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SOME PROPERTIES AND APPLICATIONS OF HARMONIC MAPPINGS (*)

BY J. H. SAMPSON

1. Introduction

If $f: M \to Y$ is a C^2 mapping of Riemannian manifolds, whose metric tensors we denote by $g_{ij}\,dx^i\,dx^j$ (resp. $g'_{\alpha\beta}\,dy^\alpha\,dy^\beta$), then the *tension field* τf of f is the section of $f^*\,T(Y)$ defined as follows: In local coordinates $x \mapsto y$, df has matrix $y^\alpha_i = \partial y^\alpha/\partial x^i$; and τf has components

$$\Delta y^\alpha + \Gamma'^\alpha_{\beta\gamma}\, y^\beta_i\, y^\gamma_j\, g^{ij},$$

where the $\Gamma'^\alpha_{\beta\gamma}$ are the Christoffel symbols on Y and where Δy^α is the Laplacian of the local function $y^\alpha(x)$ on M. The mapping f is called *harmonic* if it satisfies the elliptic system

$$(1) \qquad \tau f = 0,$$

which is precisely the Euler-Lagrange equation for a stationary value of the *energy* integral

$$(2) \qquad E(f) = \int_M e_f, \qquad \text{where} \quad e_f = \frac{1}{2}|\,df\,|^2 = \frac{1}{2} g'_{\alpha\beta}\, y^\alpha_i\, y^\beta_j\, g^{ij},$$

integration being with respect to the Riemannian volume element on M.

Existence of harmonic mappings was established under general circumstances in [9] by use of the associated parabolic system (heat equation):

$$(3) \qquad \tau f = \frac{\partial f}{\partial t}$$

for a family of mappings $M \to Y$ depending on a real parameter t. In [15] some important simplifications of the conditions required in [9] were given.

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Throughout this papeı all ingredients are supposed C^∞. Many of the results were obtained some years ago and have been informally circulated to some extent.

At this point we are very happy to express our indebtedness to Professor James Eells Jr. for many important comments, and especially his suggestions for great improvements in paragraph 9. And we are glad once more to record our gratitude to the Mathematics Institute of the University of Warwick.

In paragraph 2 we apply the Aronszajn-Carleman unique continuation theorem for elliptic equations to obtain a unique continuation theorem for harmonic mappings In paragraph 3 we prove a theorem for harmonic mappings analogous to the maximum principle for harmonic functions.

In paragraph 4 we discuss harmonic mappings into totally geodesic submanifolds, and we obtain unique continuation results for the differentiable case (i. e. non-analytic), including generalizations of results of [15]. In paragraph 5 we deal with harmonic immersions and their relative curvatures, and we obtain extensions of results of [3].

Paragraphs 6 and 7 are concerned with deformations of harmonic mappings $M \to Y$ associated with variations of the metrics on either manifold. Y being assumed of strictly negative curvature, we prove the C^∞ continuity of harmonic mappings for variations of C^∞ metrics.

Paragraphs 8 and 9 contain a brief account of our application of harmonic mappings to compact Riemann surfaces and to the study of their moduli, making use of the fact that a harmonic mapping of surfaces $M \to Y$ gives rise to a holomorphic quadratic differential on M which vanishes if an only if the mapping is (anti-) holomorphic. In this connexion we have recently learned that Theorem 11 of paragraph 9 has also been obtained by R. Schoen and S. T. Yau, using the same techniques. Their work will appear in [25].

Finally, in paragraph 10, we mention some simple applications of harmonic mapping theory to complex submanifolds of a complex manifold of negative curvature. The results are of interest in connexion with automorphic varieties. We end with a generalization of Mordell's well-known conjecture for algebraic curves of genus $\geqq 2$.

2. A unique continuation theorem

We recall here, in slightly altered form, Aronszajn's generalization of Carleman's unique continuation theorem ([1], esp. p. 248): *Let* A *be a linear elliptic second-order differential operator defined on a domain* D *of* R^n. *In* D *let* $u = (u^1, \ldots, u^r)$ *be functions satisfying the differential inequalities*

$$|A u^\alpha| \leqq \text{Const.}\left\{\sum_{i,\beta}\left|\frac{\partial u^\beta}{\partial x^i}\right| + \sum_\beta |u^\beta|\right\}.$$

If $u = 0$ *in an open set, then* $u = 0$ *throughout* D. (The conclusion holds if $u = 0$ to infinitely high order at a single point, as explained in [1].)

Our main result in this paragraph is the following:

THEOREM 1. — *Let f, f' be two harmonic mappings* $M \to Y$. *If they agree on an open set then they are identical* (M *being assumed connected, naturally*)*; and indeed the conclusion holds if f and f' agree to infinitely high order at some point. In particular, a harmonic mapping which is constant on an open set is a constant mapping.*

Proof. — Let U be a coordinate ball on M such that $f = f'$ in some open subset. We can take U small enough that both f and f' map U into a coordinate ball on Y. Write $y^\alpha(x)$ for $y^\alpha(f(x))$ and $v^\alpha(x)$ for $y^\alpha(f'(x))$. We apply the theorem cited above to the functions $u^\alpha = y^\alpha - v^\alpha$.

From equation (1) we have

$$\Delta u^\alpha = \Delta y^\alpha - \Delta v^\alpha = -\Gamma'^\alpha_{\beta\gamma} y^\beta_i y^\gamma_j g^{ij} + \Gamma''^\alpha_{\beta\gamma} v^\beta_i v^\gamma_j g^{ij}.$$

Here $\Gamma''^\alpha_{\beta\gamma}$ stands for $\Gamma'^\alpha_{\beta\gamma}(v)$. Rewrite the night-hand side as

$$-\Gamma'^\alpha_{\beta\gamma}(y^\beta_i - v^\beta_i)(y^\gamma_j + v^\gamma_j) g^{ij} + (\Gamma''^\alpha_{\beta\gamma} - \Gamma'^\alpha_{\beta\gamma}) v^\beta_i v^\gamma_j g^{ij}.$$

In U, slightly shrunk if necessary, the derivatives y_j, v_j are bounded; and $\Gamma''^\alpha_{\beta\gamma} - \Gamma'^\alpha_{\beta\gamma}$ can be estimated by the mean-value theorem. It is easy to see that we have then:

$$|\Delta u^\alpha| \leqq \text{Const.}\{\sum_{i,\beta} |u^\beta_i| + \sum_\beta |u^\beta|\}$$

in U. As the u^α vanish in an open set, we have $u = 0$ throughout the neighbourhood U. Our conclusion follows from the connectedness of M.

Q.E.D.

In particular, if a harmonic mapping f has rank 0 in an open set, i. e. is constant in an open set, it must have rank 0 everywhere. In the case of a harmonic mapping of analytic Riemannian manifolds, the mapping is also analytic ([9], p. 117), and it follows that, if it has rank r in an open set, then it has rank r in an open, dense set.

