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ON FAMILIES OF PISOT E-SEQUENCES

BY DAVID G. CANTOR (*)

In his investigations of the fractional parts of the sequence $\{ \lambda\theta^n \mid n = 0, 1, 2, \ldots \}$, Pisot [10] introduced "E-sequences". These are sequences of integers $e_0 > 0, e_1, e_2, \ldots$ with the property that $e_n = \mathrm{N}(e_{n-1}^2/e_{n-2})$ for all integral $n \geqq 2$. [Here, $\mathrm{N}(x)$ is the "nearest" integer to the real number x; i. e. $x-1/2 < \mathrm{N}(x) \leqq x+1/2$.] If $\lambda \neq 0$ and $\theta > 1$ are real numbers, then Pisot showed that if

$$\limsup_{n\to\infty} \| \lambda\theta^n \| < \frac{1}{2(1+\theta)^2},$$

then the $e_n = \mathrm{N}(\lambda\theta^n)$ form an E-sequence for n sufficiently large. [$\| x \|$ denotes $|x-\mathrm{N}(x)|$, the "distance" from x to the nearest integer.] Conversely, he showed that each E-sequence, except for certain trivial exceptions, gives rise to $\theta = \lim_{n\to\infty} e_n/e_{n-1}$ and $\lambda = \lim_{n\to\infty} (e_{n-1}^n/e_n^{n-1})$. He further showed that each E-sequence with $e_0 = 2$ or $e_0 = 3$ satisfies a linear recurrence relation with constant coefficients. The form of these relations seemed to depend, in a mysterious way, on $e_1 \pmod{e_0^2}$. Flor [6] analysed the structure of possible recurrence relations for E-sequences. Very recently Boyd [2] proved the remarkable theorem that there exist E-sequences which do not satisfy any such recurrence relation and showed explicitly that $e_0 = 14$, $e_1 = 23$ begins such an E-sequence.

From another viewpoint Bateman and Duquette [1], and then Grandet-Hugot [8] investigated the formal analogues of Pisot's E-sequences over the field of formal Laurent series in one variable. They proved theorems analogous to those already known for E-sequences and analogues of some conjectured, but not proven, properties of E-sequences.

Here we will combine these viewpoints. First, we study the formal analogues of E-sequences over the field of Laurent series in one variable, using methods unique to Laurent series and not having analogues over the fields of real or complex numbers. We obtain many new results and recover all results of Bateman and Duquette, and Grandet-Hugot in a more precise form.

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Even more interesting are the applications of these results to Pisot's E-sequences. We prove, for example, that if $p(t)$ and $q(t)$ are polynomials with integral coefficients such that $q(0) = 1$, $p(0) > 0$, and all zeros of $p(t)$ have absolute value > 1, then the sequence of polynomials $\{ C_n(x) \mid n = 0, 1, 2, \ldots \}$ defined by

$$(1-xt\,p(t)/q(t))^{-1} = 1+xt\sum_{n=0}^{\infty} C_n(x)\,t^n,$$

form an E-sequence for each sufficiently large integer x, from a certain n_0 (not depending on x) on. Pisot's sequences with $e_0 = 2$ or 3 are of this type.

These "families" of E-sequences all have the property that

$$C_1(x) \qquad (\operatorname{mod} C_0(x)^2)$$

is constant, thus putting Pisot's calculations in a much more general context. Moreover, $C_n(x) - C_{n-1}(x)^2/C_{n-2}(x) \to p_0^2\, d_n$, where $\sum_{n=0}^{\infty} d_n t^n = q(t)/p(t)$ and $p_0 = p(0)$. The principal results here are Theorems 5.1, 5.3, 5.4, and 5.5.

In Section 1, we introduce certain formal identities among Laurent series which are central to all results of this paper. Section 2 is on the formal analogues of E-sequences, and section 3 covers those which are rational. Section 4 is a collection of identities and other results preparatory for Section 5, the main section of this paper.

The author would like to thank his student P. Galyean [7] who did many of the calculations from which the results of this paper were conjectured.

In what follows K denotes an arbitrary field, $\tilde{K}$ is its algebraic closure; $K[1/x]$ denotes, as usual, the ring of polynomials in $1/x$ with coefficients from K; $K[[1/x]]$ denotes the ring of formal power series in $1/x$ with coefficients from K; $K\{1/x\}$ denotes the field of formal Laurent series of the form $a(x) = \sum_{i=i_0}^{\infty} a_i/x^i$, where i_0 may be < 0. Of course $K[[1/x]]$ is a subring of $K\{1/x\}$ and $K\{1/x\}$ is its quotient field. We define *the integral part* of $a(x)$ as $[a(x)] = \sum_{i=i_0}^{0} a_i/x^i$; it is a polynomial in x, and the *fractional part* of $a(x)$ as $\{a(x)\} = \sum_{i=1}^{\infty} a_i/x^i$; it is in $K[[1/x]]$ and has constant term 0. If $a_{i_0} \neq 0$, define the degree of $a(x)$ by $\deg(a(x)) = -i_0$. The zero series has degree $-\infty$. This definition of degree coincides with the customary one for polynomials in $K[x]$. We make $K\{1/x\}$ into a topological field by defining sets of the form

$$\{a(x) \in K\{1/x\} \mid \deg a(x) \leqq i\}$$

to be a fundamental basis of open (and closed) neighborhoods of 0. We denote the field of real numbers by **R** and the field of complex numbers by **C**. If $a(x) \in \mathbf{C}\{1/x\}$ and $b(x) = \sum_{i=i_0}^{\infty} b_i/x^i \in R\{1/x\}$, we write $a(x) \ll b(x)$ if $|a_i| \leqq b_i$ for all i. If $x \in \mathbf{R}$ denote by $[x]$ the greatest integer $\leqq x$.

1. The basic identities

Suppose that $a(x)=\sum_{i=1}^{\infty} a_i/x^i$ and $u(x)=\sum_{i=0}^{\infty} u_i/x^i$ are formal power series in $1/x$ with coefficients from a field K, and suppose that $a_1 \neq 0$. Under these conditions there exists a unique formal power series $b(y)=\sum_{i=1}^{\infty} b_i/y^i$ with coefficients from the field K satisfying $b(1/a(x))=1/x$, or equivalently $a(1/b(y))=1/y$ [9]. These relations imply that $a_1 b_1 = 1$. Now put

(1.1) $$v(y)=-yu(1/b(y))\,b'(y)/b(y),$$

and write $v(y)=\sum_{i=0}^{\infty} v_i/y^i$. Substituting $y=1/a(x)$ into (1.1) and simplifying yields the (equivalent) formula

(1.2) $$u(x)=-xv(1/a(x))\,a'(x)/a(x).$$

Substituting $y=\infty$ into (1.1) or $x=\infty$ into (1.2) yields $u_0=v_0$.

We now define two sequences $\{\mathrm{A}_n(x) \mid n=0, 1, 2, \ldots\}$, $\{\mathrm{B}_n(y) \mid n=0, 1, 2, \ldots\}$ of polynomials by

(1.3) $$\sum_{n=0}^{\infty} \mathrm{B}_n(y)/x^n=\frac{u(x)}{1-ya(x)}$$

and

(1.4) $$\sum_{n=0}^{\infty} \mathrm{A}_n(x)/y^n=\frac{v(y)}{1-xb(y)}.$$

It is easy to verify that the polynomials $\mathrm{A}_n(x)$ and $\mathrm{B}_n(y)$ have degree $\leqq n$ and that the coefficient of x^n in $\mathrm{A}_n(x)$ [respectively, y^n in $\mathrm{B}_n(y)$] is $v_0 b_1^n$ (respectively $u_0 a_1^n$). Some specific values are

(1.5) $$\begin{cases} \mathrm{A}_0=v_0, & \mathrm{B}_0=u_0,\\ \mathrm{A}_1=v_0 b_1 x+v_1, & \mathrm{B}_1=u_0 a_1 y+u_1,\\ \mathrm{A}_2=v_0 b_1^2 x^2+(v_0 b_2+v_1 b_1)x+v_2, & \mathrm{B}_2=u_0 a_1^2 y^2+(u_0 a_2+u_1 a_1)y+u_2. \end{cases}$$

The basic identity is given in the following theorem.

1.6. THEOREM. — *We have*

(1.7) $$\begin{cases} \mathrm{A}_n(x)=u(x)/a(x)^n+\alpha_n(x),\\ \mathrm{B}_n(y)=v(y)/b(y)^n+\beta_n(y), \end{cases}$$

where $\alpha_n(x)=\sum_{i=1}^{\infty}\alpha_{ni}/x^i$ *and* $\beta_n(y)=\sum_{i=1}^{\infty}\beta_{ni}/y^i$, $n=1, 2, 3, \ldots$ *are formal power series with coefficients from* K. *Furthermore*

(1.8) $$\frac{u(x)}{1-ya(x)}+\frac{v(y)}{1-xb(y)}=u_0+\sum_{n=1}^{\infty}\alpha_n(x)/y^n=v_0+\sum_{n=1}^{\infty}\beta_n(y)/x^n$$

and

$$\alpha_0(x) = u_0 - u(x), \qquad \beta_0(y) = v_0 - v(y). \tag{1.9}$$

Proof. — By (1.3), $B_n(y)$ is the coefficient of $1/x^n$ in $u(x)/(1-ya(x))$; equivalently $B_n(y)$ is the residue of $x^{n-1}u(x)\,dx/(1-ya(x))$. Substitute $x = 1/b(z)$, $dx = -b'(z)\,dz/b(z)^2$. Then $B_n(y)$ is the residue of

$$\frac{-zu(1/b(z))\,b'(z)}{b(z)^{n+1}(1-y/z)}\frac{dz}{z} = \frac{v(z)}{b(z)^n(1-y/z)}\frac{dz}{z} = \frac{v(z)}{b(z)^n}\sum_{j=0}^{\infty}\frac{y^j}{z^j}\frac{dz}{z}.$$

Thus $B_n(y) = [v(y)/b(y)^n]$. Define

$$\beta_n(y) = -\{v(y)/b(y)^n\} = B_n(y) - v(y)/b(y)^n.$$

Interchanging the roles of x and y, define

$$\alpha_n(x) = -\{u(x)/a(x)^n\} = A_n(x) - u(x)/a(x)^n.$$

This shows (1.7). Next

$$\begin{aligned}\frac{u(x)}{1-ya(x)} &= \sum_{n=0}^{\infty} B_n(y)/x^n \\ &= \sum_{n=0}^{\infty}\frac{v(y)}{x^n b(y)^n} + \sum_{n=0}^{\infty}\frac{\beta_n(y)}{x^n} \\ &= \frac{v(y)}{1-1/(xb(y))} + \beta_0(y) + \sum_{n=1}^{\infty}\frac{\beta_n(y)}{x^n} \\ &= \frac{-v(y)\,b(y)\,x}{1-xb(y)} - v(y) + v_0 + \sum_{n=1}^{\infty}\frac{\beta_n(y)}{x^n} \\ &= \frac{-v(y)}{1-xb(y)} + v_0 + \sum_{n=1}^{\infty}\frac{\beta_n(y)}{x^n}.\end{aligned}$$

Thus

$$\frac{u(x)}{1-ya(x)} + \frac{v(y)}{1-xb(y)} = v_0 + \sum_{n=1}^{\infty}\beta_n(y)/x^n = u_0 + \sum_{n=1}^{\infty}\alpha_n(x)/y^n,$$

where the last equality is obtained by interchanging the roles of x and y. □

2. Formal E-sequences

In analogy to Pisot's definition of an E-sequence, let us define a *formal* E-*sequence* to be a sequence of polynomials $\{C_0(x), C_1(x), C_2(x), \ldots\}$ such that $\deg(C_n) = n$ and $\deg(C_{n+1}(x) - C_n(x)^2/C_{n-1}(x)) \leqq 1$. As opposed to Pisot's E-sequence, two consecutive elements of a formal E-sequence do not determine the rest of the formal E-sequences.

