_2 disjoint
3	l_1+l_2, l_1 infinitely near l_2

Here p and q_i are points in $\mathbb{P}^3$, the l_j are lines, and "infinitely near" means on the appropriate blowup of $\mathbb{P}^3$. We observe that in the first three cases the Gauss mapping is degenerate, since all quadrics are singular at p [*cf.* (2.6)]. We also note that, by (1.31), in the fourth case M has the same second fundamental form of the Segre variety $\mathbb{P}^2\times\mathbb{P}^2\subset\mathbb{P}^8$ [we suspect that under these circumstances M must be a piece of the Segre variety—*cf.* (6.18) below].

In summary:

(5.61) *If* $M\subset\mathbb{P}^N$ *is a fourfold having a degenerate tangential variety, then either the structure of* M *is given by* (5.34) *above or else* M *has the same second fundamental form as the Segre variety* $\mathbb{P}^2\times\mathbb{P}^2\subset\mathbb{P}^8$ (*including a degeneration of this second fundamental form*).

In case M is a complete algebraic variety we may draw a global conclusion by observing that if M is described by (5.34) then either M lies in a $\mathbb{P}^7$ or else it is singular [*cf.* the proof of (2.29)]. Consequently:

(5.62) *If* $M\subset\mathbb{P}^N$ *is a four-dimensional algebraic variety having a degenerate tangential variety, then either* M *lies in a* $\mathbb{P}^7$, *or it is singular, or finally it has the same second fundamental form as the Segre variety* (*including degenerations of this*).

6. Varieties with degenerate secant varieties

(*a*) THE SECANT VARIETY. — The final type of degeneracy we shall consider is that of the secant variety $\sigma(M)$ of $M\subset\mathbb{P}^N$. Intuitively, $\sigma(M)$ is the union of all lines $\overline{pq}$ where $p, q\in M$. In general

$$\dim\sigma(M)=\min(N, 2n+1),$$

and we shall say that M has a degenerate secant variety in case $\dim\sigma(M)<\min(N, 2n+1)$.

To formulate this more precisely we denote by $\widetilde{M\times_\Delta M}$ the blow-up of $M\times M$ along the diagonal $\Delta\subset M\times M$. A point in $\widetilde{M\times_\Delta M}$ consists either of a pair (p, q) of distinct points of M, or else in the limiting case as $q \to p$ of a point $p \in M$ and tangent direction $v \in \mathbb{P}T_p(M)$. The involution $(p, q) \to (q, p)$ extends to $\widetilde{M \times_\Delta M}$ with the blown up diagonal as fixed point divisor, and we let $\tilde{M}^{(2)}$ be the quotient by this action.

We may view $\tilde{M}^{(2)}$ as a smooth model of the 2-fold symmetric product $M^{(2)}$ of M. Associated to each point in $\tilde{M}^{(2)}$ there is an obvious projective line in $\mathbb{P}^N$, and we let $\tilde{S}(M)$ be the abstract union of these $\mathbb{P}^1$'s. Then $\tilde{S}(M)$ is a complex manifold of dimension $2n+1$, that we may think of as the abstract secant variety of M. There is a tautological map

$$\sigma: \tilde{S}(M) \to \mathbb{P}^N, \tag{6.1}$$

and the image is the *secant variety* $\sigma(M)$. We shall now express the condition that M have degenerate secant variety in terms of the differential σ_* of the mapping (6.1).

(6.2) *Assuming that* $N \geqq 2n+1$ *the submanifold* $M \subset \mathbb{P}^N$ *has a degenerate secant variety if, and only if,*

$$\tilde{T}_p(M) \cap \tilde{T}_q(M) \neq \emptyset \tag{6.3}$$

for any $p, q \in M$.

Proof. — Over $(p, q) \in M \times M$ with $p \neq q$ we consider frames

$$\{A_i; B_j\},$$

where $\{A_i\}$ is a Darboux frame over p and $\{B_j\}$ is a Darboux frame lying over q. The secant map is described by

$$\sigma(p, q, t) = A_0 + t B_0.$$

We shall use the notations

$$\left\{\begin{array}{ll} & dA_i = \sum_j \omega_{ij} A_j, \\ & dB_j = \sum_k \varphi_{jk} B_k, \\ \omega_{0\alpha} = \omega_\alpha, & \varphi_{0\beta} = \varphi_\beta, \qquad 1 \leqq \alpha, \beta \leqq n. \end{array}\right.$$

Then

$$\begin{aligned} d(A_0 + tB_0) &= dA_0 + dt B_0 + t dB_0 \\ &= \omega_0 A_0 + (dt + t\varphi_0) B_0 + \sum_\alpha \omega_\alpha A_\alpha + \sum_\beta \varphi_\beta B_\beta \\ &\equiv (dt - t\omega_0 + t\varphi_0) B_0 + \sum_\alpha \omega_\alpha A_\alpha + \sum_\beta \varphi_\beta B_\beta \bmod (A_0 + tB_0). \end{aligned}$$

Since the $2n+1$ one-forms

$$dt - t\omega_0 + t\varphi_0; \quad \omega_\alpha; \quad \varphi_\beta$$

are independent on $\tilde{S}(M)$, the mapping (6.1) fails to have maximal rank exactly when the $2n+2$ vectors

$$A_0 + tB_0; \quad B_0; \quad A_\alpha; \quad B_\beta$$

fail to be linearly independent, which is just the condition (6.3).

Q.E.D.

For future use it will be convenient to write (6.3) in the form

$$(6.4) \qquad A_0 \wedge A_1 \wedge \ldots \wedge A_n \wedge B_0 \wedge B_1 \wedge \ldots \wedge B_n \equiv 0.$$

Example. — The most famous example of a degenerate secant variety is the *Veronese surface* $\mathbb{P}^2 \subset \mathbb{P}^5$. We recall from paragraph 1 (*c*) that the embedding is given by the complete linear system $|\mathcal{O}_{\mathbb{P}^2}(2)|$ of plane conics, and that the 2nd fundamental form at $p \in \mathbb{P}^2$ is the linear system of conics that are singular at p, which is then identified with the conics on $\mathbb{P}\,T_p(\mathbb{P}^2) \cong \mathbb{P}^1$. It follows, e.g., from (5.7), that the tangential map is everywhere of maximal rank and $\tau(M)$ is a fourfold in $\mathbb{P}^5$.

On the other hand, for $p \neq q$ the conic $2\,\overline{pq}$ is a tangent hyperplane to $\mathbb{P}^2$ at both p and q, so that $\tilde{T}_p(\mathbb{P}^2)$ and $\tilde{T}_q(\mathbb{P}^2)$ both lie in a $\mathbb{P}^4$ and (6.3) is satisfied. Consequently, $\sigma(\mathbb{P}^2)$ is degenerate.

(*b*) CONDITIONS FOR THE DEGENERACY OF THE SECANT VARIETY. — We assume that $N \geqq 2n+1$ and that $M \subset \mathbb{P}^N$ has a degenerate secant variety. We will expand the condition (6.4) in a power series around the diagonal. More precisely, for any $p \in M$ and any analytic arc $\{p(t)\} \subset M$ with $p(0) = p$ we choose Darboux frames $\{A_i(t)\}$ lying over $p(t)$ and set $A_i = A_i(0)$. Then by (6.4):

$$(6.5) \qquad (A_0 \wedge A_1 \wedge \ldots \wedge A_n) \wedge (A_0(t) \wedge A_1(t) \wedge \ldots \wedge A_n(t)) \equiv 0.$$

The first term on the left that is not identically zero is the coefficient of t^{n+3}, and we shall geometrically interpret its vanishing. The conclusion is given in (6.12) below.

For this some preliminary discussion is necessary. Recall that for any two vectors in $T_p(M) \cong \tilde{T}_p(M)/\mathbb{C}.A_0$, say A_1 and A_2, the 2nd fundamental form

$$II(A_1, A_2) \in \mathbb{C}^{N+1}/\tilde{T}_p(M)$$

has the interpretation ([62]):

$$(6.6) \qquad II(A_1, A_2) = \frac{dA_1}{dv_2} \equiv \frac{dA_2}{dv_1} \bmod \tilde{T}_p(M) \equiv \frac{d^2 A_0}{dv_1\, dv_2} \bmod \tilde{T}_p(M).$$

([62]) Recall that $v_\alpha \in T_p(M)$ corresponds to $A_\alpha \in \tilde{T}_p(M)$.

Precisely, this has the following meaning:

First we extend v_1 and v_2 to vector fields in a neighborhood of p. Then, by construction

$$(6.7)\qquad \begin{cases} \dfrac{dA_0}{dv_1} \equiv A_1 \bmod A_0, \\ \dfrac{dA_0}{dv_2} \equiv A_2 \bmod A_0, \end{cases}$$

and

$$(6.8)\qquad \frac{dA_1}{dv_2} \equiv \frac{dA_2}{dv_1} \bmod \tilde{T}_p(M).$$

This last relation follows from noting that

$$\frac{dA_1}{dv_2} - \frac{dA_2}{dv_1} = v_2(v_1 A_0) - v_1(v_2 A_0) = [v_2, v_1] A_0 \in \tilde{T}_p(M),$$

where $[v_2, v_1]$ is the Poisson bracket of vector fields. It follows that the conditions expressed by (6.7) and (6.8) depend only on the values of the vector fields v_1 and v_2 at p, and together they establish the symmetry (6.6) of the 2nd fundamental form.