It is natural to think that the Aronszajn theorem, applied to the minors of degree $r+1$ of df for a harmonic f, might yield the same conclusion in the differentiable case. However we have been able to obtain only very incomplete results along this line, except for the case $r = 0$ and $r = 1$, which is treated in a different manner in paragraph 4 below.

For unique continuation results pertinent to the parabolic system (3), *see* for example [22].

3. A maximum principle

Here we consider a harmonic mapping $f : M \to Y$ in the vicinity of a point $p \in M$ and its image $q = f(p)$. We prove:

THEOREM 2. — *Let* S *be a piece of* C^2 *hypersurface in* Y *passing through q, at which point we assume that the second fundamental form is definite. If f is not a constant mapping, then no neighbourhood of p is mapped entirely to the concave side of* S.

Proof. — Fix a geodesic coordinate system (y^α) at q, and let (x^i) be a coordinate system at p. We may assume that S is defined by an equation $u(y) = 0$, the concave side being that for which $u \leqq 0$. Putting $U(x) = u(f(x))$, we have

$$\Delta U = u_{\alpha\beta}\, y_i^\alpha\, y_j^\beta\, g^{ij} + u_\alpha\, \Delta y^\alpha,$$

where $u_\sigma = \partial u/\partial y^\alpha$, $y_i^\alpha = \partial y^\alpha/\partial x^i$, etc. By our assumption on S, the matrix $(u_{\alpha\beta})$ is positive definite in the tangent hyperplane to S at Q (*cf.* [11], § 43). Moreover, $\Delta y^\alpha = 0$ there, by (1), since the y-system is geodesic at q. If $df_p \neq 0$, i. e. if some $y_i^\alpha \neq 0$ at p, then we have $\Delta U > 0$ at p, hence in a neighbourhood of p.

On the other hand, if f maps a neighbourhood of p to the concave side of S, then $U(x)$ has a maximum at p, contradicting the maximum principle for Δ ([2], Part II, Chap. II).

If $df_p = 0$, we let S′ be a piece of y-sphere tangent to S at q and chosen so that S lies on the concave side of S′ (near q). Then S′ is given by an equation $u'(y) = 0$, analogous to that for S; and if f maps a neighbourhood of p to the concave side of S, hence also of S′, then the function $U'(x) = u'(f(x))$ has a maximum at p. As with U we have

$$\Delta U' = u'_{\alpha\beta}\, y_i^\alpha\, y_j^\beta\, g^{ij} + u'_\alpha\, \Delta y^\alpha,$$

where $u'_\alpha = \partial u'/\partial y^\alpha$ and $u'_{\alpha\beta} = \partial^2 u'/\partial y^\alpha\, \partial y^\beta$. We claim that $\Delta U' \geqq 0$ near p. For otherwise we could find a sequence of points $x_\nu \to 0\ (= p)$ with

$$u'_{\alpha\beta}\, y_i^\alpha\, y_j^\beta\, g^{ij} < u'_\alpha . \Gamma'^\alpha_{\beta\gamma}\, y_i^\beta\, y_j^\gamma\, g^{ij} \quad \text{at } x_\nu,$$

using equation (1). But $\Gamma'^\alpha_{\alpha\beta}(y(x_\nu)) \to 0$, since $\Gamma'^\alpha_{\beta\gamma} = 0$ at q, whereas the matrix $(u'_{\alpha\beta})$ is positive definite at p (and not merely in the tangent hyperplane), in virtue of our choice of S′. The contradiction here is evident, and so we must have $\Delta U' \geqq 0$ near p. From the maximum principle for Δ we deduce that $U'(x)$ is constant near p. By varying the choice of the sphere S′ we then conclude that in fact $f(x)$ is constant near p. We may now apply Theorem 1 to complete the proof.

Q.E.D.

Remark 1. — A similar theorem holds for the parabolic system (3), by the maximum principle for $\Delta - \partial/\partial t$, *see* [23].

Remark 2. — The "concave" hypothesis cannot be greatly weakened. A beautiful example due to T. Smith gives a harmonic mapping of a 2-torus, with flat metric, to the 2-sphere, in such a way that the image is the region between the Tropic of Cancer and the Tropic of Capricorn, *see* [28].

4. Mappings into totally geodesic submanifolds

We first prove a C^∞ version of a theorem related to [15], (Th. H and Cor.) for the analytic case. Applications are given in Theorems 4, 5. In Theorem 6 below, we prove an extension of Theorem 3 to higher rank.

THEOREM 3. — *If $f: \mathrm{M} \to \mathrm{Y}$ is harmonic and* M *connected, and if df has rank* 1 *in an open set, then f maps* M *into a geodesic arc in* Y, *and df has rank* 1 *in an open, dense set. If* M *is compact, then the geodesic arc is closed.*

We note the following immediate corollary:

COROLLARY. — *If* M *has dimension* 2 *and if $f: \mathrm{M} \to \mathrm{Y}$ has rank* 2 *in an open set, then it has rank* 2 *in an open, dense set.*

Proof. — If df has rank < 1 in an open set, then f is constant, in consequence of Theorem 1. Suppose now that df has rank 1 in an open set U. Then every $p \in \mathrm{U}$ has a neighbourhood which is mapped into a regular arc c in Y. As usual, let x resp. y denote local coordinates at p [resp. $f(p)$]. Since c is regular, we may assume that the y-system is chosen so that c is the coordinate curve defined by $y^\alpha = 0$ for $\alpha > 1$. Accordingly we have $y^\alpha(x) = 0$ for $\alpha > 1$, x near p. The harmonic equation (1) then shows that $\Gamma'^{\alpha}_{11} = 0$ along c near $f(p)$ for $\alpha > 1$. Now it is quickly seen that an appropriate change of variable of the form $y^1 \to y'^1 = \varphi(y^1)$ yields a coordinate system of the same type as before, but with $\Gamma'^{1}_{11} = 0$ along the curve c. Therefore c is a geodesic arc, which can be prolonged to a maximal geodesic. Our coordinates are in fact Fermi coordinates along c.

Fixing the maximal geodesic c, along any simple portion of it we can introduce Fermi coordinates, still called y. Let V be a connected open set of M which is mapped by f into the Fermi system. Suppose that in some part of V we have $\Delta y^1 = 0$ and $y^\alpha = 0$ $(\alpha > 1)$, as for the system just considered. Now the mapping $x \to (y^1, 0, \ldots, 0)$ is a harmonic mapping of V into Y, agreeing with f in an open set. From the Aronszajn-Carleman theorem (*see* 2) we conclude that the two mappings coincide on V. We can then reason from the connectedness of M to conclude that f maps M into the maximal geodesic arc c.

Now assume that M is closed. The image is a connected and compact piece of c. If c were not closed, there would be a point $q \in c$ in the image of f such that a part of c on one side of q contains no image points. But we then easily draw a contradiction from Theorem 2 above. Hence we conclude that c is closed and that $f(\mathrm{M}) = c$.

Q.E.D.

We shall apply the foregoing to prove a sharpened form of a result of Hartmann (*cf.* [15], Th. H, Cor. 2).

