ALAIN CONNES

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OUTER CONJUGACY CLASSES OF AUTOMORPHISMS OF FACTORS

BY ALAIN CONNES

INTRODUCTION

Two automorphisms α and β of a von Neumann algebra M are called outer conjugate when their classes $\varepsilon(\alpha)$, $\varepsilon(\beta)$ modulo inner automorphisms of M, are conjugate in the group Out M = Aut M/Int M.

The outer period $p_0(\alpha)$ of an automorphism α of M is by definition the period of $\varepsilon(\alpha)$ in Out M, and is equal to 0 if no power $\varepsilon(\alpha)^n$, $n \neq 0$ is equal to 1.

The obstruction $\gamma(\alpha)$ of an automorphism α of M is the root of 1, γ in **C** such that $\alpha^{p_0(\alpha)} = \operatorname{Ad} U \Rightarrow \alpha(U) = \gamma U$ for U unitary in M. This definition makes sense when M is a factor, moreover $\gamma(\alpha)^{p_0(\alpha)} = 1$ and $\gamma(\alpha) = 1$ if $p_0(\alpha) = 0$.

In [8], theorem 1.5, we showed that p_0 and γ are complete invariants of outer conjugacy for automorphisms of the hyperfinite factor of type II_1 : R, which are periodic. In this paper we shall show that the restriction of periodicity is unnecessary, that is: Any two automorphisms α and β of R such that $p_0(\alpha) = p_0(\beta) = 0$ are outer conjugate.

It shows that Out R is a simple group with only countably many conjugacy classes.

In [4] we showed that the classification of factors of type III_λ $\lambda \in]0, 1[$ is the classification of outer conjugacy classes of automorphisms θ of factors of type II_∞ : N, which multiply the trace of N by the scalar λ. In [5] we gave an example where for fixed N and λ there was more than one such outer conjugacy class of $\theta' s$.

Here we prove, using the study of automorphisms of R, that for $N = R \otimes I_\infty$, where I_∞ stands for the algebra of all bounded operators in a Hilbert space, one has: For each $\lambda \in]0,1[$ there is, up to conjugacy, only one automorphisms θ_λ of N such that θ multiplies the trace by λ. This implies that the Powers' factors are the only factors of type III_λ whose corresponding factor of type II_∞ is $R_{0,1}$. (The above $N = R \otimes I_\infty$ is the only factor of Araki-Woods of type II_∞, we take the notation $R_{0,1}$ for it, as in [1].) We shall in another paper discuss the implications of this fact on the study of hyperfinite factors and also apply theorems 1 and 2 below to get the list, up to outer conjugacy, of all automorphisms of Krieger's factors. Also we refer the reader to [9] for the applications of the above results to hyperfiniteness of representations of arbitrary solvable groups.

The content of this paper is essentially the proof of two theorems, that we now state.

We take the same notations as in [8] for periodic automorphisms of R. In particular for $p \in \mathrm{N}$, $p \geqq 1$ we let s_p be the automorphism of R (unique up to conjugacy) such that $(s_p)^p = 1$ and $p_0(s_p) = p$. For $p = 1$, $s_1 = 1$. Also we let s_0 be the infinite tensor product of all the s_p, $p \geqq 1$ on $\bigotimes_{p=1}^{\infty} (\mathrm{R}_p, \tau_p)$ where R_p is isomorphic to R and τ_p the canonical trace on R_p. By definition the asymptotic period $p_a(\theta)$ of an automorphism θ of M is the period of θ in the quotient group Aut M/CtM, where CtM is the normal subgroup of centrally trivial automorphisms (*see* [7]), i. e., those θ such that $\theta(x_n) - x_n \to 0^*$ strongly for any bounded sequence $(x_n)_{n \in \mathrm{N}}$ of elements of M such that $\| [x_n, \varphi] \| \to 0$, φ in the predual M_* of M.

As Int M $\subset$ CtM, we see that Aut M/CtM is a quotient of Out M and that $p_a(\theta)$ divides $p_0(\theta)$ for any θ.

THEOREM 1. — *Let* M *be a factor with separable predual, isomorphic to* $\mathrm{M} \otimes \mathrm{R}$. *Let* $p \in \mathrm{N}$ *and* $\theta \in$ Aut M, *then* ($\theta \otimes s_p$ *outer conjugate to* θ) $\Leftrightarrow p_a(\theta) = 0$ *modulo* p.

Take $p = 1$, then for any $\theta \in$ Aut M, one has $p_a(\theta) = 0\,(p)$ so $\theta \otimes 1_{\mathrm{R}}$ is outer conjugate to θ.

If $p_a(\theta) = 0$, then $\theta \otimes s_p$ is outer conjugate to θ for all p. Moreover we shall prove that the condition "M is isomorphic to $\mathrm{M} \otimes \mathrm{R}$" is equivalent to the *non*-commutativity of the group $\varepsilon(\overline{\mathrm{Int}}\ \mathrm{M}) = \overline{\mathrm{Int}}\ \mathrm{M}/\mathrm{Int}\ \mathrm{M}$, where the closure is taken in the natural topology of Aut M : the topology of pointwise norm convergence in M_*. This fact is a simple generalization of results of D. McDuff [11] who proved that when M is of type II_1 then "M is isomorphic to $\mathrm{M} \otimes \mathrm{R}$" is equivalent to the *non*-commutativity of the algebra of central sequences. Moreover we shall see that as soon as M is isomorphic to $\mathrm{M} \otimes \mathrm{R}$ we have

$$\varepsilon(\mathrm{C}t\mathrm{M}) = (\varepsilon(\overline{\mathrm{Int}}\ \mathrm{M}))',$$

where the prime indicates the commutant. (More explicitely a $\theta \in$ Aut M is centrally trivial iff $\varepsilon(\theta)$ commutes with any $\varepsilon(\alpha)$, $\alpha \in \overline{\mathrm{Int}}\ \mathrm{M}$.)

The basis of the proof of theorem 1 is to use for each ultrafilter (free on N), say ω, the functor $\mathrm{M} \to \mathrm{M}_\omega$ defined in [5] from the category of von Neumann algebras in the category of finite von Neumann algebras. For each ω and $\theta \in$ Aut M one shows that $p_0(\theta_\omega) = p_a(\theta)$ and then one applies a generalization of the tower theorem of Rokhlin (*see* 1.2.5). The next theorem studies the outer conjugacy problem for the approximately inner automorphisms, i. e., those which belong to the closure $\overline{\mathrm{Int}}$ M of Int M in Aut M with the same topology as above, Observe also that for $\theta \in \overline{\mathrm{Int}}$ M, $p_a(\theta)$ is the period of $\varepsilon(\theta)$ in $\varepsilon(\overline{\mathrm{Int}}\ \mathrm{M})$/Center $\varepsilon(\overline{\mathrm{Int}}\ \mathrm{M})$.

THEOREM 2. — *Let* M *be a factor with separable predual, isomorphic to* $\mathrm{M} \otimes \mathrm{R}$, *take* $\theta_1, \theta_2 \in \overline{\mathrm{Int}}$ M.

If $p_a(\theta_1) = p_a(\theta_2) = 0$ *there is a* $\sigma \in \overline{\mathrm{Int}}\,\mathrm{M}$ *such that*

$$\varepsilon(\theta_2) = \varepsilon(\sigma\theta_1\sigma^{-1}).$$

(*In particular* θ_2 *is outer conjugate to* θ_1).

If $p_a(\theta_j) > 0$, $p_a(\theta_1) = p_a(\theta_2)$ *and* $\theta_j^{p_a(\theta_j)} = 1$ *then* θ_2 *is conjugate to* θ_1.

The second part of the theorem is an easy adaptation of our previous argument in [8] and we shall omit it here.

COROLLARY 3. — *Two automorphisms* $\alpha, \beta \in \mathrm{Aut}\,\mathrm{R}$ *are outer conjugate iff* $p_0(\alpha) = p_0(\beta)$ *and* $\gamma(\alpha) = \gamma(\beta)$.

Proof. — If $p_0(\alpha) > 0$ use [8] (th. 1.5). Otherwise by [8], Lemma 3.4, $p_a(\alpha) = p_a(\beta) = 0$ and theorem 2 applies, as $\overline{\mathrm{Int}}\,\mathrm{R} = \mathrm{Aut}\,\mathrm{R}$.

COROLLARY 4. — *The group* Out R *is a simple group with countably many conjugacy classes.*

Proof. — By corollary 3 the conjugacy classes of Out R are parametrized by couples (p, γ), $p \in \mathbf{N}$, $\gamma \in \mathbf{C}$, $\gamma^p = 1$. Choose for each p, γ, s_p^γ as defined in [8] if $p \neq 0$ and s_0 if $p = 0$. We have to show that a normal subgroup G of Aut R, containing Int R and an outer automorphism, is equal to Aut R. It is enough to show that for any (p, γ), (p', γ') as above there is, if $p \neq 1$ an equality $\alpha = \alpha_1 \ldots \alpha_m$ with α_j of the form $\sigma_j\, s_p^\gamma\, \sigma_j^{-1}$ for all $j = 1, \ldots, m$ and α outer conjugate to $s_{p'}^{\gamma'}$. If $p \neq 0$, using the construction [7] part IV we can find an automorphism β of R such that $s_p^\gamma\, \beta\, s_p^\gamma\, \beta^{-1}$ has outer period 0. So we just have to treat the case $p = 0$. As, for any countable group D, there is an action, by outer automorphisms, of D on R we easily get a product $\alpha = s_0\, \sigma\, s_0\, \sigma^{-1}$ outer conjugate to $s_{p'}^1$. But by construction $s_{p'}^{\gamma'}$ is a product of an automorphism conjugate to $s_{p'}^1$ by an automorphism conjugate to an s_q^1, $q = \mathrm{Order}\,\gamma'$.

Q. E. D.

LEMMA 5. — $\mathrm{Ct}(\mathrm{R}_{0,1}) = \mathrm{Int}(\mathrm{R}_{0,1})$, *where* $\mathrm{R}_{0,1}$ *is the tensor product of* R *by a type* I_∞ *factor* F_∞.

Proof. — Let $\theta \in \mathrm{Ct}(\mathrm{R}_{0,1})$. Then by theorem 1 we have that $\theta \otimes 1_\mathrm{R}$ is outer conjugate to θ. Let $\theta' \in \mathrm{Aut}\,\mathrm{R}_{0,1}$, $\varepsilon(\theta') = \varepsilon(\theta)$, such that $\mathrm{R}_{0,1}^{\theta'}$ contains a factor of type I_∞ (use [7], lemma 3.11). It follows that $\theta \otimes 1_{\mathrm{F}_\infty}$ is outer conjugate to θ and that $\theta \otimes 1_{\mathrm{R}_{0,1}}$ is in $\mathrm{Ct}(\mathrm{R}_{0,1} \otimes \mathrm{R}_{0,1})$. Let s be the symmetry : $s(x \otimes y) = y \otimes x$, on $\mathrm{R}_{0,1}$. One checks that $s \in \overline{\mathrm{Int}}(\mathrm{R}_{0,1} \otimes \mathrm{R}_{0,1})$ and hence that $\varepsilon(s)$ commutes with $\varepsilon(\theta \otimes 1_{\mathrm{R}_{0,1}})$. Then $\theta \otimes \theta^{-1}$ is inner and so is θ.

Q. E. D.

Let M be a factor of type II_∞ and $\theta \in \mathrm{Aut}\,\mathrm{M}$ then by mod θ we mean the scalar $\lambda \in \mathbf{R}_+^*$ by which θ multiplies an arbitrary faithful normal semi-finite trace on M.

COROLLARY 6. — *Let* $\mathrm{R}_{0,1}$ *be the tensor product of* R *by a factor of type* I_∞. *Then there is, up to conjugacy, only one automorphism* θ_λ *of* $\mathrm{R}_{0,1}$ *with* $\mathrm{mod}\,\theta_\lambda = \lambda \neq 1$.

Proof. — Put $R_{0,1} = R \otimes F_\infty$. Let η be the map $\alpha \to \alpha \otimes 1_{F_\infty}$ of Out R in Out $R_{0,1}$. By [7], lemma 3.11, this map is an isomorphism of Out R onto

$$\text{Out}_1 R_{0,1} = \{\theta \in \text{Out } R_{0,1}, \text{ mod } \theta = 1\}.$$

It follows easily that $\text{Out}_1 R_{0,1} = \varepsilon(\overline{\text{Int } R_{0,1}})$, where ε is the canonical quotient map. Let $\mathscr{B}$ be the set of outer conjugacy classes of aperiodic automorphisms of $R_{0,1}$. As $R_{0,1} \otimes R_{0,1}$ is isomorphic to $R_{0,1}$ we have a commutative law of composition $\alpha . \beta$ = class of $\alpha \otimes \beta$, which makes $\mathscr{B}$ into a group for the following reasons (*a*) (Class $s_0 \otimes 1).\alpha = \alpha$ for any $\alpha \in \mathscr{B}$ (because by lemma 5, the asymptotic period of any element of the class α, is equal to 0, so that theorem 1 applies); (*b*) $\alpha . \alpha^{-1}$ = class $(s_0 \otimes 1)$ for any $\alpha \in \mathscr{B}$. [To see this last fact, note that mod $(\alpha \otimes \alpha^{-1}) = 1$ so that corollary 3 applies to show that $\alpha \otimes \alpha^{-1}$ is outer conjugate to $s_0 \otimes 1$]. At the same time we have shown that the kernel of $\mathscr{B} \xrightarrow{\text{mod}} \mathbf{R}_+^*$ is trivial, so that as the fundamental group of R is equal to $\mathbf{R}_+^*$ ([12]) we have shown that $\mathscr{B} \xrightarrow{\text{mod}} \mathbf{R}_+^*$ is an isomorphism.

This shows the uniqueness of θ_λ modulo outer conjugacy. However using [6], III, we get back to ordinary conjugacy.

Q. E. D.

It follows that all factors of type III_λ (*) M for which the associated factor of type II_∞ is $R_{0,1}$ are isomorphic to R_λ, the Powers factors. (Apply [4] theorem 4.4.1).

For each integer $p \in \mathbf{N}$ the unique automorphism of $R_{0,1}$ with module equal to p can be described as a p-shift in the following way. Let $(\lambda_{\nu,j})_{j=1,\dots,p,\nu\in\mathbf{Z}}$ be an eigenvalue list such that the corresponding infinite tensor product of the $p \times p$ matrix algebras (M_ν, λ_ν) satisfy:

$\bigotimes_{\nu \geqq 0} (M_\nu, \lambda_\nu)$ is a factor type II_1;

$\bigotimes_{\nu < 0} (M_\nu, \lambda_\nu)$ is a factor of type I_∞.

Then $\bigotimes_{\nu\in\mathbf{Z}} (M_\nu, \lambda_\nu)$ is isomorphic to $R_{0,1}$ and the shift has module p so that by corollary 7 it is conjugate to θ_p.

It also follows from corollary 7 and the existence, proven by M. Takesaki, of a one parameter group $(\theta_\lambda)_{\lambda\in\mathbf{R}_+}$ of automorphisms of $R_{0,1}$ with mod $\theta_\lambda = \lambda$ for all λ, that each of the above shifts can be imbedded in a flow.

COROLLARY 8. — *An automorphism* $\alpha \in \text{Aut } R_{0,1}$ *is unimodular if and only if it is a commutator*: $\alpha = \beta\sigma\beta^{-1}\sigma^{-1}$ *of elements of* Aut $R_{0,1}$.

Proof. — Assume mod $\alpha = 1$, then for any $\lambda \neq 1$, mod $\alpha\theta_\lambda = \lambda$ so by 7 we have a σ with $\alpha\theta_\lambda = \sigma\theta_\lambda\sigma^{-1}$.

Q. E. D.

I. Preliminaries

I.1. ASYMPTOTIC CENTRALIZER OF FACTORS. — Let M be a von Neumann algebra and ω a free ultrafilter on N — As in [5] 2.2, a centralizing sequence $(x_n)_{n \in \mathbf{N}}$ of elements of M (resp. an ω — centralizing sequence) is an element of the C* algebra $l^\infty(\mathbf{N}, \mathbf{M})$ such that $\| [x_n, \Psi] \| \to 0$ when $n \to \infty$ (resp. $n \to \omega$), $\forall \Psi \in \mathbf{M}_*$. Let us recall a result of [5].

PROPOSITION 1.1.1. — *For* M *and* ω *as above, the* ω*-centralizing sequences form a* C* *subalgebra of* $l^\infty(\mathbf{N}, \mathbf{M})$ — *The set* $\mathcal{I}_\omega$ *of* ω*-centralizing sequences* $(x_n)_{n \in \mathbf{N}}$ *with* $x_n \underset{n \to \omega}{\longrightarrow} 0^*$ *strongly is a two sided ideal of this* C* *subalgebra.* The *quotient* C* *algebra* $\mathbf{M}_\omega$ *is a finite von Neumann algebra on which each faithful normal state* φ *of* M *defines a faithful normal trace, associating to each* ω*-centralizing sequence* $(x_n)_{n \in \mathbf{N}}$, *the scalar* $\operatorname{Lim}_{n \to \omega} \varphi(x_n)$.

We say that two ω-centralizing sequences $(x_n)_{n \in \mathbf{N}}$ and $(y_n)_{n \in \mathbf{N}}$ are equivalent when $x_n - y_n$ tends to 0^* strongly when n tends to ω. If $(x_n)_{n \in \mathbf{N}}$ is ω-centralizing and $(y_n)_{n \in \mathbf{N}}$ is a bounded sequence such that $x_n - y_n \underset{n \to \omega}{\longrightarrow} 0^*$ strongly then $(y_n)_{n \in \mathbf{N}}$ is ω-centralizing. An element x of $\mathbf{M}_\omega$ is a class of equivalence of ω-centralizing sequences $(x_n)_{n \in \mathbf{N}}$, each of them being called a representing sequence for x. If $(x_n)_{n \in \mathbf{N}}$, $(y_n)_{n \in \mathbf{N}}$ represent $x, y \in \mathbf{M}_\omega$ then $x_n + y_n$, x_n^*, $x_n y_n$ represent respectively $x + y$, x^*, xy. An $x \in \mathbf{M}_\omega$ has norm less than 1 if and only if it has a representing sequence $(x_n)_{n \in \mathbf{N}}$ with $\| x_n \| \leqq 1$ for all $n \in \mathbf{N}$.

PROPOSITION 1.1.2. — *Let* M *be a countably decomposable factor, and* ω *a free ultrafilter on* N.

(*a*) *For each* $x \in \mathbf{M}_\omega$, *the weak limit of* x_n *when* $n \to \omega$: $\tau_\omega(x)$ *is an element of the center* C *of* M, *which does not depend on the choice of the representing sequence of* x.

(*b*) *The map* $x \in \mathbf{M}_\omega \to \tau_\omega(x)$ *is a faithful normal trace on* $\mathbf{M}_\omega$ *and for any* $\varphi \in \mathbf{M}_*$, *any representing sequence* $(x_n)_{n \in \mathbf{N}}$ *of* $x \in \mathbf{M}_\omega$ *one has* $\varphi(x_n) \underset{n \to \omega}{\longrightarrow} \varphi(1)\, \tau_\omega(x)$.

Proof. — (*a*) The unit ball of M is weakly compact so that $x_n \underset{n \to \omega}{\longrightarrow} \mathrm{L}$ where $\mathrm{L} \in \mathbf{M}$. As $u x_n u^* - x_n \underset{n \to \omega}{\longrightarrow} 0$ *strongly* for any unitary $u \in \mathbf{M}$ ([5], prop. 2.8), (*a*) follows easily.

(*b*) Let φ be any linear normal functional on M, then one has $\varphi(x_n) \underset{n \to \omega}{\longrightarrow} \varphi(\tau_\omega(x))$ for any representing sequence $(x_n)_{n \in \mathbf{N}}$ of $x \in \mathbf{M}_\omega$, just by definition of the weak topology. So taking φ faithful and normal state and applying proposition 1.1.1 completes the proof.

Q. E. D.

PROPOSITION 1.1.3. — *Let* M *be a factor with separable predual and* ω *a free ultrafilter on* N.