2.1. THEOREM. — *Under the hypothesis of Theorem* 1.6, *suppose* $u_0 \neq 0$ (*equivalently* $v_0 \neq 0$). *Then the sequences of polynomials* $\{A_n(x) \mid n = 0, 1, 2, \ldots\}$ *and*

$\{ B_n(y) \mid n = 0, 1, 2, \ldots \}$ *are formal* E-*sequences. For all* $n \geqq 1$ *the coefficient of* x^n *in* $A_{n+1}(x) A_{n-1}(x) - A_n(x)^2$ *is* $u_0 c_{n-1}/a_1^{n+1}$, *where* c_{n-1} *is the coefficient of* $1/y^{n-1}$ *in* $v(y)/b(y)$.

Proof. — Since $A_n = A_n(x) = u(x)/a(x)^n + \alpha_n(x)$, we compute that

$$\begin{aligned} A_{n+1} A_{n-1} - A_n^2 &= (u/a^{n+1} + \alpha_{n+1})(u/a^{n-1} + \alpha_{n-1}) - (u/a^n + \alpha_n)^2 \\ &= \qquad u/a^{n-1} + \alpha_{n-1} u/a^{n+1} - 2\alpha_n u/a^n + \alpha_{n+1}\alpha_{n-1} - \alpha_n^2. \end{aligned}$$

Each term in the right hand side of the latter expression has degree $\leqq n-1$, except for the term $\alpha_{n-1} u/a^{n+1}$ which has degree n if $\alpha_{n-1,1} \neq 0$. Thus the coefficient of x^n in $A_{n+1} A_{n-1} - A_n^2$ is the coefficient of x^n in $\alpha_{n-1} u/a^{n+1}$ which is $\alpha_{n-1,1} u_0/a_1^{n+1}$. Then, if $n \geqq 2$, we see by (1.8) that $\alpha_{n-1,1}$ is the coefficient of $1/y^{n-1}$ in $\beta_1(y) = \{-v(y)/b(y)\}$, so that $\alpha_{n-1,1} = c_{n-1}$. If $n = 1$, then the coefficient of x in $A_2 A_0 - A_1^2$ is the coefficient of x in

$$(v_0 b_1^2 x^2 + (v_0 b_2 + v_1 b_1) x + v_2) v_0 - (v_0 b_1 x + v_1)^2$$

which is $v_0(v_0 b_2 - v_1 b_1)$. Direct computation shows that this equals $u_0 c_0/a_1^2$. Thus if $n \geqq 1$ the coefficient of x^n in $A_{n+1} A_{n-1} - A_n^2$ is $u_0 c_{n-1}/a_1^{n+1}$. Since

$$\deg(A_{n+1} A_{n-1} - A_n^2) \leqq n, \qquad \deg(A_{n+1} - A_n^2/A_{n-1}) \leqq 1. \quad \square$$

In [7], Galyean gives a proof, using algebraic function theory, that the sequence $\{ A_n(x) \}$ is a formal E-sequence.

In [10], Pisot showed that if $\{ e_0, e_1, e_2, \ldots \}$ is an E-sequence, then there exist unique real numbers $\lambda > 0$ and $\theta \geqq 1$ such that

$$\limsup_{n \to \infty} | a_n - \lambda \theta^n | \leqq 1/2(\theta - 1)^2.$$

We shall obtain analogues of these results for formal E-sequences and for some more general sequences of polynomials.

2.2. THEOREM. — *Suppose* $n_0 \geqq 1$, $s \geqq 1$, $h \geqq 0$, j *are integers and* $\{ C_n(x) \mid n = 0, 1, 2, \ldots \}$ *is a sequence of polynomials satisfying*

(i) deg $(C_n) = ns + h$, *for* $n \geqq 0$ *and*

(ii) deg $(C_{n+1} C_{n-1} - C_n^2) \leqq ns + j$ *for all* $n \geqq n_0$.

Then there exists $\theta(x) \in K\{1/x\}$ *of degree* s *and* $\lambda(x) \in K\{1/x\}$ *of degree* h *such that* deg $(C_n(x) - \lambda\theta^n) \leqq j - h - s$ *for all* $n \geqq n_0 - 1$.

Proof. — If $n \geqq n_0$, then

$$\begin{aligned} \deg(C_{n+1}/C_n - C_n/C_{n-1}) &\leqq ns + j - (ns + h) - ((n-1)s + h) \\ &= j + s - 2h - ns. \end{aligned}$$

Then if $m > n \geqq n_0$,

$$\deg(C_m/C_{m-1} - C_n/C_{n-1}) \leqq j + s - 2h - ns.$$

Hence the sequence $\{ C_n/C_{n-1} \mid n = n_0, n_0+1, n_0+2, \ldots \}$ is Cauchy and

$$\theta(x) = \lim_{n\to\infty} C_n/C_{n-1}$$

exists; furthermore if $n \geqq n_0$, then $\deg(\theta(x) - C_n/C_{n-1}) \leqq j+s-2h-ns$. Hence

$$\deg(C_n/\theta^n - C_{n-1}/\theta^{n-1}) \leqq j-h-ns \qquad \text{and} \qquad \lambda(x) = \lim_{n\to\infty} C_n/\theta^n$$

exists with $\deg(\lambda - C_{n-1}/\theta^{n-1}) \leqq j-h-ns$ or $\deg(C_n - \lambda\theta^n) \leqq j-h-s$ for $n \geqq n_0-1$. That λ and θ have the specified degrees is clear. □

As a corollary to Theorem 2.2, we obtain the analogue of Pisot's characterization of E-sequences and the converse to Theorem 2.1.

2.3. COROLLARY. — *Suppose* $\{ C_n(x) \mid n = 0, 1, 2, \ldots \}$ *is a formal* E-*sequence. Then there are unique formal power series* $v(y) = \sum_{i=0}^{\infty} v_i/y^i$ *with* $v_0 \neq 0$ *and* $b(y) = \sum_{i=1}^{\infty} b_i/y^i$ *with* $b_1 \neq 0$ *such that*

$$\frac{v(y)}{1-xb(y)} = \sum_{n=0}^{\infty} C_n(x)/y^n.$$

Proof. — The hypothesis of Theorem 2.2 are satisfied with $n_0 = s = 1$, $j = h = 0$. Hence there exist series $\lambda(x)$ and $\theta(x)$ such that $\deg(C_n - \lambda\theta^n) \leqq -1$, $\deg(\lambda(x)) = 0$ and $\deg(\theta(x)) = 1$. Put $u(x) = \lambda(x)$ and $a(x) = 1/\theta(x)$. By Theorem 1.6, the polynomials $A_n(x)$ defined in (1.4) satisfy $A_n(x) = [u(x)/a(x)^n] = [\lambda\theta^n] = C_n(x)$. Thus $\sum_{n=0}^{\infty} C_n(x)/y^n = v(y)/(1-xb(y))$, where $v(y)$ and $b(y)$ are as defined at the beginning of Section 1. □

We now give another characterization of the sequences of polynomials satisfying the hypotheses of Theorem 2.2.

2.4. LEMMA. — *Suppose* $s > 0$ *is prime to the characteristic of* K, *that* $\omega \in \tilde{K}$ *is a primitive* s^{th} *root of unity, that* $0 \leqq h \leqq s$, *and that* $\{ C_n(x) \mid n = 0, 1, 2, \ldots \}$ *is a sequence o polynomials satisfying* $\deg(C_n) = ns+h$ *for* $n \geqq 0$ *and*

$$\deg(C_{n+1}C_{n-1} - C_n^2) \leqq (n+1)s+h-1 \qquad \text{for} \quad n \geqq 1.$$

Then there exist power series $v(y) = \sum_{i=0}^{\infty} v_i/y^i$ *with* $v_0 \neq 0$ *and* $b(y) = \sum_{i=0}^{\infty} b_i/y^i$ *with* $b_1 \neq 0$, *and with coefficients in* $\tilde{K}$ *such that*

$$\sum_{n=0}^{\infty} C_n(x)/y^{ns+h} = \frac{1}{s} \sum_{j=0}^{s-1} \frac{\omega^{hj} v(\omega^j y)}{1-xb(\omega^j y)}.$$

Proof. — By Theorem 2.2 there exist λ of degree h and θ of degree s such that $\deg(C_n - \lambda\theta^n) \leqq -1$ for all $n \geqq 0$. Let $1/a(x)$ be one of the s^{th} roots of $\theta(x)$ in $\tilde{K}\{ 1/x \}$

and put $u(x) = \lambda(x)\, a(x)^h$. Then $C_n(x) = A_{ns+h}(x)$, where $A_n(x) = [u(x)/a(x)^n]$. By Theorem 1.6,

$$\sum_{n=0}^{\infty} A_n(x)/y^n = \frac{v(y)}{1-xb(y)},$$

or

$$\sum_{n=0}^{\infty} \omega^{hj} A_n(x)/(\omega^j y)^n = \frac{\omega^{hj} v(\omega^j y)}{1-xb(\omega^j y)}.$$

Summing over j completes the proof of the formula. □

2.6. THEOREM. — *Suppose* $\{ C_n(x) \mid n = 0, 1, 2, \ldots \}$ *is a sequence of polynomials and* $r \geqq 0$, $s \geqq 1$, $0 \leqq h < s$ *are integers with* s *relatively prime to the characteristic of* K. *Suppose the polynomials* $C_n(x)$ *satisfy*

(i) $\deg(C_n) = ns+h$ *for* $n \geqq 0$,

(ii) $\deg(C_{n+1} C_{n-1} - C_n^2) \leqq (n+r+1)s+h-1$

for $n \geqq r+1$. *Then* $\sum_{n=0}^{\infty} C_n(x)/z^{ns+h}$ *can be written in the form*

$$\left(\frac{x}{z}\right)^{rs} \frac{1}{s} \sum_{j=0}^{s-1} \frac{\omega^{hj} v_0(\omega^j z)}{1-xb(\omega^j z)} + \sum_{j=1}^{r} \left(\frac{x}{z}\right)^{(r-j)s} v_j(z, x), \tag{2.7}$$

where ω *is a primitive* s^{th} *root of unity,* $v_0(z)$ *is a formal power series in* $\tilde{K}[[1/z]]$ *with non-zero constant term, and for* $j \geqq 1$, *the* $v_j(z, x)$ *are formal power series of the form* $v_j(z, x) = \sum_{i=0}^{\infty} c_{ij}(x)/z^i$ *with the* $c_{ij}(x)$ *polynomials in* $\tilde{K}[x]$ *of degree* $\leqq s-1$, *and* $b(z) = \sum_{i=0}^{\infty} b_i/z^i$ *is in* $\tilde{K}[[1/z]]$ *with* $b_1 \neq 0$.