THEOREM 4. — *Let $f: \mathrm{M} \to \mathrm{Y}$ be a harmonic mapping, where* M *is assumed to be compact and* Y *of non-positive sectional curvature. If $f(\mathrm{M})$ contains a point q at which the sectional curvatures of* Y *are all* < 0, *and if $f(\mathrm{M})$ is not contained in a geodesic on* Y, *then f is the only harmonic mapping in its homotopy class.*

Proof. — Of course, the sectional curvature of Y will be strictly negative in some neighbourhood of q. Let $f(p) = q$. If df has rank $\leqq 1$ in a neighbourhood of p, then $f(\mathrm{M})$ will surely be contained in a geodesic of Y, in virtue of our theorems 1 and 3. Our assumption therefore implies that df has rank > 1 at points arbitrarily near to p. We may then apply Corollary 1 of Hartman's Theorem H to obtain our conclusion.

Q.E.D.

Of course, the essential new feature of our result is that the assumption of analyticity in not required.

The following is an application of the foregoing theorem (a similar application was noticed by Hartman).

THEOREM 5. — *Let* M *be compact and of non-positive sectional curvature which is strictly negative at some point. Then the group of isometries of* M *is finite, and no two of its elements are homotopic.*

For the isometries are harmonic mappings and are therefore unique in their homotopy classes. The finiteness of the group of isometries is known from Bochner [5]; but it is also an immediate consequence of the present argument, since obviously the group must be compact and discrete.

The conclusion of the theorem above is well known for the group of complex automorphisms of a compact Riemann surface of genus > 1 (*cf.* [18]). In the present context that can be deduced from Theorem 5 and the fact that complex automorphisms are isometries for a conformal metric. (For genus > 1 there is always a conformal metric of constant negative curvature, *cf.* § 8.)

We now prove a result similar to Theorem 3, but in the case of higher rank.

THEOREM 6. — *Let* $f : \mathrm{M} \to \mathrm{Y}$ *be a harmonic mapping, and let* V^r *be a complete, totally geodesic submanifold of* Y. *If an open set of* M *is mapped into* V, *then all of* M *is mapped into* V.

Proof. — At $q \in \mathrm{V}$ choose a local coordinate system $y^1, \ldots, y^m$ such that V near q is the locus $y^{r+1} = \ldots = y^m = 0$. The Christoffel symbols will then have $\Gamma'^{\alpha}_{\beta\gamma} = 0$ for $\alpha > r$ and $\beta, \gamma \leqq r$ along V, because of the assumption that V is totally geodesic.

Let U be a connected open set of M mapped by f into the coordinate neighbourhood of the y^α, and assume that f maps an open subset U_0 of U into V. The harmonic equation (1) in U_0 takes the form

$$\Delta y^\alpha + \sum_{\beta,\gamma \leqq r} \Gamma'^{\alpha}_{\beta\gamma} y^\beta_i y^\gamma_j g^{ij} = 0 \qquad (\alpha \leqq r)$$

and

$$y^{r+1} = \ldots = y^m = 0.$$

Now the same equations give a solution of (1) throughout U, as is quickly seen, and the Aronszajn-Carleman unique continuation theorem then allows us to conclude that there is no other solution with the given form in U_0. The theorem follows readily from the connectedness of M.

5. Harmonic immersions

Here we shall generalize some results of Bochner [3] concerning the curvature of harmonic immersions.

$f : M \to Y$ being any differentiable mapping of Riemannian manifolds, fix local coordinates (x^i) at $p \in M$ and (y^α) at $f(P) \in Y$. We set

$$y_{ik} = \frac{\partial^2 y}{\partial x^i \partial x^k} - \Gamma^j_{ik} y_j \qquad \text{and} \qquad y_{i\,|\,k} = y_{ik} + \Gamma'^\alpha_{\mu\nu} y^\mu_i y^\nu_k.$$

The latter are the components of the covariant derivative $D^2 f$ of the section

$$df = y^\alpha_i \, dx^i \otimes (\partial/\partial y^\alpha) \quad \text{of } T^*(M) \otimes f^* T(Y)$$

relative to the induced connection. $D^2 f$ is sometimes called the *second fundamental form* of f. Set

$$\varphi_{ijkl} = g'_{\alpha\beta}(y^\alpha_{i\,|\,k} y^\beta_{j\,|\,l} - y^\alpha_{i\,|\,l} y^\beta_{j\,|\,k}),$$

the *relative curvature* of f. If f is harmonic, then we have

$$g^{jl} \varphi_{ijkl} = -g'_{\alpha\beta}\, g^{jl} y^\alpha_{i\,|\,l} y^\beta_{j\,|\,k},$$

and the matrix with coefficients $a_{ik} = g^{jl}\, \varphi_{ijkl}$ is negative (and indeed < 0 unless all $y_{i\,|\,l}$ vanish, i. e. $D^2 f = 0$).

THEOREM 7. — *Let* dim M = 2 *and let* $f : M \to Y$ *be a harmonic immersion. Denoting the image by* V, *we assume that the metric carried by f from* M *to* V *is conformally equivalent to the metric induced on* V *from* Y. *Then the curvature of* V *at any points is* $\leqq$ *the Riemannian curvature of* Y *at that point on the* 2-*dimensional section defined by* V.

Proof. — Our assertion being purely local, we can simply assume that f is an embedding. Since dim M = 2, the metric $g_{ij}\, dx^i\, dx^j$ can be replaced by any conformally equivalent metric without destroying the harmonicity of f. We may therefore suppose that f is a Riemannian embedding.

(We recall that f is then a minimal embedding [9], p. 119.)

We choose local coordinates (x^i) at $p \in M$ and (y^α) at $f(p) \in V$ such that $(g_{ij}) = (\delta_{ij})$ and $(g'_{\alpha\beta}) = (\delta_{\alpha\beta})$ at p and $f(p)$. We can use the x^i as coordinates on V at $f(p)$, of course. From Gauss's equation ([11], § 43) the curvature of V for the metric $g_{ij}\, dx^i\, dx^j$ is

$$R_{ijkl} = R'_{\alpha\beta\gamma\delta}\, y^\alpha_i y^\beta_j y^\gamma_k y^\delta_l + \varphi_{ijkl},$$

φ as above. Then the Gaussian curvature at p [or $f(p)$] is R_{1212}; and on the right side we have the sectional curvature of Y along V plus φ_{1212}, which is $\leqq 0$.

Q.E.D.