Coming now to the 3rd fundamental form as defined in paragraph 1 (d), we denote by $\tilde{T}_p^{(2)}(M)$ the 2nd osculating space and recall that

$$\mathrm{III}(A_1, A_2, A_3) \in \mathbb{C}^{N+1}/\tilde{T}_p^{(2)}(M)$$

defined by

$$(6.9)\qquad \mathrm{III}(A_1, A_2, A_3) \equiv \frac{d^3 A_0}{dv_1\, dv_2\, dv_3} \bmod \tilde{T}_p^{(2)}(M)$$

represents a symmetric tri-linear form on $T_p(M)$. Here, of course we are working in an open set where $\dim T_p^{(2)}(M)$ is constant. The point we wish to make here is that when $A_1 = A_2$ we may refine (6.9). More precisely, we shall show:

For any $A_1, A_2 \in \tilde{T}_p(M)$ *the vector*

$$(6.10)\qquad \frac{d^3 A_0}{dv_1^2\, dv_2} \in \mathbb{C}^{N+1}/\tilde{T}_p(M) + \mathrm{II}(A_1, T_p(M))$$

is intrinsically defined.

In other words, if we choose vector fields v_1 and v_2 as above, then $d^3 A_0/dv_1^2\, dv_2$ considered modulo the linear space

$$\tilde{T}_p(M) + \left\{ \text{vectors } \frac{d^2 A_0}{dv_1\, dv} \text{ where } v \in T_p(M) \right\}$$

depends only on the values of v_1 and v_2 at p. This is verified in the same manner as just below (6.8).

We shall use the notation

$$(6.11)\qquad \widehat{\mathrm{III}}(\mathrm{A}_1, \mathrm{A}_1, \mathrm{A}_2) = \frac{d^3 \mathrm{A}_0}{dv_1^2\, dv_2} \bmod \tilde{\mathrm{T}}_p(\mathrm{M}) + \mathrm{II}(\mathrm{A}_1, \mathrm{T}_p(\mathrm{M})).$$

Then (6.10) has the following consequence:

(6.12) *The condition*

$$\widehat{\mathrm{III}}(\mathrm{A}_1, \mathrm{A}_1, \mathrm{A}_2) \equiv 0 \bmod \tilde{\mathrm{T}}_p(\mathrm{M}) + \mathrm{II}(\mathrm{A}_1, \mathrm{T}_p(\mathrm{M}))$$

has intrinsic meaning. If it is satisfied at a generic point of M *then the* 3*rd fundamental form* $\mathrm{III}=0$ *and* M *lies in* $\tilde{\mathrm{T}}_p^{(2)}(\mathrm{M})$ *for any* $p \in \mathrm{M}$.

We now make the series expansion of the left hand side of (6.5). For this it will be convenient to use the notations

$$\begin{cases} \Lambda_0 = \mathrm{A}_0 \wedge \mathrm{A}_1 \wedge \ldots \wedge \mathrm{A}_n, \\ \mathrm{A}_1(t) = \dfrac{d\mathrm{A}_0}{dt}, \\ \mathrm{A}_k(t) = \displaystyle\sum_\lambda \frac{d^\lambda \mathrm{A}_k}{dt^\lambda} \frac{t^\lambda}{\lambda\,!}. \end{cases}$$

where $d^\lambda \mathrm{A}_k/dt^\lambda$ is understood to be evaluated at $t=0$. Then clearly

$$\mathrm{A}_0(t) \equiv \frac{d^2 \mathrm{A}_0}{dt^2} \cdot \frac{t^2}{\alpha\,!} + \frac{d^3 \mathrm{A}_0}{dt^3} \cdot \frac{t^3}{\alpha\,!} + \ldots \bmod \Lambda_0,$$

$$\mathrm{A}_\alpha(t) \equiv \frac{d\mathrm{A}_\alpha}{dt} . t + \frac{d^2 \mathrm{A}_\alpha}{dt^2} \frac{t^2}{2\,!} + \ldots \bmod \Lambda_0.$$

Moreover, by (6.6) and (6.10):

$$\begin{cases} \dfrac{d^2 \mathrm{A}_0}{dt^2} \equiv \dfrac{d\mathrm{A}_1}{dt} \bmod \Lambda_0, \\ \dfrac{d^3 \mathrm{A}_0}{dt^3} \equiv \dfrac{d^2 \mathrm{A}_1}{dt^2} \bmod \Lambda_0 + \dfrac{d\tilde{\mathrm{T}}_0(\mathrm{M})}{dt}. \end{cases}$$

The left hand side of (6.5) is

$$(6.13)\qquad \left(\frac{1}{3\,!} - \frac{1}{2\,!}\right)\left[\Lambda_0 \wedge \frac{d^3 \mathrm{A}_0}{dt^3} \wedge \frac{d\mathrm{A}_1}{dt} \wedge \ldots \wedge \frac{d\mathrm{A}_n}{dt}\right] t^{n+3} + \ldots$$

The term in brackets is $(-1)^n$ times

$$(6.14)\qquad \tilde{T}_p(M)\wedge II(A_1, A_1)\wedge \ldots \wedge II(A_1, A_n)\wedge \widehat{III}(A_1, A_1, A_1).$$

Suppose now that we assume the tangent variety $\tau(M)$ is non-degenerate, so that by (5.3) for generic choice of A_0 and A_1 ([63]):

$$\tilde{T}_p(M)\wedge II(A_1, A_1)\wedge \ldots \wedge II(A_1, A_n)\neq 0.$$

Assuming the degeneracy of the secant variety it follows from (6.13) and (6.14) that

$$\widehat{III}(A_1, A_1, A_1)\equiv 0 \bmod \tilde{T}_p(M)+v_1.\tilde{T}_p(M).$$

By changing the arc $p(t)$ at the second order this becomes

$$\widehat{III}(A_1, A_1, A)\equiv 0 \bmod \tilde{T}_p(M)+v_1.\tilde{T}_p(M)$$

for any $A\in\tilde{T}_p(M)$. Summarizing:

(6.15) *If* $M\subset\mathbb{P}^N$ *has a degenerate secant variety but non-degenerate tangential variety, then* $\widehat{III}(A, A, B)=0$ *for all* $A, B\in\tilde{T}(M)$.

In particular, from (6.12):

(6.16) *If* $M\subset\mathbb{P}^N$ *satisfies the conditions of* (6.15), *then* M *lies in a* $\mathbb{P}^m$ *where* $m=\dim\tilde{T}^{(2)}(M)$ *for a generic* $p\in M$.

Observe that in any case $m\leqq n(n+3)/2$, so that from (6.16) and (5.37):

(6.17) *If a surface* $S\subset\mathbb{P}^N$ *has a degenerate secant variety, then either* S *is a cone, the tangential ruled surface of a curve in* $\mathbb{P}^4$, *or else* S *lies in a* $\mathbb{P}^5$.

In the next sub-section we shall characterize the last case.

(*c*) CHARACTERIZATION OF THE VERONESE SURFACE. — The conclusions (6.16) and (6.17) only made use of the 3rd fundamental form and not of the refinement $\widehat{III}$ described by (6.10). To illustrate how this may be utilized we shall prove:

(6.18) *Suppose that* $S\subset\mathbb{P}^5$ *is a surface with degenerate secant variety and non-degenerate tangential variety. Then either* S *lies in* $\mathbb{P}^4$ *or else is a piece of the Veronese surface.*

([63]) I.e., at a generic point on the blown up diagonal $\mathbb{P}(T(M))$ in $\widetilde{M\times_\Delta M}$.

The proof of this result is the most intricate of any in this paper, and will be given in several steps ([64]). The basic idea is to show that, under the assumptions (6.18) and that S does not lie in a $\mathbb{P}^4$, we may canonically introduce a projective connection whose paths turn out to be plane curves. At this point one way to proceed is by showing that this projective connection is flat so that S is an open set in $\mathbb{P}^2$ with its natural projective connection ([65]). We shall proceed somewhat differently, and shall normalize our frames so that the uniqueness proposition B.2 may be applied.

Step one. — We will compute the Maurer-Cartan matrix for Darboux frames on the Veronese surface. The special features of this matrix will then serve as a guide as to how the Maurer-Cartan matrix of our unknown surface should look. We use frames $\{A, B, C\}$ on $\mathbb{P}^2$ and shall write

$$(6.19)\qquad \begin{cases} dA \equiv \frac{1}{2}\omega_1 B + \frac{1}{2}\omega_2 C \bmod A, \\ dB \equiv \omega_{12} C \bmod A, B, \\ dC \equiv \omega_{21} B \bmod A, C. \end{cases}$$

Darboux frames for the Veronese surface $\mathbb{P}^2 \subset \mathbb{P}^5$ are described by the relations

$$(6.20)\qquad \begin{cases} A_0 = A^2, \\ A_1 = AB, \quad A_2 = AC, \\ A_3 = B^2, \quad A_4 = BC, \quad A_5 = C^2 \end{cases}$$ ([66]).

