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ON THE ENRIGHT-VARADARAJAN MODULES : A CONSTRUCTION OF THE DISCRETE SERIES

By Nolan R. WALLACH

1. Introduction

Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbf{C}$. Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. Let $\mathfrak{k}$ be the complexification of $\mathfrak{k}_0$. We assume that there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ so that $\mathfrak{h} \subset \mathfrak{k}$. Fix P a system of positive roots for $(\mathfrak{g}. \mathfrak{h})$. Let $P_k \subset P$ be the corresponding positive roots for $(\mathfrak{k}, \mathfrak{h})$. Let $\langle\, , \rangle$ denote the dual of the killing form of $\mathfrak{g}$ restricted $\mathfrak{h}$. If $\lambda \in \mathfrak{h}^*$ call λ, P_k-dominant integral if

$$2\frac{\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbf{Z}^+ = \{0, 1, \ldots, n, \ldots\}$$

for $\alpha \in P_k$.

In Enright, Varadarajan [4], a construction was given of a $\mathfrak{g}$-module $W_{P,\lambda}$ for each P_k-dominant integral form $\lambda \in \mathfrak{h}^*$. These modules have several important properties:

(1) As a $\mathfrak{k}$-module, $W_{P,\lambda} = \sum \oplus m_\lambda(\mu) V_\mu$, where the sum is over all P_k-dominant integral forms, V_μ is the irreducible finite dimensional $\mathfrak{k}$-module with highest weight μ and $0 \leqq m_\lambda(\mu) < \infty$, $m_\lambda(\mu)$ an integer.

(2) $m_\lambda(\lambda) = 1$.

(3) If $m_\lambda(\mu) \neq 0$ then $\mu = \lambda + \delta$, where δ is a sum of (not necessarily distinct) elements of P.

(4) Let $U = U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Then $U(\mathfrak{g}) V_\lambda = W_{P,\lambda}$ (here we look at V_λ as being imbedded in $W_{P,\lambda}$).

(5) Let $U^{\mathfrak{k}}$ be the centralizer of $\mathfrak{k}$ in U. Then $U^{\mathfrak{k}}$ acts by scalars on V_λ and the corresponding homomorphism $\eta_\lambda : U^{\mathfrak{k}} \to \mathbf{C}$ is computed (*see* Theorem 2.4 for the formula).

By (2) and (4), $W_{P,\lambda}$ contains a unique maximal submodule $Z_{P,\lambda}$ not containing V_λ. Set $W_{P,\lambda}/Z_{P,\lambda} = D_{P,\lambda}$. There $D_{P,\lambda}$ is clearly irreducible and inherits the multiplicity properties and η_λ.

Let now G be the connected, simply connected Lie group with Lie algebra $\mathfrak{g}$. Let $G_0 \subset G$ be the connected subgroup with Lie algebra $\mathfrak{g}_0$. If $\lambda \in \mathfrak{h}^*$ we call λ integral if

$$\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbf{Z}, \qquad \alpha \in \Delta,$$

Δ the root system of $(\mathfrak{g}, \mathfrak{h})$. We call $\lambda \in \mathfrak{h}^*$ regular if $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$.

To each regular, integral $\lambda \in \mathfrak{h}^*$, Harish-Chandra [6] has constructed a central, eigen-distribution for the center $\mathfrak{z}$ of U, θ_λ, on G_0 with the following properties :

(i) $\theta_\lambda = \theta_\mu$ if and only if there is $s \in W_k$ [the Weyl group of $(\mathfrak{k}, \mathfrak{h})$] so that $s\lambda = \mu$.

(ii) Each θ_λ is the character of an irreducible, square integrable representation of G_0.

(iii) The θ_λ exhaust the characters of the irreducible, square integrable representations of G_0.

Let $\lambda \in \mathfrak{h}^*$ be integral and regular. Let $P = \{ \alpha \in \Delta \mid \langle \lambda, \alpha \rangle > 0 \}$.

One of our results is

THEOREM 1.1. — *If* $\lambda \in \mathfrak{h}^*$ *is integral and regular and if* $P = \{ \alpha \in \Delta \mid \langle \lambda, \alpha \rangle > 0 \}$. *Then* $D_{P, \lambda - \rho_k + \rho_n}$ *is infinitesimally equivalent with the irreducible representation of* G_0 *with character* θ_λ (see Theorem 4.5).

Note. — Schmid [14] has also proved this result. Many of the ideas in the proof are due to Schmid and Zuckerman.

In light of this result, the Enright, Varadarajan module becomes very important. A purpose of this paper is to give a more canonical construction of $W_{P,\lambda}$. We actually do a bit more than this. In the Enright, Varadarajan construction there is really no use of the fact that $\mathfrak{k}$ comes from a symmetric pair $(\mathfrak{g}_0, \mathfrak{k}_0)$. Thus let $\mathfrak{g}$ be as before a semi-simple Lie algebra over C. Let $\mathfrak{k} \subset \mathfrak{g}$ be a reductive subalgebra so that there is a Cartan sulbalgebra of $\mathfrak{g}$, $\mathfrak{h}$, so that $\mathfrak{h} \subset \mathfrak{k}$. Let P be a system of positive roots for $(\mathfrak{g}, \mathfrak{h})$ and let us use the same terminology as the first part of the introduction. That is, P_k-dominant integral, etc. We construct for each λ, P_k-dominant integral a $\mathfrak{g}$-module, $W_{P,\lambda}$ satisfying 1, 2, 3, 4, 5 above. The construction is quite analogous to the Verma module construction of the irreducible finite dimensional representations of $\mathfrak{g}$. In fact, if $\mathfrak{g} = \mathfrak{k}$ then $W_{P,\lambda}$ is just the irreducible finite dimensional representation of $\mathfrak{g}$ with highest weight λ. If $\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{r}$ is a parabolic subalgebra of $\mathfrak{g}$ ($\mathfrak{r}$ the unipotent radical) and P is system of positive roots for $(\mathfrak{g}, \mathfrak{h})$ contained in the roots of $\mathfrak{p}$ and if V_λ is the irreducible representation of $\mathfrak{k}$ with highest weight λ then $W_{P,\lambda} = U(\mathfrak{g}) \underset{U(\bar{\mathfrak{p}})}{\oplus} V_\lambda$, where $\bar{\mathfrak{p}} = \mathfrak{k} \oplus \bar{\mathfrak{r}}$, the opposite parabolic, and V_λ is a $\bar{\mathfrak{r}}$ module by making $\bar{\mathfrak{r}}$ act trivially ($U(\bar{\mathfrak{p}})$ is the universal enveloping algebra of $\bar{\mathfrak{p}}$).

Also in this paper we study tensor products of the modules $W_{P,\lambda}$ with finite dimensional $\mathfrak{g}$-modules. We strengthen results of Enright [3]. These results are related to results of Schmid [14]. In section 3 we derive explicit formulae for the tensor products of $D_{P,\lambda}$ and $W_{P,\lambda}$ with finite dimensional $\mathfrak{g}$-modules. We note that Lemma 3.10 contains as a special case a result of Nicole Conze (*see* Rossi, Vergne [11]).

We would like to thank W. Schmid for many helpful and stimulating conversations about the discrete series and the role of tensoring with finite dimensional representations. Many of the ideas in § 4 are due to W. Schmid. We feel that the modules $W_{P,\lambda}$ are an important discovery and we heartily congratulate Enright and Varadarajan for their discovery.

2. The Enright, Varadarajan construction

Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbf{C}$ the complex numbers. Let $\mathfrak{k} \subset \mathfrak{g}$ be a reductive subalgebra so that there is a Cartan subalgebra, $\mathfrak{h}$, of $\mathfrak{g}$, $\mathfrak{h} \subset \mathfrak{k}$. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$, $\Delta_k \subset \Delta$ the root system of $(\mathfrak{k}, \mathfrak{h})$.

Let P be a system of positive roots for Δ and set $P_k = P \cap \Delta_k$. Let W_k denote the Weyl group of $(\mathfrak{k}, \mathfrak{h})$. Let W_k be ordered as in Dixmier [2]. Chapter 7, Section 7. That is if $w_1, w_2 \in W_k$ then we say $w_2 \xrightarrow{\alpha} w_1$ if $\alpha \in P_k$ and

(*a*) $w_1 = s_\alpha w_2$.

(*b*) $l(w_1) = l(w_2)+1$ [$l(w)$ is the number of terms in the minimal expression of w as a product of P_k-simple reflections].

If $w, w' \in W_k$ then $w \leqq w'$ if there exist $w_0, \ldots, w_k \in W_k$ and $\beta_1, \ldots, \beta_n \in P_k$ so that $w_n = w'$, $w = w_0$ and

$$w_n \xrightarrow{\beta_n} w_{n-1} \rightarrow \ldots \xrightarrow{\beta_1} w_0.$$

Relative to this order $s \leqq 1$ for all $s \in W_k$ and $s \geqq t_0$ ($t_0 \in W_k$ the unique element so that $t_0 P_k = -P_k$) for all $s \in W_k$.

If $\mu \in \mathfrak{h}^*$ let V^μ denote the $\mathfrak{k}$-Verma module with highest weight μ relative to P_k. V^μ is defined as follows: let $\mathfrak{n}_k^+ = \sum_{\alpha \in P_k} \mathfrak{g}_\alpha$,

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [h, x] = \alpha(h) X \text{ for } h \in \mathfrak{h}\}.$$

Set $\mathfrak{b}_k = \mathfrak{h} + \mathfrak{n}_k^+$. Let $\mathbf{C}_\mu$ be the $\mathfrak{b}_k$-module $\mathbf{C}$ with $(h+Z).1 = \mu(h) 1$ for $h \in \mathfrak{h}$, $Z \in \mathfrak{n}_k^+$. Then $V_\mu = U(\mathfrak{k}) \underset{U(\mathfrak{b}_k)}{\oplus} \mathbf{C}_\lambda$, where $U(\mathfrak{k})$ and $U(\mathfrak{b}_k)$ are respectively the universal enveloping algebras of $\mathfrak{k}$ and $\mathfrak{b}_k$.

The theory of Verma modules (due to Verma, Bernstein, Gelfand and Gelfand, *cf.* Dixmier [2], Chapter 7) implies the following results

(1) If $\mu_1, \mu_2 \in \mathfrak{h}^*$ then $\dim \operatorname{Hom}_{\mathfrak{k}}(V^{\mu_1}, V^{\mu_2}) \leqq 1$ [$\operatorname{Hom}_{\mathfrak{k}}(.,.)$ denotes the space of $\mathfrak{k}$-module homomorphisms]. If $A \in \operatorname{Hom}_{\mathfrak{h}}(V^{\mu_1}, V^{\mu_2})$ and $A \neq 0$ then A is injective.

(2) Let $\mathfrak{n}_k^- = \sum_{\alpha \in P_k} \mathfrak{g}_{-\alpha}$ if $X \in \mathfrak{n}_k^-$ and $v \in V^\mu$ then $X v = 0$ implies $X = 0$ or $v = 0$.

(3) If $\operatorname{Hom}_{\mathfrak{k}}(V^{\mu_1}, V^{\mu_2}) \neq 0$ we say $V^{\mu_1} \subset V^{\mu_2}$. If λ is P_k-dominant integral (*see* the introduction), if $\rho_k = (1/2) \sum_{\alpha \in P_k} \alpha$ and if $s, \tau \in W_k$ then $V^{s(\lambda+\rho_k)-\rho_k} \subset V^{\tau(\lambda+\rho_k)-\rho_k}$ if and only if $s \leqq \tau$.

The theory of Verma modules is much richer than the results described above. However, we will only need the above three properties.