For higher dimensions we have:

THEOREM 8. — *Let* $f : M \to Y$ *be a harmonic Riemannian embedding* (*or immersion*), *and again call the image* V. *If* Y *has Riemannian curvature* $\leqq 0$ *in all directions at a point of* V, *then* V *has Ricci curvature* $\leqq 0$ *at that point.*

Proof. — From the Gauss equation, the negative of the Ricci tensor on M (or V) is $R_{ik} = g^{jl} R'_{\alpha\beta\gamma\delta} y_i^\alpha y_j^\beta y_k^\gamma y_l^\delta + a_{ik}$. We have already observed that $(a_{ik}) \leqq 0$. At p [resp. $f(p)$] we can assume that $(g_{ij}) = (\delta_{ij})$, $(g'_{\alpha\beta}) = (\delta_{\alpha\beta})$ and $y_i^\alpha = \delta_i^\alpha$ for i, $\alpha = 1, \ldots, n$ (= dim M), $y_i^\alpha = 0$ otherwise. The first term in R_{ik} above is then

$$\sum_{j=1}^{n} R'_{ijkj},$$

which is clearly $\leqq 0$ (i. e. as matrix.)

Q.E.D.

As an application, let V be a complex submanifold of a Kähler manifold Y with negative sectional curvature. Since the identity map $V \subset Y$ is harmonic, we conclude that V has Ricci curvature $\leqq 0$ (that is, for the unduced metric on V). We mention that V also has non-positive holomorphic sectional curvature (*see* [4], Th. 12).

6. Deformations

We now consider the effects of a variation of the metric on M or Y, as regards harmonic mappings. We shall work things out only for variations of the Y metric; a similar analysis can be given for the other case.

J. Eells and L. Lemaire have announced similar results by another method. I take this opportunity to thank L. Lemaire for having pointed out a great error in the original manuscript.

Simple examples of geodesics on surfaces of revolution show that one cannot expect a reasonable deformation theory without negative curvature restrictions. And one can give examples of non-compact Y with variable metric g'_s of strictly negative curvature for all parameter value s and such that a given homotopy class of maps $M \to Y$ contains a harmonic map for some values of s but not for others.

We suppose then that M and Y are compact and that Y has a metric g'_s depending smoothly (i. e. C^∞) on a real parameter s, $0 \leqq s \leqq 1$, and in such a way that Y is of sectional curvature $\leqq 0$ for all s. There is no difficulty in extending our results to the case of a parameter varying in some R^k.

Fix a non-trivial homotopy class H of C^2 maps of M to Y. According to the general theory of [9], for all parameter values s there is a harmonic map f_s in H for the g'_s metric on Y. We now assume that the curvature of this metric is strictly negative at some point of the image of f_s, whereupon we are assured by Theorem H of [15] (*cf.* also Theorem 4 above), that f_s is unique in H unless $f_s(M)$ is a closed g'_s-geodesic in Y, in which case it is unique up to a „rotation'' along the geodesic (*cf.* [15], Theorem I, where it is assumed that the curvature is < 0 all along the image). We shall henceforth exclude this situation unless M is a circle, and then of course all the $f_s(M)$ are closed geodesics.

It is not difficult to see that f_s, as function of s, is continuous in the C^1 topology for maps, provided in the case M = circle that suitable "rotations" are first effected. That follows simply from the derivative bounds of [9]. And those bounds, by standard theory, imply similar bounds on derivatives of all orders, from which the continuity of f_s in the C^∞ topology follows. But in order to obtain smooth dependence on s we turn to another method.

7. Smoothness

Suppose that f_s depends smoothly on s in the sense that, referred as usual to local coordinates (x^i) and (y^α), the functions $y^\alpha(x, s)$ and their first- and second-order partial derivatives with respect to x are C^1 functions of s. Denote by ξ the deformation vector with components $\xi^\alpha = \partial y^\alpha/\partial s$. It can be shown by straightforward calculation that ξ satisfies the linear equation

$$(\Delta\xi)^\alpha + R'^\alpha_{\beta\mu\gamma} y^\beta_i y^\gamma_j g^{ij} \xi^\mu + A^\alpha_{\beta\gamma} y^\beta_i y^\gamma_j g^{ij} = 0, \tag{4}$$

where we put (*cf.* paragraph 5):

$$\begin{cases} (D\xi)^\alpha_i = \xi^\alpha_{|i} = \dfrac{\partial\xi^\alpha}{\partial x^i} + \Gamma'^\alpha_{\beta\gamma}\xi^\beta y^\gamma_i, \\ \xi^\alpha_{|ij} = \dfrac{\partial\xi^\alpha_{|i}}{\partial x^j} - \Gamma^k_{ij}\xi^\alpha_{|k} + \Gamma'^\alpha_{\beta\gamma}\xi^\beta_{|i} y^\gamma_j, \\ (\Delta\xi)^\alpha = g^{ij}\xi^\alpha_{|ij}; \qquad A^\alpha_{\beta\gamma} = \dfrac{\partial\Gamma'^\alpha_{\beta\gamma}}{\partial s}. \end{cases} \tag{5}$$

We note that this last is a tensor field on Y.

We now work backwards. Equation (4) is defined for any C^2 map $f : M \to Y$. The associated homogeneous equation is

$$(\Delta\xi)^\alpha + R'^\alpha_{\beta\mu\gamma} y^\beta_i y^\gamma_j g^{ij} \xi^\mu = 0. \tag{6}$$

For a solution ξ of (6) let us take the g'_s inner product of (6) with ξ and integrate by parts. We get

$$-\int_M |D\xi|^2 + \int_M R'_{\alpha\beta\mu\gamma}\xi^\alpha y^\beta_i \xi^\mu y^\gamma_j g^{ij} = 0. \tag{7}$$

By our curvature hypothesis, the second term is $\leqq 0$, and so both terms vanish, whence $D\xi = 0$. Therefore $d|\xi| = 0$; i. e., ξ has constant length. If there is a point in the image of f where the rank of df is greater than unity and where the curvature is strictly negative, then the second integrand above is < 0 there, if $\xi \neq 0$, which contradicts (7).

For the case M = circle and for the family f_s under consideration above, assumed differentiable, let M be parametrized by the angle t. Then (6) becomes

$$(D^2_t\xi)^\alpha + R'^\alpha_{\beta\mu\gamma}\tau^\beta\tau^\gamma\xi^\mu = 0, \tag{8}$$

D_t being the covariant derivative with respect to t, and $\tau^\alpha = \partial y^\alpha/\partial t$ being the tangent vector. This equation is satisfied by $\xi = \tau$; and that is the only solution, up to a constant factor, provided the curvature of g'_s is strictly negative somewhere along the geodesic

$c_s = f_s(\mathrm{M})$. For the argument above shows that ξ and τ must be parallel where the curvature is strictly negative, and then from (8) we see that ξ is a constant multiple of τ in such a region, hence on all of M, by uniqueness of solutions of the system (8).

Referring now to (4), in the case M = circle the last term is the vector field

$$\eta^\alpha = A^\alpha_{\beta\gamma}\tau^\beta\tau^\gamma.$$

If we take the inner product of (4) with τ and integrate, we obtain

$$\int_{\mathrm{M}} (\tau, \eta)_s\,dt = 0. \tag{9}$$

By standard theory, (9) is the necessary and sufficient condition for the solvability of (4), since in our circumstances τ is the only solution of (8), apart from constant factors.