(*a*) *Any projection* $e \in \mathbf{M}_\omega$ *can be represented by a sequence* $(e_n)_{n \in \mathbf{N}}$ *of projections of* M.

(*b*) *Let* $(e_n)_{n \in \mathbf{N}}$ *and* $(f_n)_{n \in \mathbf{N}}$ *be* ω*-centralizing sequences of projections* $e_n \sim f_n$ *of* M *representing* $e, f \in \mathbf{M}_\omega$. *Any partial isometry* $u \in \mathbf{M}_\omega$, $u^* u = e$, $uu^* = f$ *has a representing sequence of partial isometries* $(u_n)_{n \in \mathbf{N}}$ *with* $u_n^* u_n = e_n$, $u_n u_n^* = f_n$.

(*c*) *Any partition of unity* $(F_j)_{j=1,\dots,n}$ *in* M_ω *can be represented by a sequence of partitions of unity*, $(F_{j,n})$. *If the* F_j *are pairwise equivalent one can choose the* $F_{j,n}$ *pairwise equivalent for each n.*

(*d*) *Any system of* $p \times p$ *matrix units in* M_ω *can be represented by a sequence of systems of* $p \times p$ *matrix units in* M.

LEMMA 1.1.4. — *Let* M *be a countably decomposable von Neumann algebra in a space* $\mathcal{H}$ *and* $\xi \in \mathcal{H}$. *Let* e, f *be projections belonging to* M.

(*a*) *Let* $fe = w\rho$ *be the polar decomposition of* fe *then*:

$$\|(w-f)\xi\| \leqq 3\varepsilon, \qquad \|(w-e)\xi\| \leqq 4\varepsilon, \qquad \|(w^*-f)\xi\| \leqq 4\varepsilon, \qquad \|(w^*-e)\xi\| \leqq 3\varepsilon,$$

where $\varepsilon = \|(e-f)\xi\|$.

(*b*) *If* $e \sim f(\mathrm{M})$, *there exists a partial isometry* $u \in \mathrm{M}$, *such that*

$$u^*u = e, \qquad uu^* = f, \qquad \|(u-f)\xi\| \leqq 6\|(e-f)\xi\|, \qquad \|(u-f)^*\xi\| \leqq 7\|(e-f)\xi\|.$$

Proof. — (*a*) We have $\rho^2 = efe \leqq e$. Also $\|f(e-f)\xi\| \leqq \varepsilon$ hence $\|(fe-e)\xi\| \leqq 2\varepsilon$ and $\|(\rho^2-e)\xi\| \leqq 2\varepsilon$. As $\rho^2 \leqq \rho \leqq e$, we have:

$$\|(\rho-e)\xi\| \leqq \|(\rho^2-e)\xi\| \leqq 2\varepsilon \qquad \text{and} \qquad \|(w\rho-we)\xi\| \leqq 2\varepsilon.$$

As $we = w$, this gives $\|(fe-w)\xi\| \leqq 2\varepsilon$ and hence $\|(f-w)\xi\| \leqq 3\varepsilon$.

The adjoint of fe is $ef = w^*(w\rho w^*)$, which shows exchanging e and f, that $\|(w^*-e)\xi\| \leqq 3\varepsilon$ and ends the proof of (*a*).

(*b*) Let c be a central projection such that $(1-c)e$ is properly infinite and ce is finite, put $e_1 = ce$, $e_2 = (1-c)e$, $f_1 = cf$, $f_2 = (1-c)f$. We have $e_1 \sim f_1$ and $e_2 \sim f_2$. Let $\eta > 0$. Choose projections $e_2^1 \leqq e_2$, $f_2^1 \leqq f_2$ such that $e_2 - e_2^1$ and $f_2 - f_2^1$ are properly infinite with the same central support as e_2, while

$$\|(e_2-e_2^1)\xi\| \leqq \eta, \qquad \|(f_2-f_2^1)\xi\| \leqq \eta.$$

Put $e^1 = e_1 + e_2^1$, $f^1 = f_1 + f_2^1$ and let $\mathrm{E} = \text{Support } f^1e^1$, $\mathrm{F} = \text{Support } e^1f^1$. We have $\mathrm{E} \leqq e^1$, $\mathrm{F} \leqq f^1$ and $e^1f^1e^1 \leqq \mathrm{E}$ so that with $\varepsilon = \|(e-f)\xi\|$,

$$\|(e^1-\mathrm{E})\xi\| \leqq \|(e^1-e^1f^1e^1)\xi\| \leqq 2\|(e^1-f^1)\xi\| \leqq 2\varepsilon + 4\eta$$

and with $f^1e^1 = w\rho$ as above, we have $w^*w = \mathrm{E}$, $ww^* = \mathrm{F}$ and

$$\|(w-f^1)\xi\| \leqq 3\|(e^1-f^1)\xi\|, \qquad \|(w^*-f^1)\| \leqq 4\|(e^1-f^1)\xi\|.$$

In the same way we get $f^1e^1f^1 \leqq \mathrm{F}$ and $\|(f^1-\mathrm{F})\xi\| \leqq 2\varepsilon + 4\eta$. The projections $e_1 - c\mathrm{E}$ and $f_1 - c\mathrm{F}$ are equivalent because $c\mathrm{E} \sim c\mathrm{F}$ and $e_1 \sim f_1$. The projections $e_2 - (1-c)\mathrm{E}$ and $f_2 - (1-c)\mathrm{F}$ dominate respectively $e_2 - e_2^1$ and $f_2 - f_2^1$ and hence are properly infinite with same central support, so they are equivalent. It follows that

$e-\mathrm{E} \sim f-\mathrm{F}$, let $\tilde{w} \in \mathrm{M}$, $\tilde{w}^* \tilde{w} = e-\mathrm{E}$, $\tilde{w}\tilde{w}^* = f-\mathrm{F}$. Then $u = w+\tilde{w}$ satisfies $u^* u = e$, $uu^* = f$ and

$$\begin{aligned}
\|\tilde{w}\xi\| &= \|\tilde{w}\tilde{w}^*\tilde{w}\xi\| \leqq \|(e-\mathrm{E})\xi\| \leqq 2\varepsilon+5\eta,\\
\|\tilde{w}^*\xi\| &= \|\tilde{w}^*\tilde{w}\tilde{w}^*\xi\| \leqq \|(f-\mathrm{F})\xi\| \leqq 2\varepsilon+5\eta,\\
\|(w-f)\xi\| &\leqq 3(\varepsilon+2\eta)+\eta = 3\varepsilon+7\eta,\\
\|(w^*-f)\xi\| &\leqq 4(\varepsilon+2\eta)+\eta = 4\varepsilon+9\eta.
\end{aligned}$$

Which taking η small enough gives the conclusion.

Q. E. D.

LEMMA 1.1.5. — *Let* $\varepsilon \in]0, 1[$, M *be a von Neumann algebra,* φ *a state on* M *and* $\rho \in \mathrm{M}$, $0 \leqq \rho \leqq 1$ *such that* $\|\rho^2-\rho\|_\varphi \leqq \varepsilon$. *Let e be the spectral projection of* ρ *for the interval* $[1-\varepsilon^{1/2}, 1]$ *then*:

$$\|\rho-e\|_\varphi \leqq 2\varepsilon^{1/2}, \qquad \|\rho^{1/2}-e\|_\varphi \leqq 3\varepsilon^{1/2}$$

Proof. — As in [10] (p. 278-279) one has $(1-\rho)^2(1-e) \geqq \varepsilon(1-e)$ and $\varphi(\rho^2(1-\rho)^2 \leqq \varepsilon^2$ so that $\varphi(\rho^2(1-e)) \leqq \varepsilon$. As $\|\rho e-e\| \leqq \varepsilon^{1/2}$ we get

$$\|\rho-e\|_\varphi \leqq \|\rho(1-e)\|_\varphi+\|\rho e-e\|_\varphi \leqq 2\varepsilon^{1/2}.$$

Also we have

$$|\varphi(\rho-\rho^2)(1-e))| \leqq \|\rho-\rho^2\|_\varphi \leqq \varepsilon,$$

hence $\|\rho^{1/2}(1-e)\|_\varphi^2 \leqq 2\varepsilon$ and as $\|\rho^{1/2}e-e\| \leqq \varepsilon^{1/2}$ we get the second inequality.

Q. E. D.

Proof of proposition 1.1.3. — (*a*) We have $\|e\| = 1$, so let $(x_n)_{n\in\mathbf{N}}$ be a representing sequence of e with $\|x_n\| \leqq 1$ for all n. As $\rho = x_n^* x_n \in [0, 1]$ and represents e we have $\|\rho_n^2-\rho_n\|_\varphi \to 0$ when $n \to \omega$, for any faithful normal state φ on M. Fix φ and let $\varepsilon_n = \|\rho_n^2-\rho_n\|_\varphi$, e_n be the spectral projection of ρ_n for $[1-\varepsilon_n^{1/2}, 1]$. Then by 1.1.5 one has $e_n-\rho_n \to 0$ *strongly when $n \to \omega$ so that $(e_n)_{n\in\mathbf{N}}$ is ω-centralizing and represents e.

(*b*) Let $(x_n)_{n\in\mathbf{N}}$, $\|x_n\| \leqq 1$ be a representing sequence for u. As $fue = u$ the sequence $f_n x_n e_n = y_n$ represents also u. Let φ be a faithful normal state on M, $\rho_n = y_n^* y_n$, $\varepsilon_n = \|\rho_n^2-\rho_n\|_\varphi$ and g_n the spectral projection of ρ_n for $[1-\varepsilon_n^{1/2}, 1]$. As $(\rho_n)_{n\in\mathbf{N}}$ represents the projection $e = u^* u$ we have, by 1.1.5, that $\varepsilon_n \underset{n\to\omega}{\to} 0$ and that $(g_n)_{n\in\mathbf{N}}$ represents e. Let $v_n = \mathrm{k}_n g_n$, where $\mathrm{k}_n \rho_n^{1/2}$ is the polar decomposition of y_n. By construction $\|\rho_n^{1/2} g_n-g_n\| \leqq \varepsilon_n^{1/2}$ so that $\|y_n g_n-v_n\| \leqq \varepsilon_n^{1/2}$ which, as $(g_n)_{n\in\mathbf{N}}$ represents e, shows that $(v_n)_{n\in\mathbf{N}}$ is an ω-centralizing sequence and represents $ue = u$.

By construction v_n is a partial isometry with $v_n^* v_n \leqq e_n$ $v_n v_n^* \leqq f_n$, and $e_n-v_n^* v_n \underset{n\to\omega}{\to} 0$, $f_n-v_n v_n^* \underset{n\to\omega}{\to} 0^*$ strongly because $e = u^* u, f = uu^*$. If $e_n-v_n^* v_n$ is equivalent to $f_n-v_n v_n^*$ via a partial isometry w_n we see that $w_n \underset{n\to\omega}{\to} 0^*$ strongly, so that $u_n = v_n+w_n$ is the desired sequence of partial isometries. With φ as above we choose for each $n \in \mathbf{N}$ projections

$e'_n, f'_n \in M$, $e'_n \leqq e_n$, $f'_n \leqq f_n$ such that $e_n = e'_n$, $f_n = f'_n$ when e_n is finite and that $e_n - e'_n$, $f_n - f'_n$ are infinite,

$$\| e_n - e'_n \|_\varphi \leqq 1/n, \qquad \| f_n - f'_n \|_\varphi \leqq 1/n$$

when e_n is infinite. Then we do the above construction with (e'_n) and (f'_n) instead of (e_n), (f_n) and we get always, as $v'^*_n v'_n \leqq e'_n$, $v'_n v'^*_n \leqq f'_n$ that $e_n - v'^*_n v'_n$ is equivalent to $f_n - v'_n v'^*_n$.

Q. E. D.

(*c*) The first part of (*c*) is easily proven by induction on the number of elements of the partition, using lemma 1.1.5.

If M is finite and the F_j are pairwise equivalent, we get $\lim_{n \to \omega} \tau(F_{j,n}) = 1/p$, where τ is the trace on M. So one can adjust the $F_{j,n}$ so that $\tau(F_{j,n}) = 1/p$ for all n. If M is infinite, for each n there is an $F_{j_n, n}$ which is infinite and hence, with φ a faithful normal state on M, we can find $p-1$ pairwise orthogonal subprojections $f_{k,n}$ of $F_{j_n,n}$, such that each $f_{k,n}$ is infinite and $\sum_k \varphi(f_{k,n}) < 1/n$. Distributing those $f_{k,n}$ to the $F_{j,n}$ $j \neq j_n$ we replace the partition $(F_{j,n})_{j=1,\ldots,p}$ by a partition $(F'_{j,n})_{j=1,\ldots,p}$ satisfying the required conditions.

(*d*) Let $(e_{ij})_{i,j=1,\ldots,p}$ be a system of matrix units on M_ω. By (*c*) let $(F_{j,n})_{j=1,\ldots,p}$ be a sequence of partitions of unity in equivalent projections of M, with $(F_{j,n})_{n \in N}$ representing e_{jj}. By (*b*) let for $j = 1, \ldots, p-1$, $(u_{j,n})_{n \in N}$ be a sequence of partial isometries of M representing $e_{j+1,j}$ and such that for all n and j:

$$u^*_{j,n} u_{j,n} = F_{j,n}, \qquad u_{j,n} u^*_{j,n} = F_{j+1,n}.$$

Then for each n the $(u_{j,n})_{j=1,\ldots,p-1}$ generate a system of matrix units e^n_{ij} such that $e^n_{j+1,j} = u_{j,n}$ and it is the desired sequence of systems of matrix units.

Q. E. D.

I.2. Non commutative Rokhlin's theorem. — We first remind the reader that given two projections e, f in a Hilbert space $\mathscr{H}$ they generate a von Neumann algebra N of type I, in fact, more precisely:

1. $a = e \wedge f + (1-e) \wedge f + e \wedge (1-f) + (1-e) \wedge (1-f)$ is the largest projection of the center C of N such that N_a is abelian.

2. N_{1-a} is a von Neumann algebra of type I_2.

3. e and f are abelian projections of N.

We put $s(e,f) = |e-f|$ and $c(e,f) = |e \vee f - e - f| = s(e \vee f - e, f)$. We have $0 \leqq s(e,f) \leqq 1$ and $s(e,f)^2 + c(e,f)^2 = e \vee f$. Both $s(e,f)$ and $c(e,f)$ belong to the center C of N. We have

$$c(e,f)^2 e = (e \vee f - e - f)^2 e = e + e + fe - 2e - 2fe + fe + efe = efe.$$

As the central support of e is larger than the support of $c(e,f)$ we get.

4. $\| c(e,f) \| = \| ef \|$, $\| s(e,f) \| = \| e-f \|$.

Let $E = e\mathcal{H}$, $F = f\mathcal{H}$ and

$$E_1 = \{\xi \in E, \|\xi\| = 1\}, \qquad F_1 = \{\eta \in F, \|\eta\| = 1\}.$$

5. $\| ef \| = \sup_{\xi \in E_1, \eta \in F_1} |\langle \xi, \eta \rangle|$.

Let now M be a von Neumann algebra and θ an automorphism of M. As in [4] (prop. 1.1.5, p 161) we let $p(\theta)$ be the largest projection $e \in M$, $\theta(e) = e$ such that the reduced automorphism θ^e is inner.

We say that θ is properly outer when $p(\theta) = 0$.

THEOREM 1.2.1. — *Let* M *be a countably decomposable von Neumann algebra and* $\theta \in \operatorname{Aut} M$. *Then* θ *is properly outer if and only if for any non zero projection* $e \in M$ *and any* $\varepsilon > 0$ *there exists a non zero projection* $f \leqq e$ *such that*:

$$\| f\theta(f) \| \leqq \varepsilon.$$

When M is abelian, $M = L^\infty(X, \mu)$ and θ is the transpose of the transformation T of X, theorem 1.2.1 translates to (M, θ) the existence, for each subset E of X, $\mu(E) > 0$, of a subset F of E, $\mu(F) > 0$ such that $TF \cap F = \varnothing$. The non commutative case relies on the following lemmas :

LEMMA 1.2.2. — *Let* M *and* $\theta \in \operatorname{Aut} M$ *be as in* 1.2.1, *Let* $\operatorname{Sp}\theta$ *be the spectrum in the sense of* [3], [4] *of the representation* $n \to \theta^n$ *of* $\mathbf{Z}$ *on* M. *Then if* $-1 \in \operatorname{Sp}\theta$ *there exists for each* $\varepsilon > 0$ *a non zero projection* $e \in M$ *such that* $\| e\theta(e) \| \leqq \varepsilon$.

Proof. — We can assume that M acts in a Hilbert space $\mathcal{H}$ and that $\theta(x) = VxV^*$ for all $x \in M$ and some unitary V in $\mathcal{L}(\mathcal{H})$. Let $x \in M$, $\| x \| = 1$ be such that $\| \theta(x)+x \| \leqq \varepsilon/2 = \delta$. (We use the hypothesis $-1 \in \operatorname{Sp}\theta$ together with [4] 2.3.5.) Let $x = a+ib$, $a = a^*$, $b = b^*$. Then

$$\| \theta(a)+a \| \leqq \delta, \qquad \| \theta(b)+b \| \leqq \delta.$$

As $1 \leqq \| a \| + \| b \|$ we can assume that $\| a \| \geqq 1/2$, so that by a suitable choice of $\alpha = \pm 1$ we see that $\rho = \alpha\, a/\| a \|$ satisfies : $\rho = \rho^*$, $\| \theta(\rho) + \rho \| \leqq 2\delta$, 1 is in the spectrum of ρ and $\| \rho \| = 1$.

Let e be the spectral projection of ρ corresponding to the interval $[1-\delta, 1]$. We know that $e \neq 0$, we now show that $\| e\theta(e) \| \leqq \varepsilon$. Let $E = e\mathcal{H}$, $E_1 = \{\xi \in E, \| \xi \| = 1\}$. For $\xi \in E_1$ we have $\| \rho\xi - \xi \| \leqq \delta$. Let $F = \theta(e)\mathcal{H} = VeV^*\mathcal{H} = VE$, $F_1 = VE_1$. For $\eta = V\xi' \in F_1$ we get:

$$\| \theta(\rho)\eta - \eta \| = \| V\rho V^* V\xi' - V\xi' \| = \| \rho\xi' - \xi' \| \leqq \delta,$$

$$\| \rho\eta + \eta \| \leqq \| \rho + \theta(\rho) \| . \| \eta \| + \| \eta - \theta(\rho)\eta \| \leqq 3\delta.$$

So for $\xi \in E_1$, $\eta \in F_1$ we get:

$$|\langle \xi, \eta \rangle - \langle \rho\xi, \eta \rangle| \leqq \|(\rho-1)\xi\| . \|\eta\| \leqq \delta,$$
$$|\langle \xi, \rho\eta \rangle + \langle \xi, \eta \rangle| < \|\xi\| . \|\rho\eta+\eta\| \leqq 3\delta.$$

So that $|\langle \xi, \eta \rangle| \leqq 2\delta$ which, using 5., shows that $\|e\,\theta(e)\| \leqq \varepsilon$.

Q. E. D.

LEMMA 1.2.3. — *Let* M *and* $\theta \in \operatorname{Aut} M$ *be given. If* $\theta \notin \operatorname{Int} M$ *then for any* $\varepsilon > 0$ *there exists a projection* $f \in M$ *such that* $\|f\theta(f)\| \leqq \varepsilon$ *and* $f \neq 0$.

