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REPRESENTATIONS OF SEMISIMPLE GROUPS ASSOCIATED TO NILPOTENT ORBITS

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1. Introduction

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the real dual space of $\mathfrak{g}$. Then G acts on $\mathfrak{g}^*$ by the dual of the adjoint representation. The " orbit method " in group representations associates unitary representations of G to certain G-orbits on $\mathfrak{g}^*$.

Kirillov [11] was the first to use the orbit method. He applied it in the case where G is a connected simply connected nilpotent group, giving a one-to-one correspondence between the set of all G-orbits on $\mathfrak{g}^*$ and the set $\hat{G}$ of all equivalence classes of irreducible unitary representations of G. Kostant [14] extended the scope of the orbit method so that it encompassed the Bott-Borel-Weil theorem for compact Lie groups, and then Auslander and Kostant [1] applied it to solvable Lie groups.

If G is semisimple (or, more generally, reductive), we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by the Killing form $\langle\, , \rangle$ (extended to be negative definite on the center). Then the G-orbits $\mathrm{Ad}^*(G).f \subset \mathfrak{g}^*$ go over to orbits $\mathrm{Ad}(G).x \subset \mathfrak{g}$. The representations of G that occur in the Plancherel formula ([9], [10], [25]) then are associated to orbits $\mathrm{Ad}(G).x$ where x is a semisimple element of $\mathfrak{g}$. In this paper we discuss representations associated to orbits $\mathrm{Ad}(G).e$ where e is a nilpotent element of $\mathfrak{g}$.

Representations are associated to orbits by means of " polarizations ". Let G be reductive, $x \in \mathfrak{g}$ and consider the centralizers

$$\mathfrak{g}^x = \{ y \in \mathfrak{g} : [y, x] = 0 \} \qquad \text{and} \qquad G^x = \{ g \in G : \mathrm{Ad}(g)x = x \}.$$

A " real polarization " for x, i. e. for the corresponding linear functional $x^* : y \mapsto \langle x, y \rangle$, is a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ of dimension $1/2\,(\dim \mathfrak{g} + \dim \mathfrak{g}^x)$ such that $\langle x, [\mathfrak{p}, \mathfrak{p}] \rangle = 0$; then $\mathfrak{g}^x \subset \mathfrak{p}$. We call $\mathfrak{p}$ " invariant " if it is normalized by G^x, i. e. if ${}^xP = G^x.P^0$ is

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a group, where P^0 is the analytic subgroup of G for $\mathfrak{p}$. Now suppose that $\mathfrak{p}$ is an invariant real polarization for x and that

$$\mathfrak{p} \ni y \mapsto 2\pi i \langle x, y \rangle$$

exponentiates to a character χ on P^0. The associated representations of G are the

$$\pi_{x,\mathfrak{p},\xi} = \operatorname{Ind}_{^xP \uparrow G}(\xi),$$

unitarily induced, where ξ ranges over the elements of $(^xP)^\wedge$ that extend χ. In the corresponding situation for solvable groups it is known [1] that the representations obtained depend only on the orbit and not on the choice of polarization, provided that the polarizations are required to satisfy the " Pukánszky condition ". We show that if $\mathfrak{g}$ is reductive then a polarization for x satisfies this condition if and only if x is semisimple (§ 2.4). Our most striking result is an example (Theorem 4.4.1) of a nilpotent element e in the split Lie algebra of type G_2, and invariant real polarizations $\mathfrak{p}_1$ and $\mathfrak{p}_2$ for e, such that none of the $\pi_{e,\mathfrak{p}_1,\xi_1}$ has a subrepresentation in common with any $\pi_{e,\mathfrak{p}_2,\xi_2}$. The point here is that $\pi_{e,\mathfrak{p}_1,\xi_1}$ and $\pi_{e,\mathfrak{p}_2,\xi_2}$ have different infinitesimal characters. This phenomenon does not occur for solvable groups ([4], [5], [6], [17], [19], [20]).

If $x \in \mathfrak{g}$ is semisimple, one studies " complex polarizations " $\mathfrak{q} \subset \mathfrak{g}_C$ such that $\mathfrak{g} \cap (\mathfrak{q}+\bar{\mathfrak{q}})$ is a cuspidal parabolic subalgebra of $\mathfrak{g}$. Real polarizations are not available unless every eigenvalue of $\operatorname{ad}(x)$ is real, so one has to use a rather complicated holomorphic induction procedure ([2], [14], [25]) rather than Mackey's relatively simple unitary induction process. These complications are avoided in our study of representations associated to nilpotent orbits. Let $x \in \mathfrak{g}$ and let $\mathfrak{p}$ be an invariant real polarization that is a parabolic subalgebra of $\mathfrak{g}$, such that $y \mapsto 2\pi i \langle x, y \rangle$ exponentiates to a character χ on P^0. Proposition 2.5.4 describes the elements of $(^xP)^\wedge$ that extend χ, and Proposition 2.6.6 relates the corresponding representations $\pi_{x,\mathfrak{p},\xi}$ to certain representations induced from the parabolic subgroup of G with Lie algebra $\mathfrak{p}$. If $\mathfrak{p}$ is a cuspidal parabolic subalgebra of $\mathfrak{g}$ that is an invariant polarization for a nilpotent element $e \in \mathfrak{g}$, then $y \mapsto 2\pi i \langle e, y \rangle$ exponentiates to the trivial character on P^0, and Theorem 3.3.1 gives an explicit analysis of the representations $\pi_{e,\mathfrak{p},\xi}$, including the calculation of their infinitesimal character. Our counterexample to independance of polarization, mentioned above, is based on this knowledge of the infinitesimal character.

2. Representations associated to real parabolic polarizations

We look at unitary representations of reductive Lie groups constructed from polarizations that are real parabolic subalgebras.

2.1. A CLASS OF REDUCTIVE GROUPS. — Let G be a reductive Lie group. In other words its Lie algebra $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}'$ where $\mathfrak{c}$ is the center and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is semisimple. We assume

(2.1.1) if $g \in G$ then $\operatorname{Ad}(g)$ is an inner automorphism on $\mathfrak{g}_C$.

Let G^0 be the identity component of G and $Z_G(G^0)$ its G-centralizer. Thus G^0 has center $Z_{G^0} = Z_G(G^0) \cap G^0$. We will also assume that G has a closed normal abelian subgroup Z such that

(2.1.2) $Z \subset Z_G(G^0)$ with G/ZG^0 finite and $Z \cap G^0$ co-compact in Z_{G^0}.

Thus our working class of groups is the class studied in [25] and [26]. While there seems to be no special reason to restrict attention to a smaller class of groups, we mention that, in view of (2.1.1), the case $Z = \{1\}$ of (2.1.2) is : $Z_G(G^0)$ is compact.

2.2. POLARIZATIONS. — Let $\mathfrak{g}$ be a real reductive Lie algebra, and let $\langle\ ,\ \rangle$ be the direct sum of the Killing form of the derived algebra and a negative definite bilinear form on the center. If G is a Lie group with Lie algebra $\mathfrak{g}$, and if G satisfies (2.1.1), then the nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$ is G-invariant.

Every $x \in \mathfrak{g}$ now defines a linear functional $x^* \in \mathfrak{g}^*$ by

(2.2.1) $$x^*(y) = \langle x, y \rangle \quad \text{for all } y \in \mathfrak{g}_{\mathbf{C}}.$$

That in turn defines an antisymmetric bilinear form

(2.2.2) $$b_x(y, z) = x^*[y, z] = \langle x, [y, z] \rangle \quad \text{for all } y, z \in \mathfrak{g}_{\mathbf{C}}.$$

If $\mathfrak{q} \subset \mathfrak{g}_{\mathbf{C}}$ is a complex subalgebra of $\mathfrak{g}_{\mathbf{C}}$ that is maximal among the totally b_x-isotropic subspaces of $\mathfrak{g}_{\mathbf{C}}$, we will say that $\mathfrak{q}$ is a *complex polarization* for x. Here it is usual also to require that $\mathfrak{q}+\bar{\mathfrak{q}}$ be a subalgebra of $\mathfrak{g}_{\mathbf{C}}$, and the reader is warned that we are not making that requirement. By a *real polarization* for x we mean a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ such that $\mathfrak{p}_{\mathbf{C}}$ is a complex polarization for x.

Our notation for centralizers is the usual

(2.2.3) $$\mathfrak{g}^x = \{ y \in \mathfrak{g} : [x, y] = 0 \} \quad \text{and} \quad \mathfrak{g}_{\mathbf{C}}^x = \{ y \in \mathfrak{g}_{\mathbf{C}} : [x, y] = 0 \}.$$

Nondegeneracy of the Killing form implies

(2.2.4) $$\mathfrak{g}^x = \{ y \in \mathfrak{g} : b_x(y, \mathfrak{g}) = 0 \} \quad \text{and} \quad \mathfrak{g}_{\mathbf{C}}^x = \{ y \in \mathfrak{g}_{\mathbf{C}} : b_x(y, \mathfrak{g}_{\mathbf{C}}) = 0 \}.$$

In orther words,

(2.2.5) b_x induces nondegenerate bilinear forms on $\mathfrak{g}/\mathfrak{g}^x$ and $\mathfrak{g}_{\mathbf{C}}/\mathfrak{g}_{\mathbf{C}}^x$.

