ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4e série, tome 7, nº 1 (1974), p. 89-137
<http://www.numdam.org/item?id=ASENS_1974_4_7_1_89_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 7, 1974, p. 89 à 138.

COHOMOLOGY OF LINE BUNDLES ON G/B

BY LAKSHMI BAI, C. MUSILI AND C. S. SESHADRI

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0. **Introduction**

The purpose of this paper is to prove the following result (*cf.* Theorem 2.1, below) :

Let G be a semi-simple algebraic group defined over an algebraically closed field k, strictly isogenous to a product of groups of type $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ or $\mathbf{G}_2$ and L a line bundle on G/B (B a Borel subgroup of G.). Then

$$\text{(1)} \qquad H^i(G/B, L) = 0 \begin{cases} i > 0, \\ L \text{ in the dominant chamber.} \end{cases}$$

When the characteristic of k is 0, this result is a particular case of a stronger theorem of Bott (*cf.* [5], [10] or [17]) which asserts that for *any* semi-simple algebraic group G, given a line bundle L on G/B, there exists at most one i which can be computed such that $H^i(G/B, L) \neq 0$, etc. This stronger result is however now known to be false in arbitrary characteristic, as has been pointed out by Mumford [SL (3), characterisric 2].

A development which has contributed to the proof of (1) is the result proved recently by several authors (*cf.* [14], [16], [18] and [20]) that the vertex of the cone over the Grassmannian, for its canonical Plücker imbedding into a projective space, is Cohen-Macaulay; this result is easily seen to be a consequence of (1) for the case G = SL (n). A common aspect of all these proofs is that they suggest the plausibility of vanishing theorems of type (1) more generally for *Schubert varieties* (we call Schubert varieties the closures of cells in G/B, G/P, etc.) so that (1) could be proved by induction on the dimension of the Schubert varieties. In fact it has been proved by these authors that the vertex of the cone over any Schubert variety in the Grassmannian is Cohen-Macaulay. Among the several proofs of these results there are really two which are different in principle. The first one (*cf.* [14], [18] and [20]) is based on induction on the dimension of the Schubert varieties and uses a result of Hodge which gives an explicit basis for $H^0(X, L^m)$ where X is a Schubert variety and L represents the restriction to X of the hyperplane bundle on the Grassmannian for the Plücker imbedding into a projective space. The second proof is due to Kempf (*cf.* [16]) who deduces these as consequences of theorems of type (1) for a *certain class of smooth Schubert varieties in* SL (n)/B; in particular he proves (1) for the case G = SL (n). In this proof one again uses induction on the dimension of these Schubert varieties and the role of Hodge's theorem is replaced by using certain properties of a $\mathbf{P}^1$-fibration of G/B. Our proof of (1) is inspired from this proof due to Kempf for the case G = SL (n).

The proof of (1) is done by checking it separately for type $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ and $\mathbf{G}_2$; however the underlying principle of the proofs is the same in all the cases.

To give an outline of the proof of (1), let us first give our version of Kempf's proof for the case G = SL ($n+1$). Fix a maximal torus T of G, and a Borel subgroup B of G, B ⊃ T. Take the maximal parabolic subgroup P of G, P ⊃ B, corresponding to the left end root in the Dynkin diagram of G so that P\G (= space of cosets of the forme P g, $g \in$ G) is the *projective space of dimension n*. Now B determines a Bruhat decomposition in G, P\G, B\G and G/B and we call Schubert varieties the closures of the corresponding Bruhat cells. Let $P\backslash G = Y_0 \supset Y_1 \supset \ldots \supset Y_n$ (= point) be the decreasing sequence of Schubert varieties in P\G. Then Y_i is a linear subspace of codimention i in P\G and if H is the tautological line bundle on P\G, then the line bundle on Y_i determined by the codimension one subvariety Y_{i+1} is $H|_{Y_i}$. Let X'_i, $0 \leqq i \leqq n$, be the inverse images of Y_i by the canonical morphism $\pi : B\backslash G \to P\backslash G$; then X'_i are *smooth* Schubert varieties in B\G and $\pi^*(H)|_{X'_i}$ is the line bundle defined by the codimension one subvariety X'_{i+1} of X'_i. Let X_i

be the Schubert varieties in G/B (= space of cosets of the form g B, $g \in$ G) defined by X'_i (i. e. if X'_i is the image in B\G of $\overline{B w B} \subset G$, $w \in W$ = Weyl group of G, then X_i is the canonical image of $\overline{B w B}$ in G/B). We see easily that X_i are also *smooth* Schubert varieties in G/B. Further X_n = P/B-the flag variety of SL (n) (i. e. the flag variety in lower rank). Let L_i be the line bundle on X_i determined by the codimension one subvariety X_{i+1}, then it can be seen that L_i are induced from line bundles on G/B; further if $L_i = L(\chi_i)$ where χ_i is a character of the maximal torus T, the χ_i can be calculated explicitly in terms of the fundamental weights (*cf.* Proposition A.6, § 3, below). We have exact sequences

$$0 \to L_i^{-1} \to \mathcal{O}_{X_i} \to \mathcal{O}_{X_{i+1}} \to 0.$$

Let χ be a character of T, L (χ) the line bundle on G/B defined by χ and $L = L(\chi)|_{X_i}$. Tensoring the above exact sequence by L gives the exact sequence

$$(2) \qquad 0 \to L \otimes L_i^{-1} \to L \to L|_{X_{i+1}} \to 0 \qquad \text{on} \quad X_i.$$

Now it can be seen that there exists a $\mathbf{P}^1$-fibration of X_i induced by the $\mathbf{P}^1$-fibration $G/B \to G/P_{\alpha_i}$, where P_{α_i} is the minimal parabolic subgroup corresponding to a certain simple root α_i (*cf.* Remark A.3, § 3, below). If N is a line bundle on X_i, we call deg N = *degree of N with respect to this* $\mathbf{P}^1$*-fibration*, the degree of the restriction of N to any fibre of this fibration. A general reasoning shows (*cf.* Proposition 1.14, below) that if deg N = -1, then $H^j(X_i, N) = 0$, $j \geqq 0$. It can be shown that deg $L_i = 1$ so that $\deg L \otimes L_i^{-1} = \deg L - 1$. Writing the cohomology exact sequence of (2), we have

$$(3) \quad 0 \to H^0(X_i, L \otimes L_i^{-1}) \to H^0(X_i, L) \to H^0(X_{i+1}, L|_{X_{i+1}})$$
$$\to H^1(X_i, L \otimes L_i^{-1}) \to H^1(X_i, L) \to H^1(X_{i+1}, L|_{X_{i+1}}) \to H^2(X_i, L \otimes L_i^{-1}) \to \dots.$$

We have seen that

$$(4) \qquad H^1(X_i, L \otimes L_i^{-1}) = 0 \qquad \text{if} \quad \deg L = 0.$$

Let us now call L (χ) *dominant* if χ is dominant. Then from the explicit computations of χ_i the following is an immediate consequence :

$$(5) \qquad L(\chi) \text{ dominant and } \deg L \geqq 1 \, [L = L(\chi)|_{X_i}]$$
$$\Rightarrow \quad L(\chi - \chi_i) = L(\chi) \otimes L(-\chi_i) \text{ is dominant.}$$

Note that $L \otimes L_i^{-1} = L(\chi - \chi_i)|_{X_i}$. The proof of (1) now follows as a special case ($i = 0$) of the following assertion

$$(6) \qquad H^j(X_i, L) = 0, \qquad j > 0, \quad L(\chi) \text{ dominant.}$$

Since X_n = the flag variety in lower rank, by induction on the rank we can suppose (6) to be true this case. We now prove (6) by induction on dim X_i so that we can suppose (6) to be true for X_{i+1} instead of X_i. Suppose now deg L = 0. Then looking at the cohomology exact sequence (3), the assertion (6) follows in this case using (4) above [and of course (6) for the case X_{i+1}]. Again looking at the exact sequence (3), because of (5) the assertion (6) follows by induction on deg L.

It can be asked why one should work with X_i rather than X'_i which seems more natural. If we argue by induction on dimension of X'_i, we require a result similar to (4). What is required is a $\mathbf{P}^1$-fibration of X'_i such that the degree of the line bundle on X'_i which defines the codimension one subvariety X'_{i+1} is 1. Now X'_i has many $\mathbf{P}^1$-fibrations but the degree turns out to be *zero* in all these cases. This is the reason why we work with X_i. In the proofs of the Cohen-Macaulay property of cones over Grassmannians, the proof of something similar to (4) is one of the essential steps and it is achieved with the help of Hodge's basis theorem (*loc. cit.*). Perhaps a suitable generalisation would also work in this case.

It is now clear how to set about proving (1) for the cases when G is of type other than $\mathbf{A}_n$. We take for P the maximal parabolic subgroup corresponding to the left end root in the Dynkin diagram of G and we define a sequence of Schubert varieties :

$$G/B = X_0 \supset X_1 \supset \ldots \supset X_s = P/B$$

starting from Schubert varieties $Y_0 \supset \ldots \supset Y_s$ in $P\backslash G$. For the case of type $\mathbf{C}_n$ we find that $P\backslash G$ is a *projective space* and the proof goes through as for type $\mathbf{A}_n$.

For type $\mathbf{B}_n$, $P\backslash G$ is an *odd dimensional quadric* and there is a unique Schubert variety in $P\backslash G$ in each dimension. One defines the sequences Y_i, X_i as above. They are not in general smooth varieties but they are always *normal*. In this case a technical complication arises from the fact that X_n is *not* a Cartier divisor in X_{n-1}; however $2X_n$ is a Cartier divisor in X_{n-1} given by the restriction to X_{n-1} of a line bundle on G/B which can be explicitly computed. Let Z be the closed *subscheme* of X_{n-1} " defined by $2X_n$ " with underlying set as X_n. One shows that

$$\text{Vanishing theorem for } X_n \Rightarrow \text{Vanishing theorem for } Z$$

and

$$\text{Vanishing theorem for } Z \Rightarrow \text{Vanishing theorem for } X_{n-1}.$$

The rest of the proof is similar to that of type $\mathbf{A}_n$.

For the case of type $\mathbf{D}_n$, $P\backslash G$ is an *even dimensional quadric*. Here, in every codimension i except when $i = n-1$, there is precisely one Schubert variety Y_i and when $i = n-1$ there are precisely two, say Y_{n-1} and Y'_{n-1}. We define the sequence of Schubert varieties $Y_0 \supset Y_1 \supset \ldots \supset Y_{2n-2}$ in $P\backslash G$ with codim $Y_i = i$ and we define the sequence X_i as before and let us denote by Z the Schubert variety in G/B defined by Y'_{n-1}. Here again there is a technical complication; X_{n-1} is *not* a Cartier divisor in X_{n-2}. However dealing with this case turns out to be simpler than in the case $\mathbf{B}_n$. One finds that $X_{n-1} \cup Z$ is a Cartier divisor in X_{n-2} defined by a line bundle on G/B which can be explicitly computed. One finds that $X_{n-1} \cap Z = X_n$ (scheme-theoretically) and by a familiar patching up argument (*cf.* [20]) one shows that

$$\text{Vanishing theorems for } X_{n-1} \text{ and } Z \Rightarrow \text{Vanishing theorems for } X_{n-1} \cup Z$$

and

$$\text{Vanishing theorems for } X_{n-1} \cup Z \Rightarrow \text{Vanishing theorems for } X_{n-2}.$$

The rest of the argument is as for type $\mathbf{A}_n$.

For type $\mathbf{G}_2$, one finds that P\G is a *five dimensional quadric* and the proof goes through as for type B_n.

From the preceding it is clear as to what is required for the above proof to go through for the remaining types. Let P be a maximal parabolic subgroup such that P\G is the *simplest*. Let X be a Schubert variety in P\G and S a hyperplane section in P\G (for the canonical projective imbedding) such that X·S is set-theoretically a union of Schubert varieties. Then we should know the scheme-theoretic intersection X·S.