Proof. — We can assume that $\theta(x) = x$ for all x in the center C of M. For each $n \geqq 1$ let d_n be the largest projection of C such that all non zero subprojections d of d_n, $d \in C$ satisfy: $(\theta^d)^q = \operatorname{Ad} u$ for some unitary $u \in M_d$, $\theta^d(u) = u$ occurs for $q = n$ but no $q \in \{1, \ldots, n-1\}$. If $d_n \neq 0$ for some $n > 1$ we can assume that this d_n is 1. Then let v be an nth root of u in M^θ, so that $\theta = \operatorname{Ad} v.\alpha$, where $\alpha^n = 1$. Choosing a spectral projection $e \neq 0$ of v such that, for some $\lambda \in \mathbf{T}$, $\|ev - \lambda e\| \leqq \varepsilon/4$, we see that the norm distance between θ^e and $\alpha^e \in \operatorname{Aut} M_e$ is smaller than $\varepsilon/2$. So we can assume that $\theta^n = 1$. By construction of d_n we know that $\Gamma(\theta) = \{n\mathbf{Z}\}^\perp$, where Γ is as defined in [4] and [6] (3.3.3). In fact, if $d_n = 1$, with $\theta^n = \operatorname{Ad} u$, let $x = \sum a_m \bigcup_\theta^m$ be an element of $W^*(\theta, M)$ and let us assume that x is in the center of $W^*(\theta, M)$. Then each a_m belongs to M^θ and satisfies $a_m \theta^m(y) = y\,a_m$, for $y \in M$. It follows easily that the center of $W^*(\theta, M)$ is generated by the center of M (it is fixed by θ) and $u^* \bigcup_\theta^n$. By [6], theorem 3.3.2, we get $\Gamma(\theta) = \{n\mathbf{Z}\}^\perp$. If $\theta^n = 1$ we see that θ is minimal periodic and an easy adaptation of [8] (2.6 *a*) shows the existence of a unitary $X \in M$, $X^n = 1$, $\theta(X) = \lambda X$, $\lambda = \exp(i2\pi/n)$. A suitable spectral projection $f \neq 0$, of X will hence satisfy $\theta(f)f = 0$. Now assume $d_n = 0$ for $n \geqq 1$ (θ is not inner). Then the center of $W^*(\theta, M)$ is equal to the center of M and by [6] theorem 3.3.2 we have $\Gamma(\theta) = \mathbf{T}$, $\operatorname{Sp}\theta = \mathbf{T}$ so that lemma 2 applies.

Q. E. D.

LEMMA 1.2.4. — *Let* e, f *be projections in a von Neumann algebra* M *and* $\alpha > 0$.

(*a*) *Assume that for any non zero projections* $e', f' \in M$, $e' \leqq e$, $f' \leqq f$ *one has* $\|e'f\| \geqq \alpha$, $\|ef'\| \geqq \alpha$, *then* $c(e,f) \geqq \alpha(e \vee f)$.

(*b*) *If the support of* $c(e,f)$ *is* $e \vee f$, *then the partial isometry* u *of the polar decomposition of* $e \vee f - (e+f)$ *satisfies*:

$$u = u^*, \qquad u^2 = e \vee f, \qquad ueu^* = f, \qquad ufu^* = e.$$

Proof. — (*a*) We can assume that M is generated by e and f. Let then C be the center of M. Let $\bar{e}$ be the central support of e. If $c(e,f)$ is not larger than $\alpha\bar{e}$, there exists a $\beta > 0$, $\beta < \alpha$ and a non zero projection $d \in C$ such that $dc(e,f) \leqq \beta\,d\bar{e} \neq 0$. We have

$$de \neq 0, \qquad de \leqq e,$$
$$c(de,f) = |de \vee f - de - f| = d\,|e \vee f - e - f| + (1-d)\,|f-f| = dc(e,f) \leqq \beta\,d\bar{e}$$

which contradicts $\|def\| \geqq \alpha$. So $c(e,f) \geqq \alpha\bar{e}$, $c(e,f) \geqq \alpha\bar{f}$ and hence we get (a).

(*b*) The module of $e \vee f-(e+f)$ is $c(e,f)$ so $u = u^*$, $u^2 = e \vee f$ are clear. ueu^* is the projection which is the support of eu^*, hence of $e(e \vee f-(e+f)) = -ef$. But $fef = fc(e,f)^2$ has support f.

Q. E. D.

Proof of theorem 1.2.1. — Assume first that $p(\theta) \neq 0$. Say that $\theta = \mathrm{Ad}\, u$, u unitary in M. Let e be a spectral projection of u, $e \neq 0$, $\| ue-\lambda e \| \leqq 1/4$ for some $\lambda \in \mathbf{T}$. Then the norm distance in Aut M^e between Ad $ue = \theta^e$ and 1 is less than 1/2 so that for any projection $f \leqq e$ one has $\| \theta(f)-f \| \leqq 1/2$ and hence $\| \theta(f)f \| \geqq 1/2$ if $f \neq 0$.

Assume now that $p(\theta) = 0$ and let $e \in \mathrm{M}$ be a non zero projection. Let $\alpha = \operatorname*{Inf}_{f \leqq e,\, f \neq 0} \| f\theta(f) \|$ (where f varies among projections of M).

We assume that $\alpha > 0$ and derive a contradiction. Let $\varepsilon > 0$ such that $(\alpha+1)\varepsilon < \alpha$, and $f \leqq e$, $f \neq 0$ such that $\| f\theta(f) \| \leqq \alpha+\varepsilon$. For any $g \leqq f$, $g \neq 0$ we have $\| g\theta(g) \| \geqq \alpha$ hence $\| f\theta(g) \| \geqq \alpha$ and $\| g\theta(f) \| \geqq \alpha$. So by lemma 4 (*a*) we get $c(f, \theta(f)) \geqq \alpha(f \vee \theta(f))$. As $\| f\theta(f) \| \leqq \alpha+\varepsilon$ it follows that

$$\alpha(f \vee \theta(f)) \leqq c(f, \theta(f)) \leqq (\alpha+\varepsilon) f \vee \theta(f).$$

Let u be the partial isometry of the polar decomposition of $f \vee \theta(f)-f-\theta(f)$, we have by 4 (*b*):

$$u = u^*, \qquad u^2 = f \vee \theta(f), \qquad ufu^* = \theta(f), \qquad u\theta(f)u^* = f$$

and

$$\| \alpha u - uc(f, \theta(f)) \| \leqq \| \alpha(f \vee \theta(f)) - c(f, \theta(f)) \| \leqq \varepsilon$$

$$\| \alpha u - (f \vee \theta(f) - f - \theta(f)) \| \leqq \varepsilon.$$

The automorphism θ' of M_f such that $\theta'(x) = u\,\theta(x)\,u^*$, is outer because $p(\theta) = 0$. So by lemma 3, there exists a projection $g \neq 0$, $g \leqq f$ such that $\| g\theta'(g) \| \leqq \varepsilon$. We have

$$\| gu\theta(g) \| = \| gu\theta(g)u^* \| \leqq \varepsilon$$

and hence:

$$\| g(f \vee \theta(f) - f - \theta(f))\theta(g) \| < \alpha\varepsilon + \varepsilon$$

But $g(f \vee \theta(f))\theta(g) - gf\theta(g) - g\theta(f)\theta(g) = -g\theta(g)$ because $g \leqq f$, and so $\| g\theta(g) \| \leqq \alpha\varepsilon+\varepsilon < \alpha$.

Q. E. D.

By definition an automorphism θ of a von Neumann algebra M is aperiodic iff all its powers θ^n, $n \neq 0$ are properly outer. We now prove the non commutative analogue of the very useful tower theorem of Rokhlin.

THEOREM 1.2.5. — *Let* N *be a finite von Neumann algebra,* τ *a faithful normal trace on* N, $\tau(1) = 1$, *and* θ *an aperiodic automorphism of* N *which preserves* τ.

For any integer n and any $\varepsilon > 0$, *there exists a partition of unity* $(\mathrm{F}_j)_{j=1,\ldots,n}$ *in* N *such that*

$$\| \theta(\mathrm{F}_1)-\mathrm{F}_2 \|_2 \leqq \varepsilon, \quad \ldots, \quad \| \theta(\mathrm{F}_j)-\mathrm{F}_{j+1} \|_2 \leqq \varepsilon, \quad \ldots, \quad \| \theta(\mathrm{F}_n)-\mathrm{F}_1 \|_2 \leqq \varepsilon.$$

As usual we used the notation $\| x \|_2 = \tau(x^* x)^{1/2}$ for $x \in \mathrm{N}$. We first need some technical lemmas:

LEMMA 1.2.6. — *Let* M *be von Neumann algebra,* $n \in \mathbf{N}$ *and* $\varepsilon > 0$, *such that* $n!\,\varepsilon < 1$. *Let* $(f_j)_{j=1,\ldots,n}$ *be a family of n projections of* M *such that* $\| f_j f_k \| \leqq \varepsilon$ *for all* $j \neq k$. *Then there is a family of n pairwise orthogonal projections* $e_j \sim f_j$ *such that* $\| e_j - f_j \| \leqq n!\,\varepsilon$ *for all* $j = 1, \ldots, n$ *and* $\bigvee_1^n e_j = \bigvee_1^n f_j$.

Proof. — Let e, f be projections in M such that $\| ef \| < 1$ then $\| c(e, f) \| < 1$ so that $\mathrm{F} = e \vee f - e$ is equivalent to f and one has

$$\| \mathrm{F} - f \| = \| s(\mathrm{F}, f) \| = \| s(e \vee f - e, f) \| = \| ef \|.$$

Suppose now that we have proven the lemma for $n-1$ projections and take n projections $(f_j)_{j=1,\ldots,n}$ with $\| f_j f_k \| \leqq \varepsilon, j \neq k$; by our induction hypothesis we get $n-1$ projections $e_1, \ldots, e_{n-1}$, pairwise orthogonal, and such that $e_j \sim f_j$, $\| e_j - f_j \| \leqq (n-1)!\,\varepsilon$. So $\| e_j f_n - f_j f_n \| \leqq (n-1)!\,\varepsilon$ for $j = 1, \ldots, n-1$. Hence

$$\| e f_n \| \leqq (n-1)(n-1)!\,\varepsilon + (n-1)\varepsilon \leqq n!\,\varepsilon,$$

where $e = e_1 + \ldots + e_{n-1}$. As $n!\,\varepsilon < 1$, the above argument shows that $e \vee f_n - e$ is a projection, equivalent to f_n, orthogonal to all the e_j' s and such that $\| e_n - f_n \| \leqq n!\,\varepsilon$ and $e \vee e_n = e \vee f_n$.

Q. F. D.

LEMMA 1.2.7. — *Let* N *and* θ *be as in proposition* 1.2.1 *and assume that* $\theta(x) = x$, $x \in$ *Center of* N. *Then for any* $n \in \mathbf{N}$, $n > 1$, *any* $\delta > 0$, *there exists a family* $(f_j)_{j=1,\ldots,n}$ *of n non zero pairwise orthogonal projections of* N *and a unitary* $v \in \mathrm{N}$ *such that*:

$$\| v - 1 \|_1 \leqq \delta\tau(\Sigma f_j), \qquad v\theta(f_j)v^* = f_{j+1}, \qquad j = 1, \ldots, n$$

[*where* $\| x \|_1 = \tau(| x |)$ *for any* $x \in \mathrm{N}$, *and* $f_{n+1} = f_1$].

Proof. — Put $\delta' = \delta/12(n+1)$, choose $m = np$ so large that $2m^{-1/2} \leqq \delta'/2$ and then choose $\varepsilon > 0$ such that $\varepsilon < 1/(m!)$ and $2m\,m!\,\varepsilon \leqq \delta'/2$.

Choose, using the aperiodicity of θ, projections $\mathrm{E}_1, \mathrm{E}_2, \ldots, \mathrm{E}_m$ such that $\mathrm{E}_m \neq 0$, $\mathrm{E}_m \leqq \ldots \leqq \mathrm{E}_1$ and that

$$\| \theta^j(\mathrm{E}_j)\mathrm{E}_j \| \leqq \varepsilon, \qquad j = 1, \ldots, m.$$

As $\mathrm{E}_m \leqq \mathrm{E}_j$ we have:

$$\| \theta^j(\mathrm{E}_m)\mathrm{E}_m \| \leqq \varepsilon, \qquad j = 1, \ldots, m.$$

Put $e = \mathrm{E}_m$. Then we have, for any $i < j$, $i, j \in \{1, \ldots, m\}$ that

$$\| \theta^i(e)\theta^j(e) \| = \| e\,\theta^{(j-i)}(e) \| < \varepsilon.$$

Let $\mathrm{E} = \bigvee_1^m \theta^j(e)$. As $\varepsilon < 1/(m!)$ we can apply lemma 1.2.2 in N_E. It gives a family of m pairwise orthogonal projections $(e_j)_{j=1,\dots,m}$ with $e_j \sim \theta^j(e)$, $j=1,\dots,m$, $e_j \leqq \mathrm{E}$, $j=1,\dots,m$ and

$$\|\theta^j(e)-e_j\| \leqq m!\,\varepsilon \leqq \delta'/4m.$$

Also we have $\sum_{j=1}^{m} \tau(e_j)\tau(\mathrm{E})$, because $\sum_{j=1}^{m} e_j = \mathrm{E}$.

Let $\mathrm{F} = \mathrm{E} \vee \theta(\mathrm{E}) = \mathrm{E} \vee \theta^{m+1}(e)$. Anyway $\tau(\mathrm{E}) \leqq \tau(\mathrm{F}) \leqq 2\tau(\mathrm{E})$. Let $\mathrm{Q} = \mathrm{N}_\mathrm{F}$. For any $j=1,\dots,m$ one has $e_j \leqq \mathrm{E}$ hence $\theta(e_j) \leqq \theta(\mathrm{E}) \leqq \mathrm{F}$, so that $\theta(e_j) \in \mathrm{N}_\mathrm{F}$.

Let $\tau' = (1/\tau(\mathrm{F}))\,\tau$ restricted to N_F. So τ' is a trace on Q whose value on the unit F of Q is equal to 1.

For $q \in [1, +\infty[$ let, for any $x \in \mathrm{Q}$, $\|x\|'_q = (\tau'(|x|^q))^{1/q} = \tau(\mathrm{F})^{-1/q}\|x\|_q$. Note also that the C* norm $\|x\|$ of any $x \in \mathrm{Q}$ is the same as its C* norm as an element of N. Put

$$f_1 = e_1 + e_{n+1} + \dots + e_{n(p-1)+1},$$

$$f_2 = e_2 + e_{n+2} + \dots + e_{n(p-1)+2}, \qquad \dots, \qquad f_n = e_n + e_{n+n} + \dots + e_{np},$$

where $m = np$ as above.

We have $\sum_1^n f_k = \mathrm{E}$, and f_k, $\theta(f_k)$ belong to Q for all k. We want to show that $\|\theta(f_k) - f_{k+1}\|'_2 \leqq \delta'$ for all $k=1,\dots,n$ and $f_{n+1} = f_1$.

For $j=1,\dots,m-1$ we have

$$\|\theta(e_j)-e_{j+1}\| \leqq \|\theta(e_j)-\theta^{j+1}(e)\| + \|\theta^{j+1}(e)-e_{j+1}\| \leqq \delta'/2m.$$

Hence $\|\theta(e_j)-e_{j+1}\|'_2 \leqq \delta'/2m$. Then for $k=1,\dots,n-1$ we get $\|\theta(f_k)-f_{k+1}\|'_2 \leqq \delta'$.

As θ leaves the center of N fixed pointwise, one has $\theta(e) \sim e$ for any projection $e \in \mathrm{N}$. In particular the e_j are pairwise equivalent in Q, $\tau'(e_j) \leqq 1/m$, and $\tau'(\theta(e_j)) \leqq 1/m$. So $\|\theta(e_{np})\|'_2 \leqq m^{-1/2}$, $\|e_1\|'_2 \leqq m^{-1/2}$ and we get $\|\theta(f_n)-f_1\|'_2 \leqq p\,\delta'/2m + 2m^{-1/2} \leqq \delta'$.

The projection $\theta(f_k) \in \mathrm{Q}$ is equivalent to f_k and hence to f_{k+1} in Q. By lemma 1.1.4 we get partial isometries $\mathrm{V}_1, \dots, \mathrm{V}_k, \dots, \mathrm{V}_n$ in Q with $\mathrm{V}_k^*\mathrm{V}_k = \theta(f_k)$, $\mathrm{V}_k\mathrm{V}_k^* = f_{k+1}$ and $\|\mathrm{V}_k - f_{k+1}\|'_2 \leqq 6\,\delta'$. Let $\mathrm{V}_0 \in \mathrm{Q}$ satisfy $\mathrm{V}_0^*\mathrm{V}_0 = \mathrm{F}-\theta(\mathrm{E})$, $\mathrm{V}_0\mathrm{V}_0^* = \mathrm{F}-\mathrm{E}$. We have:

$\tau'(\mathrm{F}-\mathrm{E}) \leqq \tau'(\theta(e_m))$ and hence

$$\|\mathrm{V}_0\|'_2 \leqq \|\theta(e_m)\|'_2 \leqq \delta'/2.$$

Let $\mathrm{V} = \mathrm{V}_0 + \mathrm{V}_1 + \dots + \mathrm{V}_n$. It is by construction a unitary element of Q because $(\mathrm{F}-\theta(\mathrm{E})) + \theta(f_1) + \dots + \theta(f_n) = \mathrm{F}$ and $\mathrm{F}-\mathrm{E}+f_2+\dots+f_{n+1} = \mathrm{F}$. We have $\|\mathrm{V}-\mathrm{F}\|'_2 \leqq 6(n+1)\,\delta' = \delta/2$ and also:

$$\mathrm{V}\,\theta(f_k)\,\mathrm{V}^* = f_{k+1}, \qquad k=1,\dots,n.$$

Put $v = \mathrm{V} + (1-\mathrm{F})$. It is a unitary element of N such that

$$\|v-1\|_1 = \tau(|v-1|) = \tau(|\mathrm{V}-\mathrm{F}|) = \tau(\mathrm{F})\,\tau'(|\mathrm{V}-\mathrm{F}|) \leqq \tau(\mathrm{F})\,\|\mathrm{V}-\mathrm{F}\|'_2.$$

So $\|v-1\|_1 \leqq \tau(\mathrm{F})\,\delta/2 \leqq \delta\tau\left(\sum_{j=1}^{n} f_j\right)$. Finally, for all k,

$$v\,\theta(f_k)\,v^* = (\mathrm{V}+(1-\mathrm{F}))\,\theta(f_k)(\mathrm{V}^*+(1-\mathrm{F})) = \mathrm{V}\,\theta(f_k)\,\mathrm{V}^* = f_{k+1}.$$

Q.E.D.

Proof of theorem 1.2.5. – First assume that $\theta(x) = x$, for $x \in \mathrm{C}$. Fix $n \in \mathbf{N}$ and $\delta > 0$. Then let $\mathscr{R}$ be the set whose elements r are couples $((\mathrm{F}_j)_{j=1,\ldots,n}, \mathrm{V})$ where:

(*a*) $(\mathrm{F}_j)_{j=1,\ldots,n}$ is a family of n pairwise orthogonal, equivalent projections of N.

(*b*) V is a unitary in N with $\|\mathrm{V}-1\|_1 \leqq \delta\,\tau(\sum \mathrm{F}_j)$.

(*c*) $\mathrm{V}\,\theta(\mathrm{F}_j)\,\mathrm{V}^* = \mathrm{F}_{j+1}$, $j = 1, \ldots, n$ (with $\mathrm{F}_{n+1} = \mathrm{F}_1$).

Now we define an ordering on $\mathscr{R}$ by putting, for $r, r' \in \mathscr{R}$ that $r \leqq r'$ if and only if the following are satisfied:

(1) $\mathrm{F}_j \leqq \mathrm{F}'_j$, $j = 1, \ldots, n$;

(2) $\|\mathrm{V}-\mathrm{V}'\|_1 \leqq \delta\,\tau(\sum(\mathrm{F}'_j-\mathrm{F}_j))$.

It is clear that $\leqq$ is an ordering.

We want to prove that $\mathscr{R}$, $\leqq$ is inductive.

Or any totally ordered subset $\mathscr{A}$ of $\mathscr{R}$ the map $r \to \tau(\sum \mathrm{F}_j)$ is an order isomorphism of $\mathscr{A}$ on a subset of $[0,1]$. We just have to show that if $(r_m)_{m\in\mathbf{N}}$ is an increasing sequence of elements of $\mathscr{R}$, there exists an $r \in \mathscr{R}$ such that $r_j \leqq r$, $i \in \mathbf{N}$.