We now show that (9) is in fact automatically satisfied. To do so, fix s and let (y^α) be a system of Fermi coordinates along a portion of the geodesic $c_s = f_s(\mathrm{M})$, where we take y^1 to be the arclength from some point. For the components $g'_{\alpha\beta}$ of the metric g'_s we shall have $g'_{11} = 1$ and $g'_{1\alpha} = 0$ along c_s for $\alpha > 1$. Furthermore, $\Gamma'^{\alpha}_{11} = 0$ along c_s for all α; and $\tau^1 = 1$, $\tau^\alpha = 0$ for $\alpha > 1$. Then $(\tau, \eta)_s = g'_{\alpha\beta}\tau^\alpha\eta^\beta = \eta^1 = A^1_{11}$. Here:

$$A^1_{11} = \frac{\partial}{\partial s}(g'^{1\alpha}\Gamma'_{11\alpha}) = \left(\frac{\partial}{\partial s}g'^{1\alpha}\right)\Gamma'_{11\alpha} + g'^{1\alpha}\frac{\partial}{\partial s}\Gamma'_{11\alpha}.$$

Since $\Gamma'_{11\alpha} = 0$ along c_s for all α, we have along c_s:

$$A^1_{11} = \frac{\partial}{\partial s}\Gamma'_{111} = \frac{1}{2}\frac{\partial}{\partial s}\frac{\partial g'_{11}}{\partial y^1}.$$

Writing:

$$\varphi = \frac{1}{2}\frac{\partial}{\partial s}g'_{11},$$

we obtain, integrating round an arc of the circle M,

$$\int_a^b (\tau, \eta)_s\,dt = \int_a^b \frac{\partial\varphi}{\partial y^1}\frac{\partial y^1}{\partial t}\,dt = \varphi(f_s(b)) - \varphi(f_s(a)).$$

Adding up round M we obtain (9).

Hence (4) can be solved for ξ, and the solution can be made unique by the condition

$$\int_{\mathrm{M}} (\xi, \tau)_s\,dt = 0.$$

We henceforth assume that the metric g'_s has sectional curvature < 0 for each s, except perhaps at isolated points. If $f \in \mathrm{H}$ is any C^k map, we can then apply general elliptic theory to solve (4) for fixed s, obtaining a unique C^k vector field $\xi_s(f)$ along f (bearing in mind the preceding normalization in case M = circle). Thus ξ_s can be considered as a vector field defined on H, which is a Banach-type manifold when equipped with the C^2 topology. We are going to consider solutions of the non-autonomous equation

$$\frac{\partial h_s}{\partial s} = \xi_s(h_s) \tag{10}$$

in H, starting at $s = 0$ with the initial condition $h_0 = f_0$, i. e. the harmonic map in H for the g'_0 metric on Y.

Let $|f, h|_2$ be a distance function on H giving the C^2 topology; and let $|\xi, \eta|_2$ be such a function on the space C^2 (M, TY). We shall then have

$$|\xi_s(f), \xi_{s'}(h)|_2 \leqq C(|f|_2 + |h|_2) \cdot |f, h|_2, \tag{11}$$

where $|f|_2$ is the maximum of the first and second partial derivatives of f with respect to some fixed, finite covering of M and Y by charts; C is a constant independent of f, h, s, s'. The bound here is a simple consequence of the uniform ellipticity of (4). See the Remark below.

We can then solve (10), starting at $s = 0$, at least for a small s-interval. A simple calculation show that the derivative with respect to s of the tension field $\tau_s(h_s)$ vanishes (τ_s being calculated with the metric g'_s, of course). Since $\tau_0(h_0) = 0$, we conclude that the h_s are harmonic, hence must coincide with the f_s (apart from rotations in the case M = circle). Derivative bounds from [9] can therefore be invoked to show that the bounds (11) do not deteriorate along the solution path, so that the latter can be continued for $0 \leqq s \leqq 1$.

We can repeat the same procedure for any C^k topology, $k = 2, 3, \ldots$ and therefore obtain the following result:

THEOREM 9. — *The space derivatives of f_s of all orders are of class C^1 in s.*

Here we understand that we take $f_s = h_s$ in the case M = circle.

If we differentiate (4) with respect to s, we obtain a new equation for the derivative $\xi' = D_s \xi$; and the first two terms are the same except that ξ is replaced by ξ'. It follows that ξ' is C^∞ in the space variables. The foregoing analysis can be repeated, with only minor changes. In this way it is not difficult to show that f_s is in fact of class C^∞ in all arguments. We omit the details.

Remark. — To solve (4) we can use the techniques of paragraph 7 of [9] to obtain global equations on M, by embedding Y in some Euclidean space R^q. One finds easily that the deformation vector ξ is given by a q-tuple $(\xi^1, \ldots, \xi^q)$ of global functions on M which satisfy a system

$$\Delta \xi^a = U^a_{bcd} W^b_i W^c_j g^{ij} \xi^d + V^a_{bc} W^b_i \xi^c_j g^{ij} + A^a_{bc} W^b_i W^c_j g^{ij}, \tag{4 bis}$$

where the coefficients are C^∞ functions of the coordinates W^a of $f_s(p)$, and of s. It is quite simple to obtain bounds of the type (11) for any C^k topology from this system.

8. On moduli of Riemann surfaces

Here we shall sketch our original applications of harmonic mappings to Teichmüller theory. An extensive development of the topic is given in [7] along rather different lines.

We fix a closed, oriented surface X of genus $\geqq 2$, and we denote by X_g that surface with a C^∞ metric g. By the theorem of Korn-Lichtenstein, a metric g on X determines a complex structure on X_g, given locally by any coordinate system $z = x^1 + i x^2$ such that (x^1, x^2) is a positive coordinate system for the orientation of X, and for which the matrix of g is diagonal, say $= \rho . I$. In that case, $g_{ij}\, dx^i\, dx^j = \rho\, (dx^1)^2 + \rho\, (dx^2)^2 = \rho\, dz\, d\bar{z}$. z is then called a local uniformizing parameter.

Although we shall require a more controlled situation, we observe in passing that X, given a conformal structure, can always be equipped with a conformal metric of negative curvature. Namely, if $\omega_1, \ldots, \omega_p$ (p = genus) is a basis for the differentials of first kind on X, then $\sum \omega_j\, \bar{\omega}_j$ defines such a metric. That this is a true-blue metric follows from the Riemann-Roch theorem. Indeed, from that theorem it is not hard to see that one can always choose two differentials of first kind, say φ, θ, such that $\varphi\bar{\varphi} + \theta\bar{\theta}$ is a conformal metric on M.

As for the curvature, if z is a local uniformizing parameter, let $\omega_j = f_j\, dz$, so that $\sum \omega_j\, \bar{\omega}_j = \rho\, dz\, d\bar{z}$, with $\rho = \sum f_j \bar{f}_j$. The curvature $\varkappa$ is then:

$$\begin{aligned}\varkappa &= -\frac{2}{\rho}\, \frac{d^2}{dz\, d\bar{z}} \log \rho \\ &= -\frac{2}{\rho^3} [\sum f_j' \bar{f}_j \sum f_k \bar{f}_k' - \sum f_j \bar{f}_j \sum f_k' \bar{f}_k'],\end{aligned}$$

which is $\leqq 0$, by Hölder's inequality (*cf.* also [4]). $\varkappa$ can vanish at only a finite number of points, namely at certain Weierstrass points of X.