This paper is respectfully dedicated to Professor Cartan whose celebrated theorems on Stein manifolds have influenced so much work on the cohomology of coherent sheaves in analytic and algebraic geometry.

1. Preliminaries

Here we set the notation and recall some of the facts needed in the sequel. For details one may consult [1], [3], [6], [7], [8] and [21]-[25]. *It would be advisable to skip most of them in the first instance and refer to them only during the course of* paragraphs 2 and 3.

We fix an algebraically closed field k of arbitrary characteristic.

Let G denote a connected semi-simple linear algebraic group (over k) of rank n. Let T be a maximal torus of G. Let $B \supset T$ be a Borel subgroup of G and let B^u denote its unipotent part. Let N (T) be the normaliser of T in G and let $W = N(T)/T$ be the Weyl group of G (relative to T).

1. SYSTEM OF ROOTS (*cf.* [1], [3], [6], [7] and [21]). — Let X (T) denote the group of rational characters of T. This is a free abelian group of rank n (= rank G). Let V be the vector space over $\mathbf{Q}$ defined by $V = X(T) \otimes_{\mathbf{Z}} \mathbf{Q}$. Fix a system of roots $R \subset X(T)$ relative to G and T on this vector space. For each root $\alpha \in R$, let θ_α be the isomorphism of the additive groupe $\mathbf{G}_a$ onto a subgroup $H_\alpha \subset G$ defined such that

$$t\,\theta_\alpha(x)\,t^{-1} = \theta_\alpha(\alpha(t)\,x)$$

for all $t \in T$ and $x \in \mathbf{G}_a$. Let R^+ denote the set of positive roots relative to B, i. e.,

$$R^+ = \{\alpha \in R \,/\, H_\alpha \subset B^u\}.$$

Recall that R is a disjoint union of R^+ and $R^- = -R^+$. We write $\alpha > 0$ (resp. $\alpha < 0$) if $\alpha \in R^+$ (resp. R^-). Let $S = \{\alpha_1, \ldots, \alpha_n\} \subset R^+$ be the simple system of roots. For each $\alpha \in R$, let s_α denote the reflection on V with respect to α. Write $s_i = s_{\alpha_i}$, $1 \leqq i \leqq n$. Recall that the Weyl group of G (same as the Weyl group of the root system) is generated by the simple reflections $s_1, \ldots, s_n$. Let (,) be a positive definite scalar product on V invariant under W. Let $\alpha^* = 2\,\alpha/(\alpha, \alpha)$, $\alpha \in R$. Define $\tilde{\omega}_i \in V$ such that $(\tilde{\omega}_i, \alpha_j^*) = \delta_{ij}$, $1 \leqq i, j \leqq n$. The $\tilde{\omega}_1, \ldots, \tilde{\omega}_n$ are called the *fundamental weights* (relative to $\alpha_1, \ldots, \alpha_n$). Finally recall that we have $(\alpha, \beta^*) \in \mathbf{Z}$, $\chi = \sum_{i=1}^{n} (\chi, \alpha_i^*)\,\tilde{\omega}_i$ and

$s_\alpha(\chi) = \chi - (\chi, \alpha^*)\alpha$ for all $\alpha, \beta \in R$ and $\chi \in X(T)$. The integers $n_{ij} = (\alpha_i, \alpha_j^*)$ are called the *Cartan numbers* of G (relative to $\alpha_1, \ldots, \alpha_n$).

2. THE ELEMENT w_0 OF W. — Let $l : W \to \mathbf{Z}^+$ denote the usual length function on W relative to $s_1, \ldots, s_n$, i. e., for $w \in W$,

$$l(w) = \min_k \{ k/w = s_{i_1} \ldots s_{i_k},\ 1 \leqq i_1, \ldots, i_k \leqq n \}.$$

Any expression $w = s_{i_1} \ldots s_{i_k}$ with $k = l(w)$ is called a *reduced expression* for w. We have the following simple

PROPOSITION 1.1. (*cf.* [6], p. 43, 158). – *There exists a unique element of* W, *denoted by* w_0, *satisfying the following equivalent properties :*

(i) $$l(w) \leqq l(w_0) \quad \textit{for all } w \in W.$$

(ii) $$l(w_0 w) = l(w_0) - l(w) \quad \textit{for all } w \in W.$$

(iii) $$w_0(\alpha) < 0 \quad \textit{for all roots } \alpha > 0 \quad [\textit{i. e., } w_0(R^+) = R^-].$$

REMARK 1.2. — The Borel subgroup $\tilde{B} = w_0 B w_0^{-1}$ of G, called the Borel subgroup *opposite* to B (relative to T) is characterised by the following equivalent properties, namely, $\tilde{B}$ is the Borel subgroup of G such that (i) $B \cap \tilde{B} = T$ or (ii) the roots positive for B are precisely those negative for $\tilde{B}$.

3. PARABOLIC SUBGROUPS OF G CONTAINING B (*cf.* [3] and [6]). — Let P denote a parabolic subgroup of G containing B. Recall that P is associated to a (unique) subset, say S_P, of S in the sense that P is the subgroup of G generated by B and the $H_{-\alpha}$, $\alpha \in R_P^+$ where R_P^+ is the set of all positive roots spanned by the simple roots in S_P, i. e.,

$$R_P^+ = \{ \alpha \in R^+ / \alpha = \sum_{\beta \in S_P} n(\beta)\beta \}$$

(Conversely, every subset of S defines a parabolic subgroup of G containing B in an obvious way). Note that $S_B = \varnothing$ and $S_G = S$. Write $R_P^- = -R_P^+$ and $R_P = R_P^+ \cup R_P^-$. Finally, recall that we can write $P = M_P \cdot U_P$ (semi-direct product) (called a *Levi decomposition* for P) where M_P (resp. U_P) is the " reductive part " (resp. unipotent radical) of P. In fact, M_P is the subgroup of G generated by T and the H_α, $\alpha \in R_P$ and U_P is the subgroup of G generated by the H_α, $\alpha \in R^+ - R_P^+$.

For a simple root α ($\in S$), the parabolic subgroup associated to the subset $\{\alpha\} \subset S$ is denoted by P_α and is referred to as the (minimal) *parabolic subgroup associated to* α. On the other hand the parabolic subgroup associated to the subset $S - \{\alpha\} \subset S$ is denoted by $P_{\hat{\alpha}}$ and is referred to as the (maximal) *parabolic subgroup obtained by omitting* α.

For a parabolic subgroup $P \supset B$, the subgroup of W generated by the s_α, $\alpha \in S_P$ is simply the Weyl group of P (or M_P) and is denoted by W_P.

4. BRUHAT DECOMPOSITION OF G RELATIVE TO P (*cf.* [3] and [6]). — Let $P \supset B$ be a parabolic subgroup of G. For $w \in W$, let $n(w) \in N(T)$ be such that its residue class mod T

is w. Observe that the (B, P)-double coset $B\,n(w)\,P$ in G depends only on the coset $w\,W_P$ in W but not on w or $n(w)$. Write $B\,w\,P$ or $C_P(w)$ for $B\,n(w)\,P$ and call it the (*open*) *Bruhat cell* in G associated to $w\,W_P$. The Zariski closure of $C_P(w)$ in G, denoted by $X_P(w)$, is called the (*closed*) *Bruhat cell* in G associated to $w\,W_P$. The Bruhat decomposition of G (relative to P) asserts that G is the disjoint union of the (open) Bruhat cells $C_P(w)$ in G. When $P = B$, we simply write $C(w)$ and $X(w)$ for $C_B(w)$ and $X_B(w)$ respectively.

We have a similar description of the Bruhat decomposition of G in terms of the Bruhat cells $P\,w\,B$, $w \in W_P\backslash W$.

5. Cellular decomposition of G/P (*cf.* [1], [3], [7] and [8]). — Let $P \supset B$ be a parabolic subgroup of G. Let G/P denote the space of cosets of the form $g\,P$, $g \in G$, i. e., the space of orbits in G for the action (by multiplication) of P on the right. Recall that G/P is a non-singular projective variety and that the natural projection $\pi_r : G \to G/P$ is a locally trivial principal fibration under the group P (and hence in particular a smooth morphism). For $w \in W$, $\pi_r(C_P(w))$ is called the (*open*) *Schubert cell* associated to the (open) Bruhat cell $C_P(w)$. The Schubert cells in G/P provide a cellular decomposition of G/P. In other words, G/P is the disjoint union of the Schubert cells $\pi_r(C_P(w))$, $w \in W/W_P$. The Zariski closure of $\pi_r(C_P(w))$ in G/P, denoted by $X_P(w)_r$, with the canonical reduced scheme structure, is called the *Schubert variety* associated to $w\,W_P$. When $P = B$, we simply write $X(w)_r$ for $X_B(w)_r$.

Letting $P\backslash G$ to denote the space of cosets of the form $P\,g$, $g \in G$, we have a similar description of the Schubert cells in $P\backslash G$, etc. *In this case, the notations are as above with r replaced by l.*

6. Dimension of Schubert cells in G/P. — For $w \in W$, let

$$R_P(w) = \{\alpha > 0 \,/\, w^{-1}(\alpha) \in R^- - R_P^-\} \qquad \text{and} \qquad N_P(w) = \text{card}\, R_P(w),$$

and let $H_P(w)$ denote the subgroup of G generated by the H_α, $\alpha \in R_P(w)$. Recall that $H_P(w)$ is a subgroup of B^u, and as a variety it is isomorphic to an affine space of dimension $N_P(w)$. When $P = B$ (we agree to omit the suffix B) note that

$$R(w) = \{\alpha > 0 \,/\, w^{-1}(\alpha) < 0\}$$

and $H(w) = B^u \cap w\,(\tilde{B})^u\,w^{-1}$ where $\tilde{B} = w_0\,B\,w_0^{-1}$, etc. Now we prove the following simple

Proposition 1.3.

(i) *Given an element* $x \in B\,w\,P$, *there exist a* $p \in P$ *and a* unique $b \in H_P(w)$ *such that* $x = bwp$.

(ii) $$\dim(X_P(w)_r) = N_P(w).$$

Proof. – Let $\pi_r : G \to G/P$ be the natural morphism. Let $e_0 = \pi_r(P)$ denote the distinguished point in G/P. Consider the natural action of B (induced from that of G) on G/P on the left. Recall that the open Schubert cell $\pi_r(C_P(w))$ in G/P is simply the

B^u-orbit (or B-orbit) through the point we_0. It is trivial to see that the isotropy subgroup of B^u at the point we_0 is simply $B^u \cap w P w^{-1}$. But recall that $B^u \cap w P w^{-1}$ is simply the subgroup of B^u generated by the H_α,

$$\alpha \in R'_P(w) = \{ \alpha > 0 / w^{-1}(\alpha) \in R_P \cup (R^+ - R_P^+) \}.$$

Note that $R_P(w)$ and $R'_P(w)$ partition the set R^+ since $(R^- - R_P^-)$ and $R_P \cup (R^+ - R_P^+)$ give a partition of the set of all roots. We know that B^u as a variety is isomorphic to the direct product $\prod_{\alpha > 0} H_\alpha$, the product being taken in any fixed order and as a group it is the product $H_\alpha \ldots H_\beta \ldots H_\gamma$ in that order (*cf.* [1] or [7, exp. 13]). So writing

$$B^u = \prod_{\alpha \in R_P(w)} H_\alpha \times \prod_{\beta \in R'_P(w)} H_\beta = H_P(w).(B^u \cap w P w^{-1}),$$

we see that the assertions (i) and (ii) follow immediately.