Proof. — For $r = 0$, this is Lemma 2.4. Now suppose $r \geqq 1$ and that the Theorem has been proven for smaller values of r. Define $c_n(x)$ as the polynomial obtained from $C_n(x)$ by deleting all terms of degree $\geqq s$. Then

$$\deg((C_{n+1}-c_{n+1})(C_{n-1}-c_{n-1})-(C_n-c_n)^2) \leqq (n+r+1)s+h-1$$

for $n \geqq r+1$. Put $D_n = (C_{n+1}-c_{n+1})/x^s$ for $n \geqq -1$. Then

$$\begin{aligned}\deg(D_{n+1} D_{n-1} - D_n^2) &\leqq (n+1+r+1)+h-1-2s \\ &= (n+r)s+h-1\end{aligned}$$

for $n \geqq r$. Since $\deg(D_n) = ns+h$, we see by induction that

$$\sum_{n=0}^{\infty} D_n/z^{ns+h} = \left(\frac{x}{z}\right)^{(r-1)s} \frac{1}{s} \sum_{j=0}^{s-1} \frac{\omega^{hj} v_0(\omega^j z)}{1-xb(\omega^j z)} + \sum_{j=1}^{r-1} \left(\frac{x}{z}\right)^{r-1-j} v_j(z, x).$$

Now

$$\sum_{n=0}^{\infty} C_n/z^{ns+h} = x^s \sum_{n=0}^{\infty} D_{n-1}/z^{ns+h} + \sum_{n=0}^{\infty} c_n/z^{ns+h}$$

$$= \left(\frac{x}{z}\right)^s \sum_{n=0}^{\infty} D_n/z^{ns+h} + v_r(z, x),$$

since $D_{-1} = 0$, and where $v_r(z, x) = \sum_{n=0}^{\infty} c_n/z^{ns+h}$. Substituting the above expression for $\sum_{n=0}^{\infty} D_n/z^{ns+h}$ in the last formula and simplifying yields (2.7). □

3. **Rational formal** E-**sequences**

Suppose $\{ e_n \mid n = 0, 1, 2, \dots \}$ is an E-sequence of rational integers satisfying $e_n = \lambda\theta^n + \varepsilon_n$, where $\lambda \neq 0$, $\theta > 1$ are real numbers and $-1/2 \leqq \varepsilon_n < 1/2$. Pisot [10] has shown that if $\sum_{n=1}^{\infty} \varepsilon_n^2 < \infty$ then the function $\sum_{n=1}^{\infty} e_n/z^n$ is rational. Other conditions for rationality, depending upon the rate at which the ε_n approach 0 as $r \to \infty$, have been obtained by Pisot [10] and the author [4]. Here we study the analogous problem for formal E-sequences and more general sequences of polynomials. This problem has also been studied by Bateman and Duquette [1] and Grandet-Hugo [9]. We obtain more precise results and use entirely different methods of proof.

If $C_n(x)$ is a formal E-sequence and $C_n(x) = [C_{n-1}(x)^2/C_{n-2}(x)]$ for all large n, then $\deg(C_n - C_{n-1}/C_{n-2}) \leqq n-3$ for all large n and, by Theorem 2.2 there exist $\lambda(x)$, $\theta(x)$ such that $\deg(C_n - \lambda\theta^n) \leqq -3$ for all large n, and thus the hypotheses of the following theorem are satisfied.

3.1. THEOREM. – *Suppose* $\lambda(x) = \sum_{i=0}^{\infty} \lambda_i/x^i$ *with* $\lambda_0 \neq 0$ *and* $\theta(x) = \sum_{i=-1}^{\infty} \theta_i/x^i$, *with* $\theta_{-1} \neq 0$. *Suppose* $\lambda\theta^n = A_n(x) + \alpha_n(x)$ *where* $A_n(x)$ *is a polynomial and* $\deg(\alpha_n(x)) \leqq -3$ *for all large n. Then there exist polynomials* p_1 *and* p_2, *each with non-zero constant terms, such that*

$$\sum_{n=0}^{\infty} A_n(x)/y^n = \frac{p_1(1/y)^2}{p_2(1/y) - x p_1(1/y)/y}.$$

Furthermore θ *is algebraic over* $K(x)$ *and satisfies the equation*

$$p_2(1/\theta) - x p_1(1/\theta)/\theta = 0.$$

Proof. – Put $a(x) = 1/\theta(x)$ and $u(x) = \lambda(x)$. From Theorem 1.6, since $\sum_{=1}^{\infty} \alpha_n(x)/y^n = \sum_{n=1}^{\infty} \beta_n(y)/x^n$, we see that $\beta_1(y)$ and $\beta_2(y)$ are polynomials in $1/y$ Now $v(y)/b(y)^n = B_n(y) + \beta_n(y)$ and $B_n(y)$ is a polynomial of degree n. Thus

$v(y)/b(y) = B_1(y)+\beta_1(y) = yp_1(1/y)$ and $v(y)/b(y)^2 = B_2(y)+\beta_2(y) = y^2 p_2(1/y)$ define polynomials p_1 and p_2. It follows that $b(y) = p_1(1/y)/(yp_2(1/y))$ and that $v(y) = p_1^2(1/y)/p_2(1/y)$.

Substituting in equation (1.4) completes the proof. □

3.2. THEOREM. — *Under the assumptions and notation of Theorem* 3.1, θ *has degree* 1, *and its conjugates over* $K(x)$ *have degree* $\leqq 0$.

Proof. — Suppose p_1 has degree r_1 and p_2 has degree r_2. Put $r = \max(r_1+1, r_2)$. Put $q_1(z) = z^{r-1} q_1(1/z)$ and $q_2(z) = z^r p_2(1/z)$. Then $q_1(z)$ has degree $r-1$, $q_2(z)$ has degree r, and $q_2(\theta)-xq_1(\theta) = 0$. An elementary application of the theory of Newton diagrams to the polynomial $q_2(z)-xq_1(z)$ of degree r in z completes the proof. □

Let us denote the conjugates of θ over $K(x)$ by $\theta = \theta_1, \theta_2, \theta_3, \ldots, \theta_r$. Under the assumptions and notation of Theorems 3.1 and 3.2 we can write

$$\frac{v(y)}{1-xb(y)} = \sum_{m=0}^{\infty} A_n(x)/y^n,$$

where $v(y) = p_1(1/y)^2/p_2(1/y)$ and $b(y) = p_1(1/y)/(yp_2(1/y))$. Then, by (1.2),

$$\begin{aligned} \lambda(x) &= u(x) \\ &= -xv(1/a(x))\, a'(x)/a(x) \\ &= xv(\theta(x))\,\theta'(x)/\theta(x). \end{aligned} \tag{3.3}$$

Put $\lambda_i(x) = xv(\theta_i(x))\,\theta_i'(x)/\theta_i(x)$. The $\lambda_i(x)$ are the conjugates of $\lambda(x)$ over the algebraic field extension $K(x, \theta(x))/K(x)$. Now an elementary partial fraction expansion yields

$$\frac{v(y)}{1-xb(y)} = \sum_{i=1}^{r} \lambda_i(x)/(1-y\,\theta_i(x)).$$

Thus

$$A_n(x) = \sum_{i=1}^{r} \lambda_i(x)\,\theta_i(x)^n$$

$$\alpha_n(x) = \sum_{i=2}^{r} \lambda_i(x)\,\theta_i(x)^n.$$

Next p_1 and p_2 have non-zero constant terms, so

$$\deg(v(\theta(x)) = \deg(p_1^2(1/\theta(x))/p_2(1/\theta(x))) = 0.$$

If $\deg(\theta_i) \neq 0$ then $\deg(\theta_i'/\theta_i) = -1$, while if $\deg(\theta_i) = 0$ then $\deg(\theta_i'/\theta_i) < -1$. Thus if $\deg(\theta_i) \neq 0$, then $\deg(\lambda_i) = 0$, while if $\deg(\theta_i) = 0$, then $\deg(\lambda_i) < 0$. In particular, $\deg(\lambda_1) = 0$ and $\deg(\lambda_i) \leqq 0$ for $2 \leqq i \leqq r$. Summarizing we have

3.4. THEOREM. — *Under the assumptions and notation of Theorems* 3.1 *and* 3.2 *we can write*

$$A_n(x) = \sum_{i=1}^{r} \lambda_i(x)\,\theta_i(x)^n,$$

$$\alpha_n(x) = \sum_{i=2}^{r} \lambda_i(x)\,\theta_i(x)^n,$$

where the $\theta_i(x)$ *are the conjugates of* $\theta(x)$, *the* $\lambda_i(x)$ *are the conjugates of* $\lambda(x)$ [*defined by* (3.3)]. *Furthermore* $\deg(\theta_1) = 1$, $\deg(\lambda_1) = 0$ *and* $\deg(\theta_i) \leqq 0$, $\deg(\lambda_i) \leqq 0$ *for* $2 \leqq i \leqq r$.

3.5. THEOREM. — *Continuing the same assumptions and notation, suppose*

$$\alpha_n(x) \to 0 \quad \text{as } n \to \infty.$$

Then $p_2(z)/p_1(z)$ *is a polynomial* $q(z)$ *and then*

$$\sum_{n=0}^{\infty} A_n(x)/y^n = \frac{p_1(1/y)/q(1/y)}{1 - x/(yq(1/y))}.$$

Furthermore if $\deg(q) = 1$ *then* $\alpha_n(x) = 0$ *for* $n > \deg(p_1) - \deg(q)$ *while if* $\deg(q) > 1$ *then*

$$\deg(\alpha_n(x)) \leqq -\frac{n - \deg(p_1) + \deg(q)}{\deg(q) - 1},$$

with equality when the right side is an integer.

Proof. — We have $v(y)/b(y)^m = y^m p_m(1/y)$ where the p_m are polynomials with non-zero constant terms. This yields

$$p_m(1/y) = p_1(1/y)\,(p_2(1/y)/p_1(1/y))^{m-1},$$

and since all p_m are polynomials $p_2(1/y)/p_1(1/y)$ is a polynomial $q(1/y)$. Then $\beta_m(y) = \{ y^m p_1(1/y)\, q(1/y)^{m-1} \}$. If $\beta_m(y) = \sum_{n=1}^{\infty} \beta_{mn}/y^n$, then $\beta_{mn} = 0$ if

$$n > (m-1)\deg(q) + \deg(p_1) - m, \quad \text{and} \quad \beta_{mn} = p_1(0)\, q(0)^{m-1} \neq 0$$

if

$$n = (m-1)\deg(q) + \deg(p_1) - m = m(\deg(q) - 1) + \deg(p_1) - \deg(q).$$

Now $\alpha_n(x) = \sum_{m=1}^{\infty} \beta_{mn}/x^m$. If $\deg(q) = 1$, $\alpha_n(x) = 0$ if $n > \deg(p_1) - \deg(q)$. If $\deg(q) > 1$, then $\beta_{mn} = 0$ if

$$m < (n + \deg(q) - \deg(p_1))/(\deg(q) - 1);$$

this yields the upper bound for $\deg(\alpha_n(x))$. Since $\beta_{mn} \neq 0$, when

$$n = m(\deg(q) - 1) + \deg(p_1) - \deg(q),$$

we obtain the exact degree in this case. □

3.6. *Remark.* — Under the assumptions and notation of Theorem 3.5 the conjugates of θ other than θ itself, namely $\theta_2, \theta_3, \ldots, \theta_r$ all have exact degree $-1/(r-1)$.