Let $r_m = ((\mathrm{F}_j^m), \mathrm{V}_m)$. Then we have, using 2, that

$$\|\mathrm{V}_m-\mathrm{V}_{m+1}\|_1 \leqq \delta\tau\Big(\sum_j(\mathrm{F}_j^{m+1}-\mathrm{F}_j^m)\Big).$$

Moreover, using (1), there exists projections F_j, pairwise orthogonal, equivalent, such that $\mathrm{F}_j^m \to \mathrm{F}_j$ when $m \to \infty$. We have $\sum_m \|\mathrm{V}_m-\mathrm{V}_{m+1}\|_1 \leqq \delta$. This shows that V_m converges in the L^1 norm to an operator V of norm L^∞ less than 1. We see that V is unitary, because the product is bicontinuous for the L^1 norm on the unit ball of N.

So $\mathrm{V}_m \to \mathrm{V}$ strongly and $\|\mathrm{V}-\mathrm{V}_m\|_1 \leqq \delta\tau(\sum(\mathrm{F}_j-\mathrm{F}_j^m))$ for all $m \in \mathbf{N}$.

It follows that $r = ((\mathrm{F}_j), \mathrm{V})$ satisfies conditions (*a*), (*b*), (*c*), where (*b*) and (*c*) are checked by a continuity argument. Also one checks that $r_m \leqq r$ for all $m \in \mathbf{N}$. By Zorn's lemma, there exists some maximal element r of $\mathscr{R}$. We assume that $r = ((\mathrm{F}_j), \mathrm{V})$ with $\sum \mathrm{F}_j < 1$ and we derive a contradiction.

Put $\mathrm{E} = 1-\sum_{j=1}^{n}\mathrm{F}_j$, and let $\mathrm{P} = \mathrm{N}_\mathrm{E}$. As (*c*) is fulfilled we have $\mathrm{V}\,\theta(\mathrm{E})\,\mathrm{V}^* = \mathrm{E}$ and hence we can consider the restriction θ' of $\mathrm{V}\,\theta(.)\,\mathrm{V}^*$ to $\mathrm{P} = \mathrm{N}_\mathrm{E}$. As θ is aperiodic so is $\mathrm{V}\,\theta(.)\,\mathrm{V}^*$, and hence so is its restriction to N_E – [*see* definition of $p(\theta)$] – Hence lemma 1.2.1 shows the existence of $(f_j)_{j=1,\ldots,n}$, a family of n equivalent pairwise orthogonal projections of N_E and of v, unitary in N_E, such that $v\,\theta'(f_j)\,v^* = f_{j+1}$, $j = 1, \ldots, r$,

$\|v-E\|_1' \leqq \delta\tau'(\sum f_j) \neq 0$, where $\tau' = 1/\tau(E)\,\tau$ on N_E. Put $F_j' = F_j+f_j$, $j = 1, \ldots, n$ and $V' = (v+(1-E))\,V$. Condition (*a*) is clear, for $r' = ((F_j'), V')$. Moreover we have that

$$\|v+(1-E)-1\|_1 = \tau(E)\|v-E\|_1' \leqq \delta\tau(E)\tau'(\sum f_j) = \delta\tau(\sum f_j)$$

and hence $\|V'-V\|_1 \leqq \delta\tau(\sum f_j)$. This shows that r' satisfies (*b*) and (r, r') satisfies (1) and (2). We also have

$$(v+(1-E))\,F_j = F_j(v+(1-E)) = F_j, \qquad j = 1, 2, \ldots, n$$

hence for all j:

$$V'\theta(F_j)V'^* = F_{j+1},\ V'\theta(f_j)V'^* = (v+1-E)\,\theta'(f_j)(v+1-E)^* = f_{j+1}$$

so that r' also satisfies (*c*).

Thus we have shown that for any $n \in N$, any $\delta > 0$, there is a partition of unity $(F_j)_{j=1,\ldots,n}$ in N such that

$$\|\theta(F_j)-F_{j+1}\|_2^2 \leqq 2\|\theta(F_j)-V\theta(F_j)V^*\|_1 \leqq 4\delta, \qquad j = 1, \ldots, n.$$

The conclusion 1.2.1 follows hence, under the hypothesis that θ fixes pointwise the center C of N.

In the general case, let $\bar{\theta}$ = restriction of θ to C. Let then $(c_j)_{j\in N}$ be a partition of unity in C such that for all $j \geqq 1$, $\bar{\theta}(c_j) = c_j$, $(\bar{\theta}^{c_j})^j = 1$ and there is a partition $(c_j^l)_{l=1,\ldots,j}$ of c_j such that $\bar{\theta}(c_j^l) = c_j^{l+1}$, $l = 1, \ldots, j$. While for $j = 0$, $\bar{\theta}^{c_0}$ is aperiodic.

Of course, to prove 1.2.1 we can assume that $c_j = 1$ for some j. The case $j = 1$ is already treated. The case $j = 0$ follows trivially from Rokhlin's theorem [13]. Assume $j > 1$. Put $c^l = c_j^l$, $l = 1, \ldots, j$, $M = N_{c^1}$ and α = restriction of θ^j to M. [It makes sense because $\theta^j(c^1) = c^1$.] As θ is aperiodic on N wesee that α is aperiodic on M. Let $n \in N$, $n > 1$ and $\eta > 0$. As α fixes pointwise the center of M we get, from the above discussion, a partition of unity $(G_s)_{s=1,\ldots,n}$ in M with $\|\alpha(G_s)-G_{s+1}\|_2 \leqq \eta$, $s = 1, \ldots, n$. Put $H_{pj+q} = \theta^q(G_p)$, for $0 < q \leqq j$, $0 \leqq p < n$. Then the H_m, $m = 1, \ldots, nj$ form a partition of unity in N such that $\|\theta(H_m)-H_{m+1}\|_2 \leqq \eta$, $m = 1, \ldots, n-1$.

Put $F_s = H_s+H_{n+s}+\ldots+H_{n(j-1)+s}$, then we see that they form a partition of unity $(F_s)_{s=1,\ldots,n}$ in N and that

$$\|\theta(F_s)-F_{s+1}\|_2 \leqq j\eta, \qquad s = 1, \ldots, n.$$

Q.E.D.

II. Factorization of automorphisms by automorphisms of the hyperfinite factor of type II_1

Let M be a von Neumann algebra. An automorphism θ of M is called centrally trivial when for any centralizing sequence $(x_n)_{n\in N}$ one has:

$$\theta(x_n)-x_n \to 0^* \text{ strongly}, \qquad \text{when} \quad n \to \infty.$$

The set Ct (M) of centrally trivial automorphisms is a normal subgroup of Aut M.

DÉFINITION 2.1.1. — *Let* M *be a factor,* θ *an automorphism of* M, *then we let* $p_a(\theta)$ *be the period of* θ *modulo* Ct M, *in other words* $p_a(\theta) \in \mathbf{N}$ *and for any* $n \in \mathbf{Z}$ *one has* $\theta^n \in \mathrm{Ct}\,\mathrm{M}$ *iff* n *is a multiple of* $p_a(\theta)$.

In particular $p_a(\theta) = 0$ means that no nontrivial power of θ is centrally trivial.

Now let ω be a free ultrafilter on $\mathbf{N}$. If $(x_n)_{n \in \mathbf{N}}$ is an ω-centralizing sequence in M, then so is the sequence $(\theta(x_n))_{n \in \mathbf{N}}$. Also the ideal $\mathscr{I}_\omega$ of proposition 1.1.1 is globally invariant under this transformation. So there is a unique automorphism θ_ω of M_ω such that if $(x_n)_{n \in \mathbf{N}}$ represents $x \in \mathrm{M}_\omega$ then $(\theta(x_n))_{n \in \mathbf{N}}$ represents $\theta_\omega(x) \in \mathrm{M}_\omega$.

The map $\theta \to \theta_\omega$ is an homomorphism from Aut M to Aut M_ω, and in fact each $\omega \in \beta\,\mathbf{N} \setminus \mathbf{N}$ defines in this way a functor $\mathrm{M} \to \mathrm{M}_\omega$, $\theta \to \theta_\omega$.

PROPOSITION 2.1.2. — *Let* M *be a factor with separable predual,* θ *an automorphism of* M *and* ω *a free ultrafilter on* $\mathbf{N}$.

$$(\theta \notin \mathrm{Ct}\,\mathrm{M}) \Leftrightarrow (\theta_\omega \neq 1) \Leftrightarrow (\theta_\omega \text{ is properly outer}).$$

Proof. — We just have to prove that if $\theta \notin \mathrm{Ct}\,\mathrm{M}$ then θ_ω is properly outer. The other implications are easy.

By hypothesis, letting φ be a faithful normal state on M, there is a centralizing sequence $(x_n)_{n \in \mathbf{N}}$ in M such that, for some $\delta > 0$

$$\|\theta(x_n) - x_n\|_\varphi^\# \geqq \delta \quad \text{for all } n \in \mathbf{N}.$$

We have to show that the only $a \in \mathrm{M}_\omega$ such that $\theta_\omega(x)\, a = ax$ for any $x \in \mathrm{M}_\omega$, is $a = 0$.

Let $(a_n)_{n \in \mathbf{N}}$ be a representing sequence for a and $\varepsilon^2 = \tau_\omega(a^* a)$. Let M act in $\mathscr{H}$ with $\langle x\xi, \xi \rangle = \varphi(x)$ for all $x \in \mathrm{M}$. We shall assume that $\varepsilon > 0$ and derive a contradiction. We can take a_n with $\|a_n \xi\| \geqq \varepsilon$, for all n.

As any weak limit of $(\theta(x_m) - x_m)^* (\theta(x_m) - x_m)$ is larger than δ^2 we can for each n find an integer $m = m(n)$ such that

$$\|(\theta(x_m) - x_m)\, a_n \xi\| \geqq \frac{1}{2}\delta\varepsilon, \qquad \|[x_m, a_n]\xi\| \leqq \frac{1}{n}$$

$$\|[x_m, \psi_j]\| \leqq \frac{1}{n}, \qquad j = 1, \ldots, n$$

where $\psi_1, \ldots, \psi_n, \ldots$ is a preassigned norm dense sequence in M_*. Then the sequence $(\mathrm{X}_n)_{n \in \mathbf{N}}$, $\mathrm{X}_n = x_{m(n)}$ is ω-centralizing and the corresponding $\mathrm{X} \in \mathrm{M}_\omega$ commutes with a, while $(\theta_\omega(\mathrm{X}) - \mathrm{X})\, a \neq 0$ which is a contradiction.

Q. E. D.

THEOREM 2.1.3. — *Let* M *be a factor with separable predual,* θ *an automorphism of* M *with* $p_a(\theta) = 0$ *and* ω *a free ultrafilter on* $\mathbf{N}$. *Then* θ_ω *is a* stable *automorphism: for any* u *unitary in* M_ω *there is a unitary* $v \in \mathrm{M}_\omega$ *such that*

$$\theta_\omega(v) = uv.$$

LEMMA 2.1.4. — *Let* M, θ *and* ω *be as in theorem* 2.1.3. *Then for any* $n \in \mathbf{N}$, $n > 1$, *and any countable subset* $(x^j)_{j \in \mathbf{N}}$ *of* M_ω *there exists a partition of unity* $(F_k)_{k=1,\ldots,n}$ *in* M_ω *such that each* F_k *commutes with all* x^j *and that* $\theta_\omega(F_k) = F_{k+1}$, $k = 1, \ldots, n$, *where* $F_{n+1} = F_1$.

Proof. — By theorem 1.2.5. and proposition 2.1.2., we can for each $\delta > 0$ find a partition of unity $(\tilde{F}_j)_{j=1,\ldots,n}$ in M_ω such that $\| \theta_\omega(\tilde{F}_j) - \tilde{F}_{j+1} \|_2 < \delta$ for $j = 1, \ldots, n$, where $\| \quad \|_2$ is the L^2 norm corresponding to τ_ω. Let φ be a faithful normal state on M, and $(\psi_\nu)_{\nu \in \mathbf{N}}$ be a dense sequence in M_*.

By induction on $\nu \in \mathbf{N}$ we can construct a sequence of partitions of unity $(F_j^\nu)_{j=1,\ldots,n}$ in M, such that for all $\nu \in \mathbf{N}$.

(*a*) $\| [\psi_l, F_j^\nu] \| \leqq 1/\nu$, $l = 1, \ldots, \nu$, $j = 1, \ldots, n$.

(*b*) $\| [x_\nu^k, F_j^\nu] \|_\varphi^\# \leqq 1/\nu$, $k = 1, \ldots, \nu$, $j = 1, \ldots, n$.

(*c*) $\| \theta(F_j^\nu) - F_{j+1}^\nu \|_\varphi^\# \leqq 1/\nu$, $j = 1, \ldots, n$.

Where $(x_\nu^k)_{\nu \in \mathbf{N}}$ is a representing sequence for x^k. (To get $(F_j^\nu)_{j=1,\ldots,n}$, apply the above discussion with $2\delta < 1/\nu$ and get $(\tilde{F}_j)_{j=1,\ldots,n}$. Then by proposition 1.1.3 choose a representing sequence $(\tilde{F}_j^m)_{m \in \mathbf{N}}$ for the $\tilde{F}_j$, such that for each m, $(\tilde{F}_j^m)_{j=1,\ldots,n}$ is a partition of unity in M. Take then m such that $(\tilde{F}_j^m)_{j=1,\ldots,n}$ satisfies conditions (*a*), (*b*), (*c*). Put $F_j^\nu = \tilde{F}_j^m$).

Then by (*a*) $(F_\nu^j)_{\nu \in \mathbf{N}}$ is for each j a centralizing sequence of projections of M. Let $(F_j)_{j=1,\ldots,n}$ be the corresponding partition of unity in M_ω. By (*b*) it commutes with all x^k, and by (*c*) we have $\theta_\omega(F_j) = F_{j+1}$, $j = 1, \ldots, n$.

Q. E. D.

Proof of Theorem 2.1.3. — Let u be a unitary in M_ω. Let $\varepsilon > 0$ and take $n \in \mathbf{N}$ such that $2n^{-1/2} \leqq \varepsilon$. Let $(F_j)_{j=1,\ldots,n}$ be a partition of unity in the relative commutant of u and such that $\theta_\omega(F_j) = F_{j+1}$, $j = 1, \ldots, n$. We have $\tau_\omega(F_j) = 1/n$ for all j, so that $\| F_j \|_2 \leqq \varepsilon/2$, $j = 1, \ldots, n$. Put

$$v_0 = F_n, \qquad v_1 = \theta_\omega^{-1}(uv_0), \ \ldots, \ v_{k+1} = \theta_\omega^{-1}(uv_k), \ \ldots, \ v_{n-1} = \theta_\omega^{-1}(uv_{n-2}).$$

We have, by induction, $v_j v_j^* = v_j^* v_j = F_{n-j}$, because assuming this true for $j = k$ we get :

$$v_{k+1} v_{k+1}^* = \theta_\omega^{-1}(uv_k v_k^* u^*) = \theta_\omega^{-1}(u F_{n-k} u^*) = \theta_\omega^{-1}(F_{n-k}) = F_{n-(k+1)},$$
$$v_{k+1}^* v_{k+1} = \theta_\omega^{-1}(v_k^* v_k) = \theta_\omega^{-1}(F_{n-k}) = F_{n-(k+1)}.$$

So the v_k are normal partial isometries with pairwise orthogonal supports, their sum $V = \sum_0^{n-1} v_k$ is a unitary in M_ω and we have:

$$\theta_\omega(V) = \sum_{k=0}^{n-1} \theta_\omega(v_k) = \theta_\omega(v_0) + \sum_{k=0}^{n-2} uv_k \quad [\text{because } \theta_\omega(v_{k+1}) = uv_k],$$

$$uV = \sum_{k=0}^{n-1} uv_k = \sum_{k=0}^{n-2} uv_k + uv_{n-1}.$$

As

$$\|\theta_\omega(v_0)\|_2 \leqq \varepsilon/2 \qquad \text{and} \qquad \|uv_{n-1}\|_2 \leqq \varepsilon/2 \quad (\text{because } \|F_j\|_2 < \varepsilon/2)$$

we see that $\|\theta_\omega(V)-uV\|_2 \leqq \varepsilon$.

We now repeat the same procedure as in the above lemma. Let $\varphi \in M_*^+$, $\varphi(1) = 1$, φ faithful, $(\psi_\nu)_{\nu \in N}$ be a dense sequence in M_*. Let $(u_\nu)_{\nu \in N}$ be a representing sequence of unitaries for u. For each $\nu \in N$, let V^ν be a unitary in M_ω such that $\|\theta_\omega(V^\nu)-uV^\nu\|_2 \leqq 1/2\nu$ and let (V_j^ν) be a representing sequence of unitaries for V^ν. Then there is for each ν a subset A_ν of N whose closure in β N contains ω, such that

(*a*) $\|[\psi_k, V_j^\nu]\| \leqq 1/\nu$, $k = 1, \ldots, \nu$, $j \in A_\nu$.

(*b*) $\|\theta(V_j^\nu)-u_j V_j^\nu\|_\varphi^\# \leqslant 1/\nu$, $j \in A_\nu$.

Choose the A_ν decreasing and with $N \cap (\bigcap_\nu A_\nu) = \varnothing$, and define $v_j = V_j^{\nu(j)}$ where $j \in A_{\nu(j)} \setminus A_{\nu(j)+1}$ determines $\nu(j)$. By condition (*a*) and the fact that $\nu(j) \to \infty$ when $n \to \omega$ we see that $(v_j)_{j \in N}$ is an ω-centralizing sequence. In the same way condition (*b*) shows that $\|\theta(v_j)-u_j v_j\|_\varphi^\# \to 0$ when $j \to \omega$ so that the element of M_ω represented by $(v_j)_{j \in N}$ satisfies $\theta_\omega(v) = uv$.

Q. E. D.

2.2 Factorizations of M by the hyperfinite factor of type II_1. — In this section we extend results of McDuff [11] and Araki [2]. We apply them to the group of automorphisms of factors. As always Aut M is gifted with the topology of pointwise norm convergence in the predual M_* of M.

Theorem 2.2.1. — *Let* M *be a factor with separable predual then the following are equivalent*, (*where* $\omega \in \beta N/N$).

(*a*) M *is isomorphic to* $M \otimes R$ (R *the hyperfinite* II_1 *factor*).

(*b*) $\overline{\text{Int}}\, M/\text{Int}\, M$ *is not abelian.*

(*c*) $\overline{\text{Int}}\, M \not\subset \text{Ct}\, M$.

(*d*) M_ω *is not abelian.*

(*e*) M_ω *is a von Neumann algebra of type* II_1.

Proof. — (*d*) $\Rightarrow$ (*e*). Let $\varphi \in M_*^+$, $\varphi(1) = 1$. Choose ω-centralizing sequences $(x_n)_{n \in N}$, $(y_n)_{n \in N}$ such that $\|[x_n, y_n]\|_\varphi^\#$ does not tend to 0 when $n \to \omega$, let $\lim_{n \to \omega} \|[x_n, y_n]\|_\varphi^\# = 2\alpha > 0$.

Let $f \in M_\omega$ be a non zero projection. We just have to show that $(M_\omega)_f$ is not abelian. Let $\beta = (\tau_\omega(f))^{1/2}$ and $(f_n)_{n \in N}$ representing f as in proposition 1.1.3 (*a*) with $\varphi(f_n) \geqq \beta^2$ for all $n \in N$. Let $(\psi_\nu)_{\nu \in N}$ be a dense sequence in M_*. Then for each $n \in N$ there is a $k_n \in N$ such that:

$$(1) \qquad \|[x_{k_n}, \psi_j]\| < \frac{1}{n}, \qquad \|[y_{k_n}, \psi_j]\| < \frac{1}{n}, \qquad j = 1, \ldots, n$$

$$(2) \qquad \|[f_n x_{k_n} f_n, f_n y_{k_n} f_n]\|_\varphi^\# \geqq \frac{\alpha\beta}{2}.$$

(Because when $k \to \omega$ one has $|[f_n x_k f_n, f_n y_k f_n]|^2 - |[x_k, y_k]|^2 f_n$ which converges strongly to 0 because $(x_k)_{k \in \mathbf{N}}$ and $(y_k)_{k \in \mathbf{N}}$ are ω-central in particular. One then uses proposition 1.1.2 to compute

$$\lim_{k \to \omega} \varphi(|[x_k, y_k]|^2 f_n) = \varphi(f_n) \tau_\omega(|[x, y]|^2) \geqq (\beta\alpha)^2).$$

Let X (resp. Y) be represented by $(x_{k_n})_{n \in \mathbf{N}}$ (resp. $(y_{k_n})_{n \in \mathbf{N}}$) then $[f \mathrm{X} f, f \mathrm{Y} f] \neq 0$ which gives the conclusion.