We begin the construction of a family of $\mathfrak{g}$-modules one for each $s \in W_k$; Fix $\lambda \in \mathfrak{h}^*$, P_k-dominant integral. Then if $s, \tau \in W_k$, $s \leqq \tau$ we clearly have

(I) $$U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} V^{s(\lambda+\rho_k)-\rho_k} \subset U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} V^{\tau(\lambda+\rho_k)-\rho_k}.$$

Let $W_{t_0,\lambda}$ denote the Verma module for $\mathfrak{g}$ with highest weight $t_0(\lambda+\rho_k)-\rho_k$ relative to $-t_0 P$ ($t_0 P_k = -P_k$, $t_0 \in W_k$). That is if $\tilde{\mathfrak{b}} = \mathfrak{h} + \sum_{\alpha \in -t_0 P} \mathfrak{g}_\alpha$ and $\mathbf{C}_{t_0(\lambda+\rho_k)-\rho_k}$ is the $\tilde{\mathfrak{b}}$-module $\mathbf{C}$ with $\mathfrak{h}$ acting by the linear form $t_0(\lambda+\rho_k)-\rho_k$ then

$$W_{t_0,\lambda} = U(\mathfrak{g}) \underset{U(\tilde{\mathfrak{b}})}{\otimes} \mathbf{C}_{t_0(\lambda+\rho_k)-\rho_k}.$$

Let $w_{t_0,\lambda} = 1 \otimes 1$ in $W_{t_0,\lambda}$. Then $U(\mathfrak{k})\, w_{t_0,\lambda}$ is $\mathfrak{k}$-isomorphic with $V^{t_0(\lambda+\rho_k)-\rho_k}$. We therefore have a surjective $\mathfrak{g}$-module homomorphism

$$U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} V^{t_0(\lambda+\rho_k)-\rho_k} \to W_{t_0,\lambda}.$$

Let I_λ denote the kernel of this $\mathfrak{g}$-homomorphism.

Then $I_\lambda \subset U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} V^{s(\lambda+\rho_k)-\rho_k}$ for all $s \in W_k$.

(II) If $s \in W_k$ define $M_{s,\lambda} = U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} V^{s(\lambda+\rho_k)-\rho_k}/I_\lambda$.

Clearly $M_{t_0,\lambda} = W_{t_0,\lambda}$. $M_{s,\lambda} \subset M_{\tau,\lambda}$ if $s \leqq \tau$.

Let $\tilde{\mu}_{s,\lambda} : U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} V^{s(\lambda+\rho_k)-\rho_k} \to M_{s,\lambda}$ be the natural map. Let $v_{s,\lambda}$ be the fundamental generator of $V^{s(\lambda+\rho_k)-\rho_k}$ $(1 \otimes 1)$. Then

(III) $$\tilde{\mu}_{s,\lambda} : \ U(\mathfrak{k}).(1 \otimes v_{s,\lambda}) \to M_{s,\lambda}$$

is injective. Set $m_{s,\lambda} = \tilde{\mu}_{s,\lambda}(1 \otimes v_{s,\lambda})$ then $M_{s,\lambda} = U(\mathfrak{g}).m_{s,\lambda}$.

(III) is clear from the definitions.

We now come to the "strange" part of the Enright, Varadarajan construction. We phrase it as a lemma.

LEMMA 2.1. — *Let* M *be a* $\mathfrak{g}$-*module. Suppose that* $M = U(\mathfrak{g})\, m$ *and that the map* $U(\mathfrak{n}_k^-) \to M$, $x \mapsto x.m$ *is injective. Then there exists a* $\mathfrak{g}$-*submodule* M_1 *of* M *so that* $U(\mathfrak{n}_k^-).m \cap M_1 = (0)$ *and*

(1) *If* $v \in M/M_1$, *if* $X \in \mathfrak{n}_k^-$ *and if* $Xv = 0$ *then* $X = 0$ *or* $v = 0$.

(2) *If* U *is a* $\mathfrak{g}$-*module such that if* $X \in \mathfrak{n}_k^-$, $u \in U$ *and if* $Xu = 0$ *then* $X = 0$ *or* $u = 0$ *then if* $\psi : M \to U$ *is a* $\mathfrak{g}$-*module homomorphism*, $\operatorname{Ker}\psi \supset M_1$.

Proof. — Let for each $X \in \mathfrak{n}_k^-$, $X \neq 0$,

$$J_{X,0} = \{ v \in M \mid X^k v = 0 \text{ for some } k \}.$$

If $Y \in \mathfrak{g}$ and $v \in J_{X,0}$ then $X^k Y v = \sum_{j=0}^{k} \binom{k}{j} (\operatorname{ad} X)^j . Y X^{k-j} . v$. Hence if $(\operatorname{ad} X)^l . Y = 0$, $X^r . v = 0$ then $X^{l+r}(Y.v) = 0$. Thus $\mathfrak{g} . J_{X,0} \subset J_{X,0}$. Define $J_0 = \sum_{\substack{X \neq 0 \\ X \in \mathfrak{n}_k^-}} J_{X,0}$. Suppose that J_i has been defined. J_i a $\mathfrak{g}$-submodule of M. Let for $X \in \mathfrak{n}_k^-$, $X \neq 0$,

$$J_{X,i+1} = \{ v \in M \mid X^n . v \in J_i \text{ for some } n \}.$$

Then as above $J_{X,i+1}$ is a $\mathfrak{g}$-submodule of M. Set

$$J_{i+1} = \sum_{\substack{X \in \mathfrak{n}_k^- \\ X \neq 0}} J_{X,i+1}.$$

Clearly $J_0 \subset J_1 \subset \ldots$. Let $J = \bigcup_{j=0}^{\infty} J_i$. Then J is a $\mathfrak{g}$-submodule of M. Set $M_1 = J$ We assert that $U(\mathfrak{n}_k^-) m \cap M_1 = (0)$. Indeed, if $v \in U(\mathfrak{n}_k^-) m \cap M_1$, $v \neq 0$ then $v \in J_i$ for some i. Hence there are elements $X_1, \ldots, X_k \in \mathfrak{n}_k^-$ so that, $X_j \neq 0$ and $v_j \in J_{X_j, i}$ so that $v = \sum_{j=1}^{k} v_j$. Now there is $k_1 \geqq 0$, $k_1 \in \mathbf{Z}$ so that $X_1^{k_1} v_1 \in J_{i-1}$. Hence

$$X_1^{k_1} v + J_{i-1} = \sum_{j=2}^{k} X_1^{k_1} v_j + J_{i-1}.$$

There is $k_2 \geqq 0$, $k_2 \in \mathbf{Z}$ so that

$$X_2^{k_2} X_1^{k_1} v + J_{i-1} = \sum_{j=3}^{k} X_2^{k_2} X_1^{k_1} v_j + J_{i-1}.$$

Continuing in this way we have $0 \neq v' \in U(\mathfrak{n}_k^-) m \cap J_{i-1'}$. Thus by recursion we find $U(\mathfrak{n}_k^-) m \cap J_0 \neq 0$. But this is impossible by hypothesis. Hence $U(\mathfrak{n}_k^-) m \cap M_1 = (0)$.

Let U and ψ be as in (2). Then if $v \in M$ and $X \neq 0$, $X \in \mathfrak{n}_k^-$ and $X^k . v = 0$ then if $k > 0$, $X . X^{k-1} v = 0$. Thus $\psi(X^{k-1} v) = 0$. But then $X . \psi(X^{k-2} v) = 0$ hence $\psi(X^{k-2} v) = 0$.

Continuing in this way we see $\psi(v) = 0$. Hence $\operatorname{Ker} \psi \supset J_0$. Suppose that we have shown that $\ker \psi \supset J_i$. Then the above argument shows that $\ker \psi \supset J_{i+1}$. Hence $\ker \psi \supset M_1$. The last assertion is also clear.

Q. E. D.

Now the pair $M_{s,\lambda}$ and $m_{s,\lambda}$ satisfy the hypothesis of lemma 2.1. Hence there is a minimal submodule $J_{s,\lambda} \subset M_{s,\lambda}$ so that $U(\mathfrak{n}_k^-) . m_{s,\lambda} \cap J_{s,\lambda} = (0)$ and if $v \in M_{s,\lambda}$ $X \in \mathfrak{n}_k^-$, $X \neq 0$ if then $X v \in J_{s,\lambda}$, $v \in J_{s,\lambda}$. We note that $U(\mathfrak{n}_k^-) m_{s,\lambda} = U(\mathfrak{k}) m_{s,\lambda}$.

Set $W_{s,\lambda} = M_{s,\lambda}/J_{s,\lambda}$. We note that $J_{t_0,\lambda} = (0)$. Thus the notation is consistent.

(IV) If $\tau \geqq s$ then $J_{\tau,\lambda} \cap M_{s,\lambda} = J_{s,\lambda}$. Clearly, lemma 2.1 implies that $J_{s,\lambda} \subset J_{\tau,\lambda} \cap M_{s,\lambda}$.

(*a*) $J_{s,\lambda} \supset (J_{\tau,\lambda})_0 \cap M_{s,\lambda}$. This is clear from the definition [here we use the notation $(J_{\tau,\lambda})_i$ for the J_i for $M_{\tau,\lambda}$]. Suppose that we have shown that $J_{s,\lambda} \supset (J_{\tau,\lambda})_i \cap M_{s,\lambda}$. If $v \in (J_{\tau,\lambda})_{i+1} \cap M_{s,\lambda}$ then there exists $X_1, \ldots, X_k \in \mathfrak{n}_k^-$, $X_1 \neq 0$ and $l_1, \ldots, l_k \in \mathbf{Z}$, $l_i \geqq 0$ so that $X_1^{l_1} \ldots X_k^{l_k} . v \in (J_{\tau,\lambda})_i \cap M_{s,\lambda}$.

Hence $X_1^{l_1} \dots X_k^{l_k}.v \in J_{s,\lambda}$. But then arguing as above we can "peel off" the X_i 's to find $v \in J_{s,\lambda}$.

We therefore have

(V) If s, $\tau \in W_k$ and $s \leqq \tau$ then $W_{s,\lambda} \subset W_{\tau,\lambda}$.

Let $\hat{\mu}_{s,\lambda} : M_{s,\lambda} \to W_{s,\lambda}$ be the canonical $\mathfrak{g}$-module-homomorphism. Set $w_{s,\lambda} = \hat{\mu}_{s,\lambda}(m_{s,\lambda})$.

(VI) $U(\mathfrak{k})\, w_{s,\lambda}$ is isomorphic as a $\mathfrak{k}$-module with $V^{s(\lambda+\rho_k)-\rho_k}$ and if $\tau \leqq s$ then $w_{\tau,\lambda} \in U(\mathfrak{k})\, w_{s,\lambda}$.

This is clear from Lemma 2.1 and the preceding constructions.

LEMMA 2.2. — *If* $s \xrightarrow{\gamma} \tau$ *and* $\gamma \in P_k$ *is* P_k*-simple then*

$$2\langle s(\lambda+\rho_k), \gamma\rangle/\langle \gamma, \gamma\rangle = n > 0 \qquad n \in \mathbf{Z}$$

and if $X \in \mathfrak{g}_{-\gamma}$, $X \neq 0$, $X^n w_{s,\lambda} = c w_{\tau,\lambda}$ *with* $c \neq 0$.

Proof. — It is easily checked that $n > 0$ (*cf.* Dixmier [2], Chapter 7, Section 7) and if $Y \in \mathfrak{n}_k^+$, $YX^n w_{s,\lambda} = 0$ and if $h \in \mathfrak{h}$ then

$$h X^n w_{s,\lambda} = (\tau(\lambda+\rho_k)-\rho_k)(h)\, X^n w_{s,\lambda}.$$

Since $w_{\tau,\lambda} \in U(\mathfrak{k})\, w_{s,\lambda}$ by (3) above and $w_{s,\lambda} \neq 0$ by construction the result follows from (1) above.