With the notations (6.19) and (6.20) the Maurer-Cartan matrix is

$$(6.21)\qquad \begin{array}{c} \begin{matrix} A^2 & AB & AC & B^2 & BC & C^2 \end{matrix} \\ \begin{bmatrix} \omega_{00} & \omega_1 & \omega_2 & 0 & 0 & 0 \\ \omega_{10} & \omega_{11} & \omega_{12} & \frac{1}{2}\omega_1 & \frac{1}{2}\omega_2 & 0 \\ \omega_{20} & \omega_{21} & \omega_{22} & 0 & \frac{1}{2}\omega_1 & \frac{1}{2}\omega_2 \\ 0 & 2\omega_{10} & 0 & \omega_{33} & 2\omega_{12} & 0 \\ 0 & \omega_{10} & \omega_{21} & \omega_{21} & \omega_{44} & \omega_{12} \\ 0 & 0 & 2\omega_{20} & 0 & 2\omega_{21} & \omega_{55} \end{bmatrix} \begin{matrix} A^2 \\ AB \\ AC \\ B^2 \\ BC \\ C^2 \end{matrix} \end{array}.$$

([64]) The referee points out that the corresponding global Theorem, stating that the only non-degenerate smooth algebraic surface in $\mathbb{P}^5$ having a degenerate secant variety is the Veronese, was proved by Severi (*Rendiconti di Palermo*, Vol. 15, 1901, pp. 33-51). His proof is easy — and correct.

([65]) The strategy of this argument would be analogous to the main Theorem in the recent paper *Abel's Theorem and Webs* (*Jahr. d. Deut. Math.-Verein.*, Vol. 80, 1978, pp. 13-100) by S. S. Chern and one of us.

([66]) Recall that $\mathbb{P}^2 = \mathbb{P}(E)$ and $\mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}^2 E)$ where E is a 3-dimensional vector space.

We note the factors of 2 and 1/2 in this matrix.

Step two. — By our assumptions both $\tau(S)$ and $\sigma(S)$ have dimension four. From (5.7) we have $\dim|II|\geqq 1$, and if $\dim|II|=1$ then from (6.15) we infer that $III\equiv 0$ and S lies in a $\mathbb{P}^4$. Therefore we may assume that $\dim|II|=2$, and the 3 quadrics spanning the 2nd fundamental form may be taken to be

$$\begin{cases} Q_1=\dfrac{1}{2}\omega_1^2, \\ Q_2=\dfrac{1}{2}\omega_1\omega_2, \\ Q_3=\dfrac{1}{2}\omega_2^2. \end{cases}$$

Choosing corresponding Darboux frames for S in $\mathbb{P}^5$ we have

$$\begin{cases} 2\omega_{13}=\omega_1, & \omega_{23}=0, \\ 2\omega_{14}=\omega_2, & 2\omega_{24}=\omega_1, \\ \omega_{15}=0, & 2\omega_{25}=\omega_2 \end{cases} \tag{6.22}$$

([67]).

Taking exterior derivatives gives

$$\begin{cases} 0=2\,d\omega_{23}=\omega_{21}\wedge\omega_1+\omega_1\wedge\omega_{43}+\omega_2\wedge\omega_{53}, \\ 0=2\,d\omega_{15}=\omega_{12}\wedge\omega_2+\omega_1\wedge\omega_{35}+\omega_2\wedge\omega_{45}, \end{cases} \tag{6.23}$$

and

$$\begin{cases} d\omega_1 \;=\omega_{00}\wedge\omega_1+\omega_1\wedge\omega_{11}+\omega_2\wedge\omega_{21}, \\ \quad\| \\ 2\,d\omega_{13}=\omega_{11}\wedge\omega_1+\omega_1\wedge\omega_{33}+\omega_2\wedge\omega_{43}, \\ \quad\| \\ 2\,d\omega_{24}=\omega_{21}\wedge\omega_2+\omega_{22}\wedge\omega_1+\omega_1\wedge\omega_{44}+\omega_2\wedge\omega_{54}; \end{cases} \tag{6.24 a}$$

$$\begin{cases} d\omega_2 \;=\omega_{00}\wedge\omega_2+\omega_1\wedge\omega_{12}+\omega_2\wedge\omega_{22}, \\ \quad\| \\ 2\,d\omega_{14}=\omega_{11}\wedge\omega_2+\omega_1\wedge\omega_{34}+\omega_{12}\wedge\omega_1+\omega_2\wedge\omega_{44}, \\ \quad\| \\ 2\,d\omega_{25}=\omega_{22}\wedge\omega_2+\omega_1\wedge\omega_{45}+\omega_2\wedge\omega_{55}. \end{cases} \tag{6.24 b}$$

If we make no further assumptions on $S\subset\mathbb{P}^5$ then we must use the Cartan Lemma in (6.23) and (6.24) to define the higher order invariants required to uniquely determine the position of S in $\mathbb{P}^5$. In paragraph B(*b*) we shall go through a simpler analogous procedure for hypersurfaces in $\mathbb{P}^{n+1}$.

([67]) Note that these relations are satisfied by the matrix (6.21).

Step three. — We now use the condition $\widehat{\mathrm{III}}=0$. With the preceding notations

$$
(6.25)\quad \begin{cases} \dfrac{dA_0}{dv_1} \equiv A_1 \bmod A_0, \\[2mm] \dfrac{2\, d^2 A_0}{dv_1^2} \equiv A_3 \bmod A_0,\ A_1,\ A_2, \\[2mm] \dfrac{2\, d^2 A_0}{dv_1\, dv_2} \equiv A_4 \bmod A_0, A_1,\ A_2. \end{cases}
$$

From (6.10) the vectors

$$
\frac{d^3 A_0}{dv_1^3}, \qquad \frac{d^3 A_0}{dv_1^2\, dv_2} \in \mathbb{C}^6 / \overline{A_0,\ A_1,\ A_2,\ A_3,\ A_4},
$$

are well-defined, and by (6.15) they are zero (this is the $\widehat{\mathrm{III}}\equiv 0$ condition). From (6.25) we obtain the list of the three equations

$$
(6.26)\quad \begin{cases} \omega_{35}=0, \\ \omega_{53}=0, \\ \omega_{33}-2\,\omega_{44}+\omega_{55}=\omega_{34}-2\,\omega_{43}-2\,\omega_{45}+\omega_{54}. \end{cases}
$$

The second follows by interchanging v_1 and v_2, and the third by using v_1+v_2 in place of v_1. Plugging the first two equations from (6.26) into (6.23) gives

$$
(6.27)\quad \begin{cases} \omega_{12}-\omega_{45}\equiv 0 \bmod \omega_2, \\ \omega_{21}-\omega_{43}\equiv 0 \bmod \omega_1. \end{cases}
$$

Taking exterior derivatives of the first two equations in (6.26) we obtain

$$
(6.28)\quad \begin{cases} 0=2\, d\omega_{35}=\omega_{32}\wedge\omega_2+2\,\omega_{34}\wedge\omega_{45}, \\ 0=2\, d\omega_{53}=\omega_{51}\wedge\omega_1+2\,\omega_{54}\wedge\omega_{43}. \end{cases}
$$

Note that (6.26) and its consequences are satisfied by the Maurer-Cartan matrix (6.21) of the Veronese surface.

Step four. — This step contains the main geometric idea. We begin by showing that:

(6.29) *On the manifold* $\mathscr{F}(S)$ *of all Darboux frames the distribution*

$$
(6.30)\quad \omega_2=\omega_{12}=0
$$

is completely integrable.

Proof. — From (6.24 *b*) we obtain

$$
d\omega_2 \equiv 0 \bmod \omega_2,\ \omega_{12}.
$$

On the other hand, using (6.22):

$$2\,d\omega_{12} = 2\,\omega_{10}\wedge\omega_2 + 2\,\omega_{11}\wedge\omega_{12} + 2\,\omega_{12}\wedge\omega_{22} + \omega_1\wedge\omega_{32} + \omega_2\wedge\omega_{42}$$
$$\equiv \omega_1\wedge\omega_{32} \bmod \omega_2,\ \omega_{12}.$$

By the first equations in (6.28) and (6.27):

$$\omega_{32}\equiv 0 \bmod \omega_2,\quad \omega_{45}\equiv 0 \bmod \omega_2,\ \omega_{12},$$

which is what we wanted to prove.

Q.E.D.