We shall need the following lemmas.

3.7. LEMMA. — *Suppose $\gamma_0, \gamma_1, \gamma_2, \ldots$ is a sequence of elements of a unique factorization domain* D *which satisfies a linear recurrence relation with constant coefficients; i. e. there exist $c_0, c_1, \ldots, c_r$ in* D *with $c_0 c_r \neq 0$ such that $\sum_{i=0}^{r} c_i \gamma_{n-i} = 0$ for all $n \geqq r$. If r is the least integer $\geqq 0$ for which such a recurrence relation exists then $c_0 \mid c_i$ for $1 \leqq i \leqq r$.*

This is the Fatou-Hurwitz Lemma. *See* [11] for a proof.

3.8. LEMMA. — *Suppose $\mu_1, \mu_2, \ldots, \mu_s$ and $\delta_1, \delta_2, \ldots, \delta_s$ are elements of a field. Put $\gamma_i = \sum_{k=1}^{s} \mu_k \delta_k^i$ for $i = 1, 2, 3, \ldots$ and define an $(s \times 1)$ by $(s+1)$ matrix* $\mathrm{H} = (h_{ij})$, *where $0 \leqq i, j \leqq s$ by putting $h_{ij} = \gamma_{i+j+1}$ for $0 \leqq i \leqq s$, $0 \leqq j \leqq s-1$ and putting $h_{is} = z^i$ for $0 \leqq i \leqq s$. Then* det (H) *is a polynomial of degree $\leqq s$ in z which vanishes when $z = \alpha_i$, $1 \leqq i \leqq s$.*

Proof. — Clearly det (H) is a polynomial of degree $\leqq s$ in z. Put $\sum_{i=0}^{s} c_i z^i = \prod_{i=1}^{s} (z-\delta_i)$. Then, if $0 \leqq i \leqq s-1$,

$$
\begin{aligned}
\sum_{i=0}^{s} c_i h_{ij} &= \sum_{i=0}^{s} c_i \sum_{k=1}^{s} \mu_k \delta_k^{i+j+1} \\
&= \sum_{k=1}^{s} \mu_k \delta_k^{j+1} \sum_{i=0}^{s} c_i \delta_k^i \\
&= 0.
\end{aligned}
$$

If now, $z = \delta_k$ so that $h_{is} = \delta_k^i$, then $\sum_{i=0}^{s} c_i h_{is} = 0$ and det (H) $= 0$. □

3.9. LEMMA. — *Suppose $\delta_1, \delta_2, \ldots, \delta_s$ and $\mu_1, \mu_2, \ldots, \mu_s$ are elements of a field, all μ_i are non-zero, and the δ_i are distinct. Suppose the $\gamma_n = \sum_{i=1}^{s} \mu_i \delta_i^n$ lie in a unique factorization domain* D *and satisfy a linear recurrence relation with constant coefficients. Then the δ_i are integral over* D.

Proof. — There exist c_i, not all zero, such that $\sum_{i=0}^{r} c_i \gamma_{n-i} = 0$ for $n \geqq r$. We may consider the recurrence relation as a system of homogeneous linear equations for the c_i with coefficients $\gamma_m \in$ D, and hence choose the c_i from D. By Lemma 3.7 we may choose $c_0 = 1$. Then

$$
\sum_{k=1}^{s} \left(\sum_{i=0}^{r} c_i \delta_k^{r-i} \right) \mu_k \delta_k^{n-r} = 0
$$

for all $n \geqq r$. It is immediate that the δ_k satisfy the monic polynomial equation over D, $\sum_{i=0}^{r} c_i x^{r-i} = 0$, and hence are integral over D. □

3.10. LEMMA. — *If* $\mu_1(x)$, $\mu_2(x)$, ..., $\mu_r(x)$ *are a complete set of conjugate elements, integral over* $K[x]$, *and all have degree* $\leqq 0$ *then they are all constant.*

Proof. — The coefficients of the monic irreducible polynomial satisfied by the $\mu_i(x)$ are symmetric functions of the $\mu_i(x)$, hence polynomials of degree $\leqq 0$. Thus these coefficients are constant and this means that the μ_i are constant. □

Suppose now that $\theta \in K\{1/x\}$ is of degree $s \geqq 1$ and $\lambda \in K\{1/x\}$ is of degree $h \geqq 0$. Suppose $\lambda\theta^n = C_n(x)+\varepsilon_n(x)$, where $C_n(x)$ is a polynomial (of degree $ns+h$) and $\varepsilon_n(x)$ has degree $\leqq -1$. Suppose further that $\varepsilon_n(x)$ has degree $\leqq -2s-1$ for all large n, say $n \geqq n_0$. Consider the equations

$$\sum_{i=0}^{d} r_i(x)\, C_{n_0+i}(x) = 0, \tag{3.11}$$

where $r_i(x) = \sum_{j=0}^{s} r_{ij}\, x^j$; (3.11) may be considered as a set of homogeneous linear equations in the unknowns r_{ij}. As such the number of equations is 1 + the degree of (3.11), hence is $1+s+(n_0+d)\,s+h$. The number of unknows is $(s+1)\,(d+1)$. If $d = n_0\,s+h+1$, then there are more variables than equations and (3.11) has a solution $r_0(x)$, $r_1(x)$, ..., $r_d(x)$, where not all of the $r_i(x)$ are 0. Suppose we have shown for some n that $\sum_{i=0}^{d} r_i(x)\, C_{n+i}(x) = 0$. Then

$$C_{n+1}(x) = \lambda\theta^{n+1}+\varepsilon_{n+1} = \theta\, C_n(x)+\varepsilon_{n+1}-\theta\varepsilon_n$$

and hence

$$\sum_{i=0}^{d} r_i(x)\, C_{n+1+i}(x) = \theta \sum_{i=0}^{d} r_i(x)\, C_{n+i}(x)+\sum_{i=0}^{d} r_i(x)\,(\varepsilon_{n+1}-\theta\varepsilon_n)$$

$$= \sum_{i=0}^{d} r_i(x)\,(\varepsilon_{n+1}-\theta\varepsilon_n).$$

If $n \geqq n_0$, then the degree of the last expression is $\leqq s+s-2s-1 = -1$, and since it is a polynomial it is 0. Thus, proceeding inductively, $\sum_{i=0}^{d} r_i(x)\, C_{n+i}(x) = 0$ for all $n \geqq n_0$. It follows that the formal power series $\sum_{n=0}^{\infty} C_n(x)/y^n$ is rational with denominator $\sum_{i=0}^{d} r_i(x)/y^{d-i}$. We may assume that the above recurrence satisfied by the C_n is that of minimal degree so that $r_0 \neq 0$ and by Lemma 3.7, we can even assume $r_d = 1$. Furthermore $\sum_{i=0}^{d} r_i\, \theta(x)^i = 0$ and hence $\theta(x)$ is integral over $K[x]$. Next,

$$\sum_{i=0}^{d} r_i\, \varepsilon_{n+i}(x) = \sum_{i=0}^{d} r_i(\lambda\theta^{n+i}-\varepsilon_{n+1}) = 0,$$

for all $n \geqq n_0$. Hence, by Lemma 3.7 applied to the unique facterization domain $K[[1/x]]$ (whose unique prime is $1/x$), there exist formal power series $0 \neq \sigma_0$, σ_1, ..., $\sigma_e = 1$

in K [[1/x]] such that $\sum_{i=0}^{e} \sigma_i \varepsilon_{n+i} = 0$, for all $n \geqq n_0$. It is immediate that

$$\sum_{i=0}^{d} r_i t^i = (t-\theta) \sum_{i=0}^{e} \sigma_i t^i,$$

hence that $e = d-1$. Thus the conjugates of θ satisfy $\sum_{i=0}^{d-1} \sigma_i t^i = 0$. Since the σ_i all have degree $\leqq 0$ the conjugates of θ all have degree $\leqq 0$. Suppose

$$\varepsilon_n(x) = \sum_{j=1}^{\infty} \varepsilon_{nj}/x^j \quad \text{and} \quad \sigma_i = \sum_{j=0}^{\infty} \sigma_{ij}/x^j.$$

If $\varepsilon_{nj} = 0$ for all large n and $j < j_0$, then $\varepsilon_{n,j_0} = \sum_{i=0}^{d-1} \sigma_{i,0}\, \varepsilon_{n-i,j_0}$. It follows that if $\varepsilon_n \to 0$ as $n \to \infty$ then all $\sigma_{i,0}$ are 0, $0 \leqq i < e$, so that all conjugates of θ have degree < 0. Following [1] let us call a non-constant $\theta \in \mathrm{K}\{1/x\}$ a PV element if it is algebraic over K [1/x] and all of its conjugates have degree < 0 and call θ a T-element if all of its conjugates have degree $\leqq 0$ and it is not PV.

We have proven:

3.12. THEOREM. — *Suppose $\theta(x)$ is of degree s and that $\lambda(x)$ has degree $h \leqq 0$. If $\{\lambda(x)\theta(x)^n\}$ has degree $\leqq -2s-1$ for all $n \geqq n_0$, then $\theta(x)$ is a* PV *or* T *element of* K $\{x\}$ *and satisfies an equation of the form $\sum_{i=0}^{d} r_i(x) z^i = 0$, where $d \leqq n_0 s+h+1$, $r_d(x) = 1$ and each $r_i(x)$ has degree $\leqq s$. If $\varepsilon_n(x) \to 0$ as $n \to \infty$ then $\theta(x)$ is a* PV *element; otherwise it is a* T *element.*

If we assume, in addition to the hypothesis of the last theorem, that s is relatively prime to the characteristic of K we can obtain explicit formulas for the $r_i(x)$. Let $a(x)$ be one of the s^{th} roots of $1/\theta(x)$ in $\tilde{\mathrm{K}}$ [[1/x]] and put $u(x) = \lambda(x)\, a(x)^h$. Then $u(x)$ has degree 0, $a(x)$ has degree -1 and $\mathrm{C}_n(x) = [u(x)/a(x)^{ns+h}]$, $\varepsilon_n(x) = \{u(x)/a(x)^{ns+h}\} = \alpha_{ns+h}$ in the notation of Theorem 1.6. It follows from the identity $\sum_{m=1}^{\infty} \alpha_m(x)/y^m = \sum_{n=1}^{\infty} \beta_n(y)/x^n$ of Theorem 1.6, that $\beta_1, \beta_2, \ldots, 2_{2s}$ have only finitely many non-zero terms of the form β_{ij}/y^j with $j \equiv h \pmod{s}$, and the sum of these terms from β_i is

$$\frac{1}{s} \sum_{j=0}^{s-1} \omega^{hj} \beta_i(\omega^j y), \tag{3.13}$$

where ω is a primitive s^{th} root of unity in $\tilde{\mathrm{K}}$. If $i \leqq 2s$ then (3.13) is a polynomial of degree $\leqq n_0 s+h$; moreover if $\varepsilon_n(x) \to 0$ as $n \to \infty$, then (3.13) is a polynomial in $1/y$ for all i. Put

$$g_i = \frac{1}{s} \sum_{j=0}^{s-1} \omega^{hj} v(\omega^j y)/b(\omega^j y)^i;$$

g_i differs from (3.13) by a polynomial in y. Each of the terms $v(\omega^j y)/b(\omega^j y)^i$ has degree i, and $y^h g_i$ is a function of y^s. Thus if $i+h \equiv 0 \pmod{s}$, then g_i has degree i, otherwise g_i has degree $< i$. We can write