Note. — In the case of P\G, we have a similar result, namely,

(i) any element $x \in P\, w\, B$ can be written as $x = pwb$ with $p \in P$ and a *unique* $b \in H_P(w^{-1})$, and

(ii) $$\dim(X_P(w)_l) = N_P(w^{-1}).$$

[We remark that in general $N_P(w) \neq N_P(w^{-1})$; however, we will see that $N(w) = N(w^{-1})$, *cf.* Remark 1.6, below.]

7. SOME RESULTS OF CHEVALLEY (*cf.* [8]).

PROPOSITION 1.4. — *Let $w \in W$ and let α $(\in S)$ be a simple root. Then the following statements are equivalent :*

(i) *The closed Bruhat cell $X(w)$ $(= \overline{B\, w\, B})$ in G is stable for multiplication on the left [resp. right] by $H_{-\alpha}$ or equivalently by the minimal parabolic subgroup P_α.*

(ii) $$l(s_\alpha w) < l(w) \quad [\text{resp. } l(w s_\alpha) < l(w)].$$

(iii) $$N(s_\alpha w) < N(w) \quad [\textit{resp. } N(w s_\alpha) < N(w)].$$

(iv) $$w^{-1}(\alpha) < 0 \quad [\textit{resp. } w(\alpha) < 0].$$

Proof. — (iii) $\Leftrightarrow$ (iv). — Recall that for $w \in W$, $R(w) = \{ \alpha > 0 / w^{-1}(\alpha) < 0 \}$ and $N(w) = \text{card } R(w)$. Suppose $N(s_\alpha w) < N(w)$. Assume if possible that $w^{-1}(\alpha) > 0$. This gives that $\alpha \in R(s_\alpha w)$ and that $\alpha \notin R(w)$. It is easy to check that the reflection s_α induces a bijection of $R(w)$ onto $R(s_\alpha w) - \{ \alpha \}$. Hence $N(w) = N(s_\alpha w) - 1$ which is a contradiction. Hence (iii) $\Rightarrow$ (iv). Conversely, suppose $w^{-1}(\alpha) < 0$. In this case, $\alpha \in R(w)$ and $\alpha \notin R(s_\alpha w)$, and consequently, s_α induces a bijection of $R(s_\alpha w)$ onto $R(w) - \{ \alpha \}$. Hence $N(s_\alpha w) = N(w) - 1$.

REMARK 1.5. — Notice that in the above proof we have also established that $w^{-1}(\alpha) > 0 \Leftrightarrow N(s_\alpha w) = N(w) + 1$. Putting together, we find that

$$N(s_\alpha w) = N(w) \pm 1, \quad \forall \alpha \in S \quad \text{and} \quad w \in W.$$

(ii) ⇔ (iii). — This is trivial in view of the following.

CLAIM. — *For all* $u \in W$, $l(u) = N(u)$.

We prove this by showing $N(u) \leqq l(u) \leqq N(u)$.

$N(u) \leqq l(u)$. — We prove this by induction on $l(u)$. If $l(u) = 0$, $u = \mathrm{Id}$, and hence $N(u) = 0$. Assume $l(u) > 0$ and the induction hypothesis that for all $v \in W$,

$$l(v) < l(u) \Rightarrow N(v) \leqq l(v).$$

Let $u = s_{i_1} \dots s_{i_r}$, $r = l(u)$, be a reduced expression for u. Define $v = s_{i_1} u$. Notice that $l(v) = l(u)-1$. Hence by induction, we have $N(v) \leqq l(v) = l(u)-1$. i. e., $N(v)+1 \leqq l(u)$. But $u = s_{i_1} v$ and so by the above remark, we have

$$N(u) = N(v) \pm 1 \leqq N(v)+1.$$

Hence $N(u) \leqq N(v)+1 \leqq l(u)$. Conversely,

$l(u) \leqq N(u)$: We prove this by induction on $N(u)$. If $N(u)=0$, $u=\mathrm{Id}$ and so $l(u)=0$. Assume $N(u)>0$ (and the induction hypothesis). Hence there exists a positive root β such that $w^{-1}(\beta)<0$. We can assume that β is simple. Now from the implication (iv)⇒(iii), we get that $N(s_\beta u) = N(u)-1$. By the induction hypothesis, we have

$$l(s_\beta u) \leqq N(s_\beta u) = N(u)-1.$$

This means that we can write $s_\beta u = s_{i_1} \dots s_{i_r}$ with $r = N(u)-1$. Hence $u = s_\beta s_{i_1} \dots s_{i_r}$ and so $l(u) \leqq r+1 = N(u)$. Hence $l(u) = N(u)$.

REMARK 1.6. — Note that we have

$$\dim(X(w)_r) = N(w) = l(w) = l(w^{-1}) = N(w^{-1}) = \dim(X(w)_l).$$

(i) ⇔ (iv). — Let $\pi_r : G/B \to G/P_\alpha$ be the canonical morphism. The fibres of π_r are $P_\alpha/B \approx \mathbf{P}^1$. Observe that for any Schubert variety $X_{P_\alpha}(w)_r$ in G/P_α, $\pi_r^{-1}(X_{P_\alpha}(w)_r)$ is irreductible in G/B and contains $X(w)_r$. It is clear that $X(w)$ is stable for the multiplication by P_α on the right

⇔ $X(W)_r$ is saturated for the $\mathbf{P}^1$-fibration π_r

⇔ $X(W)_r = \pi_r^{-1}(X_{P_\alpha}(w)_r)$

⇔ $\dim(X(w)_r) = \dim(X_{P_\alpha}(w)_r)+1$

⇔ $N(w) = N_{P_\alpha}(w)+1$

⇔ $R(w) \neq R_{P_\alpha}(w)$ [note that $R_{P_\alpha}(w) \subseteq R(w)$]

⇔ $w(\alpha) < 0$.

This completes the proof of the proposition.

8. PARTIAL ORDER ON THE WEYL GROUP W (RELATIVE TO B) (*cf.* [8]). — For $w, w' \in W$, *define* $w \leq w'$ if $X(w) \subseteq X(w')$. Obviously this defines a partial order on W. We have

PROPOSITION 1.7. — *The following statements are equivalent for any two elements w, $w' \in W$.*

(i) $$w \leqq w'.$$

(ii) *From every reduced expression for w', one can extract a sub-expression which is a reduced expression for w.*

(iii) *There exists some reduced expression for w' from which one can extract a sub-expression which is a reduced expression for w.*

Proof. — This is immediate from the following.

LEMMA 1.8. — *For any element $w \in W$, take a reduced expression for $w = r_1 \ldots r_l$ where $l = l(w)$ and $r_i \in S = \{ s_1, \ldots, s_n \}$. Define*

$$A_w = \{ w' \in W / w' = r_{i_1} \ldots r_{i_k} \text{ with } 0 \leqq i_1 < \ldots < i_k \leqq l \}.$$

Then A_w depends only on w (but not on the reduced expression taken) and we have

$$X(w) = \bigcup_{w' \in A_w} B w' B \left(= \bigcup_{w' \in A_w} X(w')\right).$$

For a proof of this lemma, *see* [4].

REMARK 1.9. — The above proposition together with the lemma enables one (in particular) to recognise the codimension 1 cells contained in $X(w)$.

REMARK 1.10. — Let $w_0 \in W$ be the element of largest length. From Propositions 1.1 and 1.7 and Remark 1.6, we deduce the following :

(i) $X(w_0)_r = G/B$ and $\dim G/B = l(w_0) = N(w_0) = \text{card } R^+ = 1/2$ number of roots. [The open Schubert cell $C(w_0)_r$ is called the *big cell.*]

(ii) For all $w \in W$, the Schubert variety $X(w_0 w)_r$ is of codimension $l(w)$ in G/B. In particular,

(iii) $X(w_0 s_i)_r$, $1 \leqq i \leqq n$, are (prime) divisors in G/B.

9. THE PICARD GROUP OF G/B (*cf.* [7], [10] and [12]). — Let $\overline{G}$ be a simply connected covering of G and let $\overline{T}$ and $\overline{B}$ denote respectively the maximal torus and the Borel subgroup in $\overline{G}$ corresponding to T and B in G. Recall that the system of roots for $\overline{G}$ (relative to $\overline{T}$) is the same as the one for G (relative to T), and that the character group $X(\overline{T})$ of $\overline{T}$ is simply the subgroup of $V = X(\overline{T}) \otimes \mathbf{Q} = X(T) \otimes \mathbf{Q}$ generated by the fundamental weights $\tilde{\omega}_1, \ldots, \tilde{\omega}_n$. Further, $\overline{G}/\overline{B}$ is canonically isomorphic to G/B. In fact, $\overline{G}/\overline{P} \approx G/P$ for any parabolic subgroup P in G, $\overline{P}$ being the corresponding parabolic subgroup of $\overline{G}$.

Assume that G is *simply connected.* For $\chi \in X(T)$, let $L(\chi)$ denote the line bundle on G/B associated to the principal B-bundle $G \to G/B$ for the character of B obtained by composing $\chi : T \to k^*$ with the natural map $B \to T = B/B^u$. This gives a homomorphism

$$L : \ X(T) \to \operatorname{Pic}(G/B)$$

and recall that this is an isomorphism (*cf.* [7]). On the other hand, consider the prime divisors $X(w_0 s_i)_r$, $1 \leqq i \leqq n$, on G/B. Let $L_i = \mathcal{O}_{G/B}(X(w_0 s_i)_r)$ be the line bundles (1) defined by $X(w_0 s_i)_r$. Recall that Pic (G/B) is a free abelian group generated by the L_i's and that the above isomorphism L is such that $L(\tilde{\omega}_i) = L_i$, $1 \leqq i \leqq n$. In otherwords, for $\chi \in X(T)$, we have

$$\chi = \sum_{i=1}^{n} (\chi, \alpha_i^*)\, \tilde{\omega}_i \qquad \text{and} \qquad L(\chi) = \bigotimes_{i=1}^{n} L_i^{\otimes (\chi, \alpha_i^*)}$$

Finally, recall that we have

(i) $H^0(G/B, L(\chi)) = \{\text{morphisms } f : G \to k / f(gb) = f(g)\,\chi(b) \text{ for all } g \in G \text{ and } b \in B\}$.

Similarly,

$H^0(B\backslash G, L(\chi)) = \{\text{morphisms } f : G \to k / f(bg) = \chi(b^{-1}) f(g) \text{ for all } g \in G \text{ and } b \in B\}$.

(ii) $H^0(G/B, L(\chi)) \neq (0) \;\Leftrightarrow\; \chi \geqq 0$, i.e., $(\chi, \alpha_i^*) \geqq 0$ for all i.

The set of χ or $L(\chi)$ such that $\chi \geqq 0$ is called the *dominant chamber* or the *positive chamber* (relative to B) and it is the « *positive* » *cone* generated by the $\tilde{\omega}_i$'s in X (T).

(iii) The vector space $H^0(G/B, L(\chi))$ is canonically a G-module and is *indecomposable* whenever $L(\chi)$ is in the dominant chamber. This is so because there exists a unique line in $H^0(G/B, L(\chi))$ stable under the unipotent subgroup B^u of B (*cf.* [7, Exp. 15], [12]). [If char $k = 0$, by a well-known theorem of H. Weyl, representations of G are completely reducible and so $H^0(G/B, L(\chi))$ is an irreductible G-module if $L(\chi) \geqq 0$.]

(iv) There exists a regular section $f \in H^0(G/B, L(\tilde{\omega}_i))$ such that $X(w_0 s_i)_r$ is the set of zeros of f and further the closed Bruhat cell $X(w_0 s_i)$ is precisely the set of zeros of the morphism $f : G \to k$ canonically associated to the section f. [Similarly, the prime divisor $X(s_i w_0)$ on B\G is the set of zeros of some regular section of $L(\tilde{\omega}_i)$ on B\G.]