By the Frobenius Theorem through each point of $\mathscr{F}(S)$ there passes a unique integral manifold of the foliation (6.30). This manifold projects onto a curve Γ in S. Moreover, given any point $p\in S$ and direction $v\in\mathbb{P}\,T_p(S)$ there is a unique such curve with initial data (p, v). The system of paths $\{\Gamma\}$ is defined by a system of 2nd order differential equations, and therefore defines a projective connection on S [68]. We next shall prove:

(6.31) *Each curve* Γ *lies in a* $\mathbb{P}^2$

Proof. — We denote by $\bar{\omega}$ a differential form ω considered modulo ω_2 and ω_{12}, and by $\bar{d}$ the restriction of differentiation to the integral manifolds of (6.30). Then

$$\bar{d}A_0\equiv\bar{\omega}_1 A_1 \bmod A_0;$$
$$\bar{d}^2 A_0\equiv\bar{\omega}_1\,\bar{d}A_1 \bmod A_0,\ A_1\equiv\frac{1}{2}\bar{\omega}_1\,\bar{\omega}_{13}\,A_3 \bmod A_0,\ A_1;$$

and

$$\bar{d}^3 A_0\equiv\frac{1}{2}\bar{\omega}_1\,\bar{\omega}_{13}\,\bar{d}A_3 \bmod A_0,\ A_1,\ A_3\equiv 0 \bmod A_0,\ A_1,\ A_3$$

since by (6.26), (6.28), and (6.27):

$$\omega_{35}=0,\qquad \bar{\omega}_{34}=\bar{\omega}_{32}=0.$$

It follows that the 2-plane $\overline{A_0 A_1 A_3}$ is constant along the leaves of the foliation (6.30).

Q.E.D.

Now, as previously indicated, we could at this point seek to establish by a computation that the projective connection is flat, so that S is a piece of $\mathbb{P}^2$ with the curves Γ corresponding to lines. However, we shall proceed somewhat differently using a very special case of the Cartan-Kähler Theorem to construct a submanifold $\tilde{\mathscr{F}}\subset\mathscr{F}(S)$ along which the Maurer-Cartan matrix has the form (6.21).

[68] *cf.* the reference cited in footnote [65] for further discussion of projective connections.

Step five. — We recall that $\mathscr{F}(S)$ is an integral manifold of the differential system.

$$(6.32)\quad \begin{cases} 2\,\omega_{13}=\omega_1, \quad \omega_{23}=0; \\ 2\,\omega_{14}=\omega_2, \quad 2\,\omega_{24}=\omega_1; \\ \omega_{15}=0, \quad 2\,\omega_{25}=\omega_2; \\ \omega_{35}=\omega_{53}=0, \\ \omega_{33}-2\,\omega_{44}+\omega_{55}=\omega_{34}-2\,\omega_{43}-2\,\omega_{45}+\omega_{54}. \end{cases}$$

We will now prove that:

(6.33) *Over any point $p\in S$ there are frames satisfying*

$$\omega_{32}=0, \quad \omega_{51}=0,$$
$$\omega_{45}=\omega_{12}, \quad \omega_{34}=2\,\omega_{12},$$
$$\omega_{43}=\omega_{21}, \quad \omega_{54}=2\,\omega_{21}.$$

Proof. — Frames satisfying only (6.32) are not uniquely determined by specifying A_0, A_1, A_2, but rather the normal vectors A_3, A_4, A_5 are determined only modulo A_0, A_1, A_2 subject to preserving the last two equations in (6.32). From (6.28) and (6.27) we have

$$\omega_{32}=\alpha_1\,\omega_2+\beta_1\,\omega_{12}.$$

A substitution

$$(6.34\,a)\quad A_3\to A_3-\alpha_1\,A_0-\beta_1\,A_1$$

preserves the condition $\omega_{35}=0$ since $\omega_{15}=0$, and if we make this substitution then at the new frame $\omega_{32}=0$. If we also multiply A_2 and A_4 by a suitable common factor then the last equation in (6.32) will be preserved. Similarly, a substitution

$$(6.34\,b)\quad A_5\to A_5-\alpha_2\,A_0-\beta_2\,A_2$$

preserves the condition $\omega_{53}=0$, and since $\omega_{51}=\alpha_2\,\omega_1+\beta_2\,\omega_{21}$ we may arrange that $\omega_{51}=0$. From (6.27) we have

$$\begin{cases} \omega_{45}=\omega_{12}+\gamma_1\,\omega_2, \\ \omega_{43}=\omega_{21}+\gamma_2\,\omega_1, \end{cases}$$

so that under a substitution

$$(6.34\,c)\quad A_4\to A_4-2\,\gamma_1\,A_2-2\,\gamma_2\,A_1,$$

we obtain from $2\,\omega_{25}=\omega_2$ and $2\,\omega_{13}=\omega_1$ that $\omega_{45}=\omega_{12}$ and $\omega_{43}=\omega_{21}$. The proof will be completed by showing that the remaining equations

$$\begin{cases} \omega_{34}=2\,\omega_{12}, \\ \omega_{54}=2\,\omega_{21}, \end{cases}$$

are automatically satisfied.

By (6.28) we have

$$\omega_{34}\wedge\omega_{45}=0 \quad\Rightarrow\quad \omega_{34}=\lambda\omega_{12},$$
$$\omega_{54}\wedge\omega_{43}=0 \quad\Rightarrow\quad \omega_{54}=\mu\omega_{21}.$$

Multiplying the last two equations in (6.24 *b*) by ω_2 gives

$$\omega_1\wedge\omega_2\wedge(\omega_{34}-\omega_{12}-\omega_{45})=0,$$

which then implies that $\lambda=2$. Similarly, $\mu=2$ follows from (6.24 *a*).

Q.E.D.

At this point the frames A_3, A_4, A_5 are uniquely specified by A_0, A_1, A_2 up to a substitution

(6.35) $$A_4\rightarrow A_4+\gamma A_0.$$

In other words we have found a 10-dimensional integral manifold $\hat{\mathscr{F}}(S)$ of the differential system (6.32) and (6.33). On $\hat{\mathscr{F}}(S)$ we have

(6.36) $$\begin{cases} 0=d\omega_{32}=\omega_{30}\wedge\omega_2+\omega_{31}\wedge\omega_{12}+2\,\omega_{12}\wedge\omega_{42}, \\ 0=d\omega_{51}=\omega_{50}\wedge\omega_1+\omega_{52}\wedge\omega_{21}+2\,\omega_{21}\wedge\omega_{41}, \end{cases}$$

in addition to the equations obtained by differentiation of the last two relations in (6.33), which are

(6.37) $$\begin{cases} 2\,d\omega_{12}=2\,\omega_{10}\wedge\omega_2+2\,\omega_{11}\wedge\omega_{12}+2\,\omega_{12}\wedge\omega_{22}+\omega_2\wedge\omega_{42}, \\ \quad\| \\ 2\,d\omega_{45}=\omega_{42}\wedge\omega_2+2\,\omega_{44}\wedge\omega_{45}+2\,\omega_{45}\wedge\omega_{55}, \\ \quad\| \\ d\omega_{34}=\dfrac{1}{2}\,\omega_{31}\wedge\omega_2+\omega_{33}\wedge\omega_{34}+\omega_{34}\wedge\omega_{44}, \end{cases}$$

and with a set involving ω_{21}, ω_{43}, and ω_{54}. If we subtract the third equation from the second we obtain

(6.38) $$(4\,\omega_{44}-2\,\omega_{55}-2\,\omega_{33})\wedge\omega_{12}+(2\,\omega_{42}-\omega_{31})\wedge\omega_2=0.$$

By (6.33) and the last equation in (6.26) the first term in (6.38) is zero; i.e.,

$$2\,\omega_{42}-\omega_{31}\equiv 0 \mod \omega_2.$$

From this relation, the first equation in (6.36), and the Cartan Lemma we have

$$\omega_{30}\equiv 0 \mod \omega_{12}.$$

It follows that $\omega_{30}=0$, and then again by the first equation in (6.36):

$$2\,\omega_{42}=\omega_{31}. \tag{6.39}$$

Similarly we hawe

$$\begin{cases} \omega_{50}=0, \\ 2\,\omega_{41}=\omega_{52}. \end{cases} \tag{6.40}$$

Taking the exterior derivatives of these equations and using the first two equations in (6.37) gives using (6.40):

$$\begin{cases} \omega_{31}=2\,\omega_{42}=2\,\omega_{10}, \\ \omega_{52}=2\,\omega_{41}=2\,\omega_{20}. \end{cases} \tag{6.41}$$

Finally, using a substitution (6.35) we may eliminate ω_{40}.

Summarizing, we have determined a 9-dimensional submanifold $\tilde{\mathscr{F}}$ (S) of $\mathscr{F}$ (S) on which the Maurer-Cartan matrix satisfies (6.32), (6.33), (6.41), and their consequences obtained by exterior differentiation. In other words, the Maurer-Cartan matrix now has exactly the form (6.21), and then our result follows from the uniqueness result B.2.

Q.E.D.

APPENDIX A

SOME RESULTS FROM ALGEBRA AND ALGEBRAIC GEOMETRY

We shall collect here the results from multi-linear algebra and algebraic geometry that are needed in our study.