Q. E. D.

LEMMA 2.3. — *Let* $s \xrightarrow{\gamma} \tau$, $\gamma \in P_k$, γ *simple relative to* P_k. *Let* $X \in \mathfrak{g}_{-\gamma}$, $X \neq 0$. *If* $v \in W_{s,\lambda}$ *then there is* $k \geqq 0$, $k \in \mathbf{Z}$ *so that* $X^k v \in W_{s,\lambda}$. *If* $v \in W_{s,\lambda}$ *and* $h.v = \mu(h)\, v$, $\mathfrak{n}_k^+.v = 0$ *and if* $v \notin W_{\tau,\lambda}$ *then* $2\langle \mu, \gamma\rangle/\langle \gamma, \gamma\rangle = k \geqq 0$ *and* $X^{k+1} v \in W_{\tau,\lambda}$, $\mathfrak{n}_k^+.X^{k+1} v = 0$ *and* $h.X^{k+1}.v = (s_\gamma(\mu+\rho_k)-\rho_k)(h)\, X^{k+1}.v$, $h \in \mathfrak{h}$.

Proof. — By lemma 2.2, if $n = 2\langle s(\lambda+\rho_k), \gamma\rangle/\langle \gamma, \gamma\rangle$ then $X^n.w_{s,\lambda} = c w_{\tau,\lambda}$, $c \neq 0$. Hence if $U = W_{s,\lambda}/W_{\tau,\lambda}$ and $\bar{v}$ denotes the projection of $v \in W_{s,\lambda}$ onto U then $X^n w_{s,\lambda} = 0$. But then by the arguments proving Lemma 2.1 if $\bar{v} \in U$ then there is $l \geqq 0$ so that $X^l v = 0$. This follows since $U = U(\mathfrak{g})\, \bar{w}_{s,\lambda}$.

Let $Y \in \mathfrak{g}_\gamma$ and $H \in \mathfrak{h}$ be so that $[Y, X] = H$, $[H, Y] = 2Y$, $[H, X] = -2X$. Suppose that $v \in W_{s,\lambda}$ satisfies the hypothesis of the second assertion of the lemma. Then $Hv = kv$ with $k = 2\langle \mu, \gamma\rangle/\langle \gamma, \gamma\rangle$. Hence $H\bar{v} = k\bar{v}$. Also $Y\bar{v} = 0$. Hence if $X^l \bar{v} = 0$ for some l. Then we would have $\dim U(\mathfrak{r})\,\bar{v} < \infty$, $\mathfrak{r} = \mathbf{R}X + \mathbf{R}H + \mathbf{R}Y$. Thus $k \geqq 0$. But then $X^{k+1}\bar{v} = 0$. The rest of the lemma is even more standard.

Q. E. D.

THEOREM 2.4. — *Define* $W_{P,\lambda} = W_{1,\lambda}/\sum_{s<1} W_{s,\lambda}$. *Then* $W_{P,\lambda} \neq 0$ *and*

(1) *As a* $\mathfrak{k}$*-module,* $W_{P,\lambda} = \sum \oplus\, m_\gamma(\mu)\, V_\mu$ *the sum taken over* $\mu \in \mathfrak{h}^*$, μ, P_k*-dominant integral and* $0 \leqq m_\lambda(\mu) < \infty$ *is an integer,* V_μ *is the irreducible, finite dimensional* $\mathfrak{k}$*-module with highest weight* μ.

(2) *Set* $w_{P,\lambda}$ *equal to the image of* $w_{P,\lambda}$ *in* $W_{P,\lambda}$. *Then* $U(\mathfrak{k})\,w_{P,\lambda}$ *is equivalent with* V_λ *as a* $\mathfrak{k}$*-module, Furthermore,* $m_\lambda(\lambda) = 1$.

(3) *If* $m_\lambda(\mu) \neq 0$ *then* $\mu = \lambda + \delta$, δ *a sum of elements of* P.

(4) *Let* $U^{\mathfrak{h}}$ *be the centralizer of* $\mathfrak{h}$ *in* U. *Let* $\tilde{\mathfrak{n}}^+ = \sum_{\alpha \in -t_0 P} \mathfrak{g}_\alpha$. *If* $z \in U^{\mathfrak{h}}$ *then*

$$z \equiv z_0 \bmod U\tilde{\mathfrak{n}}^+, \qquad z_0 \in U(\mathfrak{h}).$$

If $z, z' \in U^{\mathfrak{h}}$ *then* $zz' \equiv z_0 z'_0 \bmod U\tilde{\mathfrak{n}}^+$. *Define*

$$\eta_{P,\lambda}(z) = (t_0(\lambda+\rho_k)-\rho_k)(z_0) \qquad \textit{for} \quad z \in U^{\mathfrak{k}}.$$

Then if $z \in U^{\mathfrak{h}} \cap U^{\mathfrak{h}}$ *and* $v \in U(\mathfrak{k})\,w_{P,\lambda}$ *then* $z.v = \eta_{P,\lambda}(z).v$.

The proof of this theorem rests on the following lemma of Enright, Varadarajan [4] which we prove for the sake of completeness.

LEMMA 2.5. — *Let* M *be a* $\mathfrak{k}$*-module such that if* $m \in M$ *then* $\dim U(\mathfrak{b}_k).m < \infty$ $(\mathfrak{b}_k = \mathfrak{h}+\mathfrak{n}_k^+)$ *and such that* M *splits into a direct sum of weight spaces relative to* $\mathfrak{h}$.

Let $N \subset M$ *be a* $\mathfrak{k}$*-submodule. Suppose that* $\bar{v} \in M/N$ *and* $\mathfrak{n}_k^+.\bar{v} = 0$, $h.\bar{v} = \mu(h)\,v$, $h \in \mathfrak{h}$ *with* μ, P_k*-dominant integral. Then there is* $v \in M$ *so that* $\mathfrak{n}_k^+ v = 0$ *and* $h.v = \mu(h)v$ *for* $h \in \mathfrak{h}$ *so that* $v+N = \bar{v}$.

Proof. — Since for every $m \in M$, $m = \sum_{\xi \in \mathfrak{h}^*} m_\xi$, $h.m_\xi = \xi(h)\,m_\xi$, $h \in \mathfrak{h}$ we see that if $\mathfrak{z}_k$ is the center of $U(\mathfrak{k})$ and if for $\chi : \mathfrak{z}_k \to C$ a homomorphism of $\mathfrak{z}_k$,

$$M_\chi = \{ m \in M \mid (z-\chi(z))^k\, m = 0,\ z \in \mathfrak{z}_k \text{ for some } k \}$$

then $M = \sum \oplus M_\chi$. Now if $z \in \mathfrak{z}_k$ then $z.\bar{v} = \chi(z)\,\bar{v}$ with $\chi = \chi_\mu$ defined by

$$z \equiv z_0 \bmod U(\mathfrak{k})\,\mathfrak{n}_k^+, \quad z_0 \in U(\mathfrak{h}) \qquad \text{and} \qquad \chi_\mu(z) = \mu(z_0).$$

Now $\chi_\mu = \chi_{\mu'}$ if and only if $\mu' = s(\mu+\rho_k)-\rho_k$ for some $s \in W_k$ (*cf.* Dixmier [2], Chapter 7). Now $M/N = \sum \oplus (M/N)_\chi$ and let $P_\chi : M/N \to (M/N)_\chi$ be the $\mathfrak{k}$-invariant projection. Then $P_\chi(\bar{v}) = 0$ if $\chi \neq \chi_\mu$. Thus there is $v_1 \in M$ so that $z.v_1 = \chi_\mu(z)\,v_1$ for $z \in \mathfrak{z}_k$ and $v_1+N = v$. Arguing similarly for the action of $\mathfrak{h}$, we may assume $h.v_1 = \mu(h)\,v_1$ for $h \in \mathfrak{h}$.

Now $\dim U(\mathfrak{n}_k^+)\,v_1 < \infty$. The weights of $U(\mathfrak{n}_k^+)\,v_1$ are of the form $\mu+\delta$ with δ a sum of elements of P_k. Let δ be maximal such that there is $v \neq 0$, $v \in U(\mathfrak{n}_k^+)\,v_1$ and $h.v = (\mu+\delta)(h).v$. Then $\mathfrak{n}_k^+.v = 0$. Hence if $z \in \mathfrak{z}_k$, $z.v = \chi_{\mu+\delta}(z)\,v$. But

$$U(\mathfrak{n}_k^+)\,v_1 \subset (M)_{\chi_\mu}.$$

Hence $\chi_{\mu+\delta} = \chi_\mu$. But then there is $s \in W_k$ so that $s(\mu+\rho_k) = \mu+\delta+\rho_k$. But this is possible ($\mu$ is P_k-dominant integral) only if $\delta = 0$ and $s = 1$. Thus $v = v_1$.

Q. E. D.

Proof of Theorem 2.4. — (i) $w_{1,\lambda} \notin \sum_{s<1} W_{s,\lambda}$. Indeed, if $M = \sum_{s<1} \oplus W_{s,\lambda}$ let $M \xrightarrow{\psi} W_1$, under $\sum_{s<1} \oplus w_s \to \sum w_s$. Let $N = \ker \psi$. If $w_{1,\lambda} \in \psi(M)$ then since λ is P_k-dominant integral we see that $w_{1,\lambda} = \psi(\sum_{s<1} \oplus w_s)$ with $h.w_s = \lambda(h) w_s$, $\mathfrak{n}_k^+ . w_s = 0$. We show that this is impossible. Suppose that $s < 1$ and there is $w_s \in W_{s,\lambda}$ so that $\mathfrak{n}_k^+ w_s = 0$ and $h.w_s = \lambda(h) w_s$, $h \in \mathfrak{h}$. Let

$$s = s_0 \xrightarrow{\gamma_1} s_1 \xrightarrow{\gamma_2} \ldots \xrightarrow{\gamma_p} s_p = t_0 \qquad \text{with} \quad \gamma_i \in P_k,$$

γ_i simple (this is always possible, *cf.* Dixmier [2], Chapter 7). Defining

$$\lambda_0 = \lambda, \qquad \nu_1 = 2\langle \lambda+\rho_k, \gamma_1 \rangle / \langle \gamma_1, \gamma_1 \rangle,$$

$$\lambda_1 = s_{\gamma_1}(\lambda+\rho_k)-\rho_k, \qquad \nu_2 = 2\langle \lambda_1+\rho_k, \gamma_2 \rangle / \langle \gamma_2, \gamma_2 \rangle, \qquad \ldots$$

and applying Lemma 2.3 we find that if $X_i \in \mathfrak{g}_{-\gamma_i}$, $X_i \neq 0$ then

$$\tilde{w} = X_p^{\nu_p} \ldots X_1^{\nu_1} . w_s \in W_{t_0,\lambda}$$

and

(*a*) $h.\tilde{w} = ((t_0 s^{-1})\ (\lambda+\rho_k)-\rho_k)(h)\,\tilde{w}$ and $\mathfrak{n}_k^+ \tilde{w} = 0$. If $s \neq 1$ then $t_0 s^{-1} > t_0$. But then

$$(t_0 s^{-1})(\lambda+\rho_k)-\rho_k = t_0(\lambda+\rho_k)-\rho_k+\delta,$$

δ a sum of elements of P_k. But $W_{t_0,\lambda}$ is the $\mathfrak{g}$-Verma module with highest weight $t_0(\lambda+\rho_k)-\rho_k$ relative to $-t_0 P$. Hence we have a contradiction.

We have shown that $W_{P,\lambda} \neq 0$.