(3.14) $$g_i = y^{[(i+h)/s]\, s-h} f_i(y^{-s}),$$

where f_i is a polynomial of degree $\leqq n_0+[(i+h)/s]$, and $f_i(0) \neq 0$ if and only if $i+h \equiv 0 \pmod{s}$. Define $\mathrm{H}(z)$ to be the $(s+1)\times(s+1)$ matrix whose (i, j) entry is g_{i+j-1} for $1 \leqq i \leqq s+1$ and $1 \leqq j \leqq s$, and is z^{i-1} for $1 \leqq i \leqq s+1$ and $j = s+1$. By Lemma 3.8 $\det(\mathrm{H}(1/b(\omega^j y))) = 0$ for $1 \leqq j \leqq s$. Multiply the j^{th} column of $\mathrm{H}(z)$ by $y^{h-[(j+h)/s]s}$ for $1 \leqq j \leqq s$. Next multiply all but the first row of the resulting matrix by y^{-s}. This yields a matrix of which every element in the first s columns is a polynomial in y^{-s} and whose determinant is a power of y times $\det(\mathrm{H}(z))$. If we expand the determinant of this new matrix by cofactors of the last column, it is not hard to verify that we obtain an expression of the form $\sum_{i=0}^{s} p_i(y^{-s}) z^i$, where the $p_i(y^{-s})$ are polynomials in y^{-s}, $p_0(0) \neq 0$, $p_i(0) = 0$ for $1 \leqq i \leqq s$ and $p_s'(0) \neq 0$. Clearly

$$\sum_{i=0}^{s} p_i(y^{-s})/b(\omega^j y)^i = 0 \qquad \text{for} \quad 1 \leqq j \leqq s.$$

Let $\delta(\mathrm{T})$ be the greatest common division of the polynomials $p_i(\mathrm{T})$ and define $q_i(\mathrm{T}) = p_i(\mathrm{T})/\delta(\mathrm{T})$ for $0 \leqq i \leqq s$. Then $q_0(0) \neq 0$, $q_i(0) = 0$ for $1 \leqq i \leqq s$, $q_s'(0) \neq 0$, and

(3.15) $$\sum_{i=0}^{s} q_i(y^{-s})/b(\omega^j y)^i = 0$$

for $1 \leqq j \leqq s$. Note that

$$y^{-i} \sum_{j=0}^{s-1} \omega^{hj} v(\omega^j y)/b(\omega^j y)^i$$

is a polynomial in $1/y$ for $1 \leqq i \leqq 2s$, and if $\varepsilon_n(x) \to 0$ as $n \to \infty$ then it is a polynomial in $1/y$ for all n. When that is so, each $1/(yb(\omega^j y))$ is integral over $\mathrm{K}[1/y]$ and satisfies the equation $\sum_{i=0}^{s} y^i q_i(y^{-s}) z^i = 0$.

Each coefficient of this equation is a polynomial in $1/y$ and the coefficient of z^s, $y^s q_i(y^{-s})$ has a non-zero constant term. By Lemma 3.7, $y^s q_s(y^{-s}) = 1$ and hence $q_s(\mathrm{T}) = \mathrm{T}$.

Now substitute $y = 1/a(x)$, $b(y) = 1/x$ into (3.15) to obtain $\sum_{i=0}^{s} q_i(a(x)^s)\, x_i = 0$ or since $a(x)^s = 1/\theta(x)$, $\sum_{i=0}^{s} q_i(1/\theta(x))\, x^i = 0$. That is $z = \theta(x)$ satisfies the equation $\sum_{i=0}^{s} q_i(1/z)\, x^i = 0$. Put $d = \max_{0 \leqq i \leqq s} \deg(q_i(\mathrm{T}))$. This equation can be written as a polynomial in z and x, $\sum_{i=0}^{s} z^d q_i(1/z)\, x^i$ and can be rewritten as $r(x, z) = \sum_{i=0}^{d} r_i(x)\, z^i$, where the $r_i(x)$ are polynomials in x of degree $\leqq s$, $r_d(x) = 1$ and $r_{d-1}(x)$ has degree s. If $\varepsilon_n(x) \to 0$ as $n \to \infty$, then $q_s(\mathrm{T}) = \mathrm{T}$ and then the $r_i(x)$ have degree $\leqq s-1$ for $0 \leqq i \leqq d-2$. It follows from an application of Newton's diagram that all conjugates of θ, other than θ itself, have degree $\leqq 0$ and, in fact, that their constant terms are the roots of $T^d q_s(1/T)$ a polynomial of degree $d-1$. In the special case that $\varepsilon_n(x) \to 0$ as $n \to \infty$, $q_s(T) = T$, $T^d q_s(T) = T^{d-1}$ and all the constant terms of the conjugates are 0. If $r(x, z)$ factored, one of the factors would have the form $\prod_{i \in I} (z-\theta_i)$, where the θ_i are roots of $r(z, x) = 0$ and all have degree $\leqq 0$. By Lemma 3.10, this implies that all θ_i are constants. This is impossible and hence $r(x, z)$ is irreducible.

In [8] Grandet-Hugot showed that the PV elements in $k\{1/x\}$, where k is a finite field, do not form a closed subset. We shall prove the much stronger.

3.16. THEOREM. — *Suppose $s \geqq 1$ is an integer relatively prime to the characteristic of* K. *The* PV *and* T *elements of degree s in* $K\{1/x\}$ *are both dense in the set of elements α of degree s in* $K\{1/x\}$.

Proof. — If we choose a PV element θ in $K\{1/x\}$ such that $\deg(\theta-\alpha^{1/s}) < -h$, then $\deg(\theta^s-\alpha) < s-1-h$ and θ^s is a PV element of $K\{1/x\}$. Thus we may assume that $s = 1$. Now choose $c_1, c_0, c_{-1}, \ldots$ from K, inductively, so that

$$c(y) = c_1 y + c_0 + c_{-1}/y + \ldots + c_{-h}/y^h$$

satisfies $\deg(c(\alpha)-x) < -h$. Then an elementary application of Newton's diagram shows that the polynomial equation $y^h(c(y)-x)$ of degree $h+1$ in y has one root θ of degree 1 and that the remaining roots have degree < 0, so that θ is a PV element. Now,

$$\deg(c(\theta)-c(\alpha)) = \deg((c(\theta)-x)-(c(\alpha)-x)) < -h,$$

and

$$c(\theta)-c(\alpha) = (\theta-\alpha)(c_1+\ldots),$$

where the expression in parentheses has degree 0. Thus $\deg(\theta-\alpha) < -h$. Similarly the polynomial $y^h(c(y)-x)-x$ has one root θ' of degree 1 which is a T element, and $\deg(\alpha-\theta') < -h+1$.

4. Formal identities for F-sequences

Suppose $c_0, c_1, c_2, \ldots$ is a sequence of complex numbers with $c_0 \neq 0$. Put

$$c(t) = \sum_{i=0}^{\infty} c_i t^i, \qquad d(t) = \sum_{i=0}^{\infty} d_i t^i = c(t)^{-1}$$

and define a sequence of polynomials $C_0(x)$, $C_1(x)$, $C_2(x)$, $\ldots$ by

$$\frac{1}{1-xt\sum_{i=0}^{\infty} c_i t^i} = 1 + xt \sum_{n=0}^{\infty} C_n(x) t^n. \tag{4.1}$$

Then $C_n(x)$ has degree n and leading coefficient c_0^{n+1}. Put $b_i = c_{i-1}$ and $b(y) = \sum_{i=1}^{\infty} b_i/y^i$; put $v(y) = 1$, define $a(x) = a_1/x + a_2/x^2 + \ldots$ by $b(1/a(x)) = 1/x$, and define

$u(x) = -xa'(x)/a(x)$. In the notation of Section 1, $C_n(x) = A_{n+1}(x)/x$. Then by Theorem 1.6,

$$\begin{aligned} C_n(x) &= -a'(x)/a(x)^{n+2} + \alpha_{n+1}(x)/x \\ &= \theta'(x)\theta(x)^n + \varepsilon_n(x), \end{aligned}$$

where $\theta(x) = 1/a(x)$ has degree 1 and leading coefficient $c_0 = 1/a_1$, $\theta'(x) = -a'(x)/a(x)^2$ has degree 0, and $\varepsilon_n(x) = \alpha_{n+1}(x)/x$ has degree $\leqq -2$.

Let us write

$$\sum_{i=0}^{\infty} c_{ri}\, t^i = \left(\sum_{i=0}^{\infty} c_i\, t^i\right)^r$$

for integral $r \geqq 0$, so that, of course, $c_{1i} = c_i$, $i = 0, 1, 2, \ldots$ Similarly write

$$\sum_{i=0}^{\infty} d_{si}\, t^i = \left(\sum_{i=0}^{\infty} d_i\, t^i\right)^s$$

for integral $s \geqq 1$. Note that c_{ri} is a polynomial in $c_0, c_1, \ldots, c_i$ and that $c_0^{s+i} d_{si}$ is a polynomial in $c_0, c_1, \ldots, c_i$. Now, by (4.1)

$$x \sum_{n=0}^{\infty} C_n(x)\, t^{n+1} = \sum_{r=0}^{\infty} \left(xt \sum_{i=0}^{\infty} c_i\, t^i\right)^{r+1},$$

or

$$\sum_{n=0}^{\infty} C_n(x)\, t^n = \sum_{r=0}^{\infty} x^r \sum_{i=0}^{\infty} c_{r+1,i}\, t^{r+i}.$$

Equating coefficients of like powers of t yields

$$C_n(x) = \sum_{r=0}^{n} c_{r+1,n-r}\, x^r. \tag{4.2}$$

Next,

$$\sum_{n=1}^{\infty} \alpha_n(x)/y^n = \sum_{n=1}^{\infty} \beta_n(y)/x^n$$

and

$$\begin{aligned} \beta_n(y) &= \{v(y)/b(y)^n\} \\ &= \{y^n/c(1/y)^n\} \\ &= \sum_{j=1}^{\infty} d_{n,n+j}/y^j. \end{aligned}$$

It follows that

$$\alpha_n(x) = \sum_{s=0}^{\infty} d_{s+1,n+s+1}/x^{s+1}$$

and that

$$\begin{aligned} \varepsilon_n(x) &= \alpha_{n+1}(x)/x \\ &= \sum_{s=0}^{\infty} d_{s+1,n+s+2}/x^{s+2}. \end{aligned} \tag{4.3}$$

4.4. THEOREM. — *We have*

$$C_n(x)\,C_{n-2}(x)-C_{n-1}(x)^2$$
$$=\sum_{r,s}(c_{r+1,n-r}\,d_{s+1,n+s}-2\,c_{r+1,n-r-1}\,d_{s+1,n+s+1}+c_{r+1,n-r-2}\,d_{s+1,n+s+2})\,x^{r-s-2}.$$

The sum is taken over integral r, s with $r-s \geqq 2$. *(We put* $c_{ri}=d_{si}=0$ *if* $i<0$.)