REMARK 1.11. — Let $f \in H^0(G/B, L(\tilde{\omega}_i))$ be as in (iv) above, i. e., such that $X(w_0 s_i)$ is the set of zeros of the morphism $f : G \to k$. Then there exists a (unique) $j = j(i)$, $1 \leqq j \leqq n$, such that f satisfies the " double " invariance property, namely,

$$f(bgb') = \tilde{\omega}_j(b^{-1}) f(g)\, \tilde{\omega}_i(b')$$

for all $b, b' \in B$ and $g \in G$.

To see this, note that we have $w_0 s_i = s_j w_0$ for a unique $j = j(i)$, $1 \leqq j \leqq n$. As in (iv) above, let $f' \in H^0(B\backslash G, L(\tilde{\omega}_j))$ be such that $X(s_j w_0)$ is the set of zeros of the morphism $f' : G \to k$. Observe that we have $f'(bg) = \tilde{\omega}_j(b^{-1}) f'(g)$ for all $b \in B$ and $g \in G$. Now $X(s_j w_0) = X(w_0 s_i)$ is precisely the set of zeros of the morphism f as well as f'. Hence the rational function f/f' on G is nowhere vanishing on G. But then by a well-

(1) By a line bundle we mean also the associated invertible sheaf.

known theorem of Rosenlicht, we get that f/f' is a (non-zero) scalar in k. (One can deduce this also from the considerations of Lemma B.10, § 3, below.)

PROPOSITION 1.12. — *Let* $\pi_i : G/B \to G/P_{\alpha_i}$, $1 \leqq i \leqq n$, *be the canonical morphism* (*note that fibres of* π_i *are* $P_{\alpha_i}/B \approx \mathbf{P}^1_k$). *Then for any line bundle* $L(\chi)$ *on* G/B, *the degree of the restriction of* $L(\chi)$ *to any* (*and hence every*) *fibre of* π_i *is precisely* (χ, α_i^*).

Proof. — Since $\chi = \sum_{j=1}^{n} (\chi, \alpha_j^*) \tilde{\omega}_j$ and " degree is additive ", the result is immediate in view of the following (well-known and easy).

ASSERTION. — *Let* π_i *be as above. For each* j, $1 \leqq j \leqq n$, *and each* $y \in G/P_{\alpha_i}$, *we have*

$$L(\tilde{\omega}_j)\big|_{\pi_i^{-1}(y)} = \begin{cases} \mathcal{O}_{\mathbf{P}^1_k}(1) & \textit{if} \quad i=j, \\ 0 & \textit{otherwise.} \end{cases}$$

10. $\mathbf{P}^1$-FIBRATIONS. — Let X and Y be two algebraic varieties and $\pi : X \to Y$ a $\mathbf{P}^1$-bundle *associated to a vector bundle of rank* 2 *on* Y (in particular, π is a locally trivial $\mathbf{P}^1$-bundle). Note that this is equivalent to saying that π is a $\mathbf{P}^1$-fibration and that there exists a line bundle, say $\mathcal{O}(1)$, on X such that its restriction to every fibre of π is of degree 1. This $\mathcal{O}(1)$ is simply the tautological line bundle on X associated to the P^1-bundle π. Let D be a divisor on X and $L = \mathcal{O}_X(D)$, the line bundle defined by D. Suppose that the restriction of L to any (and hence every) fibre of π is of degree m. We call *m the degree of* D *or* L *with respect to the* $\mathbf{P}^1$*-fibration* π. Then we *claim* that there exists a line bundle M on Y such that $L \approx \pi^*(M) \otimes \mathcal{O}(m)$ where $\mathcal{O}(m)$ denotes the line bundle $\mathcal{O}(1)^{\otimes m}$. For, by taking $L \otimes \mathcal{O}(-m)$, it suffices to prove that if L is a line bundle on X such that its restriction to all the fibres of π are trivial, then $L \approx \pi^*(M)$ for some line bundle M on Y but this is well-known (*cf.* [19]).

Suppose now that Y is *normal*. Let Y_0 be the open subscheme of smooth points of Y. Then $X_0 = \pi^{-1}(Y_0)$ is the open subscheme of smooth points of X. Let D be a *Weil divisor* on X, i. e., a formal integral linear combination of closed irreducible subvarieties of codimension 1 in X. Let $\mathcal{D}$ be the sheaf associated to D, i. e., the $\mathcal{O}_X$-submodule of the sheaf $\mathcal{R}$ of rational functions on X defined by

$$H^0(U, \mathcal{D}) = \{ f \in H^0(U, \mathcal{R}) / \operatorname{div} f + D|_U \geqq 0 \}$$

for every open subset U in X. Let $\mathcal{D}_0$ be the line bundle on X_0 associated to the divisor $D_0 = D|_{X_0}$ on X_0. We see easily that $\mathcal{D} = i_*(\mathcal{D}_0)$ where i is the open immersion $X_0 \subsetneq X$, or equivalently, $\mathcal{D}$ is the *maximal* $\mathcal{O}_X$-submodule of $\mathcal{R}$ such that $\mathcal{D}|_{X_0} = \mathcal{D}_0$. On the other hand we see that if $\mathcal{D}_0$ is the $\mathcal{O}_{X_0}$-submodule of $\mathcal{R}|_{X_0}$ associated to the divisor D_0 on X_0, then $\mathcal{D} = i_*(\mathcal{D}_0)$ is the $\mathcal{O}_X$-submodule of $\mathcal{R}$ associated to the Weil divisor $D = \overline{D}_0$ (closure of D_0 in X) on X. We define *the degree of* D (*or* $\mathcal{D}$) *with respect to the* $\mathbf{P}^1$*-fibration* π *to be the degree of* D_0 *with respect to the* $\mathbf{P}^1$*-fibration* $\pi|_{X_0}$. Let m be the degree of D with respect to π. Then we *claim* that there exists a Weil divisor E on Y such that if $\mathcal{E}$ is the sheaf on Y associated to E, then

$$\mathcal{D} \approx \pi^*(\mathcal{E}) \otimes \mathcal{O}(m).$$

To see this, note that we have an isomorphism (from what we have seen above) on X_0, namely,

$$\mathscr{D}_0 \approx \pi^*(\mathscr{E}_0) \otimes \mathscr{O}(m)$$

for some line bundle $\mathscr{E}_0$ on Y_0. Now extend the divisor E_0 on Y_0 (whose associated line bundle is $\mathscr{E}_0$) to the Weil divisor E on Y, etc. and observe that $\mathscr{D}$ and $\pi^*(\mathscr{E}) \otimes \mathscr{O}(m)$ are completely determined by their restrictions to X_0 and hence the claim follows.

Let α ($= \alpha_i$ for some i) be a simple root and let $\pi : G/B \to G/P_\alpha$ be the canonical morphism. Note that this $\mathbf{P}^1$-fibration π is a $\mathbf{P}^1$-bundle associated to a vector bundle of rank 2 on G/P_α because, by Proposition 1.12, the line bundle $\mathscr{O}(1) = L(\tilde{\omega}_i)$ on G/B is such that its restriction to all the fibres of π are of degree 1. In particular, if X is a subvariety (for example, a Schubert variety) of G/B saturated for the $\mathbf{P}^1$-fibration π and $Y = \pi(X)$, then we find that $\pi : X \to Y$ is again a $\mathbf{P}^1$-bundle associated to a vector bundle of rank 2 on Y (namely, the restriction to Y of the one defining $\pi : G/B \to G/P_\alpha$). Thus we have proved the following

PROPOSITION 1.13. — *Let* X *be a Schubert* (*resp.* normal *Schubert*) *variety in* G/B *saturated for the* $\mathbf{P}^1$*-fibration* $\pi : G/B \to G/P_\alpha$, α *a simple root, and let* $Y = \pi(X)$. *Let* D *be a divisor* (*resp. Weil divisor*) *on* X *of degree* m *with respect to* $\pi : X \to Y$. *Then there exists a divisor* (*resp. Weil divisor*) E *on* Y *such that*

$$\mathscr{D} \approx \pi^*(\mathscr{E}) \otimes \mathscr{O}(m),$$

where $\mathscr{D}$ *and* $\mathscr{E}$ *are the sheaves associated to* D *and* E *respectively.*

Now we prove the following

PROPOSITION 1.14. — *Let the notation and hypothesis be as in the above proposition. Assume that* $m = -1$. *Then*

$$H^i(X, \mathscr{D}) = 0 \quad \text{for all } i \geqq 0.$$

Proof. — Observe that we have $R^j \pi_*(\mathscr{D}) = 0$ for all $j \geqq 0$. For, the question being local with respect to the base Y and π is a locally trivial $\mathbf{P}^1$-bundle, we can assume that π is actually trivial, i. e., X is of the form $Y \times \mathbf{P}^1$. Now by the Künneth formula, we have

$$H^j(Y \times \mathbf{P}^1, \pi^*(\mathscr{E}) \otimes \mathscr{O}(-1)) \approx \sum_{j_1 + j_2 = j} H^{j_1}(Y, \mathscr{E}) \otimes H^{j_2}(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(-1)),$$

which is zero because $H^*(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(-1)) = 0$. Thus we see that $H^j(U, \mathscr{D}|_U) = 0$ for all $j \geqq 0$ and all open sets U in X. Hence it follows that $R^j \pi_*(\mathscr{D}) = 0$ for all $j \geqq 0$.

But now the Leray spectral sequence

$$H^i(Y, R^j \pi_*(\mathscr{D})) \Rightarrow H^{i+j}(X, \mathscr{D})$$

degenerates and so [since $R^0 \pi_*(\mathscr{D}) = 0$] we get

$$H^i(Y, R^0 \pi_*(\mathscr{D})) = H^i(X, \mathscr{D}) = 0$$

for all $i \geqq 0$ as required.

As an immediate consequence of Propositions 1.12 and 1.14, we have the following

COROLLARY 1.15. — *Let $w \in W$ be such that $X(w)$ is stable for multiplication on the right by P_{α_i} for some simple root α_i [i. e., the Schubert variety $X(w)_r$ is saturated for the $\mathbf{P}^1$-fibration $\pi : G/B \to G/P_{\alpha_i}$]. Then for all line bundles $L(\chi)$ on G/B such that $(\chi, \alpha_i^*) = -1$, we have*

$$H^j(X(w)_r, L(\chi)|_{X(w)_r}) = 0 \quad \textit{for all } j \geqq 0.$$

2. The Main Theorem and its Consequences

THEOREM 2.1. — *Let G be a connected semi-simple algebraic group (of rank m) strictly isogenous (cf. [24]) to a product of groups of type $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ or $\mathbf{G}_2$. Let B be a Borel subgroup of G. Then for every line bundle $L(\chi)$ on G/B belonging to the dominant chamber, we have*

$$H^i(G/B, L(\chi)) = 0 \quad \textit{for all } i \geqq 1.$$

We prove this theorem in the next article. However, we deduce some of its immediate consequences below. We keep the notation and hypothesis as in the theorem.

COROLLARY 2.2. — *Let G be as above. For every line bundle $L(\chi')$ on G/B such that $\chi' + \rho \geqq 0$, where $\rho = \tilde{\omega}_1 + \ldots + \tilde{\omega}_m$ = half sum of the positive roots (cf. [6], p. 168), we have*

$$H^i(G/B, L(\chi')) = 0 \quad \textit{for all } i \geqq 1.$$

(This corollary shows that the above theorem is valid for a wider class of line bundles on G/B, namely, the dominant chamber translated by $-\rho$.)

Proof. — Clearly it suffices to consider the case of a χ' such that $(\chi', \alpha_j^*) = -1$ for some j. In this case Corollary 1.15 (applied to the big cell) gives the required result.