(*b*) $U(\mathfrak{k})\, w_{P,\lambda}$ is equivalent with V_λ. In fact, we have a map

$$U(\mathfrak{k})\, w_{1,\lambda} / \sum_{s<1} U(\mathfrak{k})\, w_{s,\lambda} \to U(\mathfrak{k})\, w_{P,\lambda}.$$

Using Lemma 7.2.4 (p. 224) of Dixmier [2] we find $U(\mathfrak{k})\, w_{P,\lambda}$ is irreducible and finite dimensional.

Since $W_{P,\lambda} = U(\mathfrak{g}).w_{P,\lambda}$ we see that if $v \in W_{P,\lambda}$, $\dim U(\mathfrak{k})\, v < \infty$. Let for $\mu \in \mathfrak{h}^*$,

$$W_{s,\lambda}^\mu = \{ v \in W_{s,\lambda} \mid h.v = \mu(h) v,\ h \in \mathfrak{h} \text{ and } \mathfrak{n}_k^+ . v = 0 \}.$$

Define $W_{P,\lambda}^\mu$ in the same way. Of course, $W_{P,\lambda}^\mu \neq 0$ implies μ is P_k-dominant integral. Now Lemma 2.5 implies that if $\varepsilon : W_{1,\lambda} \to W_{P,\lambda}$ is the canonical map then

$$\varepsilon(W_{1,\lambda}^\mu) = W_{P,\lambda}^\mu,$$

μ, P_k-dominant integral.

Let $\gamma_1, \ldots, \gamma_n$ be simple in P_k so that

$$1 \xrightarrow{\gamma_1} s_{\gamma_1} \xrightarrow{\gamma_2} s_{\gamma_2} s_{\gamma_1} \xrightarrow{\gamma_3} \ldots \xrightarrow{\gamma_n} s_{\gamma_n} \ldots s_{\gamma_1} = t_0.$$

Define $\mu_0 = \mu$,

$$\mu_i = (s_{\gamma_n} \ldots s_{\gamma_1})(\mu + \rho_k) - \rho_k \quad \text{and} \quad \nu_i = 2 \langle \mu_{i-1} + \rho_k, \gamma_i \rangle / \langle \gamma_i, \gamma_i \rangle.$$

Let $X_i \in \mathfrak{g}_{-\gamma_i}$, $X_i \neq 0$. Then Lemma 2.3 implies that

$$X_i^{\nu_i} \ldots X_1^{\nu_1} . (W_{1,\lambda}^{\mu}) \subset W_{s_{\gamma_i} \ldots s_{\gamma_1}}^{\mu_i}.$$

In particular if $d_{t_0}(\mu) = X_n^{\nu_n} \ldots X_1^{\nu_1}$ then $d_{t_0}(\mu) : W_{1,\lambda}^{\mu} \to W_{t_0,\lambda}^{t_0(\mu+\rho_k)-\rho_k}$. Now $d_{t_0}(\mu)$ is injective by the construction of the $W_{s,\lambda}$. Hence we see

(c) $\dim W_{P,\lambda}^{\mu} \leqq \dim W_{t_0,\lambda}^{t_0(\mu+\rho_k)-\rho_k} < \infty$, This implies (1) since $m_\lambda(\mu) = \dim W_{P,\lambda}^{\mu}$ by the theorem of the highest weight.

To see (2) we note $\dim W_{t_0,\lambda}^{t_0(\lambda+\rho_k)-\rho_k} = 1$ since $W_{t_0,\lambda}$ is a Verma module with highest weight $t_0(\lambda+\rho_k)-\rho_k$. To prove (3) we note that if $W_{P,\lambda}^{\mu} \neq 0$ then $W_{t_0,\lambda}^{t_0(\mu+\rho_k)-\rho_k} \neq 0$. But then

$$t_0(\lambda+\rho_k)-\rho_k = t_0(\lambda+\rho_k)-\rho_k+t_0\,\delta$$

with δ a sum of elements of P. (Every weight of $W_{t_0,\lambda}$ is of the form $t_0(\lambda+\rho_k)-\rho_k-\delta$ with δ a sum of elements in $-t_0\,P$.) Hence $\mu+\rho_k = \lambda+\rho_k+\delta$. Thus $\mu = \lambda+\delta$.

Finally let $z \in U^{\mathfrak{k}}$. Then $z . w_{P,\lambda} = \chi(z)\, w_{P,\lambda}$ by (2). By the proof above

$$\varepsilon : \; W_{1,\lambda}^{\lambda} \to \mathbf{C}\, w_{P,\lambda}$$

is bijective. Since $\varepsilon(z . w_{1,\lambda}) = \chi(z)\, w_{P,\lambda}$, we have $z . w_{1,\lambda} = \chi(z)\, w_{1,\lambda}$. But if $d_{t_0}(\lambda)$ is as above then

$$d_{t_0}(\lambda)\, z w_{1,\lambda} = z\, d_{t_0}(\lambda)\, w_{1,\lambda} \qquad (z \in U^{\mathfrak{k}},\; d_{t_0}(\lambda) \in U(\mathfrak{k})).$$

But $d_{t_0}(\lambda)\, w_{1,\lambda} = c\, w_{t_0,\lambda}$, $c \neq 0$. Now $z . w_{t_0,\lambda} = \eta_{P,\lambda}(z)\, w_{t_0,\lambda}$ for $z \in U^{\mathfrak{k}}$.

Q. E. D.

The next result expresses the essential uniqueness of the family $W_{s,\lambda}$. We note that it is clear from the above results that if $Z_s = W_{s,\lambda}$ then the conditions of Theorem 2.6 are satisfied.

THEOREM 2.6. — *Suppose that to each $s \in W_k$ we have assigned a $\mathfrak{g}$-module Z_s so that :*

(1) *Z_{t_0} is the Verma module for $\mathfrak{g}$, $-t_0\,P$ with highest weight $t_0(\lambda+\rho_k)-\rho_k$.*

(2) *If $t \leqq s$, t, $s \in W_k$, $Z_t \subset Z_s$.*

(3) *If $X \in \mathfrak{n}_k^-$ and $\nu \in Z_t$ satisfies $X . \nu = 0$ then $\nu = 0$ or $X = 0$.*

(4) *$Z_s = U(\mathfrak{g})\, z_s$ with $\mathfrak{n}_k^+ . z_s = 0$, $h . z_s = (s(\lambda+\rho_k)-\rho_k)(h)\, z_s$.*

(5) *If* $s \to s_\gamma s$, *if* γ *is simple relative to* P_k *and if* $n = 2\langle s(\lambda+\rho_k), \gamma\rangle/\langle\gamma,\gamma\rangle$ *then* $n > 0$ *and*

$$z_{s_\gamma s} = c\,E_{-\gamma}^n z_s, \qquad c \neq 0 (E_{-\gamma} \in \mathfrak{g}_{-\gamma} - \{0\}).$$

Then there exists for each $s \in W_k$ *a bijective* $\mathfrak{g}$*-module isomorphism* $\xi_s : W_{s,\lambda} \to Z_s$ *commuting with the inclusions. Furthermore if* ξ_s' *is another family of* $\mathfrak{g}$*-module isomorphisms commuting with the inclusions of the* $W_{s,\lambda}$ *and the* Z_s' *s then* $\xi_s' = c\,\xi_s$ *with* c *independent of* s.

Proof. — (1), (3), (4), (5), imply that $U(\mathfrak{k}).z_s$ is isomorphic with the Verma module for $\mathfrak{k}$, P_k with highest weight $s(\lambda+\rho_k)-\rho_k$. (4) also implies that if $U_s = U(\mathfrak{k})\,z_s$ then $Z_s = U(\mathfrak{g}).U_s$. Hence we have

$$\breve{\xi}_s : \ U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} V^{s(\lambda+\rho_k)-\rho_k} \to Z_s$$

a surjective $\mathfrak{g}$-module homomorphism. Now $Z_{t_0} = W_{t_0,\lambda}$. Thus $\ker \tilde{\xi}_{t_0} = I_\lambda$.

(*a*) If $s \in W_k$ and $s > t_0$ there is a collection of elements $\gamma_1, \ldots, \gamma_p$ simple so that

$$s \to s_{\gamma_1} s \to s_{\gamma_2} s_{\gamma_1} s \to \ldots \to s_{\gamma_p} \ldots s_{\gamma_1} s = t_0.$$

This is easily proved by induction on the order and Lemmas 7.7.2, 7.7.5 of Dixmier [2].

(*b*) In particular implies that

$$\begin{array}{ccc} U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} V^{t_0(\lambda+\rho_k)-\rho_k} & \longrightarrow & U(\mathfrak{h}) \underset{U(\mathfrak{k})}{\otimes} V^{s(\lambda+\rho_k)-\rho_k} \\ \Big\downarrow{\scriptstyle \tilde{\xi}_{t_0}} & & \Big\downarrow{\scriptstyle \tilde{\xi}_s} \\ Z_{t_0} & \longrightarrow & Z_s \end{array}$$

commutes. Thus $\operatorname{Ker} \tilde{\xi}_s \supset I_\lambda$ for each $s \in W_k$.

This implies that $\tilde{\xi}_s$ induces $\tilde{\xi}_s : M_{s,\lambda} \to Z_s$ a surjective $\mathfrak{g}$-module homomorphism.

(3) Implies that $\ker \hat{\xi}_s \supset J_{s,X,0}$ ($J_{s,X,i}$ and $J_{s,i}$ are the $J_{X,i}$ and J_i of the proof of Lemma 2.5 for $M_{s,\lambda}$) for $X \in \mathfrak{r}_k^-$, $X \neq 0$ hence $\ker \hat{\xi}_s \supset J_{s,0}$ for $s \in W_k$. But is is also clear that if $\ker \hat{\xi}_s \supset J_{s,i}$ then $\ker \hat{\xi}_s \supset J_{s,i+1}$. Hence $\ker \hat{\xi}_s \supset J_s$. We therefore have $\hat{\xi}_s$ induces $\xi_s : W_{s,\lambda} \to Z_s$ a surjective $\mathfrak{g}$-module homomorphism.

Clearly ξ_{t_0} is injective. Suppose that we have shown ξ_t is injective for $t_0 \leqq t < s$. Let γ be simple in P_k so that $s \to s_\gamma s$. Then $W_{s,\lambda}/W_{s_\gamma s,\lambda}$ is $\mathfrak{k}^\gamma$ finite ($\mathfrak{k}^\gamma = \mathfrak{g}_\gamma + \mathfrak{g}_{-\gamma} + [\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma}]$). If $v \in W_{s,\lambda}$, $\xi_s(v) = 0$ then since

$$\begin{array}{ccc} W_{s_\gamma s,\lambda} & \longrightarrow & W_{s,\lambda} \\ \Big\downarrow{\scriptstyle \xi_{s_\gamma s}} & & \Big\downarrow{\scriptstyle \xi_s} \\ Z_{s_\gamma s} & \longrightarrow & Z_s \end{array}$$

commutes $v \notin W_{s_\gamma s,\lambda}$. There is therefore $p > 0$ so that $E_{-\gamma}^p v \in W_{s_\gamma s,\lambda}$. But then

$$\xi_{s_\gamma s}(E_{-\gamma}^p v) = E_{-\gamma}^p \xi_s(v) = 0.$$

Hence $\xi_{s_\gamma s}(E_{-\gamma}^p v) = 0$. Thus $E_{-\gamma}^p v = 0$. But then $v = 0$ by (3).