In particular the maximal totally b_x-isotropic subspaces of $\mathfrak{g}_{\mathbf{C}}$ contain $\mathfrak{g}_{\mathbf{C}}^x$ and have dimension $1/2\,(\dim \mathfrak{g}_{\mathbf{C}} + \dim \mathfrak{g}_{\mathbf{C}}^x)$. Thus a complex subalgebra $\mathfrak{q} \subset \mathfrak{g}_{\mathbf{C}}$ is a complex polarization for x if, and only if, both

(2.2.6 *a*) $$\mathfrak{g}_{\mathbf{C}}^x \subset \mathfrak{q} \quad \text{and} \quad \dim \mathfrak{g}_{\mathbf{C}} - \dim \mathfrak{q} = \dim \mathfrak{q} - \dim \mathfrak{g}_{\mathbf{C}}^x,$$

and

(2.2.6 *b*) $\mathfrak{q}$ is totally b_x-isotropic, i. e. $\langle x, [\mathfrak{q}, \mathfrak{q}] \rangle = 0$.

Note that (2.2.6 *b*) is equivalent to

(2.2.6 *c*) $x^*|_{\mathfrak{q}}$ is a Lie algebra homomorphism $\mathfrak{q} \to \mathbf{C}$.

That will be the connection with representation theory.

2.3. PARABOLIC POLARIZATIONS. — Recall that the maximal solvable subalgebras of $\mathfrak{g}_{\mathbf{C}}$ are conjugate. They are called the Borel subalgebras. A subalgebra $\mathfrak{q} \subset \mathfrak{g}_{\mathbf{C}}$ is called parabolic if it contains a Borel subalgebra. A subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is called parabolic if $\mathfrak{p}_{\mathbf{C}}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbf{C}}$.

Let $x \in \mathfrak{g}$. By complex (resp. real) *parabolic polarization* for x, we mean a parabolic subalgebra of $\mathfrak{g}_{\mathbf{C}}$ (resp. of $\mathfrak{g}$) that is a complex (resp. real) polarization for x.

Ozeki and Wakimoto have shown ([18], Theorem 2.2) that any polarization of x must be parabolic. We shall use this fact in the sequel without explicit mention.

$x \in \mathfrak{g}$ is called *semisimple* if ad x is a diagonalizable operator on $\mathfrak{g}_{\mathbf{C}}$. It is well-known (and easy to prove) that every semisimple element of $\mathfrak{g}$ has a complex parabolic polarization ([18], Proposition 2.3). $x \in \mathfrak{g}$ is called *nilpotent* if $x \in [\mathfrak{g}, \mathfrak{g}]$ and ad x is a nilpotent operator on $\mathfrak{g}_{\mathbf{C}}$. It is known that there exist nilpotents with no polarizations. *See* [27], p. 63 for an example of such a nilpotent in $\mathfrak{so}(2, 3)$.

Fix $x \in \mathfrak{g}$ and a complex (parabolic) polarization $\mathfrak{q}$ for x. We decompose

(2.3.1 *a*) $\mathfrak{q} = \mathfrak{q}_r + \mathfrak{q}_n$ where $\mathfrak{q}_n$ is the nilradical and $\mathfrak{q}_r$ is a reductive (Levi) complement,

(2.3.1 *b*) $x = x_r + x_n$ where $x_r \in \mathfrak{q}_r$ and $x_n \in \mathfrak{q}_n$.

{ Here $\mathfrak{q}_n \subset [\mathfrak{g}, \mathfrak{g}]$ and $\operatorname{ad}(\mathfrak{q}_n)$ consists of nilpotent linear transformations }. Using $\mathfrak{q}_n^{\perp} = \mathfrak{q}$ it is easy to verify :

2.3.2. PROPOSITION. — *Fix x and $\mathfrak{q}$ as above. Then x_r is central in $\mathfrak{q}_r$. In particular, x_r is semisimple and* ad (x_r) *has the same eigenvalues as* ad (x). *Further* (i) *x is semisimple if and only if we can choose $\mathfrak{q}_r$ to contain x, and* (ii) *x is nilpotent if and only if $x \in \mathfrak{q}_n$.*

2.3.3. COROLLARY. — *$x^* |_{\mathfrak{q}} : \mathfrak{q} \to \mathbf{C}$ is given by $y \to \langle x_r, y \rangle$, and its kernel contains $[\mathfrak{q}_r, \mathfrak{q}_r] + \mathfrak{q}_n$.*

Proof. — Let $y \in \mathfrak{q}$. Since $\langle \mathfrak{q}_n, \mathfrak{q} \rangle = 0$ we have

$$x^*(y) = \langle x, y \rangle = \langle x_r + x_n, y \rangle = \langle x_r, y \rangle.$$

If $y \in \mathfrak{q}_n$ now $x^*(y) \in \langle x_r, \mathfrak{q}_n \rangle = 0$. If $y \in [\mathfrak{q}_r, \mathfrak{q}_r]$ then $x^*(y) = 0$ by Proposition 2.3.2

Q. E. D.

The following is the situation of our main applications

2.3.4. COROLLARY. — *If x is nilpotent then $x^* |_{\mathfrak{q}} = 0$.*

Proof. — Proposition 2.3.2 says $x = x_n$, so $x_r = 0$, and Corollary 2.3.3 says $x^* |_{\mathfrak{q}} = 0$.

Q. E. D.

2.4. PUKÁNSZKY CONDITION. — Let $\mathfrak{g}$ be any real Lie algebra. If $f \in \mathfrak{g}^*$ then a real (resp. complex) polarization for f is a subalgebra of $\mathfrak{g}$ (resp of $\mathfrak{g}_{\mathbf{C}}$) that is maximal among the subspaces totally isotropic for the form $b_f(x, y) = f[x, y]$. If $\mathfrak{g}$ is reductive this agrees with paragraph 2.2. Let G be the simply connected group with Lie algebra $\mathfrak{g}$. If G is solvable, and if $f \in \mathfrak{g}^*$ defines representations of G by polarizations and the orbit method [1], then the " Pukánszky condition " on such polarizations for f is the usual irreducibility condition for the corresponding representations. If G is reductive and $f = x^* \in \mathfrak{g}^*$, we are going to see that the " Pukánszky condition " on a polarization for f is equivalent to semisimplicity of x, and so it will not hold for most of the representations studied in this paper.

Let $\mathfrak{p}$ (resp. $\mathfrak{q}$) be a real (resp. complex) polarization for $f \in \mathfrak{g}^*$. In the real case set $\mathfrak{e} = \mathfrak{d} = \mathfrak{p}$. In the complex case set

$$\mathfrak{e} = (\mathfrak{q}+\bar{\mathfrak{q}}) \cap \mathfrak{g} \qquad \text{and} \qquad \mathfrak{d} = (\mathfrak{q} \cap \bar{\mathfrak{q}}) \cap \mathfrak{g} = \mathfrak{q} \cap \mathfrak{g}$$

where $^-$ is complex conjugation of $\mathfrak{g}_{\mathbf{C}}$ over $\mathfrak{g}$. Let D^0 denote the analytic subgroup of G for $\mathfrak{d}$. Then (*see* [27], Proposition 3.3.1.) $\mathrm{Ad}^*(D^0) f$ is an open subset of the affine subspace $f+\mathfrak{e}^\perp \subset \mathfrak{g}^*$. The polarization is said to satisfy the *Pukánszky condition* if $\mathrm{Ad}^*(D^0) f = f+\mathfrak{e}^\perp$, i. e., if $\mathrm{Ad}^*(D^0) f$ is closed in $\mathfrak{g}^*$. This is equivalent to $f+\mathfrak{e}^\perp$ being contained in the orbit $O_f = \mathrm{Ad}^*(G) f$.

2.4.1. THEOREM. — *Let $\mathfrak{g}$ be a reductive real Lie algebra, $x \in \mathfrak{g}$, and $\mathfrak{p}$ a real (resp. $\mathfrak{q}$ a complex) polarization for x. Then $\mathfrak{p}$ (resp. $\mathfrak{q}$) satisfies the Pukánszky condition if and only if x is semisimple.*

Proof. — In the real case set $\mathfrak{q} = \mathfrak{p}_{\mathbf{C}}$, so the Pukanszky condition for $\mathfrak{q}$ agrees with that for $\mathfrak{p}$.

If x is semisimple then $\mathfrak{q} = \mathfrak{q}_r + \mathfrak{q}_n$ with $\mathfrak{q}_r = \mathfrak{g}^x_{\mathbf{C}}$. For $\mathfrak{g}^x_{\mathbf{C}} \subset \mathfrak{q}_r$, $\mathfrak{q}$ is parabolic, and $\dim \mathfrak{q} = 1/2 (\dim \mathfrak{g}^x_{\mathbf{C}} + \dim \mathfrak{g}_{\mathbf{C}})$. So

$$\mathfrak{d} = (\mathfrak{q} \cap \bar{\mathfrak{q}}) \cap \mathfrak{g} = \mathfrak{g}^x + (\mathfrak{q}_n \cap \bar{\mathfrak{q}}_n) \cap \mathfrak{g}.$$

Set $\mathfrak{u} = \mathfrak{q}_n \cap \bar{\mathfrak{q}}_n \cap \mathfrak{g}$ so $U = \exp(\mathfrak{u})$ is a unipotent subgroup of G. Then $D^0 = U.(G^x)^0$, so $\mathrm{Ad}(D^0) x = \mathrm{Ad}(U) x$, which is closed in $\mathfrak{g}$ because unipotent orbits are closed. Now $\mathrm{Ad}^*(D^0) x^*$ is closed in $\mathfrak{g}^*$. That is the Pukánszky condition.

Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{q} \cap \mathfrak{g}$. Then $\mathfrak{q} = \mathfrak{q}_r + \mathfrak{q}_n$ where $\mathfrak{h}_{\mathbf{C}} \subset \mathfrak{q}_r$, and one checks (*see* [24], Lemma 2.10) that $\mathfrak{d}_{\mathbf{C}} = \mathfrak{q} \cap \bar{\mathfrak{q}}$ has reductive and unipotent parts given by

$$\mathfrak{d}_{r\mathbf{C}} = \mathfrak{q}_r \cap \bar{\mathfrak{q}}_r \qquad \text{and} \qquad \mathfrak{d}_{n\mathbf{C}} = \mathfrak{q}_r \cap \bar{\mathfrak{q}}_n \; + \; \mathfrak{q}_n \cap \bar{\mathfrak{q}}_r \; + \; \mathfrak{q}_n \cap \bar{\mathfrak{q}}_n.$$

Thus we have

$$x_r \in \mathfrak{d}_r \qquad \text{and} \qquad x_n \in (\mathfrak{q}_n \cap \bar{\mathfrak{q}}_n) \cap \mathfrak{g} \subset \mathfrak{d}_n.$$

Now let Φ_r and Φ_n be the sets of $\mathfrak{h}_{\mathbf{C}}$-roots such that

$$\mathfrak{q}_r = \mathfrak{h}_{\mathbf{C}} + \sum_{\alpha \in \Phi_r} \mathfrak{g}^\alpha \qquad \text{and} \qquad \mathfrak{q}_n = \sum_{\alpha \in \Phi_n} \mathfrak{g}_\alpha.$$

Thus $\mathfrak{q}_n \cap \bar{\mathfrak{q}}_n$ is the sum of the $\mathfrak{g}_\alpha$ with $\alpha \in \Phi_n \cap \overline{\Phi}_n$.

Let $\mathfrak{z}$ denote the center of $\mathfrak{d}_r$. Since $\mathfrak{d}_r$ is reductive and algebraic in $\mathfrak{g}$, we can split $\mathfrak{z} = \mathfrak{t}+\mathfrak{v}$ where the $\mathfrak{h}_{\mathbf{C}}$-roots are pure imaginary on $\mathfrak{t}$ and real on $\mathfrak{v}$. Now let $\mathfrak{g}_\alpha \subset \mathfrak{q}_n \cap \bar{\mathfrak{q}}_n$, that is $\alpha, \bar{\alpha} \in \Phi_n$. Then α and $\bar{\alpha}$ are nontrivial, with nonzero sum, on the center of $\mathfrak{q}_r$. This nontriviality follows on the larger subspace $\mathfrak{z}_\mathbf{C}$ of $\mathfrak{h}_\mathbf{C}$. We conclude $\alpha|_\mathfrak{v} \neq 0$. As these α all are contained in a positive root system, now we have $y \in \mathfrak{v}$ with $\alpha(y) > 0$ whenever $\mathfrak{g}_\alpha \subset \mathfrak{q}_n \cap \bar{\mathfrak{q}}_n$. In particular $\lim_{t \to \infty} \mathrm{Ad}(\exp(-ty))\, x_n = 0$.

Let V be the analytic subgroup of G for $\mathfrak{v}$. Then

$$\mathrm{Ad}(\mathrm{V})\,x = \mathrm{Ad}(\mathrm{V})(x_r + x_n) = x_r + \mathrm{Ad}(\mathrm{V})\,x_n$$

using $x_r \in \mathfrak{d}_r$. We just saw that 0 is in the closure of Ad (V) x_n. Now x_r is in the closure of Ad (V) x, hence in the closure of Ad (D^0) x. If x is not semisimple, we conclude that the Pukánszky condition fails.

Q. E. D.

2.5. Representations associated to invariant real polarizations. — Here G is a reductive Lie group that satisfies (2.1.1) and (2.1.2). Let $x \in \mathfrak{g}$.

A polarization for x is *invariant* if it is Ad (G^x)-stable. Now fix

(2.5.1) $\mathfrak{p}$: invariant real polarization for x.

As in (2.3.1), $\mathfrak{p} = \mathfrak{p}_r + \mathfrak{p}_n$ where $\mathfrak{p}_n$ is the nilradical and $\mathfrak{p}_r$ is a maximal reductive subalgebra, and $x = x_r + x_n$ accordingly. The parabolic subgroup of G for $\mathfrak{p}$ is

(2.5.2 *a*) $$\mathrm{P} = \{ g \in \mathrm{G} : \mathrm{Ad}(g)\mathfrak{p} = \mathfrak{p} \} = \{ g \in \mathrm{G} : \mathrm{Ad}(g)\mathfrak{p}_n = \mathfrak{p}_n \}.$$

From $\mathfrak{p} = \mathfrak{p}_r + \mathfrak{p}_n$ we get a semidirect product splitting

(2.5.2 *b*) $\mathrm{P} = \mathrm{P}_r . \mathrm{P}_n$ where $\mathrm{P}_n = \exp(\mathfrak{p}_n)$ unipotent and P_r is reductive.

Here $\mathrm{P}_r = \{ g \in \mathrm{G} : \mathrm{Ad}(g)\,\mathfrak{p}_r = \mathfrak{p}_r \}$ has Lie algebra $\mathfrak{p}_r$. Identity components satisfy $\mathrm{P}^0 = \mathrm{P}_r^0 . \mathrm{P}_n$.

We now require an integrality condition for x^* : there is a well-defined character on P^0 whose restriction to exp $(\mathfrak{p})$ is given by $\exp(y) \to e^{2\pi i x^*(y)}$. We formulate that as

(2.5.3 *a*) $2\pi i x^*$ integrates to a well-defined character $\exp(2\pi i x^*)$ on P^0.

Since $\exp(2\pi i x^*)$ is unitary and its kernel must contain $[\mathrm{P}^0, \mathrm{P}^0] = [\mathrm{P}_r^0, \mathrm{P}_r^0].\mathrm{P}_n$, (2.5.3 *a*) is equivalent to

(2.5.3 *b*) $2\pi i x^*$ integrates to a unitary character on $\mathrm{P}_r^0/[\mathrm{P}_r^0, \mathrm{P}_r^0]$.

If L is a locally compact group, we write $\hat{\mathrm{L}}$ for the set of all equivalence classes $[\lambda]$ of irreducible unitary representations λ of L. If M is a closed normal subgroup and $[\mu] \in \hat{\mathrm{M}}$ we write $\hat{\mathrm{L}}_\mu = \{ [\lambda] \in \hat{\mathrm{L}} : \lambda|_\mathrm{M} \text{ contains } \mu \}$. We are especially interested in cases arising from $\mathrm{Z}_\mathrm{G}(\mathrm{G}^0)^\wedge_\zeta$ where ζ is restriction of $\exp(2\pi i x^*)$ to $\mathrm{Z}_{\mathrm{G}^0}$.

Proposition 2.5.4 describes the extensions of $\exp(2\pi i x^*)$ from P^0 to ${}^x\mathrm{P}$, giving a finite-to-finite correspondence between them and the elements of $\mathrm{Z}_\mathrm{G}(\mathrm{G}^0)^\wedge_\zeta$.

2.5.4. Proposition. — *Denote* ${}^x\mathrm{P} = \mathrm{G}^x.\mathrm{P}^0$ *and* $\mathrm{P}^\dagger = \mathrm{Z}_\mathrm{G}(\mathrm{G}^0)\,\mathrm{P}^0$. *They are subgroups of finite index in* P *with* $\mathrm{P}^\dagger$ *normal and* $\mathrm{P}^\dagger \subset {}^x\mathrm{P}$. *If* $[\xi] \in ({}^x\mathrm{P})^\wedge$ *then the following conditions are equivalent.*

1. $\exp(2\pi i x^*)$ *is weakly contained in* $\xi|_{\mathrm{P}^0}$.

2. $\xi|_{P^0}$ *is a multiple* (*type* I *primary*) *of* $\exp(2\pi ix^*)$.

3. $[\xi]$ *is a subrepresentation class of a unitarily induced class*

$$[\operatorname{Ind}_{P^\dagger \uparrow {}^x P}(\gamma \otimes \exp(2\pi ix^*))]$$

where $\gamma \in Z_G(G^0)^\wedge_\zeta$ *with* $\zeta = \exp(2\pi ix^*)|_{Z_{G^0}}$.

Proof. — The adjoint representation maps G onto the real linear algebraic group $\overline{G} = G/Z_G(G^0)$. Since $P/Z_G(G^0)$ is a parabolic subgroup of $\overline{G}$, it has only finitely many components; so $|P/P^\dagger| < \infty$. Normality of $P^\dagger$ in P is clear.

Invariance of $\mathfrak{p}$ says that ${}^xP$ is a subgroup of P. Evidently $Z_G(G^0) \subset G^x$, so $P^\dagger \subset {}^xP$ and $|P/{}^xP| < \infty$.