(*a*) VARIANTS OF THE CARTAN LEMMA. — In its simplest form this Lemma states:

If $\varphi_1, \ldots, \varphi_m$ *are linearly independent vectors in a vector space* T^*, *and if* $\psi_1, \ldots, \psi_m \in T^*$ *satisfy an equation*

$$\sum_{j=1}^{m} \varphi_j \wedge \psi_j = 0, \tag{A.1}$$

then

$$\begin{cases} \psi_j=\sum_k q_{jk}\,\varphi_k, \\ \quad\text{where} \\ q_{jk}=q_{kj}. \end{cases} \tag{A.2}$$

A simple variant of this concerns the quadratic equation (A.1) but when we only assume that $\varphi_1, \ldots, \varphi_l$ $(l \leqq m)$ are independent. The conclusion then is

$$\text{(A.3)} \qquad \psi_1, \ldots, \psi_l \equiv 0 \bmod \{ \psi_{l+1}, \ldots, \psi_m; \varphi_1, \ldots, \varphi_l \}.$$

To prove this we use the ranges of indices

$$1 \leqq j, k \leqq m; \qquad 1 \leqq \alpha, \beta \leqq l; \qquad l+1 \leqq \rho, \sigma \leqq m,$$

and write

$$\varphi_\rho = \sum_\alpha A_{\rho\alpha} \varphi_\alpha.$$

Then (A.1) becomes

$$\sum_\alpha \varphi_\alpha \wedge \theta_\alpha = 0,$$

where

$$\theta_\alpha = \psi_\alpha + \sum_\rho A_{\rho\alpha} \psi_\rho.$$

Applying the usual Cartan Lemma (A.2) to the θ_α gives (A.3).

A more interesting variant concerns a system of equations

$$\text{(A.4)} \qquad \sum \varphi_{\alpha j} \wedge \psi_j = 0, \qquad \alpha = 1, \ldots, l.$$

(A.5) *If we assume that the coefficient matrix* $\{ \varphi_{\alpha j} \}$ *has linearly independent columns in the sense that*

$$\sum_j \varphi_{\alpha j} A_j = 0 \quad \Rightarrow \quad A_1 = \ldots = A_m = 0,$$

then if the $\{ \psi_j \}$ *satisfy* (A.4) *it follows that*

$$\psi_1, \ldots, \psi_m \equiv 0 \bmod \{ \varphi_{11}, \ldots, \varphi_{l1}, \ldots, \varphi_{1m}, \ldots, \varphi_{lm} \}.$$

Proof. — Let $\omega_1, \ldots, \omega_k$ be a basis for the subspace of T^* spanned by the $\varphi_{\alpha j}$, and write

$$\varphi_{\alpha j} = \sum_{\lambda=1}^{k} g_{\alpha\lambda j} \omega_{\lambda j}.$$

By adding $\omega_{k+1}, \ldots, \omega_n$ complete $\omega_1, \ldots, \omega_k$ to a basis for T^*, and use the additional ranges of indices

$$1 \leqq \lambda, \quad \mu \leqq k; \qquad k+1 \leqq \rho \leqq n.$$

Writing

$$\psi_j \equiv \sum_\rho h_{j\rho} \omega_\rho \bmod \omega_1, \ldots, \omega_k$$

we infer from (A.4) that

$$\sum_{\lambda, j, \rho} g_{\alpha\lambda j} h_{j\rho} \omega_\lambda \wedge \omega_\rho \equiv 0 \bmod \omega_\lambda \wedge \omega_\mu$$

This implies

$$\sum_j g_{\alpha\lambda j} h_{j\rho} = 0,$$

which by our assumption gives $h_{j\rho} = 0$. Q.E.D.

(*b*) LINEAR SYSTEMS ([69]). – On a projective space $\mathbb{P}^m$ with homogeneous coordinates $[X_0, X_1, \ldots, X_m]$ a *linear system* is given by a vector space E of homogeneous polynomials F (X) of fixed degree d. The associated projective space is denoted by $|E|$; one pictures $|E|$ as the family of hypersurfaces V_F defined by the equation $F(X) = 0$ where $F \in E$.

The *base locus* $B = B(E)$ of the linear system is the intersection $\cap V_F$ of all the hypersurfaces $V_F \in |E|$. If $F_0, \ldots, F_r$ is a basis for E, then the expected dimension of the base is $m - r - 1$. In much of our study we are concerned with linear systems having non-generic behaviour, such as dim $B(E) \geqq m - r$.

An important property of linear systems is:

(A.6) *Bertini's Theorem: For any linear system* $|E|$ *the generic member* $V_F \in |E|$ *is smooth outside the base locus* B(E).

We will not prove this here, but it is useful to see just why it should be true by considering the rational mapping

(A.7) $$\iota_E: \quad \mathbb{P}^m \to \mathbb{P}^r$$

defined by the linear system. In coordinates

$$\iota_E(X) = [F_0(X), \ldots, F_r(X)],$$

where $F_0, \ldots, F_r$ is a basis of E. Geometrically, $\iota_E(X)$ is the hyperplane consisting of all $F \in E$ such that $F(X) = 0$. This makes sense only for points outside the base locus, so that (A.7) may be regarded as originating from a holomorphic mapping ([70]):

(A.8) $$\mathbb{P}^m - B(E) \to \mathbb{P}^r.$$

To prove Bertini's Theorem we consider the analogous map

$$\mathbb{P}^m - B(F, G) \to \mathbb{P}^1$$

defined by choosing two generic forms $F, G \in E$; the line they span is called a *pencil* with base $B(F, G) = \{X: F(X) = G(X) = 0\}$. Bertini's Theorems for a pencil follows from Sard's Theorem applied to the map defined by this pencil and the general form by considering the general pair (F, L) of a form in E and a pencil containing it.

([69]) As a reference for notation and terminology in algebraic geometry we take [9].

([70]) One customarily says that ι_E is defined outside the base locus B(E).

It is also useful to observe that:

(A.9) *The image of a rational mapping* (A.7) *is an algebraic subvariety of* $\mathbb{P}^r$.

This follows by considering the graph

$$\Gamma \subset \mathbb{P}^m \times \mathbb{P}^r$$

of the mapping (A.7), which is defined by

$$\Gamma = \{(X, Y) \in \mathbb{P}^m \times \mathbb{P}^r : Y_\alpha = F_\alpha(X_0, \ldots, X_n) \text{ for } 0 \leqq \alpha \leqq r\}.$$

The graph is an algebraic subvariety of the product, and so its projection $\pi_2(\Gamma)$ onto the factor $\mathbb{P}^r$ is an algebraic subvariety. On the other hand, over a point $X \in \mathbb{P}^m - B(E)$ there is a unique point of Γ, and this implies that $\pi_2(\Gamma)$ is the closure of the image of the holomorphic mapping (A.8), thereby establishing (A.9).

Q.E.D.

The argument also gives that the image is an irreducible variety.

Another useful fact for us is:

(A.10) *If the image* V *of the rational mapping* (A.7) *has dimension* $\dim V = k$, *then*

$$\dim B(E) \geqq m - k - 1 \quad ({}^{71}).$$

Proof. – If $\dim B(E) \leqq m-k-2$ then a generic intersection $B(E) \cap \mathbb{P}^{k+1}$ will be empty. In this case, restricting the rational mapping to $\mathbb{P}^{k+1}$ will give a *holomorphic* mapping

$$f: \quad \mathbb{P}^{k+1} \to V.$$

If $\omega \in H^2(V, \mathbb{Z})$ is the restriction to V of the hyperplane class on $\mathbb{P}^r$ then

$$f^* \omega = \text{Const. } \varphi$$

is a positive multiple of the hyperplane class $\varphi \in H^2(\mathbb{P}^{k+1}, \mathbb{Z})$. But then

$$0 \neq (f^* \omega)^{k+1} = f^*(\omega^{k+1}) = 0$$

contradicts our assumption $\dim B(E) \leqq m-k-2$.

Q.E.D.

A consequence of (A.10) is that if the rational mapping (A.7) has positive-dimensional fibres; i.e., if through any point of $\mathbb{P}^m$ there is at least a curve on which all the polynomials F_α

[71] This conclusion will be applied in the form: $\dim B(E) \geqq \dim F - 1$ where F is a generic fibre of the rational mapping ι_E.

are proportional, then the base is non-empty. In fact we can say more:

(A.11) *Any irreducible, positive-dimensional component* W *of a fibre meets the base locus* B(E) [72].

Proof. — We may assume that along W all $F_\alpha(X)$ are multiples of $F_0(X)$. But then since $\dim W \geqq 1$ the intersection

$$W \cap \{X: F_0(X)=0\}$$

will be non-empty and contained in the base.

Q.E.D.

There are several other notions associated to linear systems E that turn up in our discussion of local differential geometry. One is the *singular set* S(E) associated to E, defined as the set of points $X \in \mathbb{P}^m$ where *all* V_F are singular. By Bertini's Theorem we have

$$S(E) \subset B(E).$$

Especially noteworthy is the case when E consists of quadratic polynomials F(X); then the singular set of each V_F is a linear space [73] and consequently:

(A.12) *If* $|E|$ *consists of quadrics, then* S(E) *is a linear space.*

The second notion is that of a *sub-linear system*, which as the terminology suggests is the linear system defined by a linear subspace E′ of E. We note the obvious relations

$$\text{(A.13)} \qquad \begin{cases} B(E) \subset B(E'), \\ S(E) \subset S(E'), \\ \text{fibres of } \iota_E \subset \text{fibres of } \iota_{E'}. \end{cases}$$

The final and most important notion is that of the *Jacobian system* $\mathcal{G}(E)$, defined to be the linear system generated by all partial derivatives $(\partial F/\partial X_\alpha)(X)$ where $F(X) \in E$. Using Euler's formula

$$\text{(A.14)} \qquad (\deg F)\, F = \sum_{\alpha=0}^{m} X_\alpha \frac{\partial F}{\partial X_\alpha}$$

we deduce the relation

$$\text{(A.15)} \qquad B(\mathcal{G}(E)) = S(E).$$

---

[72] By definition a fibre of the rational map (A.7) is the closure in $\mathbb{P}^m$ of a fibre of the holomorphic mapping (A.8).