Proof. — Clearly

$$\begin{aligned}C_n C_{n-2}-C_{n-1}^2 &= (\theta'\theta^n+\varepsilon_n)(\theta'\theta^{n-2}+\varepsilon_{n-2})-(\theta'\theta^{n-1}+\varepsilon_{n-1})^2\\ &= \theta'\theta^n\varepsilon_{n-2}-2\theta'\theta^{n-1}\varepsilon_{n-1}+\theta'\theta^{n-2}\varepsilon_n-\varepsilon_{n-1}^2+\varepsilon_{n-2}\,\varepsilon_n\\ &= (C_n-\varepsilon_n)\,\varepsilon_{n-2}-2(C_{n-1}-\varepsilon_{n-1})\,\varepsilon_{n-1}+(C_{n-2}-\varepsilon_{n-2})\,\varepsilon_n-\varepsilon_{n-1}^2+\varepsilon_{n-2}\,\varepsilon_n\\ &= C_n\varepsilon_{n-2}-2\,C_{n-1}\,\varepsilon_{n-1}+C_{n-2}\,\varepsilon_n+\varepsilon_{n-1}^2-\varepsilon_{n-2}\,\varepsilon_n\\ &= [C_n\varepsilon_{n-2}-2\,C_{n-1}\,\varepsilon_{n-1}+C_{n-2}\,\varepsilon_n].\end{aligned}$$

Substituting (4.2) for C_n, C_{n-1}, C_{n-2} and (4.3) for ε_n, ε_{n-1}, ε_{n-2} completes the proof. □

We can obtain another convenient formula for $C_n\,C_{n-2}-C_{n-1}^2$ by the following method. We have

$$\frac{1}{1-xtc(t)}=1+xt\sum_{n=0}^{\infty}C_n(x)\,t^n;$$
$$\frac{1}{1-xsc(s)}=1=xs\sum_{n=0}^{\infty}C_n(x)\,s^n.$$

Hence if $f=(1-xtc(t))^{-1}.(1-xsc(s))^{-1}$, we see that

$$f=1+xt\sum_{n=0}^{\infty}C_n(x)\,t^n+xs\sum_{n=0}^{\infty}C_n(x)\,s^n$$
$$+x^2\,st\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}C_m(x)\,C_n(x)\,s^m\,t^n.$$

Then $x^2\,C_{n-1}(x)^2$ is the constant term in $f/s^n\,t^n$ if $n \geqq 1$, and $x^2\,C_{n-2}\,C_n$ is the constant term in $f/s^{n+1}\,t^{n-1}$ if $n \geqq 2$. Thus $x^2\,(C_n\,C_{n-2}-C_{n-1}^2)$ is the constant term in

$$\left(\frac{1}{s^{n+1}\,t^{n-1}}-\frac{1}{s^n\,t^n}\right)f=\left(\frac{1}{s}-\frac{1}{t}\right)\frac{f}{s^n\,t^{n-1}}=-\frac{(s-t)f}{s^{n+1}\,t^n}.$$

This is the coefficient of $s^{n+1}\,t^n$ in

$$-\frac{s-t}{(1-xsc(s))(1-xtc(t))}$$
$$=-\frac{1}{x}\,\frac{s-t}{sc(s)-tc(t)}\left(\frac{1}{1-xsc(s)}-\frac{1}{1-xtc(t)}\right)$$
$$=-\sum_i\sum_j h_{ij}\,s^i\,t^j\sum_{m=0}^{\infty}C_m(x)\,(s^{m+1}-t^{m+1}),$$

where

$$h(s,t)=\sum_i\sum_j h_{ij}s^i t^j$$
$$=\frac{s-t}{sc(s)-tc(t)}.$$

Thus $x^2(C_n C_{n-2}-C_{n-1}^2)$ is the coefficient of $s^{n+1}t^n$ in

$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(s\sum_{i=0}^{m}h_{in}C_{m-i}-t\sum_{j=0}^{n}h_{mj}C_{n-j}\right)s^m t^n.$$

Hence

$$x^2(C_n C_{n-2}-C_{n-1}^2)=\sum_{i=0}^{n-1}h_{n+1,i}C_{n-1-i}-\sum_{i=0}^{n}h_{i,n}C_{n-i}$$
$$=\sum_{i=0}^{n}(h_{i-1,n+1}-h_{i,n})C_{n-i},$$

where we have put $h_{-1,n+1}=0$ and used the identity $h_{ij}=h_{ji}$.

We have proven

4.5. THEOREM. — *We have*

$$x^2(C_n C_{n-2}-C_{n-1}^2)=\sum_{i=0}^{n}(h_{i-1,n+1}-h_{i,n})C_{n-i}. \quad \square$$

Next note that

$$h(s,t)=\frac{s-t}{sc(s)-tc(t)} \tag{4.6}$$
$$=\frac{(s-t)d(s)d(t)}{sd(t)-td(s)}$$
$$=\frac{(s-t)d(s)d(t)}{(s-t)d_0-st\sum_{j=2}^{\infty}d_j(s^{j-1}-t^{j-1})}$$
$$=\frac{d(s)d(t)}{d_0-st\sum_{j=2}^{\infty}\sum_{i=0}^{j-2}s^i t^{j-2-i}d_j}.$$

Expanding the last expression yields.

4.7. THEOREM. — *The h_{ij} are polynomials with non-negative integral coefficients in* $1/d_0=c_0, d_0, d_1, d_2, \ldots$ $\square$

We now consider the case when $c(t)$ is a rational function $p(t)/q(t)$ where $p(t)$ and $q(t)$ are polynomials in $\mathbf{C}[t]$ and $q(0)=1$. Suppose

$$r=\max(\deg(tp(t)),\deg(q(t)))$$

and write $tp(t) = \sum_{i=1}^{r} p_i t^i$ and $q(t) = \sum_{i=0}^{r} q_i t^i$. Now $c(1/\theta(x)) = \theta(x)/x$; this is the same as saying that $t = 1/\theta(x)$ is a root of $q(t) - xtp(t) = 0$. Call the roots of $q(t) - xtp(t)$, $1/\theta_1(x)$ $(= 1/\theta(x))$, $1/\theta_2(x), \ldots, 1/\theta_r(x)$. The $\theta_i(x)$ are Laurent series in $1/x$ (or some fractional power of $1/x$), and each of these Laurent series converges for sufficiently large x. It is easy to verify that as $x \to \infty$, the $\theta_i(x)$ approach the reciprocals of the roots of $tp(t)$. Hence since $p_1 = c_0$ is not 0, $\theta_2(x), \theta_3(x), \ldots, \theta_r(x)$ have finite limits as $x \to \infty$, hence have degree $\leqq 0$, and the number of degree 0 is $\deg(p(t))$. Furthermore if $\deg(p(t))$ is $r-2$ or $r-1$ and $p(t)$ has no repeated roots then the $\theta_i(x)$ will be Laurent series in $1/x$. We apply the classical partial fraction decomposition to obtain

$$\frac{1}{1-xtc(t)} = \sum_{i=1}^{r} \frac{\lambda_i(x)}{1-t\theta_i(x)},$$

with $\lambda_i(x) = x\,\theta_i'(x)/\theta_i(x)$. This yields

4.8. THEOREM. — *We have*

$$C_n(x) = \sum_{i=1}^{r} \theta_i'(x)\,\theta_i(x)^n$$

$$\varepsilon_n(x) = \sum_{i=2}^{r} \theta_i'(x)\,\theta_i(x)^n$$

$$1/x = \sum_{i=1}^{r} \theta_i'(x)/\theta_i(x). \quad \square$$

We will use the special case when $c_2 = c_3 = c_4 = \ldots = 0$, later.

4.9. THEOREM. — *Suppose* $c(t) = c_0 + c_1 t$, *then*

$$\theta(x) = \frac{c_0 x}{2}(1+D),$$

$$\theta'(x) = \theta(x)^2/(x^2 c_0 D),$$

$$C_n(x) = \left(\frac{c_0 x}{2}\right)^{n+2} ((1+D)^{n+2} - (1-D)^{n+2})/(x^2 c_0 D),$$

where

$$D = \sqrt{1+\frac{4c_1}{c_0^2 x}}.$$

Proof. — Clearly

$$b(y) = c_0/y + c_1/y^2 \qquad \text{and} \qquad c_0\,a(x) + c_1\,a(x)^2 = 1/x,$$

or

$$\theta(x)^2 - x c_0\,\theta(x) - x c_1 = 0.$$

Solving the above formula for $\theta(x)$, differentiating to get $\theta'(x)$, and using the formula $C_n(x) = \theta_1'(x)\,\theta_1(x)^n + \theta_2'(x)\,\theta_2(x)^n$, completes the proof. $\square$

5. F-Sequences

We call the sequence of integers $c_0, c_1, c_2, \ldots$ an F-*sequence* (*from* n_0 *on*) if $c_0 > 0$ and if for all $n \geqq n_0$, δ_{n-2}, the coefficient of x^{n-2} (which is formally the leading coefficient) in $C_n C_{n-2} - C_{n-1}^2$, has absolute value $\leqq c_0^{n-1}/2$ and if $|\delta_{n-2}| = c_0^{n-1}/2$, then the sign of the leading coefficient of $C_n C_{n-2} - C_{n-1}^2 - \delta_{n-2} C_{n-2}/C_0^{n-1}$ is $-$ sign (δ_{n-2}), unless the latter polynomial is 0 in which case we require that $\delta_{n-2} = c_0^{n-1}/2$. We shall call $c_0, c_1, c_2, \ldots$ a *proper* F-sequence (from n_0 on) if in addition for all $n \geqq n_0$ either $|\delta_{n-2}| < c_0^{n-1}/2$ or $|\delta_{n-2}| = c_0^{n-1}/2$ and either the degree of the first non-zero coefficient of

$$C_n C_{n-2} - C_{n-1}^2 - \delta_{n-2} C_{n-2}$$

is $\equiv n \pmod 2$ or the latter polynomial is 0. We shall call the sequence $c_0, c_1, c_2, \ldots$ an F-sequence *from the beginning* if n_0 can be chosen equal to 2.

In the proof of the next theorem we give a simple, recursive way of calculating F-sequences.