$(e) \Rightarrow (a)$ let $(e_{ij})_{i\,j=1,2}$ be a system of 2×2 matrix units in M_ω. Let $(e_{ij}^\nu)_{\nu \in \mathbf{N}}$ be a representing sequence as in proposition 1.1.3 (d). For each ν, $(e_{ij}^\nu)_{i,j=1,2}$ is a system of 2×2 matrix units in M. Moreover, for any $\psi_1, \ldots, \psi_q \in \mathrm{M}_*$ and $\varepsilon > 0$ we can find ν such that:

$$\| [\psi_j, e_{21}^\nu] \| < \varepsilon, \qquad j = 1, \ldots, q.$$

But this shows that M has property $\mathrm{L}'_{1/2}$ of Araki [2] and by [2], theorem 1.3, that M is isomorphic to $\mathrm{M} \otimes \mathrm{R}$.

$(a) \Rightarrow (b)$. We have to show that there are automorphisms of $\mathrm{M} \otimes \mathrm{R}$, say α, β, which are approximately inner, while $\alpha\beta\alpha^{-1}\beta^{-1}$ is not inner. Choosing α and β of the form $1_\mathrm{M} \otimes \alpha_0$, $1_\mathrm{M} \otimes \beta_0$ shows that it is enough to do it for R which is easy.

$(c) \Rightarrow (d)$. We assume that (d) is not true so that M_ω is abelian. As M_* is separable it follows that for any faithful normal state φ on M and $\varepsilon > 0$ there are elements $\psi_1, \ldots, \psi_q$ of M_* and a $\delta > 0$ such that:

$$(x, y \in \mathrm{M}, \|x\| \leqq 1, \|y\| \leqq 1, \|[x, \psi_j]\| \leqq \delta, \|[y, \psi_j]\| \leqq \delta, \forall j)$$
$$\Rightarrow \quad (\|[x, y]\|_\varphi^\# < \varepsilon).$$

Let $\theta \in \overline{\mathrm{Int}}\, \mathrm{M}$, we shall show that $\theta \in \mathrm{Ct}\, \mathrm{M}$.

With φ, ψ_j, δ, ε as above, let

$$\mathscr{V} = \{\alpha \in \mathrm{Aut}\, \mathrm{M}, \|\psi_j . \alpha - \psi_j\| < \delta \text{ for all } j\}.$$

For any $\alpha \in \mathrm{Int}\, \mathrm{M} \cap \mathscr{V}$ we have

$$(x \in \mathrm{M}, \|x\| \leqq 1, \|[x, \psi_j]\| \leqq \delta, j = 1, \ldots, q) \quad \Rightarrow \quad \|\alpha(x) - x\|_\varphi \leqq \varepsilon$$

(because $\alpha = \mathrm{Ad}\, u$ and $\|[u^*, x]\|_\varphi^\# \leqq \varepsilon$).

So this is still true for any $\alpha \in \overline{\mathrm{Int}}\, \mathrm{M} \cap \mathscr{V}$. Now write $\theta = \alpha . \mathrm{Ad}\, \mathrm{W}$ with $\alpha \in \mathscr{V}$. Choose $\psi_{q+1}, \ldots, \psi_r$ in M_* and $\delta' \leqq \delta$ such that

$$(y \in \mathrm{M}, \|y\| \leqq 1, \|[\psi_j, y]\| \leqq \delta', j = q+1, \ldots, r) \quad \Rightarrow \quad \|\alpha(\mathrm{W} y \mathrm{W}^*) - \alpha(y)\|_\varphi^\# \leqq \varepsilon.$$

(We use the fact that all centralizing sequences are central.)

Then for any $x \in M$, $\|x\| \leqq 1$, $\|[x, \psi_j]\| \leqq \delta'$, $j = 1, \ldots, r$ one has

$$\|\theta(x)-x\|_\varphi \leqq \|\alpha(W x W^*)-\alpha(x)\|_\varphi + \|\alpha(x)-x\|_\varphi \leqq 2\varepsilon.$$

This shows that $\theta \in \mathrm{Ct}\, M$.

Q. E. D.

(*b*) $\Rightarrow$ (*c*) Follows From:

LEMMA 2.2.2. — *Let* M *be a von Neumann algebra with separable predual. Then for any* $\theta \in \mathrm{Ct}\, M$, *any* $\alpha \in \overline{\mathrm{Int}}\, M$, $\varepsilon(\theta)$ *commutes with* $\varepsilon(\alpha)$.

Proof. — As θ is centrally trivial, there is for any $n \in N$ a neighborhood $\mathscr{V}_n$ of 1 in Aut M such that (u unitary in M, $\mathrm{Ad}\, u \in \mathscr{V}_n$) $\Rightarrow \|\theta(u)-u\|^{\sharp}_{\varphi.\alpha^{-1}} \leqq 2^{-n}$ and $\|\theta(u)-u\|^{\sharp}_{\varphi.\theta.\alpha^{-1}.\theta^{-1}} \leqq 2^{-n}$.

Let $(\mathscr{W}_n)_{n \in N}$ be a decreasing basis of neighborhoods of α in Aut M such that $\mathscr{W}_n \mathscr{W}_n^{-1} \subset \mathscr{V}_n$, and $\beta \in \mathscr{W}_n \Rightarrow \|\varphi.\beta^{-1}-\varphi.\alpha^{-1}\| \leqq 2^{-2n}$. Let u_n be for each $n \in N$, a unitary in M such that $\mathrm{Ad}\, u_n \in \mathscr{W}_n$. We have $\theta\alpha\theta^{-1} = \lim_{n \to \infty} \mathrm{Ad}\, \theta(u_n)$ so we just have to prove that the sequence $u_n^* \theta(u_n)$ converges * strongly to a unitary of M. Let $v_n = u_{n+1} u_n^*$ for all $n \in N$, so that $v_n \in \mathscr{W}_n \mathscr{W}_n^{-1}$ for all n. We get then $\|\theta(v_n)-v_n\|^{\sharp}_{\varphi.\alpha^{-1}} \leqq 2^{-n}$ and hence

$$\|\theta(v_n^*) v_n - 1\|_{\varphi.\mathrm{Ad}\, u_n^{-1}} \leqq 2.2^{-n}+2^{-n} = 3.2^{-n}$$

because $\beta = \mathrm{Ad}\, u_n$ belongs to $\mathscr{W}_n$.

So

$$\|\theta(v_n^*) v_n u_n - u_n\|_\varphi \leqq 3.2^{-n}$$

and

$$\|\theta(u_{n+1}^*) u_{n+1} - \theta(u_n^*) u_n\|_\varphi = \|\theta(u_n^*)\theta(v_n^*) v_n u_n - \theta(u_n^*) u_n\|_\varphi \leqq 3.2^{-n}$$

Also, using

$$\|\theta(v_{n+1})-v_{n+1}\|_{\varphi.\theta.\alpha^{-1}.\theta^{-1}} \leqq 2^{-n}$$

and

$$\|\varphi.\mathrm{Ad}\,\theta(u_n^{-1})-\varphi.\theta.\alpha^{-1}.\theta^{-1}\| \leqq 2^{-n}$$

one gets

$$\|u_{n+1}^* \theta(u_{n+1}) - u_n^* \theta(u_n)\|_\varphi \leqq 3.2^{-n}.$$

This shows that $u_n^* \theta(u_n)$ converges * strongly to a unitary X such that $\mathrm{Ad}\, X = \alpha^{-1}.\theta.\alpha.\theta^{-1}$.

Q. E. D.

Let M be a factor. We now compare modulo $\overline{\mathrm{Int}}\, M$ some factorizations of M as a tensor product $M = M_1 \otimes R_1$, R_1 hyperfinite factor of type II_1. We say for short that a subfactor A of M factorizes M when the equality $\pi(x \otimes y) = xy$, $x \in A$, $y \in A' \cap M$ defines an isomorphism of $A \otimes A'_M$ onto M. The factorizations described here are the infinite ones. We shall deal later with the finite ones.

PROPOSITION 2.2.3. — *Let* M *be a factor with separable predual,* A, B *subfactors of* M, *hyperfinite of type* II_1, *and factorizing* M. *Then if* A'_M *and* B'_M *are isomorphic to* M, *there is a* $\sigma \in \overline{\mathrm{Int}}\, M$ *such that* $\sigma(A) = B$.

Proof. — Let us first reduce the problem to the construction of a triple (C, D, σ) where $C \subset A$ is a subfactor of A, factorizing A and isomorphic to A, where D has the same relations with B and $\sigma \in \overline{\mathrm{Int}}\, M$ satisfies $\sigma(C) = D$.

If such a triple is constructed, let R be a subfactor of A'_M, isomorphic to A, factorizing A'_M. In M, R and A generate a subfactor that we can identify with $R \otimes A$ because A factorizes M. There is an automorphism of this subfactor which carries C on A. Extend this automorphism to an $\alpha \in \mathrm{Aut}\, M$ such that $\alpha(x) = x$, $\forall x \in R' \cap A' \cap M$. As $R \otimes A$ factorizes M, this is possible and moreover $\alpha \in \overline{\mathrm{Int}}\, M$ because any automorphism of $R \otimes A$ is approximately inner. In the same way one constructs a $\beta \in \overline{\mathrm{Int}}\, M$ such that $\beta(D) = B$, the conclusion follows. To get C and D we shall start from a generating pairwise commuting sequence $((e^k_{ij})_{i,j=1,2})_{k\in \mathbf{N}}$ (resp. f^k_{ij}) of matrix units in A (resp. B).

Let $(\psi_j)_{j\in\mathbf{N}}$ be a dense sequence in M_*.

We build by induction a sequence $(n_\nu)_{\nu\in\mathbf{N}}$ of integers and $(u_\nu)_{\nu\in\mathbf{N}}$ of unitaries of M such that, for all ν, with $v_\nu = u_\nu \ldots\ldots u_1$, one has:

(*a*) u_ν commutes with $f^{n_1}_{ij}, \ldots, f^{n_{\nu-1}}_{ij}$.

(*b*) $v_\nu e^{n_k}_{ij} v^*_\nu = f^{n_k}_{ij}$, $k = 1, \ldots, \nu$.

(*c*) $\| \psi_j . \mathrm{Ad}\, v_{\nu+1} - \psi_j . \mathrm{Ad}\, v_\nu \| \leqq 2^{-\nu}$,
$\| \psi_j . \mathrm{Ad}\, v^{-1}_{\nu+1} - \psi_j . \mathrm{Ad}\, v^{-1}_\nu \| \leqq 2^{-\nu}$, $j = 1, \ldots, \nu$.

Letting C (resp. D) be the subfactor of A generated by the $e^{n_\nu}_{ij}$ (resp. $f^{n_\nu}_{ij}$) and $\sigma = \lim_{\nu\to\infty} \mathrm{Ad}\, v_\nu$, it is then clear that, by (*c*), σ makes sense, and, by (*b*), that $\sigma(e^{n_\nu}_{ij}) = f^{n_\nu}_{ij}$ for all i, j, ν so that $\sigma(C) = D$.

Assume n_k and u_k are constructed for $k < \nu$. Then let P be the commutant in M of the $f^{n_k}_{ij}$, $i, j = 1, 2$, $k = 1, \ldots, \nu-1$. As $v_{\nu-1} e^{n_k}_{ij} v^*_{\nu-1} = f^{n_k}_{ij}$ for $k = 1, \ldots, \nu-1$, we see that for $n > n_\nu$ we have $v_{\nu-1} e^n_{ij} v^*_{\nu-1} \in P$ and of course $f^n_{ij} \in P$. Let then ω be a free ultrafilter and (e_{ij}), (f_{ij}) be the systems of matrix units in P_ω corresponding to the ω-centralizing sequences $(v_{\nu-1} e^n_{ij} v^*_{\nu-1})_{n\in\mathbf{N}}$, $(f^n_{ij})_{n\in\mathbf{N}}$. Using a partial isometry $u \in P_\omega$ with $u^* u = e_{11}$, $uu^* = f_{11}$ and 1.1.3 (*b*) we construct an ω-centralizing sequence $(W_n)_{n\in\mathbf{N}}$ of unitaries of P such that

$$W_n v_{\nu-1} e^n_{ij} v^*_{\nu-1} W^*_n = f^n_{ij} \quad \text{for all } n \in \mathbf{N} \text{ and } i, j = 1, 2.$$

It is then clear that for some $n = n_\nu$ and $u_\nu = W_n$ the conditions (*a*), (*b*), (*c*) are realised.

Q. E. D.

2.3. *Proof of Theorem* 1. — In this section we prove a more precise form of theorem 1 — the notations R, s_p, $p \in \mathbf{N}$ are as in the introduction.

THÉORÈME 2.3.1. — *Let* M *be a factor with separable predual, isomorphic to* $M \otimes R$. *Let* $p \in \mathbf{N}$ *and* $\theta \in \operatorname{Aut} M$. *Then the following conditions are equivalent:*

(*a*) $p_a(\theta) = 0$ *modulo* p.

(*b*) $\theta \otimes s_p$ *is outer conjugate to* θ.

(*c*) *For any* $\varphi \in M_*^+$, *any* $\delta > 0$, *there is a unitary* $P \in M$ *such that* $\| P-1 \|_\varphi^\sharp < \delta$ *and that* ${}_P\theta = \operatorname{Ad} P.\theta$ *is conjugate to* ${}_P\theta \otimes s_p$.

COROLLARY 2.3.2. — *Let* M *be a factor with separable predual. If* $\varepsilon(\overline{\operatorname{Int} M})$ *is not abelian, one has* $\varepsilon(\operatorname{Ct} M) = \varepsilon(\overline{\operatorname{Int} M})'$ (1).

Proof. — We know by lemma 2.2.2 that in general $\varepsilon(\operatorname{Ct} M) \subset \varepsilon(\overline{\operatorname{Int} M})'$. Moreover by theorem 2.2.1 that M is isomorphic to $M \otimes R$. Let $\theta \in \operatorname{Aut} M$, $p_a(\theta) \neq 1$. We have to show that there is an $\alpha \in \overline{\operatorname{Int} M}$ with $\varepsilon(\theta)\varepsilon(\alpha) \neq \varepsilon(\alpha)\varepsilon(\theta)$. By theorem 2.3.1 we can assume that M is of the form $N \otimes R$ and $\theta = \theta_1 \otimes s_p$ where $p = p_a(\theta) \neq 1$. Then let $\alpha_0 \in \operatorname{Aut} R$ be such that $s_p\,\alpha_0\,s_p^{-1}\,\alpha_0^{-1}$ is not inner as an automorphism of R. As s_p is explicit α_0 is easy to construct. We have $\alpha = 1_N \otimes \alpha_0 \in \overline{\operatorname{Int} M}$ and $\theta\alpha\theta^{-1}\alpha^{-1}$ is not inner.

Q. E. D.

We need some lemmas before starting the proof of 2.3.1.

LEMMA 2.3.3. — *Let* $p > 1$, $\lambda \in \mathbf{T}$, Q *be a von Neumann algebra of type* II_1 *and* $\alpha \in \operatorname{Aut} Q$. *Assume that* 1° α *is stable* (*as in* 2.1.3) *or* 2° α^q *is properly outer for* $1 \leqq q < n$ *and* $\alpha^n = 1$, $\lambda^n = 1$. *Then there is a system of matrix units* $(f_{kl})_{k,l=1,\ldots,p}$ *in* Q *with* $\alpha(f_{kl}) = \lambda^{k-l} f_{kl}$ *for* $k, l = 1, \ldots, p$.

Proof. — Assume 1° and let $(e_{ij})_{i,j=1,\ldots,p}$ be matrix units in Q. We have $pe_{11}^\natural = 1$ where $\natural$ is the canonical center valued trace on Q ([11], Th. 2, p. 249) hence $(\alpha(e_{11}))^\natural = e_{11}^\natural$ and there exists a partial isometry W, such that $W^*W = e_{11}$, $WW^* = \alpha(e_{11})$. Put $V = \sum \alpha(e_{j1})\,W\,e_{1j}$ then $\alpha(x) = V x V^*$ for any element x of the subfactor K generated by $(e_{ij})_{i,j=1,\ldots,p}$. Let $U = \sum_{k=1}^{p} \lambda^k e_{kk}$ then we have $UV^*\,\alpha(e_{ij})\,VU^* = \lambda^{i-j} e_{ij}$ $i, j = 1, \ldots, p$. Put $u = UV^*$, and as α is stable take v, unitary in Q, such that $v^*\alpha(v) = u$. We get $(\operatorname{Ad} v)^{-1}.\alpha.\operatorname{Ad} v = \operatorname{Ad}(UV^*).\alpha$ and as a conjugate of α satisfies the conclusion of 2.3.3, so does the automorphism α.

Assume now that for some $n > 0$, $\alpha^n = 1$ and α is properly outer for $q = 1, \ldots, n-1$. Then the corresponding action $q \to \alpha^q$ of $\mathbf{Z}/n$ on Q is stable ([6], 3.2.16), and the fixed point subalgebra Q^α is a von Neumann algebra of type II_1 ([6], 3.2.15). So let $(e_{ij})_{i,j=1,\ldots,p}$ be a system of matrix units in Q^α. Put $U = \sum \lambda^k e_{kk}$ where λ is as above. Clearly $\operatorname{Ad} U.\alpha$ satisfies the conclusion of 2.3.3, moreover $(\operatorname{Ad} U.\alpha)^n = 1$ because Ad U commutes with α and $U^n = 1$. So $\operatorname{Ad} U.\alpha$ is conjugate to α because $\operatorname{Ad} U.\alpha$ defines an action of $\mathbf{Z}/n$ on Q which is outer conjugate and hence conjugate to the stable action defined by α. As above this ends the proof.

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(1) ε is the quotient map $\operatorname{Aut} M \xrightarrow{\varepsilon} \operatorname{Out} M$ and the prime means the commutant in Out M.

LEMMA 2.3.4. — *Let* M *be a factor as in* 2.3.1, *and* p *an integer*, $\theta \in \operatorname{Aut} M$, $\lambda \in \mathbf{C}$, $|\lambda| = 1$, $\lambda^{p_a(\theta)} = 1$. *Then for any* $\psi_1 \ldots \psi_q \in M_*$ *and any faithful normal state* $\varphi \in M_*$, *any* $\varepsilon > 0$, *there exists a unitary* $P \in M$ *and a system of* $p \times p$ *matrix units* $(e_{ij})_{i,j=1,\ldots,p}$ *in* M *satisfying the following conditions:*

(*a*) $\| [\psi_l, e_{ij}] \| < \varepsilon$, $l = 1, \ldots, q$ *and* $i, j \in \{1, \ldots, p\}$.

(*b*) $(\operatorname{Ad} P.\theta)(e_{ij}) = \lambda^{i-j} e_{ij}$, $i, j \in \{1, \ldots, p\}$.

(*c*) $\| P-1 \|_\varphi^\# < \varepsilon$.

Proof. — Let ω be a free ultrafilter on N and put $Q = M_\omega$. By Theorem 2.2.1, Q is of type II_1. Let $\alpha = \theta_\omega$ then either $p_a(\theta) = 0$ and then by theorem 2.1.3 α is stable or $p_a(\theta) = n \neq 0$ and then by proposition 2.1.2, for each $q \in \{1, \ldots, n-1\}$ one knows that α^q is properly outer, that $\alpha^n = 1$ and that $\lambda^n = 1$ by hypothesis. Hence we can apply 2.3.3 and get a system $(f_{ij})_{i,j=1,\ldots,p}$ of $p \times p$ matrix units in M_ω such that $\theta_\omega(f_{ij}) = \lambda^{i-j} f_{ij}$, $i, j \in \{1, \ldots, p\}$.