Let ζ be the Z_{G^0}-restriction of $\exp(2\pi ix^*)$. The group P^0 is of type I because it is a central extension of the linear algebraic group $\operatorname{ad}_G(P^0) = (P/Z_G(G^0))^0$. Thus

$$(P^\dagger)^\wedge_\beta = \{[\gamma \otimes \psi^0] : [\gamma] \in Z_G(G^0)^\wedge_\beta \text{ and } [\psi^0] \in (P^0)^\wedge_\beta\}$$

for every $\beta \in \hat{Z}_{G^0}$, and $(P^\dagger)^\wedge$ is the union of these $(P^\dagger)^\wedge_\beta$. If $[\psi] \in (P^\dagger)^\wedge$ we now have equivalence of (i) $\exp(2\pi ix^*)$ is weakly contained in $\psi|_{P^0}$, (ii) $\psi|_{P^0}$ is a discrete multiple of $\exp(2\pi ix^*)$, (iii) $[\psi] = [\gamma \otimes \exp(2\pi ix^*)]$ for some $\gamma \in Z_G(G^0)^\wedge_\zeta$. Let $[\xi] \in ({}^xP)^\wedge$. Then $[\xi]$ is a subclass of some $[\operatorname{Ind}_{P^\dagger \uparrow {}^xP}(\psi)]$, and condition 1 (resp. 2, resp. 3) for $[\xi]$ is equivalent to condition (i) [resp. (ii), resp. (iii)] for $[\psi]$.

Q. E. D.

The *representations* of G *associated* to x and its invariant real polarization $\mathfrak{p}$ are the unitarily induced

(2.5.5) $\quad \pi_{x,\mathfrak{p},\xi} = \operatorname{Ind}_{{}^xP \uparrow G}(\xi) \quad$ where $[\xi] \in ({}^xP)^\wedge$ extends $\exp(2\pi ix^*) \in (P^0)^\wedge$.

The representations ξ are obtained from $Z_G(G^0)^\wedge_\zeta$ as in Proposition 2.5.4, and thus $Z_G(G^0)^\wedge_\zeta$ gives the $\pi_{x,\mathfrak{p},\xi}$.

2.6. Reciprocity formulæ. — We express the representations $\pi_{x,\mathfrak{p},\xi}$ in terms of representations induced from the parabolic subgroup P of G.

By *Cartan involution* of G we mean an involutive automorphism θ such that $K = \{g \in G : \theta(g) = g\}$ satisfies $Z_G(G^0) \subset K$ with $K/Z_G(G^0)$ maximal compact subgroup of $G/Z_G(G^0)$. Thus $\langle\, ,\, \rangle$ is θ-invariant and is negative (resp. positive) definite on the $+1$ (resp. -1) eigenspace of θ. *See* [25], Lemmas 4.1.1 and 4.1.2 for existence of Cartan involutions in our context.

2.6.1. Lemma. — *Let* P *be a parabolic subgroup of* G *and* P_r *its reductive part as in* (2.5.2 *b*). *Then* G *has a Cartan involution* θ *with* $\theta(P_r) = P_r$. *Define*

$$\mathfrak{a} = \{y \in (\textit{center of } \mathfrak{p}_r) : \theta(y) = -y\} \qquad \text{and} \qquad A = \exp(\mathfrak{a}).$$

Then $P_r = Z_G(A)$, *centralizer of* A *in* G, *and* P_r *has a unique closed* θ-*stable subgroup* M *such that* $P_r = M \times A$.

Proof. — The argument for the cuspidal case ([25], Lemma 4.1.5) extends without difficulty.

Q. E. D.

Lemma 2.6.1 gives us decompositions of P in the sense of smooth unique factorization :

(2.6.2) $$P = MAN \quad \text{where} \quad N = P_n \text{ unipotent radical.}$$

Now suppose that we have

(2.6.3) $$[\eta] \in \hat{M} \quad \text{and} \quad \sigma \in \mathfrak{a}^*, \qquad \text{i.e.} \quad [\eta \otimes e^{i\sigma}] \in (M \times A)^\wedge = \hat{P}_r.$$

We view $[\eta \otimes e^{i\sigma}] \in \hat{P}$ by $(\eta \otimes e^{i\sigma})(man) = e^{i\sigma}(a)\,\eta(m)$.

Then the induced representation of G is denoted

(2.6.4) $$\pi_{P,\eta,\sigma} = \mathrm{Ind}_{P\uparrow G}(\eta \otimes e^{i\sigma}).$$

The various series of unitary representations of G that occur in the Plancherel formula ([9], [10], [25]) are special cases of these $\pi_{P,\eta,\sigma}$. We now express the $\pi_{x,\mathfrak{p},\xi}$ of (2.5.5) in terms of the $\pi_{P,\eta,\sigma}$.

In analogy to the splitting P = MAN we decompose

(2.6.5 *a*) $${}^x P = {}^x M \;\; AN \qquad \text{where} \quad {}^x M = {}^x P \cap M$$

and

(2.6.5 *b*) $$P^\dagger = M^\dagger AN \qquad \text{where} \quad M^\dagger = P^\dagger \cap M = Z_G(G^0)\,M^0.$$

Let $\zeta = \exp(2\pi i x^*)\big|_{Z_{G^0}}$ and $\gamma \in Z_G(G^0)^\wedge_\zeta$. We are going to prove

2.6.6. Proposition. — *If* $[\psi] \in ({}^x M)^\wedge$ *and* $[\eta] \in \hat{M}$, *then the multiplicities*

$$n_\psi = \mathrm{mult}(\gamma \otimes \exp(2\pi i x^*)\big|_{M^\dagger}, \psi\big|_{M^\dagger}) \qquad \text{and} \qquad m_\eta = \mathrm{mult}(\gamma \otimes \exp(2\pi i x^*)\big|_{M^\dagger}, \eta\big|_{M^\dagger})$$

are finite, and

$$\sum n_\psi \cdot \pi_{x,\mathfrak{p},\exp(2\pi i x^*)|a)} = \sum m_\eta \cdot \pi_{P,\eta,2\pi i x^*|\mathfrak{a}}.$$

These sums are finite.

Proof. — Let ${}^x\varphi = \mathrm{Ind}_{P^\dagger \uparrow {}^x P}(\gamma \otimes \exp(2\pi i x^*))$ and $\varphi = \mathrm{Ind}_{P^\dagger \uparrow P}(\gamma \otimes \exp(2\pi i x^*))$. Since $\gamma \otimes \exp(2\pi i x^*)$ is a finite dimensional representation, and since

$$|P/{}^x P| \leqq |P/P^\dagger| < \infty,$$

both ${}^x\varphi$ and φ are finite dimensional. Now Frobenius' original version (block form matrices) of Frobenius' Reciprocity applies. In particular the n_ψ and the m_η are finite, and we have finite decompositions

$${}^x\varphi = \sum p_{\psi,\sigma} \cdot \psi \otimes e^{2\pi i \sigma} \qquad \text{and} \qquad \varphi = \sum q_{\eta,\sigma}\, \eta \otimes e^{2\pi i \sigma}$$

where $[\psi] \in ({}^x M)^\wedge$, $[\eta] \in \hat{M}$ and $\sigma \in \mathfrak{a}^*$.

Let $\{ b_j \}$ be a system of coset representatives of M modulo $M^\dagger$, thus also of P modulo $P^\dagger$. We break up

$$\gamma \otimes \exp(2\pi ix^*) = \eta^\dagger \otimes \exp(2\pi ix^* |_{\mathfrak{a}}) \qquad \text{where} \quad [\eta^\dagger] \in (M^\dagger)^\wedge.$$

Then

$$\varphi |_{P^\dagger} = \sum (\eta^\dagger \otimes \exp(2\pi ix^* |_{\mathfrak{a}})) \circ \mathrm{Ad}(b_j) = \sum (\eta^\dagger . \mathrm{Ad}(b_j)) \otimes \exp(2\pi ix^* |_{\mathfrak{a}}).$$

First, this shows $q_{\eta,\sigma} = 0$ unless $\sigma = x^* |_{\mathfrak{a}}$, so $\varphi = \sum q_\eta . \eta \otimes \exp(2\pi ix^* |_{\mathfrak{a}})$. Second, Frobenius' Reciprocity gives us

$$q_\eta = \mathrm{mult}(\eta, \mathrm{Ind}_{M^\dagger \uparrow M}(\eta^\dagger)) = \mathrm{mult}(\eta^\dagger, \eta |_{M^\dagger}) = m_\eta.$$

We conclude

$$\mathrm{Ind}_{P^\dagger \uparrow P}(\gamma \otimes \exp(2\pi ix^*)) = \sum m_\eta . \eta \otimes \exp(2\pi ix^* |_{\mathfrak{a}}).$$

Similarly

$$\mathrm{Ind}_{P^\dagger \uparrow {}^x P}(\gamma \otimes \exp(2\pi ix^*)) = \sum n_\psi . \psi \otimes \exp(2\pi ix^* |_{\mathfrak{a}}).$$

Combining these with induction by stages

$$\begin{aligned}\sum m_\eta . \pi_{P,\eta,2\pi x^* |\mathfrak{a}} &= \mathrm{Ind}_{P \uparrow G}(\mathrm{Ind}_{P^\dagger \uparrow P}(\gamma \otimes \exp(2\pi ix^*))) \\ &= \mathrm{Ind}_{P^\dagger \uparrow G}(\gamma \otimes \exp(2\pi ix^*)) = \mathrm{Ind}_{{}^x P \uparrow G}(\mathrm{Ind}_{P^\dagger \uparrow {}^x P}(\gamma \otimes \exp(2\pi ix^*))) \\ &= \sum n_\psi . \pi_{x,p,\psi} \otimes \exp(2\pi ix^* \mid \mathfrak{a}).\end{aligned}$$

Q. E. D.

3. Representations associated to nilpotent orbits

We study representations associated to real parabolic polarizations for nilpotent elements.

3.1. GENERALITIES ON NILPOTENT ELEMENTS. — We review the basic facts on conjugacy, centralizers and polarizations of nilpotent elements in a reductive Lie algebra. *See* [21], [15], [7], [18] and [22] for proofs.