COROLLARY 2.3. — *Let G be as above. The dimension of the vector space $H^0(G/B, L(\chi))$, $\chi \geqq 0$, is independent of the characteristic of the base field k, consequently, its value is explicitly known as given by Weyl's dimension formula in characteristic 0 (cf. [11] and [17]).*

Proof. — This is an immediate consequence of the semi-continuity theorem (*cf.* [19]) and " reduction mod p " because G/B is " defined over $\mathbf{Z}$ " (*cf.* [13]) and $H^1(G/B, L(\chi)) = 0$.

COROLLARY 2.4. — *Let G be as above. For every line bundle $L(\chi)$ with $\chi > 0$, we have*

$$H^i(G/B, L(-\chi)) = 0 \qquad \textit{for} \quad 0 \leqq i < \dim G/B$$

Proof. — This follows easily from Serre's duality theorem on G/B. It is not difficult to see that the *canonical class* K on G/B is precisely the line bundle $K = L(-2\rho)$ where $\rho \,(= \tilde{\omega}_1 + \ldots + \tilde{\omega}_m)$ is half sum of the positive roots. Now by Serre's duality theorem, we have (for all $i \geqq 0$) :

$$H^i(G/B, L(-\chi)) \approx H^{N-i}(G/B, L(\chi - 2\rho))',$$

where′ denotes the vector space dual and N = dim G/B. Thus we have only to prove that

$$H^j(G/B, L(\chi-2\rho))=0 \quad \text{for all } j>0.$$

But this follows from Corollary 2.2 because we have $(\chi-2\rho)+\rho = \chi-\rho \geqq 0$ (since $\chi > 0$).

COROLLARY 2.5. — *Let* G *be as above. For every projective imbedding of* G/B *defined by a very ample line bundle* L (χ), G/B *is arithmetically* (*i. e. the cone over* G/B *is*) *normal and Cohen-Macaulay.*

Proof. — Since G/B is non-singular, recall that the arithmetic normality of G/B for the imbedding defined by L (χ) is equivalent to proving that the natural homomorphisms

$$\varphi_r: \quad (H^0(G/B, L(\chi)))^{\otimes r} \to H^0(G/B, L(r\chi))$$

are surjective for all $r \geqq 1$. To see the surjectivity of φ_r; observe that φ_r is a G-equivariant map (for the natural action of G on $H^0(G/B, L(r, \chi))$ and the diagonal action on the tensor power of $H^0(G/B, L(\chi))$, and that the result is immediate if char $k = 0$ because $H^0(G/B, L(r\chi))$ is an irreducible G-module. Since G/B is " defined over **Z** ", etc. it follows that, when char k is arbitrary, the dimension of Im $\varphi_r = d_0$ where d_0 is the dimension of the " corresponding module " $H^0(G/B, L(r\chi))$ in char $k = 0$. But then by Corollary 2.3, we know that $d_0 = \dim H^0(G/B, L(r\chi))$ in all characteristics and hence φ_r is surjective for all $r \geqq 1$.

Now by a theorem of Serre-Grothendieck (*cf.* [20], p. 160), it follows that G/B is arithmetically Cohen-Macaulay because ($\chi > 0$ and so by Theorem 2.1 and Corollary 2.4) we have

$$H^i(G/B, L(r\chi))=0 \qquad \text{for} \quad 0<i<\dim G/B$$

and all $r \in \mathbf{Z}$.

COROLLARY 2.6. — *Let* G *be as above. For all line bundles* L (χ) *and* L (χ′) *on* G/B *belonging to the dominant chamber, the natural homomorphism*

$$\varphi_{\chi,\chi'}: \quad H^0(G/B, L(\chi)) \otimes H^0(G/B, L(\chi')) \to H^0(G/B, L(\chi+\chi'))$$

is surjective.

Proof. — This is immediate in char $k = 0$ because $\varphi_{\chi,\chi'}$ is a G-equivariant map and $H^0(G/B, L(\chi+\chi'))$ is an irreducible G-module. As in the proof of the above corollay. we easily conclude that in any characteristic $\varphi_{\chi,\chi'}$ is surjective.

Remark. — The surjectivity of the map φ_r in the proof of Corollary 2.5 is itself a particular case of the above corollary. Thus the theory of " reduction mod p " and the known information in characteristic zero are used only in the proof of the above corollary. However, it seems possible to prove the above corollary directly in any characteristic by the same procedure that we are going to adopt to prove Theorem 2.1, i. e., by induction on the rank of G as well as on the dimension of a class of Schubert varieties in G/B. But we have not attempted to carry out the details.

THE CASE OF G/P. — Let G be any semi-simple (connected) algebraic group of rank m. Let $P \supset B$ be a parabolic subgroup of G, say associated to a subset S_P of the set $S = \{ \alpha_1, \ldots, \alpha_m \}$ of simple roots. Let $\pi : G/B \to G/P$ be the natural morphism. It is easy to check that Pic (G/P) is identified (*via* π^*) with the subgroup of Pic (G/B) generated by $L(\tilde{\omega}_{i_1}), \ldots, L(\tilde{\omega}_{i_p})$ where the $i_1, \ldots, i_p$ are determined by the set

$$S - S_P = \{ \alpha_{i_1}, \ldots, \alpha_{i_p} \}$$

complementary to S_P. Let us call a line bundle M on G/P "*positive*" and write $M \geqq 0$ if $\pi^*(M)$ is positive on G/B. We write $M > 0$ if $\pi^* M = L(\chi)$ with $(\chi, \alpha_{i_k}^*) \geqq 1$ for all $k = 1, \ldots, p$. Now we have the following

THEOREM 2.7. — *Let* G *be as in Theorem* 2.1 *and* P *a parabolic subgroup of* G. *Let* M *be a line bundle on* G/P. *Then*

(i) $$H^i(G/P, M) = 0 \quad \textit{for all } i \geqq 1 \textit{ if } M \geqq 0$$

and

(ii) $$H^i(G/P, M^{-1}) = 0 \quad \textit{for } 0 \leqq i < \dim G/P \textit{ if } M > 0, \textit{ consequently,}$$

(iii) G/P *is arithmetically normal and Cohen-Macaulay for the imbedding defined by every very ample line bundle.*

Proof. — Everything follows as an immediate consequence of the above theorem and its corollaries in view of the following

CLAIM. — *For all line bundles* M *on* G/P, *we have*

$$H^i(G/P, M) \approx H^i(G/B, \pi^* M) \quad \textit{for all } i \geqq 0.$$

To see this; observe that we have : (*a*) $R^0 \pi_* \mathcal{O}_{G/B} = \mathcal{O}_{G/P}$ and (*b*) $R^j \pi_* (\mathcal{O}_{G/B}) = 0$ for all $j \geqq 1$. Since G/P is normal and π is locally trivial with fibres $\approx P/B =$ Complete varieties, we see that (*a*) is immediate. Since P is of the same type as G, we find as a particular case of Theorem 2.1 for P that

$$H^j(P/B, \mathcal{O}_{P/B}) = 0 \quad \text{for all } j \geqq 1.$$

In other words, the higher cohomology groups of the restriction of $\mathcal{O}_{G/B}$ to the fibres of π are all zero and hence by the semi-continuity theorem (or directly) (*b*) follows. Now the Leray spectral sequence of π :

$$H^i(G/P, R^j \pi_*(\pi^* M)) \Rightarrow H^{i+j}(G/B, \pi^* M)$$

degenerates because

$$R^j \pi_*(\pi^* M) \approx R^j \pi_*(\mathcal{O}_{G/B}) \otimes M = 0 \qquad (\text{for } j \geqq 1)$$

and hence we get

$$\begin{aligned} H^i(G/B, \pi^* M) &= H^i(G/P, R^0 \pi_*(\pi^* M)) \\ &= H^i(G/P, R^0 \pi_*(\mathcal{O}_{G/B}) \otimes M) \\ &= H^i(G/P, M) \end{aligned}$$

as required.

3. **Proof of the Main Theorem**

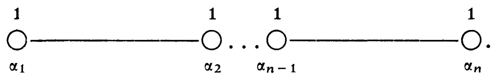
(Case by case : Type $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ or $\mathbf{G}_2$)

To prove Theorem 2.1, we can assume that G is simple, i. e., G is one of type $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ or $\mathbf{G}_2$. As is pointed out in the introduction, the method of proof is the same in all the cases. Now we carry out the details case by case (keeping the sequence of the main steps to be in the same order). We give full details in the case when G is of type $\mathbf{A}_n$. (Some propositions proved in this case do not use the fact that G is of type $\mathbf{A}_n$, i. e., they hold in the other types as well.) Whenever the proof is similar to the case of type $\mathbf{A}_n$, we omit the details in the other cases.

A. Type $\mathbf{A}_n$

1. Numerical data. — Recall the following facts for a group G of type $\mathbf{A}_n$.

Dynkin diagram :



The Cartan numbers

$$n_{ij} = (\alpha_i, \alpha_j^*) = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$$

are given by

$$n_{ij} = \begin{cases} 2 & \text{if } \; i = j, \\ -1 & \text{if } \; i = j \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

The number of roots $= n(n+1)$.

The order of the Weyl group $\mathrm{W}\,(=\mathrm{W}(\mathbf{A}_n))$ of $\mathrm{G} = (n+1)!$

2. A reduced expression for w_0.

Proposition A.1. — *A reduced expression for the element $w_0 \in \mathrm{W}$ (of largest length) is given by*

$$w_0 = s_n(s_{n-1}s_n)\ldots(s_i\ldots s_n)\ldots(s_1\ldots s_n).$$

Proof. — Write

$$u_1 = s_n(s_{n-1}s_n)\ldots(s_1\ldots s_n)$$

and note that

$$l(u_1) \leqq 1 + \ldots + n = \frac{1}{2}n(n+1).$$

We prove that $u_1(\alpha) < 0$ for all roots $\alpha > 0$. Then by Proposition 1.1 (iii), it would follow that $u_1 = w_0$. But we know that

$$l(w_0) = \frac{1}{2}n(n+1) = \frac{1}{2} \text{ number of roots}$$

which implies that the given expression for w_0 is reduced.

Now define inductively two sequences $\{v_i\}$ and $\{u_i\}$, $1 \leqq i \leqq n$, of elements in W as follows :

$$\{v_i\} : \quad v_n = s_n \quad \text{and} \quad v_i = s_i v_{i+1}, \qquad 1 \leqq i < n;$$
$$\{u_i\} : \quad u_n = v_n \quad \text{and} \quad u_i = u_{i+1} v_i, \qquad 1 \leqq i < n.$$

That $u_1(\alpha) < 0$ for all roots $\alpha > 0$ is a particular case of the following

ASSERTION I.

$$u_i(\alpha_j) = \begin{cases} -\alpha_{n+i-j} & \textit{for} \quad i \leqq j \leqq n, \\ (\alpha_{i-1} + \ldots + \alpha_n) & \textit{for} \quad j = i-1, \\ \alpha_j & \textit{for} \quad j \leqq i-2. \end{cases}$$

We prove this by decreasing induction on i. For $i = n$, we have $u_n = v_n = s_n$ and the result is trivial. Assume the assertion proved for all u_k, $k > i$. Since $u_i = u_{i+1} v_i$, it is easy to check that the result follows for u_i using the values $v_i(\alpha_j)$ as given by the following

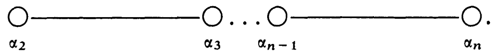
ASSERTION II. — *We have*

$$v_i(\alpha_j) = \begin{cases} -(\alpha_i + \ldots + \alpha_n) & \textit{for} \quad j = n, \\ \alpha_{j+1} & \textit{for} \quad i \leqq j \leqq n-1, \\ \alpha_{i-1} + \alpha_i & \textit{for} \quad j = i-1, \\ \alpha_j & \textit{for} \quad j \leqq i-2. \end{cases}$$

We prove this again by decreasing induction on i. For $i = n$, we have $v_n = s_n$ and the assertion is obvious. Assume the result proved for all v_k, $k > i$. Since $v_i = s_i v_{i+1}$, direct verification proves the assertion.