If $\xi'_s : W_{s,\lambda} \to Z_s$ is another such family of homomorphisms. Then clearly $\xi'_{t_0} = c\,\xi_{t_0}$ for some c. Suppose we have shown $\xi'_t = c\,\xi_t$ for $t_0 \leqq t < s$. Then again supposing γ is simple in P_k and $s > s_\gamma s$ then if $v \in W_{s,\lambda}$ there is $p \geqq 0$ so that $E^p_{-\gamma} v \in W_{s_\gamma s, \lambda}$. Hence $\xi'_s(E^p_{-\gamma} v) = c\,\xi_{s_\gamma s}(E^p_{-\gamma} v)$. Hence $E^p_{-\gamma}\,\xi'_s(v) = c\,\xi_{s_\gamma s}(E^p_{-\gamma} v)$. But then

$$E^p_{-\gamma}(\xi'_s(v) - c\,\xi_s(v)) = 0.$$

Hence $\xi'_s(v) = c\,\xi_s(v)$.

Q. E. D.

3. Tensor products of $W_{P,\lambda}$ with finite dimensional $\mathfrak{g}$-modules

In Enright [3] the tensor product of the module $D_{P,\lambda}$ with finite dimensional representations was studied. We give a proof of a sharpening of the main result on tensor products in Enright [3] our techniques are, of course, quite similar to Enright's.

Let F be an irreducible finite dimensional representation of $\mathfrak{g}$. We use the notation of Section 2. Let $\lambda \in \mathfrak{h}^*$ be P_k-dominant integral. Then we have the inclusions $W_{s,\lambda} \otimes F \subset W_{\tau,\lambda} \otimes F$ if $s \leqq \tau$.

LEMMA 3.1. — *If $\tau \xrightarrow{\gamma} s$ and $\gamma \in P_k$ is simple for P_k, if $X \in \mathfrak{g}_{-\gamma}$ and if $v \in W_{\tau,\lambda} \otimes F$ then there is $k \geqq 0$, $k \in \mathbf{Z}$ so that $X^k . v \in W_{s,\lambda} \otimes F$.*

Proof. — It is enough to prove the result for v of the form $w \otimes f$, $w \in W_{\tau,\lambda}$, $f \in F$. Now there is l so that $X^l . f = 0$. There is k so that $X^k . w \in W_{s,\lambda}$. Now

$$X^{k+l}(w \otimes f) = \sum_{j=0}^{k+l} \binom{k+l}{j} X^{k+l-j} w \otimes X^j f = \sum_{j=0}^{l-1} \binom{k+l}{j} X^{k+l-j} w \otimes X^j f.$$

But if $j \leqq l-1$, $k+l-j \geqq k$.

Hence the lemma.

LEMMA 3.2. — *If $X \in \mathfrak{n}_k^-$, $X \neq 0$ and $w \in W_{s,\lambda} \otimes F$, $X w = 0$, then $w = 0$.*

Proof. — Let $F = F_d \supset F_{d-1} \supset \ldots \supset F_1 \supset (0)$ be such that $\dim F_i = i$ and

$$\mathfrak{n}_k^+ F_i \subset F_{i-1}.$$

Let $f_1, \ldots, f_d$ be a basis of F so that $F_i = \sum_{j=1}^{i} \mathbf{C} f_j$. Then $w = \sum w_i \otimes f_i$, $w_i \in W_{s,\lambda}$,

$$0 = X w = \sum X w_i \otimes f_i + \sum w_i \otimes X f_i.$$

Since $X f_i \in F_{i-1}$ for all $i = 1, \ldots, d$. We see that $X w_d = 0$. But then $w_d = 0$. But then $X f_i \in F_{d-2}$ if $w_i \neq 0$ hence $w_{d-1} = 0$, etc.

LEMMA 3.3. — *If $v \in W_{s,\lambda} \otimes F$ and $\mathfrak{z}$ is the center of the universal enveloping algebra of $\mathfrak{g}$ then $\dim \mathfrak{z} . v < \infty$.*

Proof. — By Lemma 3.1 there exist $X_1, \ldots, X_k \in \mathfrak{n}_k^-$, $X_i \neq 0$ so that $X_1^{\nu_1} \ldots X_k^{\nu_k}$, $v \in W_{t_0, \lambda} \otimes F$. Set $u = X_1^{\nu_1} \ldots X_k^{\nu_k}$. Then

$$u : \mathfrak{z}.v \to \mathfrak{z} u.v \subset W_{t_0, \lambda} \otimes F.$$

By lemma 3.2, dim $\mathfrak{z}\, u.v = \dim \mathfrak{z}.v$. Thus it is enough to prove the result for $s = t_0$. But $W_{t_0, \lambda}$ is a Verma module relative to $-t_0$ P hence $W_{t_0, \lambda} \otimes F$ has a finite composition series by Verma modules. Hence the result is true for $W_{t_0, \lambda} \otimes F$ and therefore for $W_{s, \lambda} \otimes F$ for any $s \in W_k$.

Let for $\lambda \in \mathfrak{h}^*$, χ_λ be the infinitesimal character of the $\mathfrak{g}$-Verma module $M^{t_0, (\lambda + 2\rho_k)}$ with highest weight $t_0 (\lambda + 2 \rho_k)$ relative to $-t_0$ P.

LEMMA 3.4. — *Let $\xi_1, \ldots, \xi_q$ be the distinct weights of* F. *Let for $\chi : \mathfrak{z} \to \mathbf{C}$ a homomorphism*

$$(W_{P, \lambda} \otimes F) = \{ v \in W_{P, \lambda} \otimes F \mid \textit{there is } k > 0,\ k \in \mathbf{Z} \textit{ so that } (z - \chi(z))^k v = 0 \textit{ for } z \in \mathfrak{z} \}.$$

Then $W_{P, \lambda} \otimes F = \sum (W_{P, \lambda} \otimes F)_{\chi_\lambda + \xi_i}$.

Proof. — It is enough to prove the statement for $W_{1, \lambda} \otimes F$. The argument of Lemma 3.3 reduces this to proving the result for $W_{t_0, \lambda} \otimes F$. To prove the result for $W_{t_0, \lambda} \otimes F$ we note that Lemma 7.6.14 of Dixmier [2] implies

$$W_{t_0, \lambda} \otimes F = M_d \supset M_{d-1} \supset \ldots \supset M_1 \supset M_0 = (0)$$

with M_i a $\mathfrak{g}$-submodule and M_i/M_{i-1} is $\mathfrak{g}$-isomorphic with $M^{t_0, (\lambda + 2\rho_k + \xi_i)}$ here the weights of F are $\xi_1, \ldots, \xi_d$ counting multiplicity in a prescribed order. But now the result follows for $W_{t_0, \lambda} \otimes F$.

Q. E. D.

LEMMA 3.5. — *Let $\lambda \in \mathfrak{h}^*$ be* P*-dominant (that is $\langle \lambda, \alpha \rangle \geqq 0$, $\alpha \in$ P). If $s \in W(\nabla)$ and $s\lambda$ is* P*-dominant then $s\lambda = \lambda$.*

Proof. — Let $\alpha_1, \ldots, \alpha_l$ be the simple roots in P. Let $\lambda_1, \ldots, \lambda_l$ in $\mathfrak{h}^*$ be defined by $2 \langle \lambda_i, \alpha_j \rangle / \langle \alpha_j, \alpha_j \rangle = \delta_{ij}$. The hypotheses imply that $\lambda = \sum r_i \lambda_i$, $r_i \in \mathbf{R}$, $r_i \geqq 0$. Now

$$s\lambda_i = \lambda_i - Q_{i, s}, \qquad Q_{i, s} = \sum_{j=1}^{l} n_{i, s, j} \alpha_j, \qquad n_{i, s, j} \in \mathbf{Z}, \quad n_{i, s, j} \geqq 0.$$

Hence

$$s\lambda = \lambda - \sum_{j=1, i=1}^{l} r_i n_{i, s, j} \alpha_j = \lambda - \sum m_j \alpha_j.$$

Set $\lambda - s\lambda = u$. Then since $\lambda_i = \sum r_{ji} \alpha_j$, $r_{ji} \geqq 0$ we see

$$\langle \lambda, \lambda \rangle = \langle s\lambda + u, s\lambda + u \rangle = \langle s\lambda, s\lambda \rangle + 2 \langle s\lambda, u \rangle + \langle u, u \rangle.$$

Since $s\lambda$ is P-dominant $\langle s\lambda, u \rangle \geqq 0$. But $\langle s\lambda, s\lambda \rangle = \langle \lambda, \lambda \rangle$. Hence

$$\langle s\lambda, u \rangle = \langle u, u \rangle = 0.$$

But then $u = 0$. Q. E. D.

The following result (and its corollary) are useful in the problem of imbedding discrete series into (non-unitary) principal series.

THEOREM 3.6. — *Let* $\lambda \in \mathfrak{h}^*$ *be* P_k*-dominant integral and suppose that* $\lambda+\rho_k-\rho_n$ *is* P*-dominant and regular (that is* $\langle \lambda+\rho_k-\rho_n, \alpha \rangle > 0$ *for* $\alpha \in P$*). Let* F *be the finite dimensional irreducible representation of* $\mathfrak{g}$ *with the highest weight* μ *relative to* P. *Then* $(W_{P,\lambda} \otimes F)_{\chi_{\lambda+\mu}}$ *is* $\mathfrak{g}$*-isomorphic with* $W_{P,\lambda+\mu}$.

Proof. — As we have observed in the proof of Lemma 3.4 :

$$W_{t_0,\lambda} \otimes F = M_d \supset M_{d-1} \supset \ldots \supset M_1 \supset M_0 = (0)$$

with $M_i/M_{i-1} = M^{t_0, (\lambda+\xi_i+2\rho_k)}$ and $\xi_1, \ldots, \xi_d$ are the weights of F in a "certain order". Let us describe the order. It is any labeling of the ξ_i so that if $t_0\,\xi_j = t_0\,\xi_i - t_0\,Q$, $Q \neq 0$ (Q a sum of not necessarily distinct elements of P) then $i > j$. Hence

$$\frac{M_d}{M_{d-1}} = M^{t_0\,(\lambda+\mu+2\rho_k)}.$$

(1) If $\chi_{\lambda+\xi_i} = \chi_{\lambda+\mu}$ then $\xi_i = \mu$. Indeed if $\chi_{\lambda+\xi_i} = \chi_{\lambda+\mu}$ then there is $s \in W(\Delta)$ so that

$$s(t_0(\lambda+\mu+2\rho_k) - t_0\,\rho) = t_0(\lambda+\xi_i+2\rho_k) - t_0\,\rho.$$

That is

$$t_0\,s^{-1}\,t_0(\lambda+\rho_k-\rho_n) + t_0\,s^{-1}\,t_0\,\xi_i = \lambda+\rho_k-\rho_n+\mu.$$

If $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ are the simple roots in P. Then

$$t_0\,s^{-1}\,t_0(\lambda+\rho_k-\rho_n) = \lambda+\rho_k-\rho_n - \sum_{i=1}^{l} r_i\,\alpha_i,$$

$r_i \geqq 0$, $r_i \in \mathbf{R}$ (*see* the proof of Lemma 3.5).

$$t_0\,s^{-1}\,t_0\,\xi_i = \mu - \sum_{i=1}^{l} m_i\,\alpha_i, \qquad m_i \geqq 0, \quad m_i \in \mathbf{Z}$$

($t_0\,s^{-1}\,t_0\,\xi_i$ is a weight of F).

But then $\sum_{i=1}^{l} (r_i+m_i)\,\alpha_i = 0$. This implies $r_i+m_i = 0$. Since $r_i \geqq 0$, $m_i \geqq 0$ we see $r_i = 0$ and $m_i = 0$. Thus

$$t_0\,s^{-1}\,t_0(\lambda+\rho_k-\rho_n) = \lambda+\rho_k-\rho_n.$$

But $\lambda+\rho_k-\rho_n$ is P-dominant and regular. Hence $s = 1$. Since $t_0\,s^{-1}\,t_0\,\xi_i = \mu$. This proves (1).