5.1. THEOREM. — *Suppose $c_0 > 0$, $c_1, c_2, \ldots, c_{n_0-1}$ are integers. There exists a unique* F-*sequence from n_0 on whose initial elements are $c_0, c_1, \ldots, c_{n_0-1}$. Furthermore if c_0 is odd, or if $n_0 = 2$ and c_0 has a prime divisor not dividing c_1, this* F-*sequence will be proper.*

Proof. — Put $d_0 = 1/c_0$ and inductively for $1 \leqq n \leqq n_0 - 1$ choose d_n so that $\sum_{=0}^{n} c_i d_{n-i} = 0$. Inductively, for $n \geqq n_0$, if $\left\| c_0 \sum_{i=1}^{n-1} c_i d_{n-i} \right\| \neq 1/2$, put

$$c_n = \mathrm{N}\left(-c_0 \sum_{i=1}^{n-1} c_i d_{n-i}\right) \qquad \text{and} \qquad d_n = -d_0 \sum_{i=1}^{n} c_i d_{n-i}.$$

With this choice of c_n and d_n, we have $\sum_{i=0}^{n} c_i d_{n-i} = 0$ and $c_0^2 |d_n| < 1/2$. If, however, $\left\| -c_0 \sum_{i=1}^{n} c_i d_{n-i} \right\| = 1/2$, the following "tie-breaking" rule must be used. Put $c'_n = -c_0 \sum_{i=1}^{n-1} c_i d_{n-i}$ and define

$$\sum_{i,j=0}^{\infty} h'_{ij} s^i t^j = \frac{s-t}{\sum_{i=0}^{n-1} c_i (s^i - t^i) + c'_n (s^n - t^n)}.$$

Let i_0 be the least integer, if there are any, satisfying $1 \leqq i_0 \leqq n$ and for which $h'_{i_0-1,n+1} \neq h'_{i_0,n}$. If $h'_{i_0-1,n+1} < h'_{i_0,n}$, then put $c_n = c'_n + 1/2$; while if $h'_{i_0-1,n+1} > h'_{i_0,n}$ then put $c_n = c'_n - 1/2$. Finally if $h'_{i-1,n+1} = h'_{i,n}$ for all i satisfying $1 \leqq i \leqq n$, put $c_n = c'_n - 1/2$. As before put $d_n = -d_0 \sum_{i=1}^{n} c_i d_{n-i}$. Put $c(t) = \sum_{i=0}^{\infty} c_i t^i$ and $d(t) = \sum_{i=0}^{\infty} d_i t^i = 1/c(t)$. By Theorem 4.4, δ_{n-2} the coefficient of x^{n-2} in $C_n C_{n-2} - C_{n-1}^2$ is $d_n c_0^{n+1}$. Hence if $n \geqq n_0$, then $|\delta_{n-2}| \leqq c_0^{n-1}/2$, since $|d_n| \leqq 1/(2c_0^2)$. If

$$|\delta_{n-2}| = |c_0^{n-1}/2|, \qquad \text{then} \qquad |d_n| = 1/2 c_0^2,$$

and in this case

$$\begin{aligned} d_n &= -d_0 \sum_{i=1}^{n} c_i d_{n-i} \\ &= -d_0 \sum_{i=1}^{n-1} c_i d_{n-i} - d_0^2 c_n \\ &= d_0^2 (c_n' - c_n). \end{aligned}$$

Thus

$$\begin{aligned} \operatorname{sign}(\delta_{n-2}) &= -\operatorname{sign}(c_n - c_n') \\ &= \operatorname{sign}(h'_{i_0-1,n+1} - h'_{i_0,n}) \end{aligned}$$

if $(h'_{i_0-1,n+1} - h'_{i_0,n}) \neq 0$, while $\delta_{n-2} > 0$ if all $h'_{i-1,n+1} - h'_{i,n} = 0$. Put

$$\frac{1}{1 - xt\left(\sum_{i=0}^{n-1} c_i t^i + c_n' t^n\right)} = 1 + xt \sum_{i=0}^{n-1} C_{i-1} t^i + C_n' t^n + \ldots.$$

By the choice of c_n', $C_n' \, C_{n-2} - C_{n-1}^2$ has degree $\leqq n-3$; since by Theorem 4.5 $x^2 (C_n' \, C_{n-2} - C_{n-1}^2) = \sum_{i=0}^{n} (h'_{i-1,n+1} - h'_{i,n}) \, C_{n-i}$, we have $h'_{0,n} = 0$. Next note that $C_n' - C_n = c_n' - c_n$ and that $C_n \, C_{n-2} - C_{n-1}^2 - \delta_{n-2} \, C_{n-2}/c_0^{n-1}$ has degree $\leqq n-3$. It follows that

$$\begin{aligned} C_n' \, C_{n-2} - C_{n-1}^2 &= C_n \, C_{n-2} - C_{n-1}^2 - \delta_{n-2} \, C_{n-2}/c_0^{n-1} \\ &= \sum_{i=0}^{n} (h'_{i-1,n+1} - h'_{i,n}) \, C_{n-i}/x^2, \end{aligned}$$

and the sign of the first non-vanishing coefficient of $C_n \, C_{n-2} - C_{n-1}^2 - \delta_{n-2} \, C_{n-2}/c_0^{n-1}$ is the sign of $h'_{i_0-1,n+1} - h'_{i_0,n}$, which is $-\operatorname{sign}(\delta_{n-2})$. If all $h'_{i-1,n+1} - h'_{i,n}$ are 0 then $\delta_{n-2} = c_0^{n-1}/2$. Thus, the inductive definition of the c_n yields an F-sequence. Now δ_{n-2} is an integer, hence if c_0 is odd, $|\delta_{n-2}|$ cannot equal $c_0^{n-1}/2$, and thus the F-sequence is proper. It is easy to verify that $c_0^{n+1} d_n$ is integral and that for $n \geqq n_0$, $c_0^{n-1} c_n$ is the nearest multiple of c_0^{n-1} to $-\sum_{i=1}^{n-1} (c_0^{i-1} c_i)(c_0^{n+1-i} d_{n-i})$. Alternatively, $c_0^{n+1} d_n$ is the residue of least absolute value (modulo c_0^{n-1}) of $-\sum_{i=1}^{n-1} (c_0^{i-1} c_i)(c_0^{n+1-i} d_{n-i})$, and then

$$c_0^{n-1} c_n = -\sum_{i=0}^{n-1} (c_0^{i-1} c_i)(c_0^{n+1-i} d_{n-i}) \tag{5.2}$$

It follows that the "tie-breaking" rule need only be used when

$$c_0^{n+1} d_n \equiv \sum_{i=1}^{n-1} (c_0^{i-1} c_i)(c_0^{n+1-i} d_{n-i}) \equiv c_0^{n-1}/2 \pmod{c_0^{n-1}}.$$

If we assume that c_0 is even, $n \geqq n_0$, and take (5.2) (mod c_0) we obtain

$$(c_0^{n+1} d_n) \equiv -c_1 (c_0^n d_{n-1}) \pmod{c_0}.$$

When $n_0 = 2$, then $d_1 = -c_1/c_0^2$; hence in order that the sequence not be proper, there exists n such that

$$(c_0^{n+1} d_n) \equiv (-c_1)^n (\operatorname{mod} c_0).$$

If an odd prime divides c_0 and not c_1 then

$$(-c_1)^n \not\equiv c_0^{n-1}/2 (\operatorname{mod} c_0);$$

while if c_0 is even and c_1 is odd then $(-c_1)^n \equiv c_0^{n-1}/2$ is not possible, unless $n = 2$. When $n = 2$, $C_2(x) C_0(x) - C_1(x)^2 = c_0 c_2 - c_1^2$ and the polynomial $C_2 C_0 - C_1^2 - \delta_0 C_0$ is 0. Thus the F-sequence $c_0, c_1, c_2, \ldots$ is proper. □

The proof of Theorem 5.1 gives a simple recursive way of calculating an F-Sequence using integer computations only. Specifically, given $c_0, c_1, \ldots, c_{n-1}$ and $d_0, d_1, \ldots, d_{n-1}$, let $c_0^{n+1} d_n$ be the residue of least absolute value of

$$\gamma_n = -\sum_{i=1}^{n-1} (c_0^{i-1} c_i)(c_0^{n-i+1} d_{n-i}) (\operatorname{mod} c_0^{n-1})$$

and $c_0^{n-1} c_n = \gamma_n - c_0^{n+1} d_n$. In this calculation the c_i and $c_0^{i+1} d_i$ are integers. As we have seen, this uniquely defines the F-sequences except when $\left\| c_0 \sum_{i=1}^{n=1} c_i d_{n-i} \right\| = 1/2$. When this is the case the tie-breaking rule must be used. We must determine the sign of the first non-vanishing $h'_{i-1,n+1} - h'_{i,n}$ for sign $(c'_n - c_n) = \operatorname{sign}(h'_{i-1,n+1} - h'_{i,n})$. We can express the latter in terms of the $d'_j s$ by using (4.6). This can be simplified, and when $i = 2$, one obtains, for example,

$$\operatorname{sign}(h'_{1,n+1} - h'_{2,n}) = \operatorname{sign}\left(\sum_{j=2}^{n-1} d_j d_{n+1-j}\right).$$

Note that if c_0 is odd or has an odd prime divisor not dividing c_1, then $\left\| c_0 \sum_{j=1}^{n-1} c^j d_{n-j} \right\|$ is never 1/2 and the tie-breaking rule is not needed. When c_0 is even and c_1 is odd, then $\left\| c_0 \sum_{j=1}^{n-1} c_j d_{n-j} \right\| = 1/2$ only when $n = 2$, and then the above proof shows that $c_2 = c_1^2/c_0 + 1/2$.

When computing F-sequences, we may limit c_1 to the range $-c_0^2/2 < c_1 \leqq c_0^2/2$. Indeed, let the F-sequence from n_0 on be $c_0, c_1, c_2, \ldots$ and as usual put

$$(1 - xtc(t))^{-1} = 1 + xt \sum_{n=1}^{\infty} C_n(x) t^n.$$

Then, for integer x_0, the sequence $C_0(x_0), C_1(x_0), C_2(x_0), \ldots$ si an F-sequence with initial terms $c_0, c_1 + c_0^2 x_0$ for

$$\frac{1}{1 - xt \sum_{n=0}^{\infty} C_n(x_0) t^n} = \frac{1}{1 - xt\, c(t)/(1 - x_0\, tc(t))}$$

$$= \frac{1 - x_0\, tc(t)}{1 - (x + x_0)\, tc(t)} - \frac{x_0}{x + x_0} + \frac{x_0}{x + x_0}$$

$$= \frac{x}{(x + x_0)(1 - (x + x_0)\, tc(t))} + \frac{x_0}{x + x_0}$$

$$= \frac{x}{x+x_0}\left(1+(x+x_0)\,t\sum_{n=0}^{\infty} C_n(x+x_0)\,t^n\right)+\frac{x_0}{x+x_0}$$

$$= 1+xt\sum_{n=0}^{\infty} C_n(x+x_0)\,t^n.$$

Since for any polynomial H (x), the leading coefficient of H $(x+x_0)$ equals the leading coefficient of H (x), the definition of F-sequences shows that $C_0(x_0)$, $C_1(x_0)$, $C_2(x_0)$, ... is an F-sequence from n_0 on. Note that $\left(\sum_{n=0}^{\infty} C_n(x_0)\,t^n\right)^{-1} = d(t)-x_0\,t$. It follows from this that the sign of the first non-zero $(h'_{i-1,n+1}-h'_{i,n})$ does not depend upon d_1 (*cf.* Theorems 4.5 and 4.7). Furthermore, if $c_0, c_1, c_2, \ldots$ is a proper F-sequence, then so is the sequence $\{(-1)^i c_i \mid i = 1, 2, 3, \ldots\}$. Thus in most cases, we can limit c_1 to the range $1 \leqq c_1 < c_0^2/2$ (the cases where c_0 divides c_1 are trivial). We now consider the application of F-sequences to Pisot's E-sequences.