This completes the proof of the proposition.

3. THE PARABOLIC SUBGROUP $P (= P_{\hat{\alpha}_1})$. — Since no confusion is likely, denote by P the maximal parabolic subgroup of G associated to omitting α_1. It is easy to check that the semi-simple part of P is of type $\mathbf{A}_{n-1}$ and its Dynkin diagram can be canonically taken to be



In particular, the element say w'_0 (of largest length) in the Weyl group $W_P (= W(\mathbf{A}_{n-1}))$ of P is given by

$$w'_0 = s_n (s_{n-1} s_n) \ldots (s_i \ldots s_n) \ldots (s_2 \ldots s_n).$$

Thus we have $w_0 = w'_0 (s_1 \ldots s_n)$, $w_0 \in W$. The number of Schubert varieties in $P \backslash G = [W : W_P] = n+1$.

4. THE FAMILY OF CLOSED BRUHAT CELLS $\{X(w_i)\}$. — Define two sequences $\{\tau_i\}$ and $\{w_i\}$ of elements in W as follows,

$$\{\tau_i\} : \quad \tau_0 = \text{Id} \quad \text{and} \quad \tau_i = s_n \ldots s_{n-i+1}, \qquad 1 \leqq i \leqq n;$$
$$\{w_i\} : \quad w_i = w_0 \tau_i = w'_0 s_1 \ldots s_{n-i}, \qquad 0 \leqq i \leqq n.$$

Now we have the following

PROPOSITION A.2.

(i) $$G = X(w_0) \supset X(w_1) \supset \ldots \supset X(w_n) = P.$$

(ii) *Each* $X(w_i)$ *is of codimension* 1 *in* $X(w_{i-1})$.

(iii) $X(w_i)$ *are precisely the inverse images (under the natural morphism* $G \to P\backslash G$) *of the Schubert varieties in* $P\backslash G$.

Proof. — Since each w_i is a *segment* of w_0 and the expression for w_0 is reduced, the expression for w_i is also reduced. Since $w_{i+1} = w_i s_{n-i}$, (i) and (ii) are immediate from Proposition 1.7 and the fact that codimension of $X(w)$ in $X(u)$ [with $X(w) \subseteq X(u)$] is $l(u) - l(w)$ for all w and $u \in W$ (*cf.* Remark 1.10). Since w'_0 is the element of largest length in W_P, we have $l(s_j w'_0) < l(w'_0)$ for all $j \geqq 2$, and hence $l(s_j w_i) < l(w_i)$ for all i and $j \neq 1$. But then by Proposition 1.4, $X(w_i)$ is stable for multiplication on the left by all the (minimal) parabolic subgroups $P_{\alpha_j}, j \geqq 2$ and hence also by P. Thus each $X(w_i)$ is the inverse image of a Schubert variety in $P\backslash G$. But the number of Schubert varieties in $P\backslash G$ is $[W : W_P] = n+1$ and hence (iii) follows.

REMARK A.3.

(i) From the Bruhat decomposition of G (relative to P) and the above proposition, it follows that

$$X(w_i) = P w_i B \cup X(w_{i+1}), \qquad 0 \leqq i \leqq n-1$$

the union being set-theoretic and disjoint.

(ii) $X(w_i)$ is stable for multiplication on the right by the (minimal) parabolic subgroup $P_{\alpha_{n-i}}$. This is immediate from Proposition 1.4, because $w_{i+1} = w_i s_{n-i}$ and $l(w_{i+1}) = l(w_i) - 1$.

PROPOSITION A.4. — *We have (set-theoretically) :*

$$X(w_i) \cap X(w_1)\tau_i = X(w_{i+1}), \qquad 0 \leqq i \leqq n-1,$$

where $X(w_1)\tau_i$ *denotes the translate of* $X(w_1)$ *by an element in* N(T) *whose residue class* mod T *is* τ_i $(\tau_i = w_0 w_i)$.

Proof. — Observe the following simple facts.

(*a*) $X(w_i) \not\subset X(w_1)\tau_i$ (consequently the intersection is proper). For otherwise, we have $X(w_i) \subseteq X(w_1)\tau_i$ i.e., $X(w_i)\tau_i^{-1} \subseteq X(w_1)$ which gives in particular that $w_0 = w_i \tau_i^{-1} \in X(w_1)$ and hence $G = X(w_0) \subseteq X(w_1)$ which is a contradiction. Now it follows that (*b*) $Z_i = X(w_i) \cap X(w_1)\tau_i$ is of pure codimension 1 in $X(w_i)$ [because $X(w_1)$ is irreductible and of codimension 1 in G]. (*c*) $P w_i B \cap X(w_1)\tau_i = \varnothing$. To see this, first note that $B w_i B \cap X(w_1)\tau_i = \varnothing$. Otherwise, we have $x\tau_i^{-1} \in X(w_1)$ for some $x \in B w_i B$. By Proposition 1.3, we can write $x = b_1 w_i b_2$ with $b_1 \in B$ and a unique $b_2 \in B^u \cap w_i^{-1} \tilde{B}^u w_i$. Write $b_2 = w_i^{-1} c w_i$ with $c \in \tilde{B}^u$ and also write $c = w_0 d w_0$ for some $d \in B^u$ (since $\tilde{B} = w_0 B w_0$). Thus we have

$$x \tau_i^{-1} = b_1 w_i w_i^{-1} w_0 d w_0 w_i w_i^{-1} w_0 = b_1 w_0 d \in X(w_1)$$

and hence $w_0 \in X(w_1)$, a contradiction.

Now suppose that $P w_i B \cap X(w_1) \tau_i \neq \emptyset$. Then some $x = p w_i b \in X(w_1) \tau_i$. Since $X(w_1) \tau_i$ is P-stable on the left, we get that $p^{-1} x = w_i b \in X(w_1) \tau_i$ which means $B w_i B \cap X(w_1) \tau_i \neq \emptyset$, a contradiction.

Now the proof of the proposition is immediate in view of Proposition A.2, Remark A.3 and the facts (*b*) and (*c*) above.

PROPOSITION A.5. — *Let* $\chi_i = \tau_i^{-1}(\tilde{\omega}_n)$, $0 \leqq i \leqq n-1$. *For each i, there exists an element* $f_i \in H^0(X(w_i)_r, L(\chi_i)|_{X(w_i)_r})$ *such that (set-theoretically) the set of zeros of* f_i *in* $X(w_i)_r$ *is* $X(w_{i+1})_r$. (A more precise description of the f_i's is available in the proof below.)

Proof. — Since $w_1 = w_0 s_n = s_1 w_0$, we know by Remark 1.11 that there exists a section $f \in H^0(G/B, L(\tilde{\omega}_n))$ such that $X(w_1)$ is the set of zeros of the morphism $f: G \to k$ and f satisfies the double invariance property, namely,

$$f(bgb') = \tilde{\omega}_1(b^{-1}) f(g) \tilde{\omega}_n(b')$$

for all $b, b' \in B$ and $g \in G$.

For a fixed $h \in G$, consider the function $f_h: G \to k$ defined by $f_h(g) = f(gh)$ for all $g \in G$. [Observe that $f_h \in H^0(B \backslash G, L(\tilde{\omega}_1))$ but need not define a section of any line bundle on G/B.] We have

$$f_h(g) = 0 \quad \Leftrightarrow \quad gh \in X(w_1) \quad \Leftrightarrow \quad g \in X(w_1) h^{-1},$$

i. e., the set of zeros of f_h is precisely $X(w_1) h^{-1}$. Note that for $t \in T$, we have

$$f_{ht}(g) = f(ght) = f(gh) \tilde{\omega}_n(t) = f_h(g) \tilde{\omega}_n(t),$$

i. e., the functions f_h and f_{ht} differ by multiplication by a non-zero scalar. Since this ambiguity of a non-zero scalar multiple does not change our future calculations, we write $f_i = f_{\tau_i^{-1}}$, $0 \leqq i < n$, and find that $f_0 = f$ and the set of zeros of f_i is $X(w_1) \tau_i$. Now the result follows, in view of Proposition A.4, once we prove that f_i defines a section of the line bundle $L(\chi_i)|_{X(w_i)_r}$, i. e. it suffices to show that

$$f_i(gb) = f_i(g) \chi_i(b)$$

for all $g \in X(w_i)$ and $b \in B$.

Recall that $\chi_i = \tau_i^{-1}(\tilde{\omega}_n): T \to k^*$ is the character defined by $\chi_i(t) = \tilde{\omega}_n(\tau_i t \tau_i^{-1})$ for $t \in T$. Let $b \in B$ and write $b = b^u . t$ for some $b^u \in B^u$ and $t \in T$. We have

$$\begin{aligned} f_i(gb) &= f(gb . \tau_i^{-1}) = f((gb^u \tau_i^{-1})(\tau_i t \tau_i^{-1})) \\ &= f(gb^u \tau_i^{-1}) \tilde{\omega}_n(\tau_i t \tau_i^{-1}) = f_i(gb^u) \chi_i(t) = f_i(gb^u) \chi_i(b). \end{aligned}$$

Thus it suffices to prove that $f_i(gb'') = f_i(g)$ for $g \in X(w_i)$ and $b'' \in B''$. We see easily that it suffices to verify this for $g = w_i$, i. e. we have only to check that $f_i(w_i b'') = f_i(w_i)$ for all $b'' \in B''$. For this, as in (c) of Proposition A.4, write $w_i b'' = b_1 w_i b_2$ with $b_2 = w_i^{-1} w_0 d w_0 w_i$ where $b_1 \in B$ and $d \in B''$, etc. We can assume that $b_1 \in B''$ as T fixes elements of W. We have

$$\begin{aligned} f_i(w_i b'') &= f_i(b_1 w_i b_2) = f_i(b_1 w_i w_i^{-1} w_0 d w_0 w_i) = f(b_1 w_0 d w_0 w_i \tau_i^{-1}) \\ &= f(b_1 w_0 d)(\text{since } \tau_i = w_0 w_i) = \tilde{\omega}_1(b_1^{-1}) f(w_0)\tilde{\omega}_n(d) \\ &= f(w_0)(\text{since } b_1, d \in B'') = f(w_i w_i^{-1} w_0) = f(w_i \tau_i^{-1}) \\ &= f_i(w_i) \end{aligned}$$

as required.

5. THE FAMILY OF CHARACTERS $\{\chi_i\}$.

PROPOSITION A.6. — *We have* $\chi_0 = \tau_0^{-1}(\tilde{\omega}_n) = \tilde{\omega}_n$ *and*

$$\chi_i = \tau_i^{-1}(\tilde{\omega}_n) = \tilde{\omega}_{n-i} - \tilde{\omega}_{n-i+1} \qquad \textit{for} \quad 1 \leqq i \leqq n-1.$$

Proof. — We prove this by increasing indiction on i. $\left(\text{Recall that } \alpha_j = \sum_{k=1}^{n} n_{jk}\tilde{\omega}_k\right.$ where n_{jk} are the Cartan numbers.$\left.\right)$ We have for $i = 1$, $\tau_1 = s_n$ and

$$\begin{aligned} s_n(\tilde{\omega}_n) &= \tilde{\omega}_n - (\tilde{\omega}_n, \alpha_n^*)\alpha_n = \tilde{\omega}_n - \alpha_n \\ &= \tilde{\omega}_n - (-\tilde{\omega}_{n-1} + 2\tilde{\omega}_n) = \tilde{\omega}_{n-1} - \tilde{\omega}_n. \end{aligned}$$

Assume the result proved for all $k < i$. Recall that $\tau_i = \tau_{i-1} s_{n-i+1}$ and hence

$$\begin{aligned} \tau_i^{-1}(\tilde{\omega}_n) &= s_{n-i+1}(\tau_{i-1}^{-1}(\tilde{\omega}_n)) = s_{n-i+1}(\tilde{\omega}_{n-i+1} - \tilde{\omega}_{n-i+2}) \quad \text{(by induction)} \\ &= \tilde{\omega}_{n-i+1} - \alpha_{n-i+1} - \tilde{\omega}_{n-i+2} \\ &= \tilde{\omega}_{n-i+1} - (-\tilde{\omega}_{n-i} + 2\tilde{\omega}_{n-i+1} - \tilde{\omega}_{n-i+2}) - \tilde{\omega}_{n-i+2} \\ &= \tilde{\omega}_{n-i} - \tilde{\omega}_{n-i+1} \end{aligned}$$

as required.