The Jacobson-Morosov embedding theorem : if $e \in \mathfrak{g}$ is nonzero nilpotent, then there exist $h, f \in \mathfrak{g}$ such that

$$(3.1.1) \qquad [h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

The real span $\{ h, e, f \}_{\mathbf{R}}$ is a three-dimensional simple subalgebra (TDS) of $\mathfrak{g}$, isomorphic to $\mathfrak{sl}(2; \mathbf{R})$ under

$$(3.1.2) \qquad h \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We call h a *neutral element* for e. Given e, any two TDS $\{ h, e, f \}_{\mathbf{R}}$ are Int $(\mathfrak{g})$-conjugate. If $e' \in \mathfrak{g}$ is another nilpotent element with the same neutral element h, then e and e' are Int $(\mathfrak{g}_{\mathbf{C}})$-conjugate but not necessarily Int $(\mathfrak{g})$-conjugate.

Fix a nilpotent element $e \in \mathfrak{g}$ and a TDS $\{h, e, f\}_{\mathbf{R}}$ as in (3.1.1). Decompose $\mathfrak{g} = \sum \mathfrak{g}_j$ where the $\mathfrak{g}_j$ are the irreducible $\{h, e, f\}_{\mathbf{R}}$-modules under its adjoint action on $\mathfrak{g}$. Then $\mathfrak{g}^e \cap \mathfrak{g}_j$ is the 1-dimensional space of highest eigenvectors of ad (h) on $\mathfrak{g}_j$. Denote c_+ (resp. c_-) as the number of even (resp. odd) dimensional $\mathfrak{g}_j$.

Writing $\mathfrak{g}^{h,\lambda}$ for the λ-eigenspace of ad (h), now

$$(3.1.3) \qquad c_- = \dim \mathfrak{g}^{h,0} = \dim \mathfrak{g}^h, \qquad c_+ = \dim \mathfrak{g}^{h,1}, \qquad c_+ + c_- = \dim \mathfrak{g}^e.$$

Choose a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$. Thus there is a Cartan involution θ of a group G with Lie algebra $\mathfrak{g}$, such that $\mathfrak{k}$ and $\mathfrak{s}$ are the $(+1)$-and (-1)-eigenspaces of θ on $\mathfrak{g}$. Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{s}$ and choose

$$(3.1.4\,a) \qquad \{\alpha_1, \ldots, \alpha_r\} : \text{ simple } \mathfrak{a}\text{-root system on } \mathfrak{g},$$

$$(3.1.4\,b) \qquad \{a_1, \ldots, a_r\} : \text{ dual basis of } \mathfrak{a}, \qquad \text{i.e. } \alpha_j(a_k) = \delta_{jk}.$$

Then h is Int $(\mathfrak{g})$-conjugate to just one $\sum n_i a_i$ where each n_i is 0, 1 or 2. Note dim $\mathfrak{g}_j$ even $\Leftrightarrow \mathfrak{g}^h \cap \mathfrak{g}_j = 0 \Leftrightarrow \mathfrak{g}^{h,1} \cap \mathfrak{g}_j \neq 0$. Thus we have equivalence of

(3.1.5 *a*) each n_i is even, i. e. is 0 or 2;

(3.1.5 *b*) each irreducible $\{h, e, f\}_{\mathbf{R}}$-module $\mathfrak{g}_j$ has odd dimension;

$$(3.1.5\,c) \qquad c_+ = 0, \qquad \text{i.e. } \mathfrak{g}^{h,1} = 0, \qquad \text{i.e. } \dim \mathfrak{g}^e = \dim \mathfrak{g}^h.$$

Under conditions (3.1.5) we say that e is *even*.

Now let $e \in \mathfrak{g}$ be an even nilpotent element, h a neutral element for e, and $\mathfrak{p}$ the sum of the non-negative eigenspaces of ad h. Clearly $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, and it is known ([22], Proposition 2.2) that $\mathfrak{p}$ is actually an invariant polarization for e. This polarization, which is unique up to conjugacy, will be called the *natural polarization* for e. Since the nilpotent elements whose orbits maximal possible dimension (regular nilpotents) are all even [15], it follows that every reductive Lie algebra (with non-trivial semisimple part) contains a non-zero nilpotent e with a real invariant polarization.

3.2. Generalities on characters. — Let G be a reductive Lie group of the class described in paragraph 2.1. We recall some basic facts from Harish-Chandra's general character theory. *See* [25], paragraph 3.2 for more details, [23] for complete details.

If $[\pi] \in \hat{G}$ and $f \in C_c^\infty(G)$ then $\pi(f) = \int_G f(g)\,\pi(g)\,dg$ is a trace class operator on the representation space H_π, and

$$(3.2.1) \qquad \Theta_\pi : C_c^\infty(G) \to C \quad \text{by } \Theta_\pi(f) = \operatorname{trace} \pi(f)$$

is a Schwartz distribution on G. Θ_π is the *global* or *distribution character* of $[\pi]$. Classes $[\pi] = [\pi']$ if and only if $\Theta_\pi = \Theta_{\pi'}$.

Let $\mathfrak{G}$ be the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$ and let $\mathfrak{Z}$ be the center of $\mathfrak{G}$. Hypothesis (2.1.1) says that $\mathfrak{Z}$ is the algebra of bi-invariant differential operators on G.

$\mathfrak{Z}$ acts on distributions by $(z\,\Theta)(f) = \Theta({}^t z\,.\,f)$ where ${}^t z$ is transpose. If $[\pi] \in \hat{G}$, then Θ_π is an eigendistribution of $\mathfrak{Z}$, and

$$\text{(3.2.2)} \qquad \chi_\pi : \mathfrak{Z} \to \mathbf{C} \quad \text{by } \chi_\pi(z)\,\Theta_\pi = z\,\Theta_\pi$$

is an associative algebra homomorphism called the *infinitesimal character* of $[\pi]$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathscr{I}(\mathfrak{h}_{\mathbf{C}})$ the polynomials on $\mathfrak{h}_{\mathbf{C}}^*$ invariant by the complex Weyl group $W(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. There is an isomorphism $\gamma : \mathfrak{Z} \to \mathscr{I}(\mathfrak{h}_{\mathbf{C}})$ such that the homomorphisms $\mathfrak{Z} \to \mathbf{C}$ are precisely the

$$\text{(3.2.3)} \qquad \chi_\lambda : \mathfrak{Z} \to \mathbf{C} \quad \text{by } \chi_\lambda(z) = [\gamma(z)](\lambda), \qquad \lambda \in \mathfrak{h}_{\mathbf{C}}^*.$$

Further $\chi_\lambda = \chi_{\lambda'}$ iff $\lambda' \in W(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})(\lambda)$. If ρ is half the sum of a positive root system then $\chi_\lambda(\text{Casimir}) = \|\lambda\|^2 - \|\rho\|^2$.

The structure (3.2.3) for the differential equations (3.2.2) shows that Θ_π is a locally L_1 function analytic on the regular set of G, and that $[\pi] \to \chi_\pi$ is finite-to-one in case $|G/G^0| < \infty$.

3.3. Characters of representations associated to cuspidal parabolic polarizations. — Let G be a Lie group that satisfies (2.1.1) and (2.1.2). A parabolic subgroup $P \subset G$ is called *cuspidal* if $P_r/Z_{P_r}(P_r^0)$ has a compact Cartan subgroup. In that case, $P_r = M \times A$ where A is the split component of the center of P_r^0, $Z_G(G^0) \subset M$ $M/Z_G(G^0)$ has a compact Cartan subgroup $T/Z_G(G^0)$, and M inherits (2.1.1) and (2.1.2) from G with the same group Z. We say that P is *associated* to the conjugacy class of the Cartan subgroup $H = T \times A$ of G, and we write $P = MAN$ with $N = P_n$.

3.3.1. Theorem. — *Let $e \in \mathfrak{g}$ nilpotent. Let $P = MAN$ be a cuspidal parabolic subgroup of G associated to the conjugacy class of a Cartan subgroup $H = T \times A$. Suppose that $\mathfrak{p}$ is an invariant polarization for e. Then e satisfies the integrality condition* (2.5.3).

1. *The representations of* G *associated to e^* and $\mathfrak{p}$ are the $\pi_{e,\mathfrak{p},\xi} = \operatorname{Ind}_{{}^eP\uparrow G}(\xi)$ where $[\xi] \in ({}^eP/P^0)^\wedge$, i. e. where $[\xi] \in ({}^eP)^\wedge$ with P^0 in its kernel.*

2. *Let $[\gamma] \in (Z_G(G^0)/Z_{G^0})^\wedge = (Z_G(G^0)P^0/P^0)^\wedge$. Retain* (2.5.2) :

$$\pi_{P,\eta,\sigma} = \operatorname{Ind}_{MAN\uparrow G}(\eta \otimes e^{i\sigma}).$$

Then

$$\sum_{[\xi] \in ({}^eP/P^0)^\wedge} m(\gamma, \xi|_{Z_G(G^0)P^0})\,\pi_{e,\mathfrak{p},\xi} = \sum_{[\eta] \in (M/M^0)^\wedge} m(\gamma, \eta|_{Z_G(G^0)})\,\pi_{P,\eta,0},$$

finite sums with finite multiplicities

3. *Each $\pi_{e,\mathfrak{p},\xi}$ is a finite sum of irreducible representations.*

4. *Each $\pi_{e,\mathfrak{p},\xi}$ has infinitesimal character χ_{ρ_M} relative to $\mathfrak{h}$, as follows. $\rho_M \in i\,\mathfrak{t}^* \subset \mathfrak{h}_{\mathbf{C}}^*$ is half the sum of the elements of a positive $\mathfrak{t}_{\mathbf{C}}$-root system of $\mathfrak{m}_{\mathbf{C}}$. Thus, if ρ is half the sum over a positive $\mathfrak{h}_{\mathbf{C}}$-root system of $\mathfrak{g}_{\mathbf{C}}$, $\pi_{e,\mathfrak{p},\xi}$ sends the Casimir element of $\mathfrak{G}$ to $\|\rho_M\|^2 - \|\rho\|^2$.*

Proof. — Since e is nilpotent. Corollary 2.3.5 says $e^*|_{\mathfrak{p}} = 0$, so $\exp(2\pi i e^*)$ is the trivial representation of P^0. In particular, e satisfies the integrality condition (2.5.3). Now the classes $[\xi] \in ({}^e P)^\wedge$ with $\pi_{e, \mathfrak{p}, \xi}$ associated to e^* and $\mathfrak{p}$, are just the ^{e}P-lifts of the elements of $({}^e P/P^0)^\wedge$. That proves (1), and now (2) follows from Proposition 2.6.6.