Let $c_0, c_1, c_2, \ldots$ be a sequence of integers with $c_0 > 0$ and as before put $c(t) = \sum_{i=0}^{\infty} c_i\,t^i$, and put

$$(1-xtc(t))^{-1} = 1+xt\sum_{n=0}^{\infty} C_n(x)\,t^n.$$

5.3. THEOREM. — *If c_0, c_1 c_2, ... is an F-sequence from n_0 on then for $n \geqq n_0$ and all sufficiently large integers x we have $C_n(x) = N(C_{n-1}(x)^2/C_{n-2}(x))$. Conversely if there exists n_0 such that for $n \geqq n_0$ and all sufficiently large integers x,*

$$C_n(x) = N(C_{n-1}(x)^2/C_{n-2}(x)),$$

then $c_0, c_1, c_2, \ldots$ is an F-sequence from n_0 on.

Before we give the proof we note there is no uniformity in n in the above theorem. The magnitude required for x may depend upon n. Later, in the case when $c(t)$ is a rational function, we shall obtain this result uniformly in n.

Proof of Theorem 5.3. — We can write

$$C_n(x)-C_{n-1}(x)^2/C_{n-2}(x) = \gamma_0+\gamma_1/x+\gamma_2/x^2+\ldots.$$

Then, for large x, $C_n(x) = N(C_{n-1}(x)^2/C_{n-2}(X))$ if and only if $|\gamma_0| < 1/2$), or $|\gamma_0| = 1/2$ and all remaining $\gamma_i = 0$, or $|\gamma_0| = 1/2$ and the first non-zero γ_1, $i = 1, 2, 3, \ldots$ has sign opposite of γ_0. By the definition of F-sequences, this occurs for all $n \geqq n_0$ if and only if $c_0, c_1, c_2, \ldots$ is an F-sequence from n_0 on.

We now give a simple construction for F-sequences.

5.4. THEOREM. — *Suppose $p(t)$ and $q(t)$ are polynomials with integral coefficients such that all roots of $p(t)$ have absolute value > 1, $p(0) > 0$, and $q(0) = 1$. Put*

$$\sum_{i=0}^{\infty} c_i\,t^i = p(t)/q(t).$$

Then $c_0, c_1, c_2, \ldots$ is an F-sequence.

Proof. — Suppose that $c(t) = p(t)/q(t)$, where $p(t)$ has all of its zeros outside the unit circle. Then $d(t) = q(t)/p(t)$ is regular in the closed unit disk and the d_i go to 0 geometrically. □

Note that if $\sum_{i=0}^{\infty} d_i^2 < \infty$, then by [3], $d(t)$ is a rational function as above and hence the d_i go to 0 geometrically. Under these circumstances, there is uniformity in Theorem 5.3.

5.5. THEOREM. — *Suppose* $|d_n| \leqq \mathrm{MR}^n$ *for all* $n \geqq 0$, *where* $\mathrm{R} < 1$, *and suppose that* $c_0, c_1, c_2, \ldots$ *is an* F*-sequence from* n_0 *on. Then there exists* x_0 *such that if* x *is an integer* $\geqq x_0$, *then* $\mathrm{C}_{n_0-2}(x)$, $\mathrm{C}_{n_0-1}(x)$, $\mathrm{C}_{n_0}(x), \ldots$ *is an* E*-sequence.*

Proof. — Put, as before, $b_n = c_{n-1}$, $n = 1, 2, 3, \ldots$ and then $1/b(y) = y \sum_{n=0}^{\infty} d_n/y^n$ and $\beta_n(y) = -\{ y^n (\sum_j d_j/y^j)^n \}$. Now

$$\begin{aligned}\sum_{n=1}^{\infty} \alpha_n(x)/y^n &= \sum_{n=1}^{\infty} \beta_n(y)/x^n \\ &\ll \sum_{n=1}^{\infty} \left\{ y^n \left(\sum_{j=0}^{\infty} |d_j|/y^j \right)^n \right\} \Big/ x^n \\ &\ll \sum_{n=1}^{\infty} \left\{ y^n \left(\sum_{j=0}^{\infty} \mathrm{MR}^j/y^j \right) \right\}.\end{aligned}$$

We can compute explicity the last sum by considering the case when

$$d_j = \mathrm{MR}^j, \qquad d(t) = \mathrm{M}/(1-\mathrm{R}t), \qquad c(t) = 1/\mathrm{M} - (\mathrm{R}/\mathrm{M})t,$$
$$c_0 = 1/\mathrm{M}, \; c_1 = -\mathrm{R}/\mathrm{M} \qquad \text{and} \qquad c_2 = c_3 = c_4 = \ldots = 0.$$

From Theorem 4.8 and 4.9 (the quadratic case), we obtain

$$\begin{aligned}\varepsilon_n(x) &\ll \mathrm{M}(x/2\mathrm{M})^{n+2}(1-\mathrm{D})^{n+2}/(x^2\mathrm{D}) \\ &= \mathrm{M}(x/2\mathrm{M})^{n+2}(4\mathrm{RM}/x)^{n+2}/(x^2\mathrm{D}(1+\mathrm{D})^{n+2}) \\ &= \mathrm{M}(2\mathrm{R}/(1+\mathrm{D}))^{n+2}/(x^2\mathrm{D}),\end{aligned}$$

where $\mathrm{D} = \sqrt{1-4\mathrm{RM}/x}$. When x is sufficiently large, then $2\mathrm{R}/(1+\mathrm{D}) < 1$ and $\varepsilon_n(x) \to 0$ as $n \to \infty$. Now $\theta(x)$ satisfies

$$1 - xp(1/\theta)/q(1/\theta) = 0 \qquad \text{or} \qquad x = \theta(q_0 + q_1/\theta + \ldots)/(p_1 + p_2/\theta + \ldots),$$

where $p(t) = p_1 t + p_2 t^2 + \ldots$ and $q(t) = q_0 + q_1 t + \ldots$ hence $x \sim (q_0/p_0)\theta$; equivalently, $\theta \sim (p_0/q_0)x$. It follows that there exists a constant μ such that $\theta(x) \leqq \mu x$ for sufficiently large x. Then $|\theta^2(x)\varepsilon_n(x)| < \mu^2 \mathrm{M}(2\mathrm{R}/(1+\mathrm{D}))^{n+2}/\mathrm{D}$. Hence there exists x_0 such that $\theta^2(x)\varepsilon_n(x) \to 0$ as $n \to \infty$, uniformly in $x \geqq x_0$. Now

$$\mathrm{C}_n - \mathrm{C}_{n-1}^2/\mathrm{C}_{n-2} = \frac{\theta^2 \varepsilon_{n-2} - 2\theta\varepsilon_{n-1} + \varepsilon_n}{1 + \varepsilon_{n-2}/\lambda\theta^{n-2}} + \frac{\varepsilon_{n-2}\varepsilon_n - \varepsilon_{n-1}^2}{\lambda\theta^{n-2} + \varepsilon_{n-2}}.$$

Thus $C_n - C_{n-1}^2/C_{n-2} \to 0$ as $n \to \infty$ uniformly for $x \geqq x_0$. So there exists n_1 such that $C_{n_1}(x), C_{n_1+1}(x), C_{n_1+2}(x), \ldots$ is an E-sequence for $n \geqq n_1$. By Theorem 5.3, we can choose $n_1 = n_0$, possibly at the expense of increasing x_0. □

The above proof gives a simple and effective way of calculating x_0. There is a simpler proof of Theorem 5.5 in the case when $q(t)$ has no repeated roots, but it does not lead to quite so simple a way of estimating x_0. When $q(t)$ has no repeated roots, we can write

$$\theta_i(x) = \sum_{j=0}^{\infty} \theta_{ij}/x^j, \qquad 2 \leqq i \leqq r,$$

where the $1/\theta_{i0}$ are the roots of $q(t)$. Then each $\theta_i'(x)$ has degree $\leqq -2$ and $\theta^2(x)\,\varepsilon_n(x) = \theta^2(x) \sum_{i=2}^{r} \theta_i'(x)\,\theta_i(x)^n$ has degree $\leqq 0$ and goes to 0 as $n \to \infty$. Since the degree is $\leqq 0$, this convergence to 0 is uniform in x when x is so large that $|\theta_i(x)| < 1$ for $2 \leqq i \leqq r$.

In this thesis [7], Galyean did extensive computation of E-sequences using methods similar to those described here. The following is a short table of known rational F-sequences, from the beginning for $2 \leqq c_0 \leqq 5$, based upon computations of Galyean and the author [5]. Except for 4, 2 and 4, −2, all of these F-sequences are proper. In each case we give c_0, c_1, the rational function $c(t)$ and the range of x for which $C_0(x), C_1(x), C_2(x), \ldots$ is a Pisot E-sequence. We omit pairs c_0, c_1 for which $c_0 \mid c_1$ and, with one exception, limit c_1 to the range $c_0 < c_1 < c_0^2/2$. Other omitted cases are those in which we do not know if $c(t)$ is rational.

5.6. TABLE OF F-SEQUENCES:

c_0	c_1	$c(t)$	Range of x
2	1	$(2-t)/(1-t)$	all x
3	1	$3+t$	all x
3	2	$3+2t+t^2$	$\lvert x\rvert \geqq 1$
3	4	$(3-2t)/(1-2t+t^2)$	all x
4	1	$4+t$	all x
4	2	$(4-2t-t^2)/(1-t)$	$x \geqq 0$
4	−2	$4-2t+t^2$	$x \leqq -1$
4	5	$(4-3t)/(1-2t+t^2)$	all x
4	7	$(4-t+2t^2)/(1-2t+t^2-t^3)$	all x
5	1	$5+t$	all x
5	2	$5+2t+t^2$	$\lvert x\rvert \geqq 1$
5	3	$5+3t+2t^2+t^3$	all x
5	6	$(5-4t)/(1-2t+t^2)$	all x
5	8	$(5+3t)/(1-t-t^2)$	all x
5	9	$(5-t+3t^2)/(1-2t+t^2-t^3)$	all x
5	11	$(5-4t+t^2+2t^3)/(1-3t+2t^2-t^4)$	all x
5	12	$(5+2t)/(1-2t-t^2)$	all x

As a final remark we note that Pisot's definition of E-sequences was somewhat arbitrary. He could, for example, have required that $a_{n+1} = [a_n^2/a_{n-1}]$ and this definition [which is equivalent to $a_{n+1} = N(a_n^2/a_{n-1} - 1/2)$] would lead to a different, but quite similar theory.

More generally, given any sequence $\{\gamma_n\}$ of real numbers one could require that $a_n = \mathrm{N}(a_{n-1}^2/a_{n-2}+\gamma_n)$. A similar comment applies to F-sequences. We could modify their definition to require that $\mathrm{C}_n(x) = \mathrm{N}(\mathrm{C}_{n-1}(x)^2/\mathrm{C}_{n-2}(x)+\gamma_n)$ for sufficiently large x. Since the leading coefficient of $\mathrm{C}_n(x)-\mathrm{C}_{n-1}(x)^2/\mathrm{C}_{n-1}(x)$ is $d_n c_0^2$, the definition of the c_n would have to be modified to require that $\| d_n c_0^2 - \gamma_n \|$ be $\leqq 1/2$ with a special tie-breaking rule in case of equality. If γ_n is irrational then, of course, no tie can occur.

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