Metrics of the above type have certain advantages, since they can be constructed rationally from an equation $f(z, w) = 0$ defining X as an algebraic curve. We note that from Cartan's theorem ([17]. Chap. 1, § 13) and the negative curvature of our metrics, that the universal covering surface $\tilde{X}$ of X is a cell.

In fact, of course $\tilde{X}$ is conformally equivalent to the upper half-plane H, and the Poincaré metric on H induces on X a metric of constant curvature -1. This metric is of particular importance to us. It is however highly transcendental. The following result shows that the curvature determines the metric.

THEOREM 10. — *Let $\rho\, dz\, d\bar{z}$ and $\rho'\, dz\, d\bar{z}$ be two conformal metrics on the compact Riemann surface* X, *and suppose that they have the same curvature $\varkappa \leqq 0$ ($\varkappa \not\equiv 0$). Then they are equal.*

Proof. — From standard calculations $\Delta \log \rho = -\varkappa/2$ and $\Delta' \log \rho' = -\varkappa/2$, where

$$\Delta = 4\rho^{-1} \frac{\partial^2}{\partial z\, \partial \bar{z}} \qquad \text{and} \qquad \Delta' = 4\rho'^{-1} \frac{\partial^2}{\partial z\, \partial \bar{z}} = \frac{\rho}{\rho'} \Delta.$$

Thus, with $c = -\varkappa/2$,

$$\Delta \log \rho - \Delta \log \rho' = c - \frac{\rho'}{\rho} c,$$

i. e.:

$$\Delta \log \frac{\rho}{\rho'} = c\left(1 - \frac{\rho'}{\rho}\right).$$

Set $f = \log(\rho/\rho')$ (this is a globally defined function on X, although ρ and ρ' are only local functions). We have

$$\Delta f = c(1 - e^{-f}).$$

If f is not constant, then it has an absolute maximum at some point p, and $\Delta f \leqq 0$ at p. The right member must also have an absolute maximum at p, and consequently $\Delta f \leqq 0$ everywhere, whence $f = \text{Const.}$ But then $\Delta f = 0$, and so $f = 0$ if $c \neq 0$.

Q.E.D.

Thus, in particular, the conformal metric of curvature -1 on a Riemann surface X is uniquely determined by the conformal structure.

9. Quadratic differentials

For the present we consider harmonic mappings $f : X \to Y$ of any two compact Riemann surfaces with C^∞ conformal metrics. If our metrics are $\rho\, dz\, d\bar{z}$ (resp. $\lambda\, dw\, d\bar{w}$) near a point $p \in X$ [resp. $f(p) \in Y$], then the harmonic equation (1) takes the form

$$w_{z\bar{z}} + \frac{\lambda_w}{\lambda} w_z w_{\bar{z}} = 0. \tag{13}$$

We note that the harmonic property is not altered if the metric on X is replaced by any conformally equivalent metric. But that is not so for Y (except when f is holomorphic or antiholomorphic).

With any C^2 mapping $f :\ X \to Y$ we associate the *quadratic differential*

$$Q = \varphi\, dz^2 = \lambda\, w_z \bar{w}_z\, dz^2 \tag{14}$$

on X. If f is harmonic, one shows easily from (13) that Q is holomorphic (*cf.* [24] and [27]). It vanishes only when f is holomorphic or anti-holomorphic. Observe that Q is just the (2,0) part of the form $f^*(\lambda\, dw\, d\bar{w})$ on X.

Conversely, assume that Q is holomorphic. Then $Q_{\bar{z}} = 0$, which is

$$\lambda_w w_{\bar{z}} w_z \bar{w}_z + \lambda_{\bar{w}} \bar{w}_{\bar{z}} w_z \bar{w}_z + \lambda\, w_{z\bar{z}} \bar{w}_z + \lambda\, \bar{w}_{z\bar{z}} w_z = 0,$$

whence

$$\bar{w}_z P + w_z \bar{P} = 0,$$

where P is the left member of (13). If $P \neq 0$ near a point p, then we have $|w_z| = |\bar{w}_z| = |w_{\bar{z}}|$. The Jacobian of $z \to w$ is equal to $|w_z|^2 - |w_{\bar{z}}|^2$, and we see that the Jacobian vanishes near p.

We now require some computations. For a harmonic f set

$$(15) \qquad H = \lambda w_z \bar{w}_{\bar{z}}, \qquad L = \lambda w_{\bar{z}} \bar{w}_z, \qquad J = H - L.$$

These are of course local functions on X. But the quantities

$$(16) \qquad \mathcal{H} = \rho^{-1} H, \qquad \mathcal{L} = \rho^{-1} L, \qquad \mathcal{J} = \rho^{-1} J$$

are globally defined functions, and $\mathcal{J}$ is the Jacobian of f.

The curvature of Y is computed from

$$(\log \lambda)_{w\bar{w}} = -\frac{\varkappa}{2}\lambda.$$

i. e.:

$$(17) \qquad \lambda_{w\bar{w}} = -\frac{\varkappa}{2}\lambda^2 + \frac{\lambda_w \lambda_{\bar{w}}}{\lambda}.$$

Using this in conjunction with (13), one finds:

$$(18) \qquad \begin{cases} H_z = (\lambda_w w_z^2 + \lambda w_{zz}) \bar{w}_{\bar{z}}, \\ H_{z\bar{z}} = -\dfrac{\varkappa}{2} H^2 + \dfrac{\varkappa}{2} \varphi\bar{\varphi} + \dfrac{H_z H_{\bar{z}}}{H}, \end{cases}$$

where

$$\varphi = \lambda w_z \bar{w}_z,$$

as in (14). Note that $\varphi\bar{\varphi} = HL$. Since φ is holomorphic, if it is not zero we conclude that *$\mathcal{H}$ and $\mathcal{L}$ can have only isolated zeroes of finite order* (*see* Eells-Wood [10]).