If $[\eta] \in \hat{M}$ and $\sigma \in \mathfrak{a}^*$ then ([25], Theorem 4.3.8) $\pi_{P, \eta, \sigma}$ is a finite sum of irreducible representations. Now (3) follows from (2).

Let χ_ν be the infinitesimal character of $[\eta] \in \hat{M}$ relative to $\mathfrak{t}$. If $\sigma \in \mathfrak{a}^*$ then ([25], Theorem 4.3.8) $\pi_{P, \eta, \sigma}$ has infinitesimal character $\chi_{\nu + i\sigma}$ relative to $\mathfrak{h}$. In our case, η annihilates M^0 and $\sigma = 0$. That η kills M^0 means that $\chi_\nu = \chi_\eta$ kills all elements of positive degree in the center of the universal enveloping algebra of $\mathfrak{m}_{\mathbf{C}}$. In other words, χ_ν would be denoted χ_0 in Harish-Chandra's earlier work ([8], Theorem 5). As we are using Harish-Chandra's more recent convention ([9], [10]), now $\nu = \rho_M$. Thus $\chi_{\pi_{e, \mathfrak{p}, \xi}} = \chi_{\rho_M}$ as asserted.

Q. E. D.

3.3.2. Corollary. — *Let $\mathfrak{p}$ be an invariant cuspidal parabolic polarization for a nilpotent element $e \in \mathfrak{g}$.*

If $[\xi] \in ({}^e P/P^0)^\wedge$ then the representation $\pi_{e, \mathfrak{p}, \xi}$ is CCR, *i. e. it sends every $f \in L_1(G)$ to a compact operator.*

Proof. — Every class $[\pi] \in \hat{G}$ is CCR; this is how one shows that G is of type I. *See* [25], paragraph 3.2 for a discussion. Theorem 3.3.1 shows that $[\pi_{e, \mathfrak{p}, \xi}]$ is a finite sum of irreducible classes, so it is CCR.

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Corollary 3.3.2 gives examples of CCR representations associated to non-closed Ad* (G)-orbits in $\mathfrak{g}^*$. This contrasts with the case of solvable groups, where one expects [17] that the representations associated to a co-adjoint orbit should be CCR if and only if the orbit is closed.

4. An example of dependance on polarization

In this section we give an example of a nilpotent element e with two invariant polarizations $\mathfrak{p}\{\alpha_i\}$ which are cuspidal parabolic subalgebras, such that the representations associated to $\mathfrak{p}\{\alpha_1\}$ and $\mathfrak{p}\{\alpha_2\}$ respectively have different infinitesimal characters, and in particular have no equivalent subquotients. This gives an example, in the setting of semisimple groups, in which the representations depend very strongly on the choice of polarization. By contrast, results of Dixmier, Kirillov, Pukánskzy and Duflo (*see* [4], [5], [6], [17], [19], and [20]) show that such a phenomenon cannot occur in the setting of solvable groups.

We also show independence of polarization for a closely related nilpotent.

4.1. THE COUNTER-EXAMPLE. — Let $\mathfrak{g}$ be the normal form of G_2, and $\mathfrak{h}$ a split Cartan subalgebra, whose Dynkin diagram is $\overset{\alpha_1}{\circ}\!\Lleftarrow\!\!\equiv\!\overset{\alpha_2}{\bullet}$ with $\langle \alpha_1, \alpha_1 \rangle = 3$, $\langle \alpha_2, \alpha_2 \rangle = 1$. Choose root vectors e_{α_1} and $e_{\alpha_1+\alpha_2}$ in the real form $\mathfrak{g}$ and let $e_{\alpha_1+2\alpha_2} = [e_{\alpha_2}, e_{\alpha_1+\alpha_2}]$, $e_{\alpha_1+3\alpha_2} = [e_{\alpha_2}, e_{\alpha_1+2\alpha_2}]$. Write a_1, a_2 for the duals of α_1, α_2 as in (3.1.4 *b*). Let $\mathfrak{p}\{\alpha_i\}$ be the parabolic subalgebra spanned by the positive root vectors, the Cartan subalgebra $\mathfrak{h}$ (spanned by a_1, a_2) and the semisimple subalgebra $\mathfrak{g}[\alpha_i]$ spanned by e_{α_i}, $e_{-\alpha_i}$ and $h_{\alpha_i} = [e_{\alpha_i}, e_{-\alpha_i}]$. Let $G_{\mathbf{C}}$ be the simply connected complex group corresponding to $\mathfrak{g}_{\mathbf{C}}$. Then $G_{\mathbf{C}}$ is also the adjoint group, ([3], § 8), and G, the connected subgroup of $G_{\mathbf{C}}$ corresponding to $\mathfrak{g}$, is the adjoint group of $\mathfrak{g}$. Write $P\{\alpha_i\}$ for the parabolic subgroup of G corresponding to $\mathfrak{p}\{\alpha_i\}$.

4.1.1. THEOREM. — *In the notation as above, let $\mathfrak{g}$ be the normal real form of* G_2 *and*

$$e = e_{\alpha_1+\alpha_2} + e_{\alpha_1+3\alpha_2} \in \mathfrak{g}.$$

Then $\mathfrak{p}\{\alpha_1\}$ and $\mathfrak{p}\{\alpha_2\}$ are both invariant cuspidal parabolic polarizations for e. For any $\xi_i \in ({}^e P\{\alpha_i\}/P\{\alpha_i\}^0)^\wedge$, $i = 1, 2$, the representations $\pi_{e,\mathfrak{p}\{\alpha_i\},\xi_i}$ have infinitesimal character $\chi_{(1/2)\alpha_i}$. In particular, the representations $\pi_{e,\mathfrak{p}\{\alpha_1\},\xi_1}$ and $\pi_{e,\mathfrak{p}\{\alpha_2\},\xi_2}$ are disjoint (*have no equivalent subquotients*) *for any ξ_1, ξ_2.*

The theorem will be proved in 4.2 and 4.3. Note first that $h = 2a_1$ is a neutral element for e, so that $\mathfrak{p}\{\alpha_2\}$ is the natural polarization for e and is therefore invariant. The main part of the proof of Theorem 4.1.1 consists of showing that $\mathfrak{p}\{\alpha_1\}$ is also invariant (*see* § 4.2). The claim regarding the infinitesimal character is proved in paragraph 4.3 using Theorem 3.3.1 (4).

4.2. CALCULATION OF G^e. — We show that G^e normalizes $\mathfrak{p}\{\alpha_1\}$. Since $(G^e)^0$ normalizes $\mathfrak{p}\{\alpha_1\}$ because $\mathfrak{p}\{\alpha_1\} \supset \mathfrak{g}^e$, it suffices to prove the claim for representatives of $G^e/(G^e)^0$. The following general lemma shows that these representatives may be chosen in $G^h \cap G^e$.

4.2.1. LEMMA. — *Let $\mathfrak{g}$ be a semisimple Lie algebra with G a corresponding connected group. Then if $e \in \mathfrak{g}$ is nilpotent and h is a neutral element for e, then*

$$G^e = (G^h \cap G^e).(G^e)^0.$$

Proof. – *See* [18], Lemma 3.2.

Since h is semisimple, $G^h_{\mathbf{C}}$ is connected ([13], § 2, Lemma 5), and its Lie algebra is $\mathfrak{g}^h_{\mathbf{C}} = a_1\mathbf{C} \oplus \mathfrak{g}_{\mathbf{C}}[\alpha_2]$. Therefore, $G^h_{\mathbf{C}} = G_{\mathbf{C}}[\alpha_2].\exp(\mathbf{C}a_1)$, where $G_{\mathbf{C}}[\alpha_2]$ is the connected subgroup of $G_{\mathbf{C}}$ corresponding to $\mathfrak{g}_{\mathbf{C}}[\alpha_2]$.