Let (prop. 1.1.3) $(f_{ij}^k)_{k\in\mathbf{N}}$ be a system of representing sequences, where $(f_{ij}^k)_{k\in\mathbf{N}}$ represents f_{ij} and for each k, $(f_{ij}^k)_{i,j}$ is a system of $p \times p$ matrix units in M.

For each k, f_{11}^k is necessarily equivalent to $\theta(f_{11}^k)$ (because $(\theta(f_{ij}^k))$ is also a system of $p \times p$ matrix units) and, as $\theta_\omega(f_{11}) = f_{11}$ we get (lemma 1.1.4) a sequence $(u_k)_{k\in\mathbf{N}}$ of partial isometries such that $u_k^* u_k = f_{11}^k$, $u_k u_k^* = \theta(f_{11}^k)$ and that $u_k - f_{11}^k \xrightarrow[k\to\omega]{} 0$ * strongly. Put $v_k = \sum_{j=1}^{p} \lambda^{1-j} \theta(f_{j1}^k) u_k f_{1j}^k$. Then we see that the sequence $(v_k)_{k\in\mathbf{N}}$ is ω-centralizing and represents

$$\sum_j \lambda^{1-j} \theta_\omega(f_{j1}) f_{11} f_{1j} = ..$$

So we have shown that $v_k \xrightarrow[k\to\omega]{} 1$ * strongly.

Also v_k is a unitary for all k and

$$v_k^* \theta(f_{ij}^k) v_k = f_{i1}^k u_k^* \theta(f_{1i}^k) \theta(f_{ij}^k) \theta(f_{j1}^k) u_k f_{1j}^k \lambda^{i-j}.$$

And, as $u_k^* \theta(f_{11}^k) u_k = u_k^* u_k = f_{11}^k$, one gets

$$v_k^* \theta(f_{ij}^k) v_k = \lambda^{i-j} f_{ij}^k, \qquad \forall\, i, j \in \{1, \ldots, p\}, \quad \forall\, k \in \mathbf{N}.$$

As each sequence $(f_{ij}^k)_{k\in\mathbf{N}}$ is ω-centralizing, and as $v_k^* \xrightarrow[k\to\omega]{} 1$ * strongly, one gets the conclusion of 2.3.4 with $P = v_k^*$.

Q. E. D.

For the next lemma we take the following notation, where M is a von Neumann algebra, K a type I_n subfactor. For each $\psi \in M_*$ we let $\psi/K' \otimes \tau_K$ be the element of M_*, which when M is identified with $K' \otimes K$ (K' = relative commutant of K) is equal to the tensor product of the restriction of ψ to K' by the normalized trace τ_K of K.

LEMMA 2.3.5. — *Let* M *be a von Neumann algebra*, $(e_{ij})_{i,j=1,\ldots,n}$ *a system of* $n\times n$ *matrix units in* M. *Then for any* $\psi \in M_*$ *one has* $\| \psi - \psi/K' \otimes \tau_K \| \leqq n^2 \operatorname{Sup}_{i,j} \| [e_{ij}, \psi] \|$ *where* K *is the subfactor generated by the* (e_{ij}), $i, j = 1, \ldots, n$.

Proof. — Let $\varepsilon = \operatorname{Sup}_{i,j} \| [e_{ij}, \psi] \|$. Let $x \in K'$, and $i, j \in \{1, \ldots, n\}$, $i \neq j$. We have

$$| \psi(xe_{ij}) - \psi(e_{ij} x e_{ii}) | \leqq \varepsilon \| x \|$$

and, as $e_{ij} x e_{ii} = 0$ we get

$$| \psi(xe_{ij}) | \leqq \varepsilon \| x \| \qquad \text{for} \quad i \neq j.$$

Also

$$| \psi(xe_{ii}) - \psi(xe_{jj}) | = | \psi(xe_{ij} e_{ji}) - \psi(e_{ji} x e_{ij}) | \leqq \varepsilon \| x \|$$

so

$$\left| n \psi(xe_{ii}) - \sum_{j=1}^{n} \psi(xe_{jj}) \right| \leqq n \varepsilon \| x \|$$

and we get:

$$\left| \psi(xe_{ii}) - \frac{1}{n} \psi(x) \right| \leqq \varepsilon \| x \| \quad \text{for all } i \text{ and } x \in K'.$$

Put $x \in M$, $\| x \| \leqq 1$, $x = \sum x_{ij} e_{ij}$, with $x_{ij} \in K'$. One has $\| x_{ij} \| \leqq 1$ and

$$(\psi/K' \otimes \tau_K)(x) = \frac{1}{n} \sum_{j=1}^{n} \psi(x_{jj}).$$

$$(\psi - \psi/K' \otimes \tau_K)(x) = \sum_{i \neq j} \psi(x_{ij} e_{ij}) + \sum_{j} \left(\psi(x_{jj} e_{jj}) - \frac{1}{n} \psi(x_{jj}) \right).$$

So the above inequalities show that

$$| (\psi - \psi/K' \otimes \tau_K)(x) | \leqq n(n-1)\varepsilon + n\varepsilon = n^2 \varepsilon.$$

Q. E. D.

LEMMA 2.3.6. — *Let* M *be a von Neumann algebra*, $(n_\nu)_{\nu \in \mathbf{N}}$ *be a sequence of positive integers* (²), $(K_\nu)_{\nu \in \mathbf{N}}$ *a sequence of pairwise commuting subfactors of* M *with* K_ν *of type* I_{n_ν} *for all* $\nu \in \mathbf{N}$. *Let* $(\psi_j)_{j \in \mathbf{N}}$ *be a countable total subset of* M_*.

Assume that for all $j \in \mathbf{N}$ *one has:*

$$\sum_{\nu} \| \psi_j - \psi_j/K'_\nu \otimes \tau_{K_\nu} \| < \infty.$$

Then the K_ν *generate a subfactor* K *of type* II_1 *of* M *and* M *is equal to the tensor product of* K *by its relative commutant* K'.

(²) We assume $n_\nu \geqq 2$ for all ν.

Proof. — For each ν, let m_ν be the haar measure on the unitary group of K_ν such that that $m_\nu(1) = 1$. For $\nu \in N$, $x \in M$ define $E_\nu(x) = \int uxu^* \, dm_\nu(u)$. Then E_ν is a faithful normal conditional expectation of M on the relative commutant K'_ν of K_ν, and when identifying M with $K'_\nu \otimes K_\nu$, it coincides with $1 \otimes \tau_{K_\nu}$. The transposed E^*_ν of E_ν in M_* is the projection of norm 1 which to each $\psi \in M_*$ associates $\psi \circ E_\nu = \psi/K'_\nu \otimes \tau_{K_\nu}$.

So we can rewrite the hypothesis of the lemma as

$$\sum_\nu \| E^*_\nu \psi_j - \psi_j \| < \infty, \qquad \forall j \in N. \tag{2.3.7}$$

Now the E_ν, $\nu \in N$ obviously commute pairwise because Ad u and Ad v commute for u unitary in K_ν, v unitary in $K_{\nu'}$, $\nu \neq \nu'$. Hence the E^*_ν also commute pairwise, and condition 2.3.7 shows that the product $P = \prod_1^\infty E^*_\nu$ converges pointwise in norm. [For any j the sequence $\left(\prod_1^m E^*_\nu\right)\psi_j = \psi^m_j$ satisfies

$$\sum_{m=1}^\infty \| \psi^{m+1}_j - \psi^m_j \| \leq \sum_{m=1}^\infty \| E^*_{m+1} \psi_j - \psi_j \| < \infty.]$$

It follows that the product $\prod_1^\infty E_\nu$ converges pointwise weakly to the transpose E of P. By construction E is weakly continuous. For $x \in M$ and $\nu < m$ we know that $\left(\prod_1^m E_j\right) x$ belongs to the commutant of K_ν, and we see that the range of E is contained in $K' = \bigcap_\nu K'_\nu$. For $x \in K'$ we have $E_\nu x = x$ for all ν and hence $E\,x = x$. We have shown that E is a weakly continuous projection of norm 1 of M onto K'. We have by construction that $E(uxu^*) = E(x)$, $\forall x \in M$, $\forall u$ unitary in K, because this holds for $\left(\prod_1^m E_\nu\right)$ provided u is a unitary in the algebra generated by $K_1, \ldots, K_m$. Now for any faithful normal state φ on K', $\psi = \varphi \circ E$ is a normal state on M such that $\psi(uxu^*) = \psi(x)$, $x \in M$, u unitary in K. So the support $e = s(\psi)$ of ψ must belong to the relative commutant K'. As then $E(e) = e$ we get $\psi(1-e) = \varphi(1-e) = 0$ and $e = 1$. We have shown that E is faithful and that K is a finite factor. (($\varphi \circ E$)/K is a faithful normal trace on K so [10], prop. 1, p. 271, shows that K is a factor.)

Choose a faithful normal $\psi = \varphi \circ E$ as above, then σ^ψ leaves K pointwise fixed and hence K' globally invariant. So by [14], corollary 1, to check that $M = K \otimes K'$ we just have to show that K and K' generate the von Neumann algebra M.

Let $x \in M$, then x is the weak limit of the sequence $x_m = \left(\prod_m^\infty E_\nu\right)(x)$. For each m, $\left(\prod_m^\infty E_\nu\right)(x)$ belongs to the von Neumann algebra generated by $K_1, \ldots, K_{m-1}$ and K'.

Q. E. D.

Proof of theorem 2.3.1. — $(a) \Rightarrow (c)$. Let $p \in \mathrm{N}$ and $\delta > 0$ be as in 2.3.1 (c). If $p = 0$ let $(n_\nu)_{\nu \in \mathrm{N}}$ be a sequence of integers $n_\nu > 1$ where each $q > 1$ appears infinitely many times. Put $\lambda_\nu = \exp(i\, 2\, \pi/n_\nu)$ for all ν. If $p = 1$ let $n_\nu = 2$ for all ν and $\lambda_\nu = 1$ for all ν. If $p > 1$ take $n_\nu = p$ for all ν and $\lambda_\nu = \exp(i\, 2\, \pi/p)$.

Let φ be a faithful normal state on M and $(\psi_j)_{j \in \mathrm{N}}$ a sequence dense in M_*.
We construct by induction on ν a sequence $(\mathrm{P}_\nu)_{\nu \in \mathrm{N}}$ of unitaries of M and $(e^\nu_{ij})_{i,j=1,\ldots,n_\nu}$ of systems of matrix units in M which for each ν satisfy the following conditions.

(α) The factor K_ν generated by the $(e^\nu_{ij})_{i,j=1,\ldots,n_\nu}$ commutes with $\mathrm{K}_1, \ldots, \mathrm{K}_{\nu-1}$.

(β) $\| [e^\nu_{ij}, \psi_k] \| \leqq n_\nu^{-2}\, 2^{-\nu}$ for $k \leqq \nu$ and any $i, j = 1, \ldots, n_\nu$.

(γ) $\mathrm{P}_\nu \in (\mathrm{K}_1 \cup \ldots \cup \mathrm{K}_{\nu-1})'$.

(δ) $\theta_\nu = \mathrm{Ad}(\mathrm{P}_\nu \mathrm{P}_{\nu-1} \ldots \mathrm{P}_1)\, \theta$ satisfies $\theta_\nu(e^k_{ij}) = \lambda_k^{i-j}\, e^k_{ij}$ for $k \leqq \nu$.

(ε) $\| (\mathrm{P}_\nu \mathrm{P}_{\nu-1} \ldots \mathrm{P}_1) - (\mathrm{P}_{\nu-1} \ldots \mathrm{P}_1) \|^\#_\varphi < \delta . 2^{-\nu}$.

Assume the construction is done up to ν, let us construct $\mathrm{P}_{\nu+1}$, $e^{\nu+1}_{ij}$. Let $\tilde{\mathrm{M}}$ be the relative commutant in M of $(\mathrm{K}_1 \cup \ldots \cup \mathrm{K}_\nu)'' = \mathrm{K}^\nu$ the factor generated by $\mathrm{K}_1, \ldots, \mathrm{K}_\nu$. As M is identical with $\mathrm{K}^\nu \otimes \tilde{\mathrm{M}}$ we get from $\psi_1, \ldots, \psi_{\nu+1} \in \mathrm{M}_*$, elements $\tilde{\psi}_1, \ldots, \tilde{\psi}_r$ of $\tilde{\mathrm{M}}_*$ and an $\varepsilon > 0$ such that:

$$(2.3.8) \qquad \begin{aligned} &(x \in \tilde{\mathrm{M}},\ \|x\| \leqq 1,\ \|[x, \tilde{\psi}_j]\| < \varepsilon,\ j = 1, \ldots, r) \\ &\quad \Rightarrow \quad (\|[x, \psi_j]\| \leqq n_{\nu+1}^{-2}\, 2^{-(\nu+1)} \text{ for } j = 1, \ldots, \nu+1). \end{aligned}$$

Also as the restriction of φ to $\tilde{\mathrm{M}}$ is faithful, there is an $\eta > 0$ with:

$$(2.3.9)\ (\mathrm{P} \text{ unitary in } \tilde{\mathrm{M}},\ \|\mathrm{P}-1\|^\#_\varphi \leqq \eta) \ \Rightarrow\ (\|\mathrm{P}(\mathrm{P}_\nu \ldots \mathrm{P}_1) - \mathrm{P}_\nu \ldots \mathrm{P}_1\|^\#_\varphi \leqq \delta . 2^{-(\nu+1)}).$$

Let $\tilde{\theta} = \theta_\nu / \tilde{\mathrm{M}}$. It makes sense by ($\delta$). One has $p_a(\tilde{\theta}) = p_a(\theta)$ by an immediate computation. Then by the choice of $n_{\nu+1}$ and $\lambda_{\nu+1}$ and lemma 2.3.4, there exists a system of $n_{\nu+1} \times n_{\nu+1}$ matrix units $(e_{ij})_{i,j=1,\ldots,n_{\nu+1}}$ and a unitary $\tilde{\mathrm{P}}$ in $\tilde{\mathrm{M}}$ such that:

(a) $\| [\tilde{\psi}_k, e_{ij}] \| < \varepsilon$, $k = 1, \ldots, r$; $i, j = 1, \ldots, n_{\nu+1}$.

(b) $\mathrm{Ad}\, \tilde{\mathrm{P}} \circ \tilde{\theta}(e_{ij}) = \lambda_{\nu+1}^{i-j}\, e_{ij}$; $i, j = 1, \ldots, n_{\nu+1}$.

(c) $\| \tilde{\mathrm{P}} - 1 \|^\#_\varphi < \eta$.

Taking $e^{\nu+1}_{ij} = e_{ij}$ and using 2.3.8 and (a) we check (β). Conditions (α) and (γ) are clearly verified. Condition (δ) for $k = \nu+1$ follows from (b) and $\mathrm{P}_{\nu+1} = \tilde{\mathrm{P}}$, $\mathrm{Ad}\, \mathrm{P}_{\nu+1} \circ \theta_\nu = \theta_{\nu+1}$. For $k \leqq \nu$ one has

$$\theta_{\nu+1}(e^k_{ij}) = \mathrm{P}_{\nu+1}\, \theta_\nu(e^k_{ij})\, \mathrm{P}^*_{\nu+1} = \theta_\nu(e^k_{ij})$$

because by construction $\mathrm{P}_{\nu+1}$ commutes with $\theta_\nu(e^k_{ij}) = \lambda_k^{i-j}\, e^k_{ij}$. Finally condition ($\varepsilon$) follows from (c) and 2.3.9.

Now $(\mathrm{P}_\nu \mathrm{P}_{\nu-1} \ldots \mathrm{P}_1)_{\nu \in \mathrm{N}}$ converges * strongly [by (ε)] to a unitary $\mathrm{P} \in \mathrm{M}$ such that $\| \mathrm{P} - 1 \|^\#_\varphi \leqq \delta$. Let $\theta_\infty = \mathrm{Ad}\, \mathrm{P} \circ \theta$ so that $\theta_\nu \to \theta_\infty$ when $\nu \to \infty$ and, by (δ) we get (δ') $\theta_\infty(e^k_{ij}) = \lambda_k^{i-j}\, e^k_{ij}$ for all i, j, k.

Combining (β) and lemmas 2.3.5, 6 we see that the K_ν, $\nu \in N$ generate a subfactor K of type II_1 of M which factorizes M in $M = K \otimes K'$. By (δ') the restriction of θ_∞ to K is conjugate to s_p and as $s_p \otimes s_p$ is conjugate to s_p we get 2.3.1 (*c*).

(*c*) $\Rightarrow$ (*b*) is obvious.

(*b*) $\Rightarrow$ (*a*) follows by constructing explicitely for $q \neq 0\,(p)$ a central sequence $(x_n)_{n \in N}$ in R such that

$$\|(s_p)^q(x_n) - x_n\|_2 \not\to 0 \qquad \text{when} \quad n \to \infty.$$

Q. E. D.

III. Proof of Theorem 2

We recall the theorem for convenience.

THEOREM 2. — *Let* M *be a factor with separable predual, isomorphic to* $M \otimes R$ *and* θ_1, θ_2 *be automorphisms of* M *such that*

$$\theta_j \in \overline{\text{Int}}\, M, \qquad p_a(\theta_j) = 0, \qquad j = 1, 2.$$

Then there exists a $\sigma \in \overline{\text{Int}}\, M$ *such that*

$$\varepsilon(\theta_2) = \varepsilon(\sigma\theta_1\sigma^{-1}).$$

On $M \otimes R$ the automorphism $\theta = 1 \otimes s_0$ satisfies the conditions of the theorem.

To prove the theorem we let θ be an element of $\overline{\text{Int}}\, M$ such that $p_a(\theta) = 0$ and we construct a factorization $M = K \otimes K'_M$ of M, with K isomorphic to R, and an automorphism α of K such that $\theta^{-1}(\alpha \otimes 1)$ is inner. By construction α will be an infinite tensor product of automorphisms of finite dimensional factors and will not depend, up to conjugacy, on θ.

The proof is divided in two parts. In the first one our aim is the technical lemma 3.1.4 which will be repeatedly applied in the second part.

LEMMA 3.1.1. — *Let* M *as above,* $\theta \in \overline{\text{Int}}\, M$, $p_a(\theta) = 0$. *Then there exists a sequence* $(Y_p)_{p \in N}$ *of unitaries in* M *such that:*

(*a*) Ad $Y_p \to \theta$ *in* Aut M *when* $p \to \infty$.

(*b*) $\theta(Y_p^k) - Y_p^k \to 0$ * *strongly when* $p \to \infty$, *for any* $k \in Z$.

Proof. — As $\theta \in \overline{\text{Int}}\, M$ there is a sequence $(V_p)_{p \in N}$ of unitaries of M satisfying 3.1.1 (*a*). We have $\theta \circ \text{Ad}\, V_p \circ \theta^{-1} = \text{Ad}\, \theta(V_p)$ for all $p \in N$, and hence Ad $\theta(V_p) \to \theta$ in Aut M, when $p \to \infty$.

It follows that Ad $(V_p^* \theta(V_p)) \to 1$ in Aut M, when $p \to \infty$. Put $W_p = V_p^* \theta(V_p)$, then one has $\|\varphi \circ \text{Ad}\, W_p - \varphi\| \to 0$ when $p \to \infty$, for any $\varphi \in M_*$. This shows that $(W_p)_{p \in N}$ is a centralizing sequence.

Let ω be a free ultrafilter on N and let W be the unitary element of M_ω represented by the sequence $(W_p)_{p \in N}$. As θ_ω is a stable automorphism of M_ω (theorem 2.1.3) we can find a unitary $X \in M_\omega$ such that:

$$W = X^* \theta_\omega(X).$$

Let $(X_p)_{p \in N}$ be a representing sequence for X, where each X_p is unitary. We have: $X_p^* \theta(X_p) - W_p \to 0$ * strongly when $p \to \omega$, Ad $X_p \to 1$ in Aut M when $p \to \omega$.

It follows that Ad $V_p X_p^* \to \theta$ in Aut M, when $p \to \omega$ and that

$$\theta(V_p X_p^*) - V_p X_p^* = V_p(V_p^* \theta(V_p) - X_p^* \theta(X_p)) \theta(X_p^*)$$

tends to 0, * strongly, when p tends to ω (3).