[73] Recall that in a suitable coordinate system any quadratic polynomial has the form

$$F(X) = X_0^2 + \ldots + X_{\rho-1}^2,$$

where ρ is the *rank* of the quadric Q defined by F. From this equation we see that Q is a cone over a smooth quadric in $\mathbb{P}^{\rho-1}$ with the linear space $\{X_0 = \ldots = X_{\rho-1} = 0\} \cong \mathbb{P}^{m-\rho}$ as vertex. This vertex is also the singular locus of Q.

Proof. — The left hand side is defined by the equations

$$\frac{\partial F}{\partial X_\alpha}(X)=0,$$

which by (A.14) are the same as those defining S(E).

Q.E.D.

APPENDIX B

Some observations on moving frames

(*a*) Regarding the general philosophy. — Underlying the use of moving frames are two elementary Lemmas and a general algorithmic procedure, both of which we shall now briefly comment on. In the next section we will illustrate the algorithmic procedure by some examples.

The elementary Lemmas concern mappings of a connected manifold B into a Lie group G ([73]). They are valid in a C^∞ or holomorphic setting, and so it is not necessary to be specific on this point. We view the Maurer-Cartan forms on G collectively as a 1-form φ with values in the Lie algebra $\mathcal{G}$ of G and write the Maurer-Cartan equation as

$$d\varphi=\frac{1}{2}[\varphi, \varphi]. \tag{B.1}$$

Given a mapping

$$f:\quad B\rightarrow G$$

the pullback form $\varphi_f=f^*\varphi$ determines f, up to a rigid motion, in the following sense:

(B.2) *For a pair of mappings* $f, \tilde{f}$: $B\rightarrow G$

$$f(x)=g.\tilde{f}(x),\qquad x\in B,$$

for some fixed $g\in G$ *if, and only if,*

$$\varphi_f=\varphi_{\tilde{f}}.$$

In addition to the uniqueness there is the following existence statement:

(B.3) *On a simply-connected manifold* B, *a* $\mathcal{G}$*-valued* 1*-form* ψ *is* $f^*\varphi$ *for some* f: $B\rightarrow G$ *if, and only if,*

$$d\psi=\frac{1}{2}[\psi, \psi].$$

([73]) The proofs are given in [7].

Given a closed subgroup $H \subset G$ we consider mappings

$$f: \quad M \to G/H$$

into the corresponding homogeneous space. The method of moving frames seeks to canonically associate an essentially unique lifting, to be called a moving frame,

$$\text{(B.4)} \qquad \begin{array}{ccc} & & G \\ & \overset{F}{\nearrow} & \downarrow \\ f: & M \xrightarrow{f} & G/H \end{array}$$

to B. To this lifting we may then apply (B.2) to obtain a complete set of invariants for f. Assuming, for example, that f is real-analytic then it may be proved that such a lifting exists in a neighborhood of a general point $p \in M$. In fact, in [6] and [10] this method of moving frames is extensively discussed from both a theoretical and practical point of view when M is one-dimensional. In this Appendix we should like to comment on the case when M is higher-dimensional, so that integrability conditions intervene, and when G/H is a projective space with G being the full projective group.

(*b*) We shall show how to attach a moving frame to a hypersurface $M \subset \mathbb{P}^{n+1}$ in a neighborhood where the 2nd fundamental form is non-degenerate. The idea is at each step to normalize the Maurer-Cartan matrix of a lifting F as much as possible, and then take the exterior derivative of the equations expressing this normalization, thereby leading to the next step.

The first step in finding the lifting (B.4) comes by restricting to Darboux frames (1.12), which we recall are defined by [*cf.* (1.16)],

$$\omega_{n+1} = 0.$$

Taking the exterior derivative of this equation and using the Cartan Lemma as in paragraph 1 (*a*) gives

$$\text{(B.5)} \qquad \omega_{\alpha, n+1} = \sum_{\beta} q_{\alpha\beta}\, \omega_\beta, \qquad q_{\alpha\beta} = q_{\beta\alpha}.$$

This leads to the 2nd fundamental form

$$Q = \sum_{\alpha, \beta} q_{\alpha\beta}\, \omega_\alpha\, \omega_\beta.$$

A general change of Darboux frame is given by

$$\tilde{A}_0 = \lambda\, A_0,$$
$$\tilde{A}_\alpha = \sum_{\beta} g_{\alpha\beta}\, A_\beta + h_\alpha\, A_0,$$
$$\tilde{A}_{n+1} = \mu\, A_{n+1} + \sum_{\alpha} \xi_\alpha\, A_\alpha + \xi\, A_0,$$

and it is easy to see that under this change

$$\tilde{Q}=\gamma . G Q^t G,$$

where $G=(g_{\alpha\beta})$ and $\gamma \neq 0$ [*cf.* the proof of (B.10) below]. It follows that the only invariant of Q is its rank $\rho(Q)$. If at a generic point $\rho(Q)<n$, then we have seen in (3.14) that M is the dual of a lower-dimensional submanifold in $\mathbb{P}^{n+1^*}$. So we assume that $\rho(Q)=n$, and now normalize our frame so that

$$Q=\sum_{\alpha} \omega_{\alpha}^{2}; \tag{B.6}$$

i. e., by (B.5):

$$\omega_{\alpha, n+1}=\pm \omega_{\alpha}. \tag{B.7}$$

Restricting to such frames constitutes the second step in finding the lifting (B.4).

We remark that if $\{A_i\}$ and $\{\tilde{A}_i\}$ are two frames for which (B.7) is satisfied, then [*cf.* the proof of (B.10)]:

$$(\sum_{\alpha} \omega_{\alpha}^{2})=\lambda(\sum_{\alpha} \tilde{\omega}_{\alpha}^{2}), \qquad \lambda \neq 0,$$

so that at this stage what is intrinsically defined is a field of non-degenerate quadrics in $\mathbb{P} T_p(M)$, or equivalently a conformal structure.

Taking the exterior derivative of the equations (B.7) using the plus sign there and the structure equations (1.5) gives

$$d\omega_{\alpha}=\omega_{00} \wedge \omega_{\alpha}+\sum_{\beta} \omega_{\beta} \wedge \omega_{\beta\alpha}=\sum_{\beta}(\delta_{\alpha\beta} \omega_{00}-\omega_{\beta\alpha}) \wedge \omega_{\beta};$$

$$d\omega_{\alpha, n+1}=\sum_{\beta} \omega_{\alpha\beta} \wedge \omega_{\beta, n+1}+\omega_{\alpha, n+1} \wedge \omega_{n+1, n+1}=\sum_{\beta}(\omega_{\alpha\beta}-\delta_{\alpha\beta} \omega_{n+1, n+1}) \wedge \omega_{\beta}.$$

Setting $d\omega_{\alpha}=d\omega_{\alpha, n+1}$ and subtracting we obtain

$$\sum_{\beta}[\delta_{\alpha\beta}(\omega_{00}+\omega_{n+1, n+1})-(\omega_{\alpha\beta}+\omega_{\beta\alpha})] \wedge \omega_{\beta}=0.$$

By the Cartan Lemma (A.2) this implies

$$\delta_{\alpha\beta}(\omega_{00}+\omega_{n+1, n+1})-(\omega_{\alpha\beta}+\omega_{\beta\alpha})=\sum_{\gamma} v_{\alpha\beta\gamma} \omega_{\gamma}, \tag{B.8}$$

where $v_{\alpha\beta\gamma}=v_{\alpha\gamma\beta}$. But the left hand side of (B.8) is also symmetric in α and β, and consequently

$$V=\sum_{\alpha, \beta, \gamma} v_{\alpha\beta\gamma} \omega_{\alpha} \omega_{\beta} \omega_{\gamma} \tag{B.9}$$

is a cubic differential form.

The next step is to see how V transforms when we change to another admissible frame, and for this we shall prove:

If $\{\tilde{A}_i\}$ *is another Darboux frame satisfying* (B.7), *then*

$$\tilde{V}=\mu V+LQ, \qquad \mu\neq 0, \tag{B.10}$$

where L *is a linear form:*

Proof. — One way is to proceed by a "straightforward" calculation, but this seems to be rather messy. So we shall proceed somewhat differently in a manner that should shed additional light on the cubic form V.

In $\mathbb{P}^{n+1}$ we consider homogeneous coordinate systems $[X_0, \ldots, X_{n+1}]$ such that

$$\begin{cases} p=[1, 0, \ldots, 0], \\ \tilde{T}_p(M)=\{X_{n+1}=0\}. \end{cases} \tag{B.11}$$

The associated affine coordinate system is

$$x_1=X_1/X_0, \ \ldots, \ x_{n+1}=X_{n+1}/X_0;$$

in this coordinate system p is the origin and $\tilde{T}_p(M)$ is the hyperplane $x_{n+1}=0$.