6. STRUCTURE OF SCHUBERT VARIETIES IN P\G. — For the purpose of this section, we take $G = SL(n+1)$ and fix T and B as usual (i. e., the diagonal and upper triangular matrices). It is easy to see that $P = P_{\hat{\alpha}_1}$ is the subgroup of matrices of the form (g_{ij}), $0 \leqq i, j \leqq n$, with $g_{i0} = 0$ for $i \geqq 1$, and its " semi-simple part " is the set of matrices of the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & X & \\ 0 & & & \end{pmatrix} \approx SL(n).$$

We identify $P\backslash G$ with the projective space $\mathbf{P}(V)$ where V is a vector space of dimension $n+1$, with coordinates $(x_0, \ldots, x_n)$. The action of G on the right can be identified with the usual multiplication of matrices, namely,

$$(x_0, \ldots, x_n)(g_{ij}) = \left(\ldots, \sum_{i=0}^{n} x_i g_{ij}, \ldots\right).$$

It is clear that the linear subspaces of $\mathbf{P}(V)$ defined by $(0, *, \ldots, *)$, $\ldots$, $(0, \ldots, 0, *)$ are B-stable i. e., the Schubert varieties in $\mathbf{P}(V)$ are obtained by taking

$$x_0 = 0; \qquad x_0 = x_1 = 0; \qquad \ldots; \qquad x_0 = x_1 = \ldots = x_{n-1} = 0.$$

In particular (these are non-singular and) each is obtained from the previous one with the (scheme-theoretic) intersection of a hyper-plane in $\mathbf{P}(V)$.

7. THE IDEAL SHEAF OF $X(w_i)_r$ IN $X(w_{i-1})_r$.

PROPOSITION A.7. — *The sheaf of ideals defining* $X(w_i)_r$ *in* $X(w_{i-1})_r$ *is precisely* $L(-\chi_{i-1})|_{X(w_{i-1})_r}$ *(i. e., the equalities in Proposition* A.4 *and A.5 are scheme-theoretic).*

Proof. — Let M denote the tautological line bundle on the projective space $\mathbf{P}(V) = P\backslash G$ where V is a vector space of dimension $n+1$ with coordinates $(x_0, \ldots, x_n)$. By Proposition A.2, we know that $Y_i = X_P(w_i)_l$, $0 \leqq i \leqq n$, are the Schubert varieties in $P\backslash G$. We have seen that Y_i is scheme-theoretically the intersection of Y_{i-1} and the hyper-plane whose equation is $x_{i-1} = 0$ [i. e., the set of zeros of the section $x_{i-1} \in H^0(P\backslash G, M)$].

Recall that, if $\pi_l : B\backslash G \to P\backslash G$ is the natural morphism, we have $\pi_l^* M = L(\tilde{\omega}_1)$ and that

$$H^0(P\backslash G, M) = H^0(B\backslash G, L(\tilde{\omega}_1)).$$

Hence, we see that the functions f_i, $0 \leqq i < n$, as in Proposition A.5, are elements in $H^0(P\backslash G, M)$ and have the property that Y_i is set-theoretically the intersection of Y_{i-1} and the set of zeros of f_{i-1}. Hence it follows that each f_i is a non-zero scalar multiple of x_i, consequently, we get that $X(w_i)_l$ [resp. $X(w_i)$] is scheme-theoretically, the intersection of $X(w_{i-1})_l$ [resp. $X(w_{i-1})$] and the set of zeros of the section

$$f_{i-1} \in H^0(B\backslash G, L(\tilde{\omega}_1))$$

(resp. the function $f_{i-1} : G \to k$) because the morphism $B\backslash G \to P\backslash G$ (resp. $G \to P\backslash G$) is smooth. Since the morphism $G \to G/B$ is smoth, it follows that $X(w_i)_r$ is scheme-theoretically the intersection of $X(w_{i-1})_r$ and the set of zeros of the section

$$f_{i-1} \in H^0(X(w_{i-1})_r, L(\chi_{i-1})|_{X(w_{i-1})_r})$$

and hence the result.

8. VANISHING THEOREM.

THEOREM A.8. — *For every line bundle* $L(\chi)$ *on* G/B *belonging to the dominant chamber* [*i. e.*, $(\chi, \alpha_j^*) \geqq 0$ *for all* j], *we have*

$$H^p(X(w_i)_r, L(\chi)|_{X(w_i)_r}) = 0 \quad \textit{for all } p > 0$$

and $i = 0, \ldots, n$. *In particular for* $i = 0$, *we have*

$$H^p(G/B, L(\chi)) = 0 \quad \textit{for all } p > 0.$$

Proof. — Write $X_i = X(w_i)_r$, $0 \leqq i \leqq n$. We prove the result by induction on n = rank of G as well as on the dimension of X_i. If $n = 1$, X_1 is a point and $X_0 = G/B \approx \mathbf{P}^1$ and so the result follows. Assume $n > 1$ and the result true for the groups of type $\mathbf{A}_m$, $m < n$.

We have $\dim X_i \geqq \dim X_n$ for all i and $X_n = P/B$. We know that the semi-simple part of P is of type $\mathbf{A}_{n-1}$, and so the result is true for X_n by the induction hypothesis. Now assume the second induction hypothesis namely that the result is true for all X_k, $k > i$. *To prove the result for* X_i. — By Proposition A.7, the sheaf of ideals defining X_{i+1} in X_i is

$$\approx L(-\chi_i)\,|_{X_i} = L(\tilde{\omega}_{n-i+1} - \tilde{\omega}_{n-i})\,|_{X_i}.$$

This gives the exact sequence

$$0 \to L(\tilde{\omega}_{n-i+1} - \tilde{\omega}_{n-i})\,|_{X_i} \to \mathcal{O}_{X_i} \to \mathcal{O}_{X_{i+1}} \to 0.$$

Tensoring by $L(\chi)$, we get the exact sequence

$$0 \to L(\chi')\,|_{X_i} \to L(\chi)\,|_{X_i} \to L(\chi)\,|_{X_{i+1}} \to 0,$$

where $\chi' = \chi + \tilde{\omega}_{n-i+1} - \tilde{\omega}_{n-i}$.

This gives the cohomology exact sequence

$$\to H^p(X_i, L(\chi')) \to H^p(X_i, L(\chi)) \to H^p(X_{i+1}, L(\chi)) \to$$

By the second induction hypothesis, this sequence reduces to the exact sequences

$$(\bigstar) \qquad H^p(X_i, L(\chi')) \to H^p(X_i, L(\chi)) \to 0$$

for all $p \geqq 1$.

Now we complete the proof of the theorem by increasing induction on the integer (χ, α^*_{n-i}) i.e., the degree of $L(\chi)$ with respect to the $\mathbf{P}^1$-fibration $G/B \to G/P_{\alpha_{n-i}}$. Note that we have

$$\text{(i)} \qquad (\chi', \alpha^*_{n-i}) = (\chi, \alpha^*_{n-i}) - 1 \geqq -1$$

and

$$\text{(ii)} \qquad (\chi', \alpha^*_j) \geqq (\chi, \alpha^*_j) \quad \text{for all } j \neq n-i.$$

Suppose $(\chi, \alpha^*_{n-i}) = 0$. Then, since $(\chi', \alpha^*_{n-i}) = -1$, by Remark A.3 (ii) and Corollary 1.15, we have

$$H^p(X_i, L(\chi')) = 0 \quad \text{for all } p \geqq 1.$$

In this case $(\bigstar)$ implies the required result. Assume now $(\chi, \alpha^*_{n-i}) \geqq 1$ and the induction hypothesis [that for all $\chi'' \geqq 0$ such that $(\chi'', \alpha^*_{n-i}) < (\chi, \alpha^*_{n-i})$, we have $H^p(X_i, L(\chi'')) = 0$

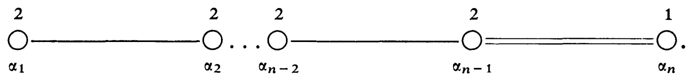
for all $p \geqq 1$]. But then in view of (i) and (ii) above we have $\chi' \geqq 0$ and so by the induction hypothesis we find that $H^p(X_i, L(\chi') = 0)$ for all $p \geqq 1$ and hence (★) implies the required result.

This completes the proof of the theorem.

B. Type B_n ($n \geqq 2$)

1. Numerical data. — Recall the following facts for a group G of type B_n.

Dynkin diagram :



The Cartan numbers

$$n_{ij} = (\alpha_i, \alpha_j^*) = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$$

are given by

$$n_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1 \text{ and } i, j \leqq n-1, \\ -2 & \text{if } (i, j) = (n-1, n), \\ -1 & \text{if } (i, j) = (n, n-1), \\ 0 & \text{otherwise.} \end{cases}$$

The number of roots $= 2n^2$.

The order of the Weyl group W (= W (B_n)) of G $= 2^n.n!$

2. A reduced expression for w_0.

Proposition B.1. — *A reduced expression for the element* $w_0 \in W$ (*of largest lenght*) *is given by*

$$w_0 = s_n(s_{n-1}s_ns_{n-1})\ldots(s_i\ldots s_n\ldots s_i)\ldots(s_1\ldots s_n\ldots s_1).$$

Proof. — This is similar to the proof of Proposition A.1. In this case we define the sequences $\{v_i\}$ and $\{u_i\}$ as follows :

$$\{v_i\}: \quad v_n = s_n, \quad v_i = s_i v_{i+1} s_i, \qquad 1 \leqq i < n;$$
$$\{u_i\}: \quad u_n = v_n, \quad u_i = u_{i+1} v_i, \qquad 1 \leqq i < n.$$

The following assertion for u_1 together with Proposition 1.1 (iii) proves the proposition.

Assertion I.

$$u_i(\alpha_j) = \begin{cases} -\alpha_j & \textit{for } i \leqq j \leqq n, \\ (\alpha_{i-1} + 2\alpha_i + \ldots + 2\alpha_n) & \textit{for } j = i-1, \\ \alpha_j & \textit{for } j \leqq i-2. \end{cases}$$

(*In particular* $u_1 = -\mathrm{Id}$.)

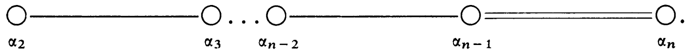
We prove this by decreasing induction on i using the fact that $u_i = u_{i+1} v_i$ and the values of $v_i(\alpha_j)$ from the following

ASSERTION II. — *We have*

$$v_i(\alpha_j) = \begin{cases} \alpha_j & \text{for } j \neq i-1 \text{ and } i, \\ -(\alpha_i + 2\alpha_{i+1} + \ldots + 2\alpha_n) & \text{for } j = i, \\ (\alpha_{i-1} + 2\alpha_i + \ldots + 2\alpha_n) & \text{for } j = i-1. \end{cases}$$

We prove this again by decreasing induction on i noting that $v_i = s_i v_{i+1} s_i$.