Using (1) it is easy to see that

(2) $(W_{t_0,\lambda} \otimes F)_{\chi_{\lambda+\mu}} \equiv M^{t_0, (\lambda+\mu+2\rho_k)} = W_{t_0,\lambda+\mu}$.

Let $P_\mu : W_{t_0,\lambda} \otimes F \to (W_{t_0,\lambda} \otimes F)_{\chi_{\lambda+\mu}}$ be the corresponding projection.

Let $\mathfrak{z}_k$ be the center of the universal enveloping algebra of $\mathfrak{k}$. Let for $\lambda \in \mathfrak{h}^*$, η_λ be the infinitesimal character of V^λ, the $\mathfrak{k}$-Verma module with highest weight λ relative to P_k.

$$U(\mathfrak{k})\, w_{t_0,\lambda} \otimes F = V_d \supset V_{d-1} \supset \ldots V_1 \supset V_0 = (0)$$

with $V_i/V_{i-1} = V^{t_0(\lambda+\xi_i+2\rho_k)}$ ($\xi_1, \ldots, \xi_d$ ordered as above). Arguing as above we find

(3) $(U(\mathfrak{k})\, w_{t_0,\lambda} \otimes F)_{\eta_{\lambda+\mu}} \equiv V^{t_0(\lambda+\mu+2\rho_k)}$.

(4) $P_\mu(V_{d-1}) = 0$.

First of all we show $P_\mu V_1 = 0$. Indeed if $P_\mu V_1 \neq 0$ then $P_\mu(W_{t_0,\lambda} \otimes F)$ must have the weight $t_0(\lambda+\xi_1+2\rho_k)$ with positive multiplicity. Since $\xi_1 \neq \mu$, $\xi_1 = \mu-\delta$, δ a sum of elements of P. Hence

$$t_0(\lambda+\xi_1+2\rho_k) = t_0(\lambda+\mu+2\rho_k) - t_0\delta.$$

But every weight of $M^{t_0(\lambda+\mu+2\rho_k)}$ is of the form $t_0(\lambda+\mu+2\rho_k)+t_0\delta'$, δ' a sum of positive roots. Hence $P_\mu V_1 = 0$. Suppose $P_\mu V_i = 0$, and $i \leqq d-2$. Then, arguing as above, we find $P_\mu V_{i+1} = 0$. This proves (4).

We note that $P_\mu(U(\mathfrak{k})\, w_{t_0,\lambda} \otimes F) \neq 0$ since $P_\mu(W_{t_0,\lambda} \otimes F) = U(\mathfrak{g}).P_\mu(U(\mathfrak{k})\, w_{t_0,\lambda} \otimes F)$. We therefore have

(5) $P_\mu(U(\mathfrak{k})_{t_0,\lambda} \otimes F) \equiv V^{t_0(\lambda+\mu+2\rho_k)}$.

We extend P_μ to $W_{1,\lambda} \otimes F$ by noting that

$$W_{1,\lambda} \otimes F = (W_{1,\lambda} \otimes F)_{\chi_{\lambda+\mu}} + \sum_{\chi_\delta \neq \chi_{\lambda+\mu}} (W_{1,\lambda} \otimes F)_\chi$$

(6) If $w \in (W_{1,\lambda} \otimes F)_{\chi_{\lambda+\mu}}$ then $zw = \chi_{\lambda+\mu}(z)\, w$ for all $z \in \mathfrak{z}$.

This follows since there exist $X_1, \ldots, X_n \in \mathfrak{n}_k^-$, $X_i \neq 0$ so that if $u = X_1 \ldots X_n$ then $u.w \in W_{t_0,\lambda} \otimes F$. Hence in $u.w \in (W_{t_0,\lambda} \otimes F)_{\chi_{\lambda+\mu}}$. But then $z.u.w = \chi_{\lambda+\mu}(z)\, u.w$, $z \in \mathfrak{z}$. Hence $u.(z-\chi_{\lambda+\mu}.(z))\, w = 0$. This implies $z.w = \chi_{\lambda+\mu}(z)\, w$.

(7) $P_\mu(U(\mathfrak{k})\, W_{s,\lambda} \otimes F) \equiv V^{s(\lambda+\mu+\rho_k)-\rho_k}$. To prove this we note that if $\eta \neq \eta_{\lambda+\mu}$ then $P_\mu((U(\mathfrak{k})\, w_{s,\lambda} \otimes F)_\eta) = 0$. Indeed if $v \in (U(\mathfrak{k})\, w_{s,\lambda} \otimes F)_\eta$ then there are $X_1 \ldots, X_m \in \mathfrak{n}_k^- - \{0\}$ so that if $X_1 \ldots X_m = u$ then $u.v \in (U(\mathfrak{k})\, w_{t_0,\lambda} \otimes F)_\eta$. But $\eta \neq \eta_{\lambda+\mu}$ Hence $P_\mu(u.v) = 0$. Hence $P_\mu v = 0$.

Since $P_\mu(W_{s,\lambda} \otimes F) = U(\mathfrak{g})\ P_\mu(U(\mathfrak{k})\, w_{s,\lambda} \otimes F)$ we see $P_\mu(U(k)\, w_{s,\lambda} \otimes F) \neq 0$. Hence (7). Using these observations we see that if $Z_s = P_\mu(W_{s,\lambda} \otimes F)$.

(8) Z_s, $s \in W_k$ satisfy (1)-(5) of Theorem 2.6.

Let $\varepsilon : W_{1,\lambda} \to W_{P,\lambda}$ be the natural projection. Then $(\varepsilon \otimes I)(W_{1,\lambda} \otimes F) = W_{P,\lambda} \otimes F$. Hence $(\varepsilon \otimes I)(Z_1) = (W_{P,\lambda} \otimes F)_{\chi_{\lambda+\mu}}$. But

$$\operatorname{Ker}(\varepsilon \otimes I)\Big|_{Z_1} = \Big(\sum_{s<1} W_{s,\lambda} \otimes F\Big) \cap Z_1 = \sum_{s<1} Z_s.$$

Hence $Z_1/\sum_{s<1} Z_s$ is g-isomorphic with $(W_{P,\lambda} \otimes F)_{\chi_{\lambda+\mu}}$. Theorem 2.6 now implies our theorem.

Q. E. D.

COROLLARY 3.7. — *Let the hypotheses be as in Theorem* 3.6. *Let* $D_{P,\lambda}$ *be the non-zero irreducible quotient of* $W_{P,\lambda}$. *Then* $(D_{P,\lambda} \otimes F)_{\chi_{\lambda+\mu}} \equiv D_{P,\lambda+\mu}$.

Proof. — Let $\operatorname{Hom}_{\mathfrak{k}}(U(\mathfrak{g}), V_\lambda)$ denote the space of all $f : U(\mathfrak{g}) \to V_\lambda$ such that $f(kg) = k.(f(g))$, $k \in U(\mathfrak{k})$, $g \in U(\mathfrak{g})$. Define $(g.f)(x) = f(xg)$, $g \in U(\mathfrak{g})$, $x \in U(\mathfrak{g})$ Then $\operatorname{Hom}_{\mathfrak{k}}(U(\mathfrak{g}), V_\lambda)$ is a g-module.

Let $A : W_{P,\lambda} \to V_\lambda$ be a non-zero $\mathfrak{k}$-module homomorphism. We note that since $m_\lambda(\lambda) = 1$, A is unique up to scalar multiple. Let

$$\psi_{P,\lambda} : \quad W_{P,\lambda} \to \operatorname{Hom}_{\mathfrak{k}}(U(\mathfrak{g}), V_\lambda)$$

be defined as follows: $\psi_{P,\lambda}(w)(g) = A(g.w)$. Clearly $\psi_{P,\lambda}(W_{P,\lambda}) \subset \operatorname{Hom}_{\mathfrak{k}}(U(\mathfrak{g}), V_\lambda)$ If $x \in U(\mathfrak{g})$ then

$$\psi_{P,\lambda}(x.w)(g) = A(gx.w) = \psi_{P,\lambda}(w)(gx) = (x.\psi_{P,\lambda}(w))(g).$$

Hence $\psi_{P,\lambda} : W_{P,\lambda} \to \operatorname{Hom}_{\mathfrak{k}}(U(\mathfrak{g}), V_\lambda)$ is a g-module homomorphism.

(1) Let $Q_{P,\lambda} \subset W_{P,\lambda}$ be the g-module so that $W_{P,\lambda}/Q_{P,\lambda} = D_{P,\lambda}$. Then $\ker \psi_{P,\lambda} = Q_{P,\lambda}$.

In fact, let $\eta : W_{P,\lambda} \to D_{P,\lambda}$ be the g-module projection. Let $\tilde{A} : D_{P,\lambda} \to V_\lambda$ be a non-zero $\mathfrak{k}$-module homomorphism (again $\tilde{A}$ is unique up to scalar multiple and $\tilde{A}$ exists). By the above observations about A, $A = c\,\tilde{A} \circ \eta$, $c \neq 0$. (1) is now clear.

Let now

$$h : \quad \operatorname{Hom}_{\mathfrak{k}}(U(\mathfrak{g}), V_\lambda) \otimes F \to \operatorname{Hom}_{\mathfrak{k}}(U(\mathfrak{g}), V_\lambda \otimes F)$$

be defined by $h(f \otimes v)(g) = (\delta \otimes I)(g.(f \otimes v))$, where $\delta(f) = f(1)$. Then h is clearly a g-module homomorphism.

(2) h is injective.

Let $v_1, \ldots, v_d$ be a basis of F. Suppose $h(\sum f_i \otimes v_i) = 0$. If $h(\sum f_i \otimes v_i) = 0$ then clearly $h(\sum f_i \otimes v_i)(1) = \sum f_i(1) \otimes v_i = 0$. Thus $f_i(1) = 0$, $i = 1, \ldots, d$. Let $U^j(\mathfrak{g}) \subset U^{j+1}(\mathfrak{g})$ be the standard filtration of $U(\mathfrak{g})$. Suppose that we have shown that $f_i(g) = 0$ for $g \in U^j(\mathfrak{g})$. If $g \in U^{j+1}(\mathfrak{g})$ then

$$0 = h(\sum f_i \otimes v_i)(g) = \sum_i f_i(g) \otimes v_i$$

by the inductive hypothesis. Thus $f_i(g) = 0$, $i = 1, \ldots, d$. (2) is now proved.

Now $(V_\lambda \otimes F)_{\eta_{\lambda+\mu}} V_{\lambda+p}$. Let Q be the projection of $V_\lambda \otimes F$ into $V_{\lambda+\mu}$ Set $\psi = h \circ (\psi_{P,\lambda} \otimes I)$. Define for $f \in \operatorname{Hom}_{\mathfrak{k}}(U(\mathfrak{g}), V_\lambda \otimes F)$, $(Qf)(g) = Q(f(g))$. Then $Q \circ \psi : W_{P,\lambda} \otimes F \to \operatorname{Hom}_{\mathfrak{k}}(U(\mathfrak{g}), V_{\lambda+\mu})$

Now $W_{P,\lambda} \otimes F = W_{P,\lambda+\mu} \oplus H$, H a g-submodule with $(H)_{\chi_{\mu+\lambda}} = 0$

(3) $((I-Q)\circ\psi)(W_{P,\lambda+\mu}) = 0$. Indeed, $W_{P,\lambda+\mu}$ contains only $\mathfrak{k}$-types of the form $V_{\lambda+\mu+\delta}$, δ a sum of elements of P. On the other hand $(I-Q)(V_\lambda\otimes F)$ contains only $\mathfrak{k}$-types of the form $V_{\lambda+\mu-\delta'}$, δ' a non zero sum elements of P.