We have shown how to construct a sequence $Y_p = V_p X_p^*$ satisfying 3.1.1 (*a*) and $\theta(Y_p) - Y_p \to 0$ * strongly.

Let $l \in N$, assume that $\theta(Y_p^l) - Y_p^l \to 0$ strongly when $p \to \infty$. Then $\theta(Y_p^{-l}) Y_p^l \to 1$ strongly, $Y_p \theta(Y_p^{-1}) \theta(Y_p^{-l}) Y_p^l \to 1$ strongly (because $Y_p \theta(Y_p^{-1}) \to 1$ strongly). As for any $\varphi \in M_*$ we have $\varphi \circ \mathrm{Ad}\, Y_p^{-1} \to \varphi \circ \theta^{-1}$, we get that $\varphi(\theta(Y_p^{-(l+1)}) Y_p^{l+1}) \to \varphi(1)$. We have shown that $\theta(Y_p^{-(l+1)}) Y_p^{l+1} \to 1$ weakly hence that $\theta(Y_p^{l+1}) - Y_p^{l+1} \to 0$ strongly. Condition 3.1.1 (*b*) follows by induction.

Q. E. D.

LEMMA 3.1.2. – *Let* M *be as above*, $\theta \in \overline{\mathrm{Int}\, M}$, $p_a(\theta) = 0$, *let* φ *be a faithful normal state on* M *and* $\psi_1, \ldots, \psi_q \in M_*^+$. *Then for any* $n \in N$, *any* $\delta > 0$, *any* $k \in N$, *there exists a partition of unity* $(F_j)_{j=1,\ldots,n}$, *unitaries* u, $W \in M$ *such that:*

(*a*) $\| [\psi_s, F_j] \| \leq \delta$, $s = 1, \ldots, q$, $j = 1, \ldots, n$.

(*b*) $u F_j u^* = F_{j+1}$, $j = 1, \ldots, n$, $(F_{n+1} = F_1)$.

(*c*) $\| \psi_s \circ \theta^{-1} - \psi_s \circ \mathrm{Ad}\, u^{-1} \| \leq \delta$, $s = 1, \ldots, q$.

(*d*) *With* $\theta' = \mathrm{Ad}\, W \circ \theta$ *one has* $\| \varphi \circ \theta' - \varphi \circ \mathrm{Ad}\, u \| < \delta$.

(*e*) $\| \theta'(u^l) - u^l \|_\varphi \leq \delta$, for $|l| < k$.

(*f*) $\theta'(F_j) = F_{j+1}$, $j = 1, \ldots, n$, $(F_{n+1} = F_1)$,

(*g*) $\| W - 1 \|_\varphi^\# \leq \delta$.

Proof. – For any sequence $(W_m)_{m \in N}$ of unitaries of M we have

$$(W_m \to 1 \text{ strongly}) \Rightarrow \mathrm{Ad}\, W_m \to 1 \text{ in Aut M}.$$

So there exists an $\eta > 0$ such that, for any unitary W in M,

$$(\| W - 1 \|_\varphi^\# \leq \eta) \quad \Rightarrow \quad \| \psi_s \circ \theta^{-1} \circ \mathrm{Ad}\, W^{-1} - \psi_s \circ \theta^{-1} \| \leq \delta/4, \qquad \forall s \leq q.$$

Take such an η, with $\eta < \delta$.

By theorem 2.3.1, applied with the above φ, we let W be a unitary in M, $\| W - 1 \|_\varphi^\# \leq \eta$, such that $\theta' = \mathrm{Ad}\, W \circ \theta$ is of the form $\theta_1 \otimes s_0$ in a factorization $M = Q \otimes R$ of M as a tensor product of a factor Q by the hyperfinite factor of type II_1: R. Once θ' is fixed this way we first choose a partition of unity $(F_j)_{j=1,\ldots,n}$ of M satisfying (*a*), (*f*). (Choose

(3) If $Z_p \to 0$ strongly then for $\varphi \in M_*$, $\varphi((Z_p V_p^*)^* (Z_p V_p^*)) - \varphi.\theta(Z_p^* Z_p) \to 0$, so that $Z_p V_p^* \to 0$ strongly.

$F_j \in 1 \otimes R$ in the above factorization of M.) Then we let, for $l \in \mathbf{Z}$, $|l| \leqq k$, $\varphi_l = \varphi \circ \theta'^{-l}$ and also $\psi = (2k+1)^{-1} \sum_{|l| \leqq k} \varphi_l$.

Choose $\varepsilon < \delta/2$ such that $3\varepsilon + 2k(2\varepsilon + (2k+1)^{1/2}\, 7n\varepsilon) \leqq \delta$ and that

$$(2k+1)^{1/2}\, 7n\varepsilon \leqq \eta'$$

where for any unitary $X \in M$:

$$(\|X-1\|_\varphi^\# \leqq \eta') \Rightarrow \|\psi_s \circ \theta'^{-1} \circ \operatorname{Ad} X^{-1} - \psi_s \circ \theta'^{-1}\| \leqq \delta/4, \qquad \text{for} \quad s = 1, \ldots, q.$$

By lemma 3.1.1 there exists a unitary $Y \in M$ such that:

$$\begin{aligned} &\|\psi_s \circ \theta'^{-1} - \psi_s \circ \operatorname{Ad} Y^{-1}\| \leqq \varepsilon, \qquad s = 1, \ldots, q, \\ &\|\varphi_l - \varphi \circ \operatorname{Ad} Y^{-l}\| \leqq \varepsilon^2, \qquad |l| \leqq k, \\ &\|YF_jY^* - \theta'(F_j)\|_\psi^\# \leqq \varepsilon, \qquad j = 1, \ldots, n, \\ &\|\theta'(Y^l) - Y^l\|_\varphi \leqq \varepsilon, \qquad |l| \leqq k. \end{aligned}$$

As $\theta'(F_j) = F_{j+1}$ for all j, we get by lemma 1.1.4 a partial isometry $X_j \in M$, with initial support YF_jY^* and final support F_{j+1}, such that $\|X_j - F_{j+1}\|_\psi^\# \leqq 7\varepsilon$. Then $X = \sum_{j=1}^{n} X_j$ is a unitary such that $\|X-1\|_\psi^\# \leqq 7n\varepsilon$, and that

$$XYF_jY^*X^* = F_{j+1}, \qquad j = 1, \ldots, n.$$

For each l, $|l| \leqq k$, we have $(\|X-1\|_{\varphi_l}^\#)^2 \leqq (2k+1)(\|X-1\|_\psi^\#)^2$, and hence $\|X-1\|_{\varphi_l}^\# \leqq (2k+1)^{1/2}\, 7n\varepsilon$. As $\|\varphi_l - \varphi \circ \operatorname{Ad} Y^{-l}\| \leqq \varepsilon^2$ and $\|X-1\| \leqq 2$ we get

$$\|(X-1)Y^l\|_\varphi \leqq 2\varepsilon + (2k+1)^{1/2}\, 7n\varepsilon = \alpha.$$

For $l > 0$ we have

$$\|(XY)^{l+1} - Y^{l+1}\|_\varphi \leqq \|(X-1)Y^{l+1}\|_\varphi + \|(XY)^l - Y^l\|_\varphi \quad (^4)$$

so that for $0 \leqq l \leqq k$ we get $\|(XY)^l - Y^l\|_\varphi \leqq l\alpha$. In the same way $\|(XY)^l - Y^l\|_\varphi \leqq |l|\alpha$ for all l, $|l| \leqq k$ and $\|Y(XY)^lY^* - Y^l\|_\varphi \leqq |l|\alpha$ for all l, $|l| < k$. The last conclusion implies, using $\|\varphi \circ \theta' - \varphi \circ \operatorname{Ad} Y\| \leqq \varepsilon^2$, that, for $|l| < k$,

$$\|\theta'(XY)^l - \theta'(Y^l)\|_\varphi \leqq 2\varepsilon + \|\operatorname{Ad} Y((XY)^l - Y^l)\|_\varphi \leqq 2\varepsilon + |l|\alpha.$$

Put $u = XY$. We have shown that for any l, $|l| < k$ one has

$$\|\theta'(u^l) - u^l\|_\varphi \leqq 2\varepsilon + |l|\alpha + \varepsilon + |l|\alpha \leqq 3\varepsilon + 2k\alpha \leqq \delta.$$

We just have to check conditions $(c)(d)$. We have $\|X-1\|_\varphi^\# \leqq \eta'$, because $(2k+1)^{1/2}\, 7n\varepsilon \leqq \eta'$. So $\psi_s \circ \operatorname{Ad} u^{-1} = \psi_s \circ \operatorname{Ad} Y^{-1} \circ \operatorname{Ad} X^{-1}$ is at less than $\varepsilon + \delta/4$

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(4) $(XY)^{l+1} - Y^{l+1} = (X-1)Y^{l+1} + XY((XY)^l - Y^l)$.

of $\psi_s \circ \theta'^{-1}$, hence at less than δ of $\psi_s \circ \theta^{-1}$. Finally

$$\|\varphi \circ \mathrm{Ad}\, X - \varphi\| \leqq 2\|X-1\|_\varphi^\# \leqq 2(2k+1)^{1/2} 7 n \varepsilon$$

and as $\varepsilon^2 + 2(2k+1)^{1/2} 7 n \varepsilon \leqq \delta$, we get *d*).

Q. E. D.

LEMMA 3.1.3. — *Let* M *be a von Neumann algebra,* φ *a state on M and* $u \in$ M *a unitary with projection valued spectral measure denoted by* J $\to e$ (J) (J *borel subset of* **T**).

Then $\Lambda(\varphi, u) = \{ \lambda \in \mathbf{T}, \varphi(e_{J_{\lambda,q}}) \leqq 2^{-q}, \forall q \in \mathbf{N}, q > 2 \}$ *is not empty, where* $J_{\lambda,q}$ *is the interval in* **T**, *of center* λ *and haar measure* 2^{-2q}.

Proof. — We let m be the (normalized) haar measure of **T**. For each $q \in \mathbf{N}$ we have $m\{ \lambda \in \mathbf{T}, \varphi(e(J_{\lambda,q})) > 2^{-q} \} \leqq 3.2^{-q}$. In fact, otherwise we could find a disjoint collection of $J_{\lambda_s,q}$ $s = 1, \ldots, l$ whose union has a haar measure larger than 2^{-q} while, for each s, one has $\varphi(e(J_{\lambda_s,q})) > 2^{-q}$. As each of those intervals has haar measure 2^{-2q}, one has $l \geqq 2^q$ and we get a contradiction because $\varphi(1) = 1 < \sum_{s=1}^{l} \varphi(e(J_{\lambda_s,q}))$. Now $m\{ \lambda \in \mathbf{T}, \exists q > 2, \varphi(e(J_{\lambda,q})) > 2^{-q} \}$ is smaller than $\sum_{q>2} 3.2^{-q} = 3/4$, hence

$$m(\Lambda(\varphi, u)) \geqq 1/4.$$

Q. E. D.

In the rest of this section we denote by f_n, for each $n \in \mathbf{N}$, the borel function from **T** to **T** such that:

$$f_n(e^{i\theta}) = e^{i\theta/n}, \qquad \forall \theta, \quad -\Pi < \theta \leqq \Pi.$$

LEMMA 3.1.4. — *Let* M *be a factor, with separable predual, isomorphic to* M $\otimes$ R, *let* $\theta \in \overline{\mathrm{Int}\, M}$, $p_a(\theta) = 0$, *and let* φ *be a faithful normal state on M, and* $\psi_1, \ldots, \psi_q \in M_*$. *Then for any* $n \in \mathbf{N}$, *any* $\varepsilon > 0$ *there exists a partition of unity* $(F_j)_{j=1,\ldots,n}$ *in* M *and unitaries* $u, v \in$ M *such that:*

(1) $$\|[\psi_k, F_j]\| < \varepsilon, \qquad k = 1, \ldots, q, \qquad j = 1, \ldots, n.$$

(2) $$u F_j u^* = F_{j+1}, \qquad j = 1, \ldots, n, \qquad (F_{n+1} = F_1).$$

(3) $$\|\psi_k \circ \theta^{-1} - \psi_k \circ \mathrm{Ad}\, u^{-1}\| < \varepsilon, \qquad k = 1, \ldots, q.$$

(4) $$-1 \in \Lambda(\varphi, u^n).$$

(5) $\mathrm{Ad}\, v \circ \theta(x) = \tilde{u} x \tilde{u}^*$ *for any* x *in the type* I_n *factor generated by* $(F_j)_{j=1,\ldots,n}$ *and* $\tilde{u} = u f_n(u^n)^*$.

(6) $$\|v - 1\|_\varphi^\# < \varepsilon.$$

Proof. — Choose $m \in \mathbf{N}$ such that $3(2^{-m})^{1/2} \leqq \varepsilon/8n$. Then for $p = 1, \ldots, n$ choose polynomials (of z and z^{-1}), $R_p(z) = \sum_{|t| \leqq k} a_{p,t} z^t$ such that:

(7) $$\begin{cases} |R_p(z) - (z f_n(z^n)^{-1})^p| \leqq \varepsilon/8n, & \forall z \in \mathbf{T}, \quad z^n \notin J_{-1,m}, \\ |R_p(z)| \leqq 2, & \forall z \in \mathbf{T}. \end{cases}$$

Let $A = \sum_{p,t} | a_{p,t} |$ and take $\delta < \varepsilon$, $\varepsilon/4n + ((\varepsilon/4n)^2 + 9\delta)^{1/2} + A\delta \leqq \varepsilon/n$. Applying lemma 3.1.2 with this δ we get a partition of unity $(F_j)_{j=1,\ldots,n}$ and unitaries u, $W \in M$. By lemma 3.1.3 we can assume that $-1 \in \Lambda(\varphi, u^n)$. Put $\theta' = \operatorname{Ad} W \circ \theta$. Let e be the spectral projection of u^n for $J_{-1,m}$. As $\varphi(e) \leqq 2^{-m}$, it follows from 7) that:

$$(8) \qquad \| R_p(u) - \tilde{u}^p \|_\varphi \leqq \varepsilon/4n, \qquad p = 1, \ldots, n.$$

It follows that $\| R_p(u) - \tilde{u}^p \|_{\varphi_1} \leqq \varepsilon/4n$, $p = 1, \ldots, n$ where $\varphi_1 = \varphi \circ \operatorname{Ad} u$, using the commutativity of u with both $R_p(u)$ and $\tilde{u}^p$. The condition (d) of lemma 3.1.2 and the inequality $\| R_p(u) - \tilde{u}^p \| \leqq 3$ show that

$$(9) \qquad \| R_p(u) - \tilde{u}^p \|_{\varphi \circ \theta'} \leqq ((\varepsilon/4n)^2 + 9\delta)^{1/2}, \qquad p = 1, \ldots, n.$$

Moreover the condition (e) of lemma 3.1.2 shows that

$$\| \theta'(u^l) - u^l \|_\varphi \leqq \delta, \qquad | l | \leqq k$$

and hence, by the choice of A, that

$$(10) \qquad \| R_p(\theta'(u)) - R_p(u) \|_\varphi \leqq A\delta, \qquad p = 1, \ldots, n.$$

From (8), (9) and (10) we get:

$$\| \tilde{u}^p - \theta'(\tilde{u}^p) \|_\varphi \leqq \varepsilon/4n + ((\varepsilon/4n)^2 + 9\delta)^{1/2} + A\delta,$$

and hence

$$(11) \qquad \| \tilde{u}^p - \theta'(\tilde{u}^p) \|_\varphi \leqq \varepsilon/n, \qquad p = 1, \ldots, n,$$

by the choice of δ.

As $\delta < \varepsilon$ the conditions (1) to (4) of the lemma are fulfilled. We shall now construct $v = VW$ satisfying conditions (5), (6). By construction we have $\tilde{u}^n = 1$, and as u^n commutes with the F_j's, so does $f_n(u^n)^*$. It follows that

$$\tilde{u} F_j \tilde{u}^* = u F_j u^* = F_{j+1}, \qquad j = 1, \ldots, n, \quad F_{n+1} = F_1$$

and hence that $\tilde{u}$, F_j generate a type I_n subfactor K of M. A system of matrix units $(e_{ij})_{i,j=1,\ldots,n}$ in K is given in particular by $e_{ij} = \tilde{u}^{i-j} F_j$, $i, j = 1, \ldots, n$. Moreover u^n and $f_n(u^n)$ belong to K'.

Note that $\tilde{u} e_{ij} \tilde{u}^* = e_{i+1, j+1}$ for all i and j and that $u e_{ij} u^* = e_{i+1, j+1}$ for all i and j.

Take $V = \sum_{j=1}^{n} e_{j+1, 2} \theta'(e_{1j})$. Then one checks that

$$V \theta'(e_{s,t}) V^* = e_{s+1, 2} \theta'(e_{11}) e_{2, t+1} = u e_{s,t} u^* \qquad \text{for} \quad s, t = 1, \ldots, n.$$

Because $\theta'(e_{11}) = \theta'(F_1) = e_{22}$. With $v = VW$ this proves the condition (5) of the lemma. We have, for $j = 1, \ldots, n$, that

$$e_{j+1,2} \theta'(e_{1,j}) = e_{j+1,2} \theta'(F_1) \theta'(\tilde{u}^{1-j}) = e_{j+1,2} \theta'(\tilde{u}^{1-j})$$

hence, by (11):

$$\| e_{j+1,2}\,\theta'(e_{1,j}) - e_{j+1,2}\,\tilde{u}^{1-j} \|_\varphi \leqq \varepsilon/n$$

and as the term with a minus sign in the last inequality is equal to F_{j+1} we have shown that $\| V-1 \|_\varphi \leqq \varepsilon$. As $\| W-1 \|_\varphi < \varepsilon$ (because $\delta < \varepsilon$), we get $\| VW-1 \|_\varphi \leqq 2\,\varepsilon$. Now one has to estimate $\| V^*-1 \|_\varphi$. We have, for all j,

$$(e_{j+1,2}\,\theta'(e_{1,j}) = \tilde{u}^{j-1}\,\theta'(\tilde{u}^{1-j})\,F_{j+1}$$

hence

$$\begin{aligned}&\| (e_{j+1,2}\,\theta'(e_{1,j}))^* - F_{j+1} \|_\varphi \\ &\quad\leqq \| \tilde{u}^{1-j} - \theta'(\tilde{u}^{1-j}) \|_\varphi \leqq \varepsilon/n\end{aligned}$$

so that, as above, $\| V^*-1 \|_\varphi \leqq \varepsilon$, $\| (VW)^*-1 \|_\varphi \leqq 2\,\varepsilon$.

Q. E. D.

3.2. *Second part of the proof.* — We fix a factor M with separable predual, isomorphic to $M \otimes R$ and a $\theta \in \operatorname{Aut} M$, $\theta \in \overline{\operatorname{Int} M}$, $p_a(\theta) = 0$.

We choose a sequence of positive integers $(n_\nu)_{\nu \in \mathbf{N}}$ such that

$$\sum_{\nu=1}^{\infty} 1/n_\nu < \infty. \tag{3.2.1}$$

In the next two lemmas we determine two sequences $(\delta_\nu)_{\nu \in \mathbf{N}}$, $(\varepsilon_\nu)_{\nu \in \mathbf{N}}$ of positive reals.

LEMMA 3.2.2. — *For each* $\nu \in \mathbf{N}$ *there exists a* $\delta_\nu > 0$ *such that if* $(F_j)_{j=1,\ldots,n_\nu}$ *is a partition of unity in* M *and* $u \in M$ *a unitary with* $u^{n_\nu} = 1$, $u\,F_j\,u^* = F_{j+1}$, $j = 1, \ldots, n_\nu$ *then:*

$$(\psi \in M_*,\ \| [\psi, u] \| < \delta_\nu,\ \| [\psi, F_j] \| < \delta_\nu,\ j = 1, \ldots, n_\nu)$$

implies $\| \psi - \psi/K' \otimes \tau_K \| < 2^{-\nu}$ *with the notations of* 2.3.5 *where* K *is the subfactor of* M *generated by* u *and the* F_j'*s.*

Proof. — A system of matrix units in K is given by $e_{ij} = u^{i-j}\,F_j$. If $\| [\psi, u] \| < \delta$ we have, for $k > 0$, $\| [\psi, u^k] \| \leqq k\,\delta$, hence with $\| [\psi, F_j] \| < \delta$ for all j, we get $\| [\psi, e_{ij}] \| \leqq n_\nu\,\delta + \delta$ for all $i, j \in \{1, \ldots, n_\nu\}$. Applying lemma 2.3.5 we just have to require

$$n_\nu^2\,(n_\nu + 1)\,\delta_\nu \leqq 2^{-\nu}.$$

Q. E. D.

Throughout we let $\delta_\nu = 2^{-\nu}\,n_\nu^{-2}\,(n_\nu + 1)^{-1}$.