3. THE PARABOLIC SUBGROUP $P (= P_{\hat{\alpha}_1})$. — As before, it can be seen that the semi-simple part of P is of type B_{n-1} (or A_1 if $n = 2$) and its Dynkin diagram can be taken to be



In particular, the element w'_0 in the Weyl group $W_P (= W(B_{n-1}))$ of P is given by

$$w'_0 = s_n (s_{n-1} s_n s_{n-1}) \ldots (s_i \ldots s_n \ldots s_i) \ldots (s_2 \ldots s_n \ldots s_2).$$

Thus $w_0 = w'_0 (s_1 \ldots s_n \ldots s_1)$. The number of Schubert varieties in

$$P \backslash G = [W : W_P] = 2n.$$

4. THE FAMILY OF CLOSED BRUHAT CELLS $\{ X(w_i) \}$. — Define the sequences $\{ r_i \}$, $\{ \tau_i \}$ and $\{ w_i \}$ of elements in W as follows :

$$\{r_i\} : \quad r_0 = \text{Id}, \quad r_1 = s_1, \ \ldots, \ r_n = s_n,$$
$$r_{n+1} = s_{n-1}, \ \ldots, \ r_{n+i} = s_{n-i}, \ \ldots, \ r_{2n-1} = s_1;$$
$$\{\tau_i\} : \quad \tau_0 = \text{Id}, \quad \tau_i = r_{2n-1} \ldots r_{2n-i}, \qquad 1 \leqq i \leqq 2n-1;$$
$$\{w_i\} : \quad w_i = w_0 \tau_i = w'_0 r_1 \ldots r_{2n-1-i}, \qquad 0 \leqq i \leqq 2n-1.$$

PROPOSITION B.2.

(i) $$G = X(w_0) \supset X(w_1) \supset \ldots \supset X(w_{2n-1}) = P.$$

(ii) *Each* $X(w_i)$ *is of codimension* 1 *in* $X(w_{i-1})$.

(iii) $X(w_i)$ *are the inverse images* (*under the natural morphism* $G \to P\backslash G$) *of the Schubert varieties in* $P\backslash G$.

Proof. — Similar to that of Proposition A.2.

REMARK B.3. — We have

(i) $$X(w_i) = P w_i B \cup X(w_{i+1}), \qquad 0 \leqq i \leqq 2n-2$$

the union being the set-theoretic and disjoint.

(ii) $X(w_i)$ is stable for multiplication on the right by the (minimal) parabolic subgroup $P_{\alpha_{i+1}}$ if $0 \leqq i \leqq n-1$ or $P_{\alpha_{n-j-1}}$ if $i = n+j$, $0 \leqq j \leqq n-2$.

Proof. — Similar to that of Remark A.3.

PROPOSITION B.4. — *We have (set-theoretically) :*

$$X(w_i) \cap X(w_1)\tau_i = X(w_{i+1}), \qquad 0 \leqq i \leqq 2n-2$$

(where $\tau_i = w_0 w_i$*),*

Proof. — Similar to that of Proposition A.4.

PROPOSITION B.5. — *Let* $\chi_i = \tau_i^{-1}(\tilde{\omega}_1)$, $0 \leqq i \leqq 2n-2$. *For each i, there exists an element* $f_i \in H^0(X(w_i)_r, L(\chi_i)|_{X(w_i)_r})$ *such that (set-theoretically) the set of zeros of* f_i *in* $X(w_i)_r$ *is* $X(w_{i+1})_r$. (A more precise description of the f_i*'s* is available in the proof below).

Proof. — Since $w_1 = w_0 s_1 = s_1 w_0$, we know by Remark 1.11 that there exists a section $f \in H^0(G/B, L(\tilde{\omega}_1))$ such that $X(w_1)$ is the set of zeros of the morphism $f: G \to k$ and f satisfies the double invariance property, namely,

$$f(bgb') = \tilde{\omega}_1(b^{-1}) f(g) \tilde{\omega}_1(b')$$

for all $b, b' \in B$ and $g \in G$. As in the proof of Proposition A.5, we define $f_i = f_{\tau_i^{-1}}$, $0 \leqq i < 2n-1$ and prove that the f_i*'s* are the required ones. [In fact, in proving that f_i defines a section of $L(\chi_i)|_{X(w_i)_r}$, we do not use the fact that G is of type $\mathbf{A}_n$ or $\mathbf{B}_n$].

5. THE FAMILY OF CHARACTERS $\{\chi_i\}$.

PROPOSITION B.6. — *The characters* $\chi_i (=\tau_i^{-1}(\tilde{\omega}_1))$, $0 \leqq i \leqq 2n-2$, *are given by* $\chi_0 = \tilde{\omega}_1$ *and*

(*a*) $$\chi_i = \tilde{\omega}_{i+1} - \tilde{\omega}_i \quad \textit{for} \quad 1 \leqq i \leqq n-2.$$

(*b*) $$\chi_{n-1} = -\chi_n = 2\tilde{\omega}_n - \tilde{\omega}_{n-1}.$$

(*c*) $$\chi_{n+j} = \tilde{\omega}_{n-j-1} - \tilde{\omega}_{n-j} \quad \textit{for} \quad 1 \leqq j \leqq n-2.$$

Proof. — Recall that $\tau_0 = \mathrm{Id}$ and $\tau_i = r_{2n-1} \ldots r_{2n-i}$ for $1 \leqq i \leqq 2n-1$ (*see* the previous section for the notation), i. e. we have

$$\tau_i = s_1 \ldots s_i = \tau_{i-1} s_i \qquad \text{for} \quad 1 \leqq i \leqq n$$

and

$$\tau_{n+j} = \tau_{n+j-1} s_{n-j}, \qquad 1 \leqq j \leqq n-1.$$

Proof of (a). — We proceed by increasing induction on *i*. For $i = 1$, we have $\tau_1 = s_1$ and

$$\begin{aligned} s_1(\tilde{\omega}_1) &= \tilde{\omega}_1 - (\tilde{\omega}_1 - \alpha_1^*)\alpha_1 = \tilde{\omega}_1 - \alpha_1 \\ &= \tilde{\omega}_1 - \left(\sum_{j=1}^{n} n_{1j}\tilde{\omega}_j\right) = \tilde{\omega}_1 - (2\tilde{\omega}_1 - \tilde{\omega}_2) = \tilde{\omega}_2 - \tilde{\omega}_1 \end{aligned}$$

as required. Assume the result true for all $k < i$. Now

$$\begin{aligned} \chi_i = \tau_i^{-1}(\tilde{\omega}_1) &= s_i \tau_{i-1}^{-1}(\tilde{\omega}_1) = s_i(\tilde{\omega}_i - \tilde{\omega}_{i-1}) = \tilde{\omega}_i - \tilde{\omega}_{i-1} - \alpha_i \\ &= \tilde{\omega}_i - \tilde{\omega}_{i-1} - \left(\sum_{j=1}^{n} n_{ij}\tilde{\omega}_j\right) = \tilde{\omega}_i - \tilde{\omega}_{i-1} - (-\tilde{\omega}_{i-1} + 2\tilde{\omega}_i - \tilde{\omega}_{i+1}) \\ &= \tilde{\omega}_{i+1} - \tilde{\omega}_i \quad \text{as required.} \end{aligned}$$

Proof of (*b*). — We have

$$\tau_{n-1} = \tau_{n-2}\, s_{n-1} \qquad \text{and} \qquad \tau_n = \tau_{n-1}\, s_n.$$

By (*a*), we have

$$\chi_{n-2} = \tilde{\omega}_{n-1} - \tilde{\omega}_{n-2}.$$

Now

$$\chi_{n-1} = s_{n-1}(\chi_{n-2}) = \chi_{n-2} - (\chi_{n-2},\, \alpha^*_{n-1})\, \alpha_{n-1} = \tilde{\omega}_{n-1} - \tilde{\omega}_{n-2} - \alpha_{n-1}.$$

But

$$\alpha_{n-1} = \sum_{j=1}^{n} n_{n-1\,j}\, \tilde{\omega}_j = -\tilde{\omega}_{n-2} + 2\,\tilde{\omega}_{n-1} - 2\,\tilde{\omega}_n$$

and hence

$$\chi_{n-1} = 2\,\tilde{\omega}_n - \tilde{\omega}_{n-1}\, (= \alpha_n)$$

as required. Further, we have

$$\chi_n = s_n(\chi_{n-1}) = s_n(\alpha_n) = -\alpha_n = -\chi_{n-1}$$

and hence the result.

Proof of (*c*). — We proceed by increasing induction on *j*. Recall that

$$\tau_{n+j} = \tau_{n+j-1}\, s_{n-j}, \qquad 1 \leqq j \leqq n-2.$$

For $j = 1$, we have

$$\begin{aligned} \chi_{n+1} &= s_{n-1}(\chi_n) = \tilde{\omega}_{n-1} - 2\,\tilde{\omega}_n - \alpha_{n-1} \quad [\text{by } (b)] \\ &= \tilde{\omega}_{n-2} - \tilde{\omega}_{n-1} \quad (\text{since } \alpha_{n-1} = -\tilde{\omega}_{n-2} + 2\,\tilde{\omega}_{n-1} - 2\,\tilde{\omega}_n) \end{aligned}$$

as required. Now the result follows with the obvious formal step.

This completes the proof of the proposition.

6. Structure of Schubert varieties in P\G. — Recall (*cf.* § 1,8 above) that for the purpose of this section we can take any connected semi-simple group of type $\mathbf{B}_n$ (simply connected or not). For instance, we take $G = SO(2n+1)$, called the (*odd*) *Orthogonal group in* $2n+1$ *variables* (*cf.* [2], [7], [23] and [25]). Recall that $SO(2n+1)$ is realised as a subgroup of $GL(2n+1)$ as follows :

Let V be a vector space of rank $(2n+1)$ over the ground field k. Let us choose a basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n+1}$ of V. Let us represent a point v of V by

$$v = (x_1, \ldots, x_n, z, y_1, \ldots, y_n)$$

with respect to this basis. Let $Q\,(Q = Q_n)$ be the quadratic form

$$Q(v) = z^2 + (x_1 y_n + x_2 y_{n-1} + \ldots + x_n y_1)$$

with respect to this basis. Then $SO(2n+1)$ *is defined as the subgroup of* $SL(2n+1)$ *which leaves* Q *invariant, i.e.*,

$$SO(2n+1) = \{ A \in SL(2n+1) / Q(Av) = Q(v) \text{ for all } v \in V \}.$$

For this imbedding $SO(2n+1) \subsetneq GL(2n+1)$, the subgroup T (resp. B) of $SO(2n+1)$ consisting of the diagonal (resp. upper triangular) matrices is a maximal torus (resp. Borel subgroup). Fixing these, it is easy to check that $P = P_{\hat{\alpha}_1}$ is precisely the subgroup of $SO(2n+1)$ consisting of the set of matrices of the form

$$\begin{pmatrix} * & * & \dots & * & * \\ 0 & * & \dots & * & * \\ \vdots & \vdots & & \vdots & \vdots \\ \cdot & * & & * & * \\ 0 & 0 & \dots & 0 & * \end{pmatrix}$$

and its " semi-simple part " is the set of matrices of the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \boxed{X} & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \approx SO(2n-1).$$

Let P′ be the (maximal) parabolic subgroup of $GL(2n+1)$ consisting of the set of matrices of the form

$$\begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \dots & * \end{pmatrix}.$$

It is seen easily that $P = SO(2n+1) \cap P'$ (as group schemes).

As usual identifying $P' \backslash GL(2n+1)$ canonically with $\mathbf{P}(V)$, we see easily that the canonical closed immersion $P\backslash G \subsetneq P'\backslash GL(2n+1) = \mathbf{P}(V)$ identifies $P\backslash G$ with the quadraic $Q = 0$.

Notice that the points on $P\backslash G$ fixed by the maximal torus T are precisely the points $(0, \dots, 0, 1, 0, \dots, 0)$, 1 at the i-th place, $i \neq n+1$. Hence it is easy to see that the Schubert varieties in