(3) Implies that $\psi(W_{P,\lambda+\mu}) = Q(\psi(W_{P,\lambda+\mu}))$. Now $w\mapsto(Q\circ\psi)(w)(1)$ maps $W_{P,\lambda+\mu}$ to $V_{\lambda+\mu}$. Hence (1) implies $\psi(W_{P,\lambda+\mu}) = D_{P,\lambda+\mu}$. Now

$$\psi(W_{P,\lambda}\otimes F) = h\circ(\psi_{P,\lambda}\otimes I)(W_{P,\lambda}\otimes F) = h(D_{P,\lambda}\otimes F).$$

Thus

$$(D_{P,\lambda}\otimes F)_{\chi_{\lambda+\mu}} \equiv \psi((W_{P,\lambda}\otimes F)_{\chi_{\lambda+\mu}}) = D_{P,\lambda+\mu}.$$

Q. E. D.

Actually Theorem 3.6 is not especially useful in applications to the realization of discrete series. We actually need.

THEOREM 3.8. — *Suppose that λ is P_k-dominant integral. Let μ be P-dominant integral and let F be the irreducible $\mathfrak{g}$-module with lowest weight $-\mu$. Then:*

(1) $W_{P,\lambda+\mu}\otimes F$ *contains the $\mathfrak{k}$-submodule V_λ with multiplicity* 1.

(2) *There is a surjective $\mathfrak{g}$-module homomorphism of $W_{P,\lambda}$ onto the cyclic space for* $V_\lambda\subset W_{P,\lambda+\mu}\otimes F$.

Proof. — We note that $-t_0\,\mu$ is the highest weight of F relative to $-t_0$ P. Hence the highest weight of $W_{t_0,\lambda+\mu}\otimes F$ relative to $-t_0$ P is $t_0(\lambda+2\rho_k)$. Further more, this weight space is one dimensional. Let v_0 be a non-zero element of the $t_0(\lambda+2\rho_k)$ weight space of $W_{t_0,\lambda+\mu}\otimes F$.

It is easily proved that $V_{\lambda+\mu}\otimes F$ contains the $\mathfrak{k}$-type V_λ with multiplicity 1 and that every $\mathfrak{k}$-type of $V_{\lambda+\mu}\otimes F$ is of the form $V_{\lambda+Q}$ with Q a sum of elements of P. Also, if $\xi\neq\lambda+\mu$ and if V_ξ occurs in $W_{P,\lambda+\mu}$ then every $\mathfrak{k}$-type in $V_\xi\otimes F$ is of the form $V_{\lambda+Q}$, $Q\neq 0$, Q a sum of elements of P. This proves (1).

Let v be a non-zero highest weight vector for $V_\lambda\subset W_{P,\lambda+\mu}\otimes F$. Let

$$v_1\in W_{1,\lambda+\mu}\otimes F$$

be so that $h.v_1 = \lambda(h)v_1$, $h\in\mathfrak{h}$, $\mathfrak{n}_k^+.v_1 = 0$ and if $\varepsilon: W_{1,\lambda+\mu}\to W_{P,\lambda+\mu}$ is the natural map the $\varepsilon(v_1) = v$ (this is possible by lemma 2.5). Let $s\in W_k$ and suppose

$$1\xrightarrow{\gamma_1}s_{\gamma_1}\xrightarrow{\gamma_2}s_{\gamma_2}s_{\gamma_1}\to\ldots\xrightarrow{\gamma_l}s_{\gamma_l}\ldots s_{\gamma_1} = s,$$

with γ_i simple in P_k. Let $X_i\in\mathfrak{g}_{-\gamma_i}-\{0\}$. Set

$$\nu_i = \frac{2\langle s_{\gamma_{i-1}}\ldots s_{\gamma_1}(\lambda+\rho_k),\gamma_i\rangle}{\langle\gamma_i,\gamma_i\rangle}$$

Let $v_s = X_l^{\nu_l}\ldots X_1^{\nu_1}v_1$. Then $v_s\in W_{s,\lambda+\mu}\otimes F$. Furthermore

$$h.v_s = (s(\lambda+\rho_k)-\rho_k)(h)v_s \qquad\text{and}\qquad \mathfrak{n}_k^+.v_s = 0.$$

In particular, $v_{t_0}\in\mathbf{C}\,v_0$.

Set $Z_s = U(\mathfrak{g})\, v_s$. Then $\{Z_s\}_{s \in W_k}$ satisfies the conditions of theorem 2.6. Clearly, $(\varepsilon \otimes I)(Z_1)$ is the cyclic space for V_λ in $W_{P,\lambda+\mu} \otimes F$. Furthermore,

$$\operatorname{Ker}(\varepsilon \otimes I)\big|_{Z_1} \supset \sum_{s<1} Z_s.$$

Thus theorem 2.6 implies that $(\varepsilon \otimes I)\big|_{Z_1}$ induces a $\mathfrak{g}$-module surjection of $W_{P,\lambda}$ onto $(\varepsilon \otimes I)(Z_1)$.

Q. E. D.

COROLLARY 3.9. – *Let* λ. μ *and* F *be or in theorem* 3.8. Then $D_{P,\lambda+\mu} \otimes F$ *contains* $D_{P,\lambda}$ *as a subquotient.*

Proof. – Let $\eta : W_{P,\lambda+\mu} \to D_{P,\lambda+\mu}$ be the natural map. Then $(\eta \otimes I)(V_\lambda) \neq (0)$ with $V_\lambda \subset W_{P,\lambda+\mu} \otimes F$ as in (1) of theorem 3.8. Using the notation of the proof of theorem 3.8 we see that $U = (\eta \otimes I)(\varepsilon \otimes I)(Z_1) \neq (0)$.

Since U is a non-zero homomorphic image of $W_{P,\lambda}$. U has $D_{P,\lambda}$ as a quotient.

Q. E. D.

CONJECTURE 3.10. — If λ is P_k-dominant integral and if $\lambda + \rho_k - \rho_n$ is P-dominant then $W_{P,\lambda}$ is irreducible.

We look at the special case that there is a parabolic $\mathfrak{p}$ of $\mathfrak{g}$, $\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{r} = \mathfrak{k} \oplus \sum_{\alpha \in P_n} \mathfrak{g}_\alpha$. Under these hypotheses we have

LEMMA 3.11. – *If* $2 < \lambda+\rho_k-\rho_n,\ \beta >/ < \beta, \beta > \neq -1, -2, \ldots$ *for any* $\beta \in P_n$ *then* $W_{P,\lambda}$ *is irreducible.*

Proof. – In this case the simple roots of P_k are actually simple in P. Thus it is not hard to show that $W_{s,\lambda} = M^{s(\lambda+\rho_k)-\rho_k} = M^{s(\lambda - t_0\rho)+t_0\rho}$ and hence $W_{P,\lambda} = U(\mathfrak{g}) \underset{U(\bar{\mathfrak{p}})}{\otimes} V_\lambda$ ($\bar{\mathfrak{p}}$ the opposite parabolic to $\mathfrak{p}$), where V_λ is made into a $\bar{\mathfrak{p}}$-module by taking $x V_\lambda = 0$, $x \in \bar{\mathfrak{r}}$. If $W_{P,\lambda}$ is reducible then there is $M \subsetneq W_{P,\lambda}$ a submodule. Let $\tilde{M}$ be the inverse image of M in $W_{1,\lambda}$. If $\tilde{M} = W_{1,\lambda}$ then $M = W_{P,\lambda}$. Now the weights of $\tilde{M}$ are bounded above relative to $-t_0$ P. Using 7.6.23 Dixmier [2] there is $0 \neq v \in \tilde{M}$ so that $\tilde{\mathfrak{r}}^+ . v = 0$, $h.v = \mu(h)\, v$ ($\tilde{\mathfrak{r}}^+ = \sum_{\alpha \in -t_0 P} \mathfrak{g}_\alpha$) and μ is P_k-dominant integral. Since $\tilde{M} \neq W_{1,\lambda}$, $\mu < \lambda$ relative to $-t_0$ P. Now arguing as usual $d_{t_0}(\mu)\, v \in W_{t_0,\lambda}$. If $d_{t_0}(\mu)\, v \in C\, w_{t_0,\lambda}$ then $v \in C\, w_{1,\lambda}$. But $\tilde{M} \neq M$. Hence $d_{t_0}(\mu)\, v \notin C\, w_{t_0,\lambda}$. Now the Bernstein, Gelfand, Gelfand theorem (*see* Dixmier [2], Chapter 7) implies there is $\beta \in -t_0$ P so that

$$\frac{2\langle t_0\lambda-\rho, \beta\rangle}{\langle \beta, \beta\rangle} = n, \qquad n > 0.$$

If $\beta \in P_k$ then

$$\frac{2\langle t_0\lambda-\rho, \beta\rangle}{\langle \beta, \beta\rangle} < 0.$$

Thus $\beta \in -t_0 P_n$. But $\beta = -t_0 \beta'$, $\beta' \in P_n$. Hence

$$n = \frac{2\langle t_0 \lambda - \rho, t_0 \beta' \rangle}{\langle \beta', \beta' \rangle} = -\frac{2\langle \lambda - t_0 \rho, \beta' \rangle}{\langle \beta', \beta' \rangle}.$$

Now

$$-t_0 \rho = -\rho_n + \rho_k.$$

We therefore have a contradiction, that implies the lemma.

Q. E. D.

4. Applications to the discrete series

Let G be a simply connected, complex semi-simple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be a real form. Let $G_0 \subset G$ be the connected subgroup with Lie algebra $\mathfrak{g}_0$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{k}_0$ be a Cartan decomposition of $\mathfrak{g}_0$. Let $\mathfrak{k}$ be the complexification of $\mathfrak{k}_0$. We assume that there is a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{h} \subset \mathfrak{k}$.

Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$ and let Δ_k be the roots of $(\mathfrak{k}, \mathfrak{h})$, $\Delta_k \subset \Delta$. Set $\Delta_n = \Delta - \Delta_k$. Let $\Lambda \in \mathfrak{h}^*$ be integral that is $2\langle \Lambda, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbf{Z}$, $\alpha \in \Delta$ and regular ($\langle \Lambda, \alpha \rangle \neq 0$ for $\alpha \in \Delta$). Fix $P \subset \Delta$ the system of positive roots for Δ so that $\langle \Lambda, \alpha \rangle >$ for $\alpha \in P$.

Let $H_0 = \exp(\mathfrak{h} \cap \mathfrak{g}_0)$, $H_1, \ldots, H_k$ be a complete set of non-conjugate Cartan subgroups of G_0. Let

$$\det(\operatorname{Ad}(x) - (\lambda+1) I) = \lambda^l D_l(x) + \sum_{j>l} \lambda^j D_j(x).$$

Set $D(x) = D_l(x)$. Let $G'_0 = \{ x \in G_0 \mid D_l(x) \neq 0 \}$. Let $H'_i = G'_0 \cap H_i$. Then $G'_0 = \bigcup_{i=0} \operatorname{Ad}(G_0) H'_i$. $\operatorname{Ad}(g) x = g \otimes g^{-1}$. Let for each i, $\mathfrak{h}_i$ be the complexified Lie algebra of H_i. Let $c_i : \mathfrak{h}_0 \to \mathfrak{h}_i$, $c_i \in \operatorname{Ad}(G)$. Then c_i is uniquely determined up to multiplication by an element of the Weyl group of $\mathfrak{h}_i$ on the left (equivalently up to multiplication on the right by an element of the Weyl group of $\mathfrak{h}_0 = \mathfrak{h}$). Let $\mathfrak{z}$ be the center of $U(\mathfrak{g})$. Then to Λ there is associated a homomorphism in $\chi_\Lambda : \mathfrak{z} \to \mathbf{C}$ (denoted γ_Λ in Warner [15], Section 10.1).

We recall the following theorem of Harish-Chandra [6] (*see* also Warner [15], p. 391. Theorem 10.1.1.1, p. 407, Theorem 10.2.4.1).