Using the index range $1\leqq\alpha, \beta\leqq n$, the most general linear change of homogeneous coordinates preserving (B.11) is given by

$$\begin{cases} X_{n+1}=\mu\, Y_{n+1}, \\ X_\alpha=\sum_\beta g_{\alpha\beta}\, Y_\beta+\rho_\alpha\, Y_{n+1}, \\ X_0=\xi\, Y_0+\xi\sum_\alpha \sigma_\alpha\, Y_\alpha+\tau\xi\, Y_{n+1}, \end{cases} \tag{B.12}$$

where $\mu\xi \det g_{\alpha\beta}\neq 0$. From (B.12) we obtain series expansions

$$\begin{cases} X_0=\xi\, Y_0(1+\sum_\alpha \sigma_\alpha y_\alpha+\tau\, y_{n+1}), \\ x_{n+1}=\mu\xi^{-1}\, y_{n+1}(1+\text{terms involving } y_1, \ldots, y_{n+1}), \\ x_\alpha=\xi^{-1}(\sum_\beta g_{\alpha\beta}\, y_\beta+\rho_\alpha\, y_{n+1})(1-\sum_\alpha \sigma_\alpha y_\alpha+E), \end{cases} \tag{B.13}$$

where E are terms involving $y_\alpha y_\beta$ or y_{n+1}. We may assume that M is given by an equation

$$0=x_{n+1}+\sum_\alpha x_\alpha^2+V(x_1, \ldots, x_n)+F(x), \tag{B.14}$$

where $V=\sum_{\alpha,\beta,\gamma} v_{\alpha\beta\gamma}\, x_\alpha x_\beta x_\gamma$ is a cubic form and $F(x)$ consists of terms in x_{n+1}, or quartic expressions in $x_1, \ldots, x_n$. We assume that under the coordinate change (B.13) M has an equation

$$0=y_{n+1}+\sum_\alpha y_\alpha^2+V(y_1, \ldots, y_n)+F(y).$$

Substituting (B.13) in (B.14) we infer first that

$$(\mu\xi)^{-1}\sum_{\beta} g_{\alpha\beta}\, g_{\gamma\beta}=\delta_{\alpha\gamma},$$

so that $g=(g_{\alpha\beta})$ is in the conformal group associated to the 2nd fundamental form (B.6), and then that

$$\text{(B.15)}\qquad V(y)=\mu^{-1}\xi^{-2}V(g.x)-2(\mu\xi)^{-1}(\sum_{\alpha} y_\alpha^2)(\sum_{\beta}\sigma_\beta y_\beta).$$

Upon identifying the cubic form (B.9) with that appearing in the series development (B.14) may deduce (B.10) from (B.15)([74]).

Q.E.D.

Now we arrive at an interesting point. For hypersurfaces $M\subset\mathbb{P}^{n+1}$ that are not duals to lower-dimensional submanifolds we have associated to a generic point p a quadric Q and cubic V in $\mathbb{P}^{n-1}\cong\mathbb{P}T_p(M)$ intrinsically defined up a transformation

$$\text{(B.15)}\qquad \begin{cases} Q\to\lambda Q, & \lambda\neq 0,\\ V\to\mu Q+LQ, & \mu\neq 0.\end{cases}$$

When $n=2$ we may again normalize and continue the process of isolating our moving frame. However, when $n\geqq 3$ the algebro-geometric data of the smooth intersection of a quadric and cubic $\mathbb{P}^{n-1}$ has moduli. So at this point we have already arrived at our moving frame. We shall illustrate this by considering cases.

The totally degenerate case. — We shall characterize hypersurfaces for which the cubic form is zero.

(B.16) LEMMA. — *The condition* $V\equiv 0 \bmod Q$ *at a generic point is equivalent to* M *being a piece of a smooth quadric hypersurface in* $\mathbb{P}^{n+1}$.

Proof. — The proof is similar to, but simpler than, the proof of (6.18). It will be outlined in two steps.

Step one. — We consider the smooth quadric $M_0\subset\mathbb{P}^{n+1}$ defined by $\sum_{i=0}^{n+1} X_i^2=0$. The corresponding quadratic form on $\mathbb{C}^{n+2}$ will be denoted by Q_0. If $\{A_0; A_\alpha; A_{n+1}\}$ is a Darboux frame field for M_0, then from $Q_0(A_0, A_0)=0$ we have

$$\begin{cases} Q_0(A_0, A_\alpha)=0,\\ \omega_\beta Q_0(A_\beta, A_\alpha)+\omega_{\alpha, n+1}Q_0(A_0, A_{n+1})=0.\end{cases}$$

Since Q_0 is non-singular, the first equation implies that $Q_0(A_0, A_{n+1})\neq 0$, and the second implies that $\{Q_0(A_\alpha, A_\beta)\}$ is non-singular. We normalize so that $Q_0(A_0, A_{n+1})=1$ and

([74]) The referee remarks that the intersection $Q\cap V$ is the "tritangent cone", defined to be the set of lines having contact of order $\geqq 4$ with the hypersurface at the point. This follows immediately from (B.14).

$Q_0(A_\alpha, A_\beta) = -\delta_{\alpha\beta}$, and then denote by $\hat{\mathscr{F}}(M_0) \subset \mathscr{F}((M_0)$ the submanifold of Darboux frames on which Q_0 has the standard form

$$\left[\begin{array}{c|ccc|c} 0 & 0 & \dots & 0 & 1 \\ \hline 0 & -1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & -1 & 0 \\ \hline 1 & 0 & \dots & 0 & 0 \end{array}\right]$$

On $\hat{\mathscr{F}}(M_0)$ the Maurer-Cartan matrix satisfies

$$\text{(B.16}\,a) \qquad \left\{\begin{array}{c} \omega_{n+1} = 0 = \omega_{n+1,\,0}, \\ \omega_{00} + \omega_{n+1,\,n+1} = 0, \\ \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0, \\ \omega_{\alpha,\,n+1} = \omega_\alpha = \omega_{n+1,\,\alpha}. \end{array}\right.$$

It follows from (B.16 *a*) that on M_0 the cubic form $V \equiv 0$.

Step two. — Conversely, we assume that $V \equiv 0$. By (B.8) we may find a submanifold $\mathscr{F}_1(M) \subset \mathscr{F}(M)$ lying over M and on which the equations

$$\text{(B.16}\,b) \qquad \left\{\begin{array}{c} \omega_{n+1} = 0, \\ \omega_{\alpha,\,n+1} = \omega_\alpha, \\ (\omega_{00} + \omega_{n+1,\,n+1})\,\delta_{\alpha\beta} = \omega_{\alpha\beta} + \omega_{\beta\alpha}, \end{array}\right.$$

are satisfied. Following the same procedure as in the proof of (6.18), we may then determine an integral manifold $\hat{\mathscr{F}}(M)$ of the differential system obtained by adjoining the equations

$$\left\{\begin{array}{c} \omega_{00} + \omega_{n+1,\,n+1} = 0, \\ \omega_{n+1,\,0} = 0, \end{array}\right.$$

to (B.16 *b*). On this integral manifold the equations (B.16 *a*) are satisfied, and we conclude from (B.2) that M is projectively equivalent to the quadric M_0.

Surfaces. — There is an incredibly vast classical literature on the local projective differential geometry of surface in $\mathbb{P}^3$ (*cf.* [3] and [4]), and we make no pretense to understand it in detail. Granted this, from the present point of view here are a few highlights.

The projectivized tangent spaces are $\mathbb{P}\,T_p(M) \cong \mathbb{P}^1$, and in this case instead of (B.5) it is convenient to use the normalization

$$\text{(B.17)} \qquad Q = \omega_1\,\omega_2$$

for the 2nd fundamental form. The directions $\omega_1=0$, $\omega_2=0$ are the tangents to the intersection curve $\tilde{T}_p(M)\cap M$ (this is always an analytic curve in the complex case), and are called the asymptotic directions.

Suppose that the cubic form is:

$$V(\omega)=v_{3,0}\,\omega_1^3+v_{2,1}\,\omega_1^2\,\omega_2+v_{1,2}\,\omega_1\,\omega_2^2+v_{0,3}\,\omega_2^3.$$

By subtracting multiples of the 2nd fundamental form (B.17) we may assume that $v_{2,1}=v_{1,2}=0$. If $v_{3,0}=v_{0,3}=0$ then by (B.16) M is a piece of a quadric hypersurface. If, say, $v_{3,0}=0$ then it can easily be verified that the asymptotic curves $\{\omega_1=0\}$ are straight lines so that M is a ruled surface ([75]) (recall that the quadric surface is ruled by the two families $\omega_1=0$, $\omega_2=0$ of straight lines).

Consequently, for M not a ruled surface at a generic point we may normalize so that

$$V=\omega_1^3+\omega_2^3. \tag{B.18}$$

At this juncture the Darboux frames for which (B.17) and (B.18) are valid are almost uniquely determined. Indeed, if we take into account the fact that Q is given by (B.17) instead of (B.5) the conditions on the Darboux frame for (B.17) and (B.18) to hold are

$$\text{(B.19)}\qquad \begin{cases} \omega_3=0, \\ \omega_{13}=\omega_2, \quad \omega_{23}=\omega_1, \\ \omega_{12}=\omega_1, \quad \omega_{11}+\omega_{22}-\omega_{00}-\omega_{33}=0, \quad \omega_{21}=\omega_2. \end{cases}$$

Taking the exterior derivatives of the third equations in (B.19) and applying the Cartan Lemma once again one finds that it is possible to uniquely specify the frame by suitably normalizing a certain quartic differential form (*cf.* [3], pp. 285-286).