We have also

$$(19) \qquad \begin{cases} L_z = (\lambda \bar{w}_{zz} + \lambda_{\bar{w}} \bar{w}_z^2) w_{\bar{z}}, \\ L_{z\bar{z}} = -\dfrac{\varkappa}{2} L^2 + \dfrac{\varkappa}{2} \varphi\bar{\varphi} + \dfrac{L_z L_{\bar{z}}}{L}. \end{cases}$$

On X the Laplace operator is $\Delta = 4\rho^{-1} \partial^2/\partial z\, \partial \bar{z}$. We compute $\Delta\mathcal{H}$ where $\mathcal{H}$ is defined in (16). The result is

$$\frac{1}{4}\Delta\mathcal{H} = \frac{\gamma}{2} H + \frac{\rho_z \rho_{\bar{z}}}{\rho^3} H - \rho^{-3}(\rho_z H_{\bar{z}} + \rho_{\bar{z}} H_z) + \rho^{-2} H_{z\bar{z}},$$

where γ = curvature of X. A similar equation holds for $\mathcal{L}$. We obtain:

$$(20) \qquad \Delta \log \frac{\mathcal{H}}{\mathcal{L}} = -2\varkappa\mathcal{J}.$$

Now let f be a harmonic mapping of topological degree 1. Then $\mathscr{H}$ cannot vanish identically, since otherwise we would have $\mathscr{J} \leqq 0$ everywhere. Consequently either $\mathscr{L} = 0$ (holomorphic case), and then $\mathscr{H}$ has only a finite number of zeroes, each of finite multiplicity, since w_z in (15) is holomorphic, or else $\mathrm{HL} = \varphi\bar{\varphi} \not\equiv 0$, and the same conclusion obtains, since φ is holomorphic. We are going to prove:

PROPOSITION 1. — *If* X *and* Y *have the same genus, and if* $\deg f = 1$, *then* $\mathscr{H} > 0$ *everywhere.*

Our original proof was restricted to $\varkappa < 0$. The fundamental step here is the index theorem of Eells-Wood [10]: Let $\mathscr{H}$ have zeroes at $p_1, \ldots, p_r$. Then there are integers $n_i > 0$ such that $z_i^{-n_i}\mathscr{H} \neq 0$ at p_i $(i = 1, \ldots, r)$, z_i being a local uniformizing parameter at p_i. According to [10], Proposition 1, $n_1 + \ldots + n_r = 0$. Hence $r = 0$.

Q.E.D.

We now look at the Jacobian $\mathscr{J}$. If $\mathscr{J} \leqq 0$ in a region, i. e. $\mathscr{L} \geqq \mathscr{H}$, then also $\mathscr{L} > 0$ there, and so $\log(\mathscr{H}/\mathscr{L})$ is finite. We prove:

PROPOSITION 2. — *Under the hypotheses of Proposition* 1, *if* $\varkappa \leqq 0$, *then* $\mathscr{J} \geqq 0$ *on* X.

Proof. — By (20), $\Delta \log(\mathrm{H/L}) \leqq 0$ wherever $\mathscr{J} \leqq 0$, and so $\log(\mathscr{H}/\mathscr{L})$ is superharmonic in any region where $\mathscr{J} \leqq 0$. Therefore it is not possible to have $\mathscr{J} < 0$ at a point of X.

Q.E.D.

Combining our results, we shall obtain:

THEOREM 11. — *If* $f : \mathrm{X} \to \mathrm{Y}$ *is a harmonic mapping of degree* 1 *of compact surfaces of the same genus, and if the curvature of* Y *is* $\leqq 0$, *then* f *is a diffeomorphism.*

This follows directly from Proposition 2 and from fundamental local results of Wood [29].

Let us turn briefly to the situation when X has genus 1. Then on X, up to a constant multiple, there is but one holomorphic quadratic differential, and it vanishes nowhere. Thus, if $f : \mathrm{X} \to \mathrm{Y}$ is harmonic (Y of any genus), then either f is $\pm$ holomorphic, or else H and L never vanish. In the latter case we have (20) in force over all of X. Integration and Stokes's theorem give

$$0 = \int_{\mathrm{X}} \varkappa \mathrm{J} = \mathrm{C} \int_{\mathrm{Y}} \varkappa = \mathrm{C}.2\pi\chi,$$

C being a constant which is non-zero if $\deg f \neq 0$, and χ being the Euler-Poincaré characteristic of Y (Gauss-Bonnet theorem). Thus either $\chi = 0$ or else J vanishes, whence f is constant or else maps X onto a closed geodesic of Y (*cf.* Th. 3). This is a special case of a result of [10], *see* also [30].

We now revert to the notation of paragraph 8. Namely, X_g denotes the fixed compact surface with the metric g and the consequent conformal structure. g_0 is a fixed reference metric. The main result of this paragraph is:

THEOREM 12. — *Let* g *and* g' *be two metrics on* X *of curvature* -1. *If* $f : \mathrm{X}_{g_0} \to \mathrm{X}_g$ *and* $f' : \mathrm{X}_{g_0} \to \mathrm{X}_{g'}$ *are harmonic mappings of degree* 1 *which induce the same quadratic differential on* X_{g_0}, *then* X_g *and* $\mathrm{X}_{g'}$ *are conformally equivalent.*

Proof. — Consider the functions $\mathscr{H}$, $\mathscr{L}$ (resp. $\mathscr{H}'$, $\mathscr{L}'$) of (10) for g and g'. By Proposition 1, $\mathscr{H}$ and $\mathscr{H}'$ do not vanish. We have $\mathscr{H}\mathscr{L} = \mathscr{H}'\mathscr{L}'$, from our hypotheses, and from (14) or (20) it is easily seen that

$$\Delta \log \frac{\mathscr{H}'}{\mathscr{H}} = \mathscr{J}' - \mathscr{J} = (\mathscr{H}' - \mathscr{H})\left(1 + \frac{\mathscr{L}'}{\mathscr{H}}\right).$$

Suppose now that $\mathscr{H}' > \mathscr{H}$ at some point. Then at a maximum of $\mathscr{H}'/\mathscr{H}$ the function $\log(\mathscr{H}'/\mathscr{H})$ also has a maximum, and the left side of the equation there is $\leqq 0$, whereas the right side is > 0, an evident contradiction. Thus we must have $\mathscr{H}' \leqq \mathscr{H}$, and similarly $\mathscr{H} \leqq \mathscr{H}'$. Therefore $\mathscr{H} = \mathscr{H}'$. Now $\varphi\bar{\varphi} = \rho^2 \mathscr{L}\mathscr{H} = \rho^2 \mathscr{L}'\mathscr{H}'$, and we find that $\mathscr{L} = \mathscr{L}'$. It follows at once that the diffeomorphism $f_{g'} f_g^{-1} : X_g \to X_{g'}$ is an isometry, which proves our assertion.

Harmonic mappings $X_{g_0} \to X_g$ homotopic to the identity therefore yield an injective mapping of the set of surfaces X_g with curvature -1 into the space of quadratic differentials on X_{g_0}. We shall develop this further elsewhere.

10. An application to automorphic varieties

We first mention an easy consequence of [15] (*cf.* the Theorem of Borel and Narsimhan in [20], Theorem 8.11).

THEOREM 12. — *Let* Y *be a Kähler manifold of sectional curvature* < 0 *everywhere; and let* V, V′ *be compact complex submanifolds of* Y *of dimension* > 0. *Suppose that there is a holomorphic mapping* $f : V \to V'$ *which is homotopic in* Y *to the identity mapping* $V \to Y$. *Then* $V = V'$ *and* $f =$ *identity.*

Proof. — f is harmonic as mapping of V into Y, relative to the induced Kähler metric on V ([9], p. 118), and the assertion follows from [15], Theorem B.