THEOREM 4.1. — *There exists one and only one central eigendistribution θ_Λ on G_0 so that*

(1) $z.\theta_\Lambda = \chi_\Lambda(z)\,\theta_\Lambda$.

(2) $\sup_{x \in G'_0} |D(x)|^{1/2} |\theta_\Lambda(x)| < \infty$.

(3) $\theta_\Lambda = \Delta_{H_0}^{-1} \sum_{s \in W_k} \det(s)\, e^{s.\Lambda}$ *on* H'_0 [*here* $\Delta_{H_0} = e^\rho \prod_{\alpha \in P} (1 - e^{-\alpha}) = \sum_{s \in W(\Delta)} \det(s)\, e^{s.\rho}$].

Also there exists π_Λ an irreducible square integrable representation of G_0 with character $(-1)^{\dim(G_0/K_0)/2}\,\theta_\Lambda$. The π_Λ defined as above exhaust the irreducible square integrable representations of G_0.

Fix for each $\mathfrak{h}_i$, P_i a system of positive roots for $\mathfrak{h}_i$, $P_0 = P$. Let for $\chi : \mathfrak{z} \to \mathbf{C}$ a homomorphism $\Lambda_i \in \mathfrak{h}_i^*$ be defined by $\chi = \chi_{\Lambda_i}$, where $\chi_{\Lambda_i+\rho_i}$ the infinitessimal character of the Verma module with highest weight Λ_i relative to P_i ($\rho_i = (1/2) \sum_{\alpha \in P_i} \alpha$). Λ_i is determined up to an element of the Weyl group of $W(\Delta_i)$.

THEOREM 4.2 (Harish-Chandra, *see* Warner [15], p. 136, Theorem 8.3.3.3). — *Let* T *be a central eigen-distribution on* G_0 *with* $z.T = \chi(z)T$ *for* $z \in \mathfrak{z}$. *Let* F_T *be the locally summable function on* G_0 *that gives* T. *Let* $\chi = \chi_{\Lambda_i}$, $i = 0, 1, \ldots, k$. *Let* $h \in H_i'$. *Then there is a neighborhood* $U_{h,i}$ *of* 0 *in* $\mathfrak{h}_i \cap \mathfrak{g}_0$ *and polynomial functions* $p_s(H)$, $s \in W(\Delta_i)$ *so that if* $H \in U_{h,i}$ *then*

$$F_T(h \exp H) = |D(h \exp H)|^{-1/2} \sum_{s \in W(\Delta_i)} P_s(H) e^{s\Lambda_i(H)} \xi_{s\Lambda_i}(h).$$

If Λ_i *is regular then* $p_s(H)$ *is a scalar. Here* ξ_μ *is the character of the complexified Cartan corresponding to* $\mathfrak{h}_i$.

THEOREM 4.3 (Harish-Chandra [6]). — *Let* F_Λ *be the locally integrable function on* G_0 *that gives* θ_Λ. *Then in the expression of Theorem* 4.2 *the constants* $p_s = p_s(0)$ *depend only on* $P = \{ \alpha \in \Delta^r \mid \langle \Lambda, \alpha \rangle > 0 \}$ if $\Lambda_i = \Lambda \circ c_i^{-1}$.

THEOREM 4.4 (Schmid [12], Enright, Varadarajan [4]). — *There is a constant* $C_P > 0$ *so that if* $\langle \Lambda, \alpha \rangle > C_P$ *for all* $\alpha \in P$ *and if* Λ *is integral then* $D_{P,\Lambda+\rho_n-\rho_k}$ *is equivalent with* π_Λ.

Let now $\Lambda \in \mathfrak{h}^*$ be regular and dominant integral relative to P. Let $\mu \in \mathfrak{h}^*$ be dominant integral relative to P so that $\Lambda+\mu$ satisfies the hypothesis of Theorem 4.4. Let η be the character of the irreducible finite dimensional representation, F, of G with lowest weight $-\mu$. Then Corollary 3.9 implies that $(\pi_{\Lambda+\mu} \otimes F)\chi_\Lambda$ contains $D_{P,\Lambda+\rho_n-\rho_k}$ as a subquotient. But now the character of $\pi_{\Lambda+\mu} \otimes F$ is $\eta\theta_{\Lambda+\mu}$. $\eta = \sum_{\xi \in \pi(F)} m_\xi e^\xi$, $\pi(F)$ the weights of F. Let now $h \in H_i'$. Let p_s be as above (p_s independent of Λ). Then

$$\theta_{\Lambda+\mu}(h \exp H) = |D(h \exp H)|^{-1/2} \sum_{s \in W(\Delta_i)} {}^i p_s e^{s(\Lambda_i+\mu_i)(H)} \xi_s(\Lambda_i+\mu_i)(h).$$

Thus

$$\begin{aligned}(\eta\,\theta_{\Lambda+\mu})(h \exp H) \\ = |D(h \exp H)|^{-1/2} \sum_{\gamma \in \pi_i(F)} m_\gamma \sum_{s \in W(\Delta_i)} p_s e^{s(\Lambda_i+\mu_i)(H)+\gamma(H)} \xi_\gamma(h).\xi_{s(\Lambda_i+\mu_i)(h)},\end{aligned}$$

$\pi_i(F)$ the weights of F on $\mathfrak{h}_i$. Now $\theta_{\Lambda+\mu} = \theta + T$ with $z.\theta = \chi_\Lambda(z)\theta$, $T = \sum_{i=1}^{u} T_i$ with $(z-\chi_i(z))^i T_i = 0$, $i = 1, \ldots, u$, $z \in \mathfrak{z}$, $\chi_i \neq \chi_\Lambda$. Now $\gamma \in \pi_i(F)$ is of the form $-\mu_i+\delta$, δ a sum of elements of P_i. Hence of the form $-s\mu_i + s\delta$, $s \in W(\Delta_i)$ and δ as above.

Using the arguments of the proof of Theorem 3.8 we find

$$\theta(h \exp H) = |D(h \exp H)|^{-1/2} \sum_{\substack{\gamma \in \pi_i(F),\ s \in W(\Delta_i) \\ s(\Lambda_i+\mu_i)+\gamma = s'(\Lambda_i)}} m_\gamma p_s e^{(s(\Lambda_i+\mu_i)+\gamma)(H)} \xi_{\gamma+s(\Lambda_i+\mu_i)}(h).$$

But now $\gamma = -s\,\mu_i + s\,\delta$ as above if $s(\lambda_i+\mu_i) - s\,\mu_i + s\,\delta = s'\,\Lambda_i$ then $s(\Lambda_i+\delta) = s'\,\Lambda_i$. Hence $s^{-1}\,s'\,\Lambda_i = \Lambda_i+\delta$. But Λ_i is P_i-dominant integral thus $\delta = 0$ and $s^{-1}\,s'\,\Lambda_i = \Lambda_i$. But Λ_i is regular hence $s = s'$. We therefore have

$$\theta(h \exp H) = \left| D(h \exp H) \right|^{-1/2} \sum_{s \in W(\Delta_i)} p_s\, e^{s\Lambda_i(H)}\, \xi_{s\Lambda_i}(h).$$

But then $\theta = \theta_\Lambda$. We have proved

THEOREM 4.5. — *If* $\Lambda \in \mathfrak{h}^*$ *is integral and regular and if* $P = \{ \alpha \in \Delta \mid \langle \Lambda, \alpha \rangle > 0 \}$ *then* $D_{P, \Lambda+\rho_n-\rho_k}$ *is infinitesimally equivalent with* π_Λ.

The preceeding argument to prove Theorem 4.5 is due to Zuckerman. It has also been used by W. Schmid in the course of his proof of Blattner's conjecture.

We note that Corollary 3.7 now says how discrete series tensored with finite dimensional representations decompose. This result has been proved by Hecht and Schmid by different methods.

5. Application to the realization of the discrete series

We retain the notation of Section 4. In Hotta [16] a realization of "most" of the discrete series for G_0 is given as follows. Let $\lambda \in \mathfrak{h}^*$ be regular and integral.

Let P be the system of positive roots for Δ so that $\langle \lambda, \alpha \rangle > 0$, $\alpha \in P$. Let T_λ be the representation of G_0 on the space $\mathfrak{H}_\lambda$, of all $f : G_0 \to V_{\lambda+\rho-2\rho_k}$ so that

(i) $f(gk) = k^{-1}.f(g)$ for $k \in K_0$, $g \in G_0$.

(ii) $\displaystyle\int_{G_0} |f(g)|^2\, dg < \infty$.

(iii) $\Omega f = (\langle \lambda, \lambda \rangle - \langle \rho, \rho \rangle) f$.

Ω the Casimir operator for $\mathfrak{g}_0$. $T_\lambda(g) f(x) = f(g^{-1}x)$.

We prove

THEOREM 5.1. — *Let* $\lambda \in \mathfrak{h}^*$ *be regular and integral and let* $(T_\lambda, \mathfrak{H}_\lambda)$ *be defined as above. Then* T_λ *is irreducible and has character* θ_λ.

Proof. — The Plancherel theorem for G_0 implies that $(T_\lambda, \mathfrak{H}_\lambda)$ is a finite sum of discrete series representations (*cf.* Hotta [16]). Frobenius reciprocity for multiplicites of discrete series in representations induced from K_0 to G_0 is true. Hence $T_\lambda = \sum m_i\, \pi_{\lambda_i}$ with m_i less than or equal to the multiplicity of $V_{\lambda+\rho-2\rho_k}$ in π_{λ_i}. λ_i can be taken P_k-dominant.

Hence if $m_i \neq 0$ and $s \in W$ is such that $s\,P \supset P_k$ and $\lambda_i = s\,\mu$, μ, P-dominant integral, then since $V_{\lambda+\rho-2\rho_k}$ appears in π_{λ_i} we must have $\lambda+\rho-2\,\rho_k = s\,\mu + s\,\rho - 2\,\rho_k + s\,Q$, Q a sum of elements of P. But then $\lambda+\rho = s(\mu+\rho+Q)$.

Now the action of the Casimir operator Ω [(iii) above] implies $\langle \mu, \mu \rangle = \langle \lambda, \lambda \rangle$. Hence

$$\langle s(\mu+\rho+Q)-\rho,\ s(\mu+\rho+Q)-\rho \rangle = \langle \mu, \mu \rangle.$$

But then

$$\langle \mu+\rho-s^{-1}\rho+Q,\ \mu+\rho-s^{-1}\rho+Q \rangle = \langle \mu, \mu \rangle.$$

This implies that

$$0 = 2\langle \mu, \rho - s^{-1}\rho + Q \rangle + \langle \rho - s^{-1}\rho + Q, \rho - s^{-1}\rho + Q \rangle.$$

Now $\rho - s^{-1}\rho$ is a sum of elements of P. Since μ is P-dominant integral and regular this implies that $\rho - s^{-1}\rho + Q = 0$. But then $\rho = s^{-1}\rho$ and $Q = 0$. Hence $s = 1$ and $\lambda_i = \lambda$. To complete the proof we need only show that $\mathfrak{H}_\lambda \neq 0$.

Let (π_λ, H) be a realization of π_λ. Let $P : \to H\, V_{\lambda+\rho-2\rho_k}$ be a K_0-intertwining operator. Let $v \in H$ be K_0-finite and define $f_v(g) = P(\pi_\lambda(g)^{-1} v)$. Then f_v satisfies (i) and (ii).

$\Omega f_v = \chi_{-P, \lambda+\rho}(\Omega) f_v = (\langle \lambda, \lambda \rangle - \langle \rho, \rho \rangle) f_v$ by the results of Section 4. Hence if $v \neq 0$, $f_v \in \mathfrak{H}_\lambda$.

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