Now suppose $g \in G^h$, $g = g_1 \exp c\,a_1$ where $g_1 \in G_{\mathbf{C}}[\alpha_2]$, $c \in \mathbf{C}$. Since g normalizes $\mathfrak{g}^h$ and therefore $\mathfrak{g}[\alpha_2]$ and $\exp c\,a_1$ acts trivially on $\mathfrak{g}[\alpha_2]$, g_1 normalizes $\mathfrak{g}[\alpha_2]$. Then $g_1 \in F.G[\alpha_2]$, where F is the finite group generated by $\exp[(\pi i/2)h_{\alpha_2}]$ and $G[\alpha_2]$ is the subgroup of G corresponding to $\mathfrak{g}[\alpha_2]$ (Matsumato [16]). So

$$G^h \subset \exp(\mathbf{C}a_1).F.G[\alpha_2].$$

To find $G^h \cap G^e$ it is convenient to use the Bruhat decomposition of $G[\alpha_2]$,

$$G[\alpha_2] = M_2 A_2 N_2 \cup M_2 A_2 N_2 s_2 M_2 A_2 N_2,$$

where $A_2 = \exp \mathbf{R} h_{\alpha_2}$, $N_2 = \exp \mathbf{R} e_{\alpha_2}$, $M_2 = \{ 1, \exp \pi i h_{\alpha_2} \}$, and s_2 is the Weyl group reflection around α_2. We first show that no element of the form $g = hg_1$, where $h \in \exp \mathbf{C} a_1 . F$, $g_1 \in M_2 A_2 N_2 s_2 M_2 A_2 N_2$, can centralize e. For let

$$g_1 = m (\exp t_1 e_{\alpha_2}) (\exp t_2 h_{\alpha_2}) s_2 (\exp t_3 h_{\alpha_2}) (\exp t_4 e_{\alpha_2})$$

where t_1, t_2, t_3, t_4 are real constants. (Note that M_2 is central in $G[\alpha_2]$.) Then

$$\begin{aligned} \operatorname{Ad}(g).e &= \operatorname{Ad}(hg_1).(e_{\alpha_1+\alpha_2}+e_{\alpha_1+3\alpha_2}) \\ &= \operatorname{Ad}(hm(\exp t_1 e_{\alpha_2})(\exp t_2 h_{\alpha_2}) s_2 (\exp t_3 h_{\alpha_2})).(e_{\alpha_1+\alpha_2}+t_4 e_{\alpha_1+2\alpha_2}+\frac{1+t_4^2}{2} e_{\alpha_1+3\alpha_2}) \end{aligned}$$

by the choice of $e_{\alpha_1+2\alpha_2}$ and $e_{\alpha_1+3\alpha_2}$. Then

$$\operatorname{Ad}(g).e = \operatorname{Ad}(hm(\exp t_1 e_{\alpha_2})(\exp t_2 h_{\alpha_2}) s_2)(r_1 e_{\alpha_1+\alpha_2}+r_2 e_{\alpha_1+2\alpha_2}+r_3 e_{\alpha_1+3\alpha_2})$$

where r_1 and r_3 are non-zero real numbers and $r_2 = 0$ iff $t_4 = 0$. Now

$$\operatorname{Ad}(g).e = \operatorname{Ad}(hm(\exp t_1 e_{\alpha_2})(\exp t_2 h_{\alpha_2})).(r'_1 e_{\alpha_1+2\alpha_2}+r'_2 e_{\alpha_1+\alpha_2}+r'_3 e_{\alpha_1}),$$

$r'_i \in \mathrm{R}$, $r'_1 \neq 0 \neq r'_3$. It is now clear that $\operatorname{Ad}(g) e$ will have as summand a non-zero multiple of e_{α_1}, and therefore cannot equal e.

The preceding calculation shows that if $g \in G^h \cap G^e$, then $g = hg_2$, where $g_2 \in M_2 A_2 N_2$ and $h \in \exp \mathbf{C} a_1 . F$. Since M_2, A_2, N_2, $\exp \mathbf{C} a_1$ and F all normalize $\mathfrak{p}\{\alpha_2\}$ this already shows that $\mathfrak{p}\{\alpha_2\}$ is invariant. For completeness, we shall find representatives for the connected components of G^e. If $\operatorname{Ad}(g) e = e$, then by the preceding we may write $g = g_1 g_2$, where $g_1 = \exp(c_1 a_1 + c_2 a_2)$ and $g_2 = \exp t e_{\alpha_2}$ where t is a real number. [Note that M_2 and A_2 are contained in $\exp(\mathbf{C} a_1 + \mathbf{C} a_2)$.] Then

$$\operatorname{Ad}(g) e = \operatorname{Ad}(\exp(c_1 a_1 + c_2 a_2)).\left(e_{\alpha_1+\alpha_2}+t e_{\alpha_1+2\alpha_2}+\frac{t^2}{2} e_{\alpha_1+3\alpha_2}+e_{\alpha_1+3\alpha_2}\right).$$

Since $\operatorname{Ad}(\exp(c_1 a_1 + c_2 a_2))$ can only change the coefficients of the e_α's, $\operatorname{Ad}(g) e = e$ implies $t = 0$. Now

$$\operatorname{Ad}(\exp(c_1 a_1 + c_2 a_2)) e = \exp(c_1+c_2) e_{\alpha_1+\alpha_2} + \exp(c_1 + 3 c_2) e_{\alpha_1+3\alpha_2}.$$

This shows $c_1 + c_2 = 2\pi i n$ and $c_1 + 3 c_2 = 2 \pi i m$ for integers n, m. Therefore $c_1 = \pi i k_1$, $c_2 = \pi i k_2$, k_1, k_2 integers, with k_1, k_2 either both odd or even. In other words the components of G^e are represented by 1 and $(\exp \pi i a_1)(\exp \pi i a_2)$.

4.3. INFINITESIMAL CHARACTERS OF $\pi_{e, \mathfrak{p}[\alpha_i], \xi_i}$. — By Theorem 3.3.1 (4) each $\pi_{e, \mathfrak{p}\{\alpha_i\}, \xi_i}$ has infinitesimal character $\chi_{\rho_{M_i}}$ where ρ_{M_i} is half the sum of the elements of a positive root system on $\mathfrak{m}_{\mathbf{C}}[\alpha_i]$. Since

$$\mathfrak{m}_{\mathbf{C}}[\alpha_i] = \mathfrak{g}_{\mathbf{C}}[\alpha_i],$$

$\rho_{M_i} = (1/2)\alpha_i$ as claimed. Now $\pi_{e,\mathfrak{p}\{\alpha_1\},\xi_1}$ and $\pi_{e,\mathfrak{p}\{\alpha_2\},\xi_2}$ can have equivalent subquotients only if ρ_{M_1} and ρ_{M_2} are conjugate under the complex Weyl group ([8], Theorem 5 and the remark at the end of the proof of Theorem 3.3.1 above). Since $\langle \alpha_1, \alpha_1 \rangle \neq \langle \alpha_2, \alpha_2 \rangle$, this is impossible, which completes the proof of Theorem 4.1.1.

Q. E. D.

We note here that there are many other examples of pairs of parabolic polarizations of nilpotent elements which are not conjugate, or even associated, for example in the split Lie algebras of types B_2 and F_4. However, in all cases of non-associated parabolic polarizations, other than in G_2, which we checked one of the two polarizations was not invariant. In the cases where the two parabolic polarizations are associated, Theorem 3.3.1 shows that the infinitesimal characters of the associated representations are the same.

4.4. An example of uniqueness of polarization. — We illustrate the delicacy of the argument of paragraph 4.1 for $e = e_{\alpha_1+\alpha_2}+e_{\alpha_1+3\alpha_2}$ by examining the nilpotent $e' = e_{\alpha_1+\alpha_2} - e_{\alpha_1+3\alpha_2}$. Both e and e' have $2a_1$ as neutral element, so they are $G_{\mathbf{C}}$-conjugate and they have the same natural polarization $\mathfrak{p}\{\alpha_2\}$. But they are very far from being G-conjugate :

4.4.1. Theorem. — *In the notation of paragraph* 4.1, *let* $\mathfrak{g}$ *be the normal real form of* G_2 *and let* $e' = e_{\alpha_1+\alpha_2} - e_{\alpha_1+3\alpha_2} \in \mathfrak{g}$. *Then the natural polarization* $\mathfrak{p}\{\alpha_2\}$ *for* e' *is its only invariant real parabolic polarization.*

The first step in the proof of Theorem 4.4.1 is

4.4.2. Lemma. — *The only real parabolic polarizations for* e' *are* $\mathfrak{p}\{\alpha_2\}$ *and the* $\mathrm{Ad}(g).\mathfrak{p}\{\alpha_1\}$ *with* $g \in G[\alpha_2]$.

Proof. — Let $\mathfrak{p}$ be a real parabolic polarization for e'. Then $\mathfrak{p} \supset \mathfrak{n}' \supset \mathfrak{g}^{e'}$ where $\mathfrak{n}'$ is a nilpotent subalgebra of dimension 6 in $\mathfrak{g}$. We may assume e_{α_1} chosen so that

$$[e_{\alpha_1}, e_{\alpha_1+3\alpha_2}]+[e_{\alpha_1+\alpha_2}, e_{\alpha_1+2\alpha_2}] = 0.$$

As $\dim \mathfrak{g}^{e'} = \dim \mathfrak{g}^{2a_1} = 4$, it follows that $\mathfrak{g}^{e'}$ has basis

$$\{ e_{\alpha_1+\alpha_2}, e_{\alpha_1+3\alpha_2}, e_{2\alpha_1+3\alpha_2}, e_{\alpha_1}+e_{\alpha_1+2\alpha_2} \}.$$

Let $\mathfrak{h}$ be the split Cartan subalgebra relative to which our roots are determined, and $\mathfrak{n}$ the sum of the positive $\mathfrak{h}$-root spaces. If $e_+ \in \mathfrak{n}$ and $0 \neq h' \in \mathfrak{h}$ then $e_+ + h'$ is not nilpotent, so $e_+ + h' \notin \mathfrak{n}'$. Now let $x \in \mathfrak{n}'$. If the $e_{-(2\alpha_1+3\alpha_2)}$-term of x were nonzero, then $[e_{2\alpha_1+3\alpha_2}, x]$ would be of the form $e_+ + h'$ with $h' \neq 0$ as above, contradicting $[e_{2\alpha_1+3\alpha_2}, x] \in \mathfrak{n}'$. Thus x has no $e_{-(2\alpha_1+3\alpha_2)}$-term. The same argument now shows that x has no $e_{-(\alpha_1+3\alpha_2)}$-term, then that x has no $e_{-(\alpha_1+\alpha_2)}$-term, and finally that x has no $e_{-\alpha_1}$-term. In summary, if $x \in \mathfrak{n}'$ then $x = r_1 e_+ + r_2 h' + r_3 e_{-\alpha_2}$ with $r_i \in \mathbf{R}$.