We note the following adaptation:

If there are only a finite number k of homotopy classes containing holomorphic mappings $V \to Y$, *then there are at most* $k-1$ *analytic submanifolds* V′ *of* X *which are holomorphically equivalent to* V. The assumption is true for compact Y, by [8], paragraph 9.13.

For if $f : V \to V'$ is holomorphic, then it is the only one in its class (i. e. as mapping $V \to Y$). We of course exlude the constant mapping, whence $k-1$ instead of k. The curvature assumptions can be weakened.

These considerations have a certain bearing on a conjecture which we shall now state, considerably generalizing the well-known conjecture of Mordell (*see* [21] and [21 *a*]).

Let D be a domain in C^n equivalent to a bounded domain, and let G be a discontinuous group of rational mappings of C^n which are automorphisms of D without fixed points (except the identity, of course). Then $M = D \bmod G$ is a complex manifold, which we assume compact; and it is known that M can be embedded analytically in a complex projective space P^N (*cf.* [6], [26]). Therefore M is an algebraic variety, by Chow's theorem. We assume further that D admits a Kähler metric of strictly negative curvature which is invariant under the transformations of G. Our conditions are those of the classical automorphic varieties, when D is an irreducible symmetric domain.

Now suppose that the transformations of the group and the subvariety $M \subset P^N$ are defined over some number field K. Our conjecture is that *there are only finitely many subvarieties of* M *of given degree which are defined over* K.

For M of complex dimension 1 (the subvarieties are then points), this is Mordell's conjecture.

REFERENCES

[1] N. Aronszajn, *A Unique Continuation Theorem for Solutions of Elliptic Partial Differential Equations or Inequalities* (*J. Math. pures et appl.*, T. 36, (1957, pp. 235-249).

[2] L. Bers, F. John and M. Schechter, *Partial Differential Equations*, Interscience, New York, 1964.

[3] S. Bochner, *Harmonic Surfaces in Riemannian Metric* (*Trans. Amer. Math. Soc.*, Vol. 47, 1940 pp. 146-154).

[4] S. Bochner, *Curvature in Hermitian Metric* (*Bull. Amer. Math. Soc.*, Vol. 53, 1947, pp. 179-195).

[5] S. Bochner, *Curvature and Betti Numbers* (*Ann. Math.*, Vol. 49, 1948, pp. 379-390).

[6] A. Borel, *Les fonctions automorphes de plusieurs variables complexes* (*Bull. Soc. Math.*, T. 80, 1952, pp. 167-182).

[7] C. Earle and J. Eells Jr., *A Fiber Bundle Description of Teichmüller Theory* (*Compos. Math.*, Vol. 21, 1969, pp. 155-161).

[8] J. Eells Jr. and L. Lemaire, *A Report on Harmonic Maps* (to appear in the *Bull. London Math. Soc.*).

[9] J. Eells Jr. and J. H. Sampson, *Harmonic Mappings of Riemannian Manifolds* (*Amer. J. Math.*, Vol. 86, 1964, pp. 109-160).

[10] J. Eells Jr. and J. C. Wood, *Restrictions on Harmonic Maps of Surfaces* (*Topology*, Vol. 15, 1976, 263-266).

[11] L. P. Eisenhart, *Riemannian Geometry*, Princeton, 1949.

[12] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, N. H., 1964.

[13] A. Friedman, *Interior Estimates for Parabolic Systems of Partial Differential Equations* (*J. Maths. and Mech.*, Vol. 7, 1958, pp. 393-418).

[13 *a*] M. Gerstenhaber and H. E. Rauch, *On Extremal quasi-conformal Mappings.* I and II (*Proc. Nat. Acad. Sc.*, Vol. 40, 1954, pp. 808-816 and 991-994).

[14] R. S. Hamilton, *Harmonic Mappings of Manifolds with Boundary* (*Springer Lecture Notes in Mathematics*, No. 471, 1975, New York).

[15] P. Hartman, *On Homotopic Harmonic Maps* (*Can. J. Math.*, Vol. 19, 1967, pp. 673-687).

[16] E. Heinz, *Über gewisse elliptische Systeme von Differentialgleichungen zweiter Ordnung mit Anwendung auf die Monge-Ampèresche Gleichung* (*Math. Ann.*, Vol. 131, 1956, pp. 411-428).

[17] S. Helgason, *Differential Geometry and Symmetric Spaces.* Academic Press, New York, 1962.

[18] A. Hurwitz, *Über algebraische Gebilde* u. s. w., Gesammelte Abhandlungen.

[19] J. Kazdan and F. W. Warner, *Curvature Functions for Compact 2-manifolds* (*Ann. Math.*, Vol. 99, 1974, pp. 14-47).

[20] S. Kobayashi, *Intrinsic Distances, Measures and Geometric Theory* (*Bull. Amer. Math. Soc.*, Vol. 82, 1976, pp. 357-416).

[21] S. Lang, *Integral Points on Curves* (*Publ. Math.*, No. 6, I.H.E.S., Paris, 1960, pp. 27-43).

[21 *a*] S. Lang, *Some Theorems and Conjectures in Diophantine Equations* (*Bull. Amer. Math. Soc.*, Vol. 66, 1960, pp. 240-249).

[22] M. Lees and M. H. Protter, *Unique Continuation for Parabolic Differential Equations and Inequalities* (*Duke Math. J.*, Vol. 28, 1961, pp. 369-382).

[23] L. NIRENBERG, *A Strong Maximum Principle for Parabolic Equations* (*Comm. Pure and Appl. Maths.*, Vol. 6, 1953, pp. 167-177).

[23 *a*] O. NOUHAUD, *Déformations infinitésimales harmoniques* (*C. R. Acad. Sc.*, T. 275, série A, 1972, pp. 999-1001 and 1103-1106).

[24] H. E. RAUCH, *On the Transcendental Moduli of Algebraic Riemann Surfaces* (*Proc. Nat. Acad. Sc.*, Vol. 41, 1965, pp. 42-49, 176-180 and 231-238).

[25] R. SCHOEN and S. T. YAU, *Univalent Harmonic Maps of Surfaces* [*Invent. Math.* (to appear)].

[26] J. H. SAMPSON, *A Note on Automorphic Functions* (*Proc. Nat. Acad. Sc.*, Vol. 38, 1952, pp. 895-898).

[27] K. SHIBATA, *On the Existence of a Harmonic Mapping* (*Osaka Math. J.*, Vol. 15, 1963, pp. 173-211).

[28] R. T. SMITH, *Harmonic Mappings of Spheres* (*Dissertation*, Warwick University, 1972).

[29] J. C. WOOD, *Singularities of Harmonic Maps and Applications of the Gauss-Bonnet Formula* [*Amer. J. Math.* (to appear, at last)].

[30] J. C. WOOD, *Harmonic Maps and Complex Analysis* (*Proc. of a Summer Course in Complex Analysis*, Vol. III, Trieste, 1975, pp. 289-308).

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