LEMMA 3.2.3. — *For each* $\nu \in \mathbf{N}$ *there exists an* $\varepsilon_\nu > 0$ *such that* $\varepsilon_\nu \leqq 1/n_\nu$ *and satisfying the following: Let* φ *be a faithful normal state on* M, *and* u *a unitary,* $u \in M$ *such that* $-1 \in \Lambda(\varphi, u^{n_{\nu+1}})$ *then:*

$(\psi \in M_*^+,\ \psi \leqq \varphi,\ \| [\psi, u] \| \leqq 2\,\varepsilon_\nu)$ *implies*

$$\| [\psi, \tilde{u}] \| \leqq \delta_{\nu+1} \qquad \textit{where} \qquad \tilde{u} = u\,(f_{n_{\nu+1}}(u^{n_{\nu+1}}))^*.$$

Proof. — Put $n = n_{\nu+1}$, $\delta = \delta_{\nu+1}$. Let $R(z) = \sum_{-m}^{m} a_k z^k$ be such that $|R(z)| \leqq 2$, $\forall z \in \mathbf{T}$ and

$$|R(z)-(\overline{f_n(z^n)})z|^2 \leqq \delta^2/8, \qquad z \in \mathbf{T}, \quad z^n \notin J_{-1,q}, \tag{3.2.4}$$

where $q \geqq 3$ is such that $9.2^{-q} \leqq \delta^2/8$. We have $(\| R(u)-f_n(u^n)^* u \|_\varphi^\#)^2 \leqq \delta^2/8 + 9.2^{-q}$ because $-1 \in \Lambda(\varphi, u^n)$.

It follows that $\| R(u)-\tilde{u} \|_\psi^\# \leqq \delta/2$, $\forall \psi$, $0 \leqq \psi \leqq \varphi$. Moreover $\| [\psi, u] \| < \varepsilon$ implies $\| [\psi, u^k] \| \leqq |k| \varepsilon$ for any $k \in \mathbf{Z}$ so that we just have to choose ε_ν such that $\varepsilon_\nu \leqq 1/n_\nu$ and:

$$\left(\sum_{-m}^{m} |k| |a_k|\right) 2\varepsilon_\nu \leqq \delta/2 \tag{3.2.5}$$

and check that, $0 \leqq \psi \leqq \varphi$, $\| [\psi, u] \| \leqq 2\varepsilon_\nu$ implies

$$\| [R(u), \psi] \| \leqq \delta/2, \qquad \| [\tilde{u}, \psi] \| \leqq \delta/2 + \delta/2 \quad (\textit{see } [5],\ 2.1).$$

Q. E. D.

We fix $(\varepsilon_\nu)_{\nu \in \mathbf{N}}$ once for all, with $\varepsilon_{\nu+1} \leqq \varepsilon_\nu$, $\forall \nu$.

LEMMA 3.2.6. — *Let* $M = Q \otimes N$ *be the tensor product of a finite dimensional factor* Q *by a factor* N. *Then for any* $\psi \in M_*$ *there exists* m *elements* (m = *dimension of* Q) *of* N_*, $\psi^1, \ldots, \psi^m$ *such that:*

(*a*) $\forall x \in N$, $\| [\psi, 1 \otimes x] \| \leqq \operatorname{Sup}_j \| [\psi^j, x] \|$.

(*b*) $\forall U$ *unitary in* Q, V *unitary in* N, $\theta \in \operatorname{Aut} N$, *one has*

$$\| \psi \circ ((\operatorname{Ad} U) \otimes \theta) - \psi \circ \operatorname{Ad}(U \otimes V) \| \leqq \operatorname{Sup}_j \| \psi^j \circ \theta - \psi^j \circ \operatorname{Ad} V \|.$$

Proof. — Let $(e_{ij})_{i,j=1,\ldots,m^{1/2}}$ be a system of matrix units in Q and $(\omega_j)_{j=1,\ldots,m}$ be a basis of Q_* dual to the (e_{ij}).

For each $x \in Q \otimes N$, the operator $(\omega_j \otimes 1)(x)$ is a matrix element of x (x is a matrix with coefficients in N). It follows that $\| \omega_j \otimes \omega \| \leqq \| \omega \|$ for any $\omega \in N_*$. Write $\psi = \sum_{j=1}^{m} \omega_j \otimes \psi_j$ and put $\psi^j = m\psi_j$, $j = 1, \ldots, m$; so that

$$\psi = \frac{1}{m} \sum_{j=1}^{m} \omega_j \otimes \psi^j.$$

For $x \in N$ we have:

$$[\psi, 1 \otimes x] = \frac{1}{m} \sum_{j=1}^{m} \omega_j \otimes [\psi^j, x]$$

which shows (*a*). For U, V and θ as in 3.2.6 (*b*) we have

$$\psi \circ ((\mathrm{Ad\,U}) \otimes \theta) = \frac{1}{m} \sum (\omega_j \circ \mathrm{Ad\,U}) \otimes (\psi^j \circ \theta),$$

$$\psi \circ ((\mathrm{Ad}(\mathrm{U} \otimes \mathrm{V}) = \frac{1}{m} \sum (\omega_j \circ \mathrm{Ad\,U}) \otimes (\psi^j \circ \mathrm{Ad\,V})$$

and as $\| (\omega_j \circ \mathrm{Ad\,U}) \otimes \omega \| \leqq \| \omega \|$, for any $\omega \in \mathrm{N}_*$ we get (*b*).

Q. E. D.

LEMMA 3.2.7. — *Let* M *and* θ *as above,* φ *a faithful normal state on* M, $(\psi_j)_{j=1,\ldots}$ *a sequence of elements of* $[0, \varphi]_{\mathrm{M}_*}$. *There exists a sequence* $(\mathrm{K}_\nu)_{\nu \in \mathrm{N}}$ *of subfactors of* M *and* $(\mathrm{P}_\nu)_{\nu \in \mathrm{N}}$ *of unitaries of* M *such that:*

(*a*) *For each* $\nu \in \mathrm{N}$, K_ν *commutes with* K_j, $j < \nu$.

(*b*) *For each* $\nu \in \mathrm{N}$, K_ν *is generated by a partition of unity* $(\mathrm{F}_j^\nu)_{j=1,\ldots,n_\nu}$ *and a unitary* U_ν, $\mathrm{U}_\nu^{n_\nu} = 1$,

$$\mathrm{U}_\nu \mathrm{F}_j^\nu \mathrm{U}_\nu^* = \mathrm{F}_{j+1}^\nu, \qquad \forall j = 1, \ldots, n_\nu.$$

(*c*) $\| [\psi_l, \mathrm{U}_\nu] \| \leqq \delta_\nu$, $\| [\psi_l, \mathrm{F}_j^\nu] \| \leqq \delta_\nu$ *for any* $\nu \in \mathrm{N}$, *any* $l < \nu$ *and* $j = 1, \ldots, n_\nu$.

(*d*) *For any* $\nu \in \mathrm{N}$, P_ν *commutes with* $\mathrm{K}_1, \ldots, \mathrm{K}_{\nu-1}$.

(*e*) *For any* $\nu \in \mathrm{N}$, $\| (\mathrm{P}_\nu - 1) \mathrm{P}_{\nu-1} \mathrm{P}_{\nu-2} \ldots \mathrm{P}_1 \|_\varphi^\# \leqq 8/n_\nu$.

(*f*) *Put* $\theta_\nu = \mathrm{Ad}(\mathrm{P}_\nu \mathrm{P}_{\nu-1} \ldots \mathrm{P}_1) \circ \theta$ *then each* θ_ν *leaves* K_j, $j \leqq \nu$ *globally invariant and coincides with* $\mathrm{Ad\,U}_j$ *on such a* K_j.

(*g*) *For any* $\nu \in \mathrm{N}$, $j \leqq \nu$ *one has:*

$$\| \psi_j \circ \theta_\nu^{-1} - \psi_j \circ \mathrm{Ad}(\mathrm{U}_\nu \mathrm{U}_{\nu-1} \ldots \mathrm{U}_1)^{-1} \| \leqq \varepsilon_\nu.$$

Proof. — We assume that K_j, P_j have been constructed up to $j = \nu$ and we look for $\mathrm{K}_{\nu+1}$, $\mathrm{P}_{\nu+1}$.

Let Q be the subfactor generated by the K_j, $j \leqq \nu$ and let m be the dimension of Q. Let N be the relative commutant of Q in M. The automorphism $\theta_\nu\ \iota\,^c$ M leaves Q globally invariant and coincides on Q with the inner automorphism Ad U where $\mathrm{U} = \mathrm{U}_\nu \mathrm{U}_{\nu-1} \ldots \mathrm{U}_1$ (note that the U_j commute pairwise). Let $\tilde{\theta}$ be the restriction of θ_ν to N and note that if we identify $\mathrm{Q} \otimes \mathrm{N}$ with M we get $\mathrm{Ad\,U} \otimes \theta = \theta_\nu$. Let, for $l = 1, \ldots, \nu+1$, ψ_l^s, $s = 1, \ldots, m$ be elements of N_* satisfying lemma 3.2.6 relative to ψ_l.

By theorem 2.3.1 we see that $\tilde{\theta}$ is outer conjugate to θ and hence $\tilde{\theta} \in \overline{\mathrm{Int\,N}}$ and $p_a(\tilde{\theta}) = 0$.

By lemma 3.1.4 there exists a partition of unity $(\mathrm{F}_j)_{j=1,\ldots,n_{\nu+1}}$ in N and unitaries u, $v \in \mathrm{N}$ such that:

(1) $$\| [\psi_l^s, \mathrm{F}_j] \| \leqq \delta_{\nu+1}, \qquad l = 1, \ldots, \nu, \quad \forall s, \quad \forall j.$$

(2) $$u \mathrm{F}_j u^* = \mathrm{F}_{j+1}, \qquad j = 1, \ldots, n_{\nu+1}.$$

(3) $$\|\psi_l^s \circ \tilde{\theta}^{-1} - \psi_l^s \circ \operatorname{Ad} u^{-1}\| < \varepsilon_{\nu+1}/2, \qquad l = 1, \ldots, \nu+1, \quad \forall s.$$

(4) $-1 \in \Lambda(\varphi_N, u^{n_{\nu+1}})$ where $\varphi_N = \varphi$ restricted to N.

(5) ${}_v\tilde{\theta}(x) = uxu^*$, $\forall x \in K$ where K is the factor generated by the F_j's and

$$\tilde{u} = uf_{n_{\nu+1}}(u^{n_{\nu+1}})^*.$$

(6) $$\begin{cases} \|v-1\|_\varphi^\sharp < \varepsilon_{\nu+1}/2, \qquad \|\tilde{\theta}^{-1}(v)-1\|_\varphi < \varepsilon_{\nu+1}/4 \\ \text{and} \\ \qquad \|(v-1)P_\nu P_{\nu-1}\ldots P_1\|_\varphi < \varepsilon_{\nu+1}/2. \end{cases}$$

We are applying 3.1.4 with $\varepsilon \leqq \delta_{\nu+1}$, $\varepsilon \leqq \varepsilon_{\nu+1}/2$ and ε so small that any unitary $v \in N$ such that $\|v-1\|_{\varphi_N}^\sharp < \varepsilon$ satisfies the condition (6) above. It is possible because φ_N is faithful. We have $\theta_\nu = \operatorname{Ad} U \otimes \tilde{\theta}$, hence (3) and 3.2.6 show that:

(7) $$\|\psi_l \circ \theta_\nu^{-1} - \psi_l \circ \operatorname{Ad}(u\,U)^{-1}\| \leqq \varepsilon_{\nu+1}/2, \qquad l = 1, \ldots, \nu+1.$$

But the induction hypothesis (g) shows that

$$\|\psi_l \circ \theta_\nu^{-1} - \psi_l \circ \operatorname{Ad} U^{-1}\| \leqq \varepsilon_\nu, \qquad l = 1, \ldots, \nu.$$

And, as u and U commute we get $\|\psi_l \circ \operatorname{Ad} u^{-1} - \psi_l\| \leqq \varepsilon_\nu + \varepsilon_{\nu+1}/2$

(8) $$\|[\psi_l, u]\| \leqq 2\varepsilon_\nu, \qquad l = 1, \ldots, \nu.$$

As $\psi_l \leqq \varphi$, condition (4) and lemma 3.2.3 show that

(9) $$\|[\psi_l, \tilde{u}]\| \leqq \delta_{\nu+1}, \qquad l = 1, \ldots, \nu.$$

Let $\tilde{P} = f_{n_{\nu+1}}(u^{n_{\nu+1}})^*$, then $\|\tilde{P}-1\| \leqq \pi/n_{\nu+1}$, and (6) shows that, with $P = \tilde{P}v$ we have

$$\|(1-P)P_\nu P_{\nu-1}\ldots P_1\|_\varphi \leqq \pi/n_{\nu+1} + \frac{1}{2}\varepsilon_{\nu+1},$$

$$\|P_1^*\ldots P_\nu^*(1-P^*)\|_\varphi \leqq \|(1-v^*)\|_\varphi + \pi/n_{\nu+1} \leqq 1/2\,\varepsilon_{\nu+1} + \pi/n_{\nu+1}.$$

Moreover by 3.2.3 we have $\varepsilon_{\nu+1} \leqq 1/n_{\nu+1}$ and hence

(10) $$\|(1-P)P_\nu\ldots P_1\|_\varphi^\sharp \leqq 8/n_{\nu+1}.$$

Now we have

$$\begin{aligned} &\|\psi_l \circ \theta_\nu^{-1} \circ \operatorname{Ad} v^{-1} - \psi_l \circ \theta_\nu^{-1}\| \\ &\quad = \|\psi_l \circ \operatorname{Ad}(\theta_\nu^{-1}(v^{-1})) - \psi_l\| \leqq 2\|\theta_\nu^{-1}(v)-1\|_{\psi_l} \\ &\quad \leqq 2\|\theta_\nu^{-1}(v)-1\|_\varphi = 2\|\tilde{\theta}^{-1}(v)-1\|_\varphi \leqq \varepsilon_{\nu+1}/2, \end{aligned}$$

for any l, using (6). Together with (7) we get:

$$\|\psi_l \circ \theta_\nu^{-1} \circ \operatorname{Ad} v^{-1} - \psi_l \circ \operatorname{Ad}(u\,U)^{-1}\| \leqq \varepsilon_{\nu+1}; \qquad l = 1, \ldots, \nu+1.$$

Applying $\operatorname{Ad} \tilde{P}^{-1}$ to both sides gives, using $P^{-1} = v^{-1} \tilde{P}^{-1}$, and $u^{-1} \tilde{P}^{-1} = (\tilde{u})^{-1}$ that:

$$(11) \qquad \| \psi_l \circ \theta_\nu^{-1} \circ \operatorname{Ad} P^{-1} - \psi_l \circ \operatorname{Ad} (\tilde{u}\, U)^{-1} \| \leqq \varepsilon_{\nu+1}; \qquad l = 1, \ldots, \nu+1$$

We take $F_j^{\nu+1} = F_j$, $j = 1, \ldots, n_{\nu+1}$, $U_{\nu+1} = \tilde{u}$, $K_{\nu+1} = K$, $P_{\nu+1} = P = \tilde{P}\, v$. Conditions 3.2.7 (*a*) and (*b*) are easy to check. Condition (*c*) follows from (9) and from condition (1) above and lemma 3.2.6 (*a*). Condition (*d*) is clear because $P \in N$, condition (*e*) is given by (10). To check (*f*) note that

$$\theta_{\nu+1} = \operatorname{Ad} P \circ \theta_\nu = \operatorname{Ad} U \otimes {}_P\tilde{\theta}$$

which proves (*f*) for $j = 1, \ldots, \nu$.

Moreover $K_{\nu+1} = K \subset N$ and we just have to check that ${}_P\tilde{\theta}(x) = \tilde{u} x \tilde{u}^*$, $\forall x \in K$. By (5) we have $v\, \tilde{\theta}(x)\, v^* = u x u^*$, $\forall x \in K$ and as $\tilde{P} u = \tilde{u}$, we get

$$\tilde{P}\, v\, \tilde{\theta}(x)\, v^*\, \tilde{P}^* = \tilde{P}\, u x u^*\, \tilde{P}^* = \tilde{u} x \tilde{u}^*, \quad \forall x \in K.$$

We thus have checked (*f*) for $j = 1, \ldots, \nu, \nu+1$.

To prove (*g*) note that $\theta_{\nu+1}^{-1} = \theta_\nu^{-1} \circ \operatorname{Ad} P^{-1}$ and that $U_{\nu+1} U_\nu \ldots U_1 = \tilde{u}\, U$ with the above notations. Hence (*g*) follows from inequality (11).

To end the proof of 3.2.7 we note that, for $\nu = 1$, the conditions (*c*) are vacuous because there is no ψ_l, $l < \nu$. Hence the construction of $(F_j^1)_{j=1, \ldots, n_1}$, U_1 and P_1 follows from the same argument as above, with $\nu = 0$.

Q. E. D.

End of the proof of theorem 2. — We choose a faithful normal state φ on M and a sequence $(\psi_j)_{j \in \mathbf{N}}$, of $[0, \varphi]_{M_*}$, which is *total* in M_*. Then we construct $(K_\nu)_{\nu \in \mathbf{N}}$, $(U_\nu)_{\nu \in \mathbf{N}}$ and $(P_\nu)_{\nu \in \mathbf{N}}$ as in lemma 3.2.7 and we note that:

(α) The K_ν generate a subfactor K of type II_1 in M and M is equal to the tensor product of K by its relative commutant K′. [Apply condition 3.2.7 (*c*), lemma 3.2.2 and lemma 2.3.6.]

(β) The unitaries $W_\nu = P_\nu P_{\nu-1} \ldots P_1$ converge * strongly to a unitary $W \in M$ [by condition 3.2.7 (*e*) one has $\| W_\nu - W_{\nu-1} \|_\varphi^\# \leqq 8/n_\nu$, $\nu \in \mathbf{N}$ and by hypothesis $\sum_{\nu \in \mathbf{N}} 1/n_\nu < \infty$].

Let $\theta_\infty = \operatorname{Ad} W \circ \theta = \lim_{\nu \to \infty} \theta_\nu$ in Aut M. We have

(γ) For each $j \in \mathbf{N}$, θ_∞ leaves K_j globally invariant and coincides with $\operatorname{Ad} U_j$ on K_j [Use 3.2.7 (*f*)].

Using (α) one sees that K is the infinite tensor product of the couples (K_ν, τ_ν), τ_ν = canonical trace on K_ν. Let $\alpha \in \operatorname{Aut} K$ be the infinite tensor product of the $\operatorname{Ad} U_\nu \in \operatorname{Aut} K_\nu$.

From 3.2.7 (*g*), identifying M with $K \otimes K'$ we get:

(δ) $\theta_\infty = \alpha \otimes 1_{K'}$.

By 2.3.1 α is outer conjugate to $\alpha \otimes 1_R$ so modifying α by an inner automorphism of K we can get an automorphism β of a subfactor A of K (factorizing K and such that A and $A' \cap K$ are isomorphic to R) and a unitary $v \in K$ with:

(ε) $\operatorname{Ad} v \circ \alpha = \beta \otimes 1_{A' \cap K}$.

Then $\operatorname{Ad} v \circ \theta_\infty = \beta \otimes 1_{(A' \cap K)} \otimes 1_{K'}$. Using proposition 2.2.3 one gets the desired conclusion.

Q. E. D.

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Alain Connes,
Université de Paris VI, U.E.R. 47,
4, place Jussieu,
75230 Paris Cedex 05.