If we always have $r_3 = 0$, then $\mathfrak{n}' = \mathfrak{n}$, and it follows that $\mathfrak{p}$ is $\mathfrak{p}\{\alpha_1\}$ or $\mathfrak{p}\{\alpha_2\}$.

Now suppose that we have $x \in n'$ with $r_3 \neq 0$. From $[x, e_{\alpha_1+3\alpha_3}] \in \mathfrak{n}'$ we see $e_{\alpha_1} \in \mathfrak{n}'$ and $e_{\alpha_1+2\alpha_2} \in \mathfrak{n}'$, so we may assume $x = r_1 e_{\alpha_2} + r_2 h' + r_3 e_{-\alpha_2}$. Express

$$r_2 h' = v_1 h_{\alpha_2} + v_2 a_1, \qquad v_i \in \mathbf{R},$$

so x has $\mathfrak{g}[\alpha_2]$-component $r_1 e_{\alpha_2} + v_1 h_{\alpha_2} + r_3 e_{-\alpha_2}$. That component is nilpotent in $\mathfrak{g}[\alpha_2]$, thus nilpotent in $\mathfrak{g}$, and it commutes with the semisimple element $v_2 a_1$. As x is nilpotent now

$$x = r_1 e_{\alpha_2} + v_1 h_{\alpha_2} + r_3 e_{-\alpha_2} \in \mathfrak{g}[\alpha_2],$$

and so x is $G[\alpha_2]$-conjugate to $\pm e_{\alpha_2}$. Since $G[\alpha_2]$ normalizes the space

$$\mathfrak{n}_2 = \text{real span of } \{e_{\alpha_1}, e_{\alpha_1+\alpha_2}, e_{\alpha_1+2\alpha_2}, e_{\alpha_1+3\alpha_2}, e_{2\alpha_1+3\alpha_2}\},$$

and since $\mathfrak{n}' = \mathfrak{n}_2 + (x)$, now $\mathfrak{n}'$ is $G[\alpha_2]$-conjugate to $\mathfrak{n}$. It follows that $\mathfrak{p}$ is $G[\alpha_2]$-conjugate to $\mathfrak{p}\{\alpha_1\}$ or to $\mathfrak{p}\{\alpha_2\}$. Thus we have $g \in G[\alpha_2]$ with $\mathfrak{p} = \mathrm{Ad}(g)\,\mathfrak{p}\{\alpha_1\}$ or $\mathfrak{p} = \mathrm{Ad}(g)\,\mathfrak{p}\{\alpha_2\} = \mathfrak{p}\{\alpha_2\}$.

Q. E. D.

We will show that $\mathrm{Ad}(g)\,\mathfrak{p}\{\alpha_1\}$, $g \in G[\alpha_2]$, cannot be $G^{e'}$-invariant. Let

$$q = \exp(\sqrt{2}\, e_{\alpha_2}).s_2.\exp\left(\frac{1}{\sqrt{32}} e_{\alpha_2}\right).\exp\left(\log(16\sqrt{2})\, a_1 + \log\left(\frac{1}{8}\right) a_2\right). \tag{4.4.2}$$

We will first check that $q \in G^{e'}$ and then show that q cannot normalize any of the $\mathrm{Ad}(g)\,\mathfrak{p}\{\alpha_1\}$, $g \in G[\alpha_2]$.

That $q \in G^{e'}$, is a direct calculation using

$$\mathrm{ad}(s_2) : \; e_{\alpha_1+\alpha_2} \to \frac{1}{2} e_{\alpha_1+2\alpha_2} \qquad \text{and} \qquad e_{\alpha_1+2\alpha_2} \to -2\, e_{\alpha_1+\alpha_2}. \tag{4.4.3}$$

To prove (4.4.3), note that the space of the 2-dimensional representation of $G[\alpha_2]$ has basis $\{v_1, v_2\}$ in which

$$e_{\alpha_2} \text{ has matrix } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad s_2 \text{ has matrix } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now the space of the 4-dimensional representation of $G[\alpha_2]$ has basis $\{w_1, \ldots, w_4\}$, $w_{j+1} = e_{\alpha_2}(w_j)$, where

$$w_1 = v_1 \otimes v_1 \otimes v_1, \qquad w_2 = v_2 \otimes v_1 \otimes v_1 + v_1 \otimes v_2 \otimes v_1 + v_1 \otimes v_1 \otimes v_2,$$
$$w_3 = 2(v_2 \otimes v_2 \otimes v_1 + v_2 \otimes v_1 \otimes v_2 + v_1 \otimes v_2 \otimes v_2), \qquad w_4 = 6\, v_2 \otimes v_2 \otimes v_2.$$

This 4-dimensional representation is the Ad_G action on $\mathfrak{g}^{\alpha_1} + \mathfrak{g}^{\alpha_1+\alpha_2} + \mathfrak{g}^{\alpha_1+2\alpha_2} + \mathfrak{g}^{\alpha_1+3\alpha_2}$, $w_j = \mathrm{ad}(e_{\alpha_2})^{j-1} e_{\alpha_1}$. Since $s_2 : w_2 \to (1/2)\, w_3$ and $w_3 \to -2\, w_2$ we get (4.4.3).

The next step in our proof of non-invariance is

(4.4.4) if $g \in G[\alpha_2]$ then $\mathrm{Ad}(q)\,\mathrm{Ad}(g)\,\mathfrak{p}\{\alpha_1\} \neq \mathrm{Ad}(g)\,\mathfrak{p}\{\alpha_1\}$.

This says $\mathrm{Ad}(g^{-1}qg)\,\mathfrak{p}\{\alpha_1\} \neq \mathfrak{p}\{\alpha_1\}$, hence follows from

4.4.5. Lemma. — *Let* $\mathfrak{p}\{\alpha_1\}$ *be normalized by* $q' \in G[\alpha_2]\exp \mathbf{R}a_1$. *Then*

$$q' \in M_2 A_2 N_2 . \exp \mathbf{R}\, a_1.$$

If $g \in G[\alpha_2]$ *and* q *is given by* (4.4.2), *then*

$$g^{-1} qg \notin M_2 A_2 N_2 . \exp \mathbf{R}\, a_1,$$

so $g^{-1} qg$ *does not normalize* $\mathfrak{p}\{\alpha_1\}$.

Proof. — The reductive group $G[\alpha_2].\exp \mathbf{R}a_1$ has Bruhat decomposition

$$M_2 A_2 N_2 . \exp \mathbf{R}\, a_1 \cup M_2 A_2 N_2 s_2 M_2 A_2 N_2 . \exp \mathbf{R}\, a_1.$$

If $q' \in M_2 A_2 N_2 s_2 M_2 A_2 N_2 . \exp \mathbf{R}a_1$ then a direct calculation shows that $\mathrm{Ad}(q')\, e_{-\alpha_1}$ has nonzero $e_{-(\alpha_1+3\alpha_2)}$-component. Since $e_{-\alpha_1} \in \mathfrak{p}\{\alpha_1\}$ and $e_{-(\alpha_1+3\alpha_2)} \notin \mathfrak{p}\{\alpha_1\}$, q' does not normalize $\mathfrak{p}\{\alpha_1\}$. Thus $q' \in M_2 A_2 N_2 . \exp \mathbf{R}a_1$ as asserted.

Let q be given by (4.4.2). Then the $G[\alpha_2]$-component of q, in the 2-dimensional representation used to prove (4.4.3), has matrix

$$\begin{pmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \dfrac{1}{\sqrt{32}} & 1 \end{pmatrix}.\mathrm{Exp}\frac{1}{2}\begin{pmatrix} -\log\dfrac{1}{8} & 0 \\ 0 & \log\dfrac{1}{8} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 2\sqrt{2} & 0 \\ \dfrac{1}{2} & \dfrac{1}{2\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2} & \bigstar \\ \bigstar & \dfrac{1}{2} \end{pmatrix}$$

which has trace 1. If $g \in G[\alpha_2]$ now the $G[\alpha_2]$-component of $g^{-1}qg$ goes to a matrix of trace 1 in the 2-dimensional representation. However, if $g^{-1}qg \in M_2 A_2 N_2 . \exp \mathbf{R}a_1$ then its $G[\alpha_2]$-component goes to a matrix $q'' = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$ with

$$|\,\mathrm{trace}\, q''\,| = \left| a + \frac{1}{a} \right| \geqq 2.$$

Thus $g^{-1}qg \notin M_2 A_2 N_2 . \exp \mathbf{R}a_1$.

Q. E. D.

Theorem 4.4.1 follows directly from Lemma 4.4.2, the fact $q \in G^{e'}$, and Lemma 4.4.5.

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