The main consequence of all this is that by so determining the lifting F and using (B.2) one may prove the following analogue, due to Fubini, of the classical Euclidean rigidity Theorem for surfaces in $\mathbb{R}^3$:

(B.20) *If* M, $\tilde{M}$ *are surfaces in* $\mathbb{P}^3$ *and if there is a biholomorphic mapping* $f: M\to\tilde{M}$ *such that*

$$\begin{cases} f^*\tilde{Q}=\lambda Q, \\ f^*\tilde{V}=\mu V+LQ, \end{cases}$$

then M *and* $\tilde{M}$ *are congruent by a projective transformation.*

A proof of this is given in [3], pp. 315-316, and continuing on pp. 316-317 the existence Theorem (B.3) is utilized to give necessary and sufficient conditions an abstract surface having a structure $\{Q, V\}$ to arise from an embedding $M\subset\mathbb{P}^3$. There is also apparently

([75]) If we add to this proposition (3.15) then we have completely characterized submanifolds $M_n\subset\mathbb{P}^N$ that are ruled by $\mathbb{P}^{n-1}$'s; note the distinction between the cases $n=2$ and $n\geq 3$.

some analogue of the Beez rigidity Theorem for hypersurfaces in $\mathbb{P}^n$ ($n \geqq 4$); this is given in [3], p. 345. The intuition for this result as well as the Fubini Theorem (B.20) seems to be that the connection matrix of the conformal connection essentially specifies the skew-symmetric part of $\{\omega_{\alpha\beta}\}$ while V gives the symmetric part. The structure equations then yield the rest. It would be quite interesting to make this heuristic reasoning precise, if possible.

Higher-dimensional hypersurfaces. — In higher dimensions, say for $n \geqq 4$, the intersection of the quadric $Q(\omega)=0$ and cubic $V(\omega)=0$ gives an algebraic variety

$$X_p \subset \mathbb{P}^{n-1}$$

depending on $p \in M$. Except when the original hypersurface is a piece of a quadric, $\dim X = n-3$ and this variety uniquely determines Q and V up to transformations (B.15). For example, when $n=4$ we have a canonical curve $X_p \subset \mathbb{P}^3$, one that in general will be smooth although the analysis of special cases where X has certain degenerate forms should prove interesting. (*Question:* Do the singular points of X correspond to lines in M?) Since the genus g of X is 4, the number of moduli $3g-3$ is 9. Thus we may expect that the mapping

$$M \to \{\text{moduli of curves of genus } 4\}$$

will be locally injective, although we don't know how to prove this.

When $n=3$ the projectivized tangent spaces are $\mathbb{P}^2$'s, and the pair $\{Q, V\}$, defined up transformations (B.15), amounts to giving 6 points on the standard plane conic $X_0^2+X_1^2+X_2^2=0$. This conic is biholomorphic to $\mathbb{P}^1$ and so the 6 points depend on 3 parameters, so that this situation falls somewhere in between the cases $n=2$ and $n \geqq 4$ and may be expected to have certain special features (*cf.* [3], pp. 529-538).

(*c*) Manifolds of codimension two. — For a codimension two submanifold $M_n \subset \mathbb{P}^{n+2}$ the 2nd fundamental form II is given by a *pencil* of quadrics on $\mathbb{P}T_p(M) \cong \mathbb{P}^{n-1}$. This means that there are two linearly independent quadrics Q_0 and Q_1 such that a general $Q \in |II|$ is

$$Q(t_0, t_1) = t_0 Q_0 + t_1 Q_1.$$

The discriminant $\det Q(t_0, t_1)$ is a homogeneous form of degree n, that we shall assume to have n distinct roots. Setting $t = t_1/t_0$ we take these roots to be $\lambda_1, \ldots, \lambda_n$. Then the quadric $Q(\lambda_\alpha)$ will be singular at a unique point of $\mathbb{P}^{n-1}$, and taking these points to be vertices of a coordinate simplex we will have

$$\text{(B.21)} \qquad \begin{cases} Q_0 = \sum_\alpha \omega_\alpha^2, \\ Q_1 = \sum_\alpha \lambda_\alpha \omega_\alpha^2. \end{cases}$$

Referring to (1.15) this is equivalent to selecting Darboux frames such that

$$\text{(B.22)} \qquad \begin{cases} \omega_{\alpha, n+1} = \pm \omega_\alpha, \\ \omega_{\alpha, n+2} = \pm \lambda_\alpha \omega_\alpha. \end{cases}$$

At this point we may suspect that for $n \geqq 3$ the frame has almost been determined, since in this case the automorphism group of the configuration (B.21) is a finite group. Indeed, the sets of n points $\{\lambda_\alpha\}$ on $\mathbb{P}^1$ depend on ∞^{n-3} parameters and have only permutations as symmetries. For example, for $n=4$ the base of the pencil $|\mathrm{II}|$ is the elliptic curve E_p in $\mathbb{P}^3$ defined by

$$Q_0(\omega)=0, \qquad Q_1(\omega)=0,$$

it is well known that E_p depends on 1 parameter and has only a finite group of projective automorphisms.

On the other hand, for $n=2$ or 3 we may normalize the 2nd fundamental form of $M_n \subset \mathbb{P}^{n+2}$. For a surface in $\mathbb{P}^4$ it may be assumed to be generated by

$$\begin{cases} Q_0=\omega_1^2, \\ Q_2=\omega_2^2, \end{cases}$$

while for a threefold in $\mathbb{P}^5$ it may be taken to be generated by

$$\text{(B.23)} \qquad \begin{cases} Q_0=\omega_1^2+\omega_2^2, \\ Q_1=\omega_2^2+\omega_3^2, \end{cases}$$

Proof. — The base of the pencil $|\mathrm{II}|$ consists of 4 points in $\mathbb{P}^2$, which when taken to be the points $[1, \pm i, 0]$ and $[0, 1, \pm i]$ implies (B.23).

It seems to us an interesting exercise to investigate what additional structure is needed to specify completely a framing and from this derive the corresponding rigidity Theorem.

Finally, for M_n and $\tilde{M}_n$ given in $\mathbb{P}^{n+2}$ and $n \geqq 4$ it is already a strong condition that there should exist a biholomorphic mapping $f: M \to \tilde{M}$ taking $|\tilde{\mathrm{II}}|$ to $|\mathrm{II}|$. More precisely, we consider $M \subset \mathbb{P}^{n+2}$ and let $Y_p \subset \mathbb{P}^{n-1}$ be the base of the 2nd fundamental form at $p \in M$. Then there is a holomorphic mapping

$$\text{(B.24)} \qquad M \to \{\text{moduli space of } Y_p\text{'s}\},$$

and the biholomorphic mapping f should make the diagram

$$\begin{array}{ccc} M & \searrow & \\ f\downarrow & & \{\text{moduli space of } Y_p\text{'s}\} \\ \tilde{M} & \nearrow & \end{array}$$

commutative. We suspect that in this case M is congruent to $\tilde{M}$ by a projective transformation, but we have not been able to establish this.

A final observation is that the mapping (B.24) is not arbitrary but is subject to "Gauss-Codazzi" equations obtained by exterior differentiation of (B.22) ([76]).

([76]) In this regard it would be interesting to determine those M for which the mapping (B.24) is constant.

Notations

$\{z, e_1, \ldots, e_N\}$ is a frame for $\mathbb{C}^N$.

$\{A_0, A_1, \ldots, A_N\}$ is a frame for $\mathbb{P}^N$.

$T_p(M)$ is the tangent space at the point p on the manifold M:

for $M \subset \mathbb{C}^N$, z denotes the coordinate of $p \in M$, and $T_z(M)$ is the translate to the origin of the embedded tangent space to M;

for $M \subset \mathbb{P}^N$, at each point $p \in M$ the projective tangent space is $\tilde{T}_p(M)$; it is the limiting position of chords $\overline{pq}$ where $p, q \in M$ and $q \to p$.

$\tilde{T}(M) = \bigcup_{p \in M} \tilde{T}_p(M)$ is the abstract bundle of projective tangent spaces.

$\tilde{T}_p^{(k)}(M)$ is the kth osculating space to $M \subset \mathbb{P}^N$ at the point p [thus $\tilde{T}_p^{(1)}(M) = \tilde{T}_p(M)$].

$\overline{A_1, \ldots, A_m}$ is the linear subspace of $\mathbb{P}^N$ spanned by vectors $A_1, \ldots, A_m \in \mathbb{C}^{N+1}$.

Unless mentioned otherwise, the following ranges of indices will be used throughout

$$1 \leqq a, b, c \leqq N; \qquad 0 \leqq i, j, k \leqq N,$$
$$1 \leqq \alpha, \beta, \gamma \leqq n; \qquad n+1 \leqq \mu, \nu \leqq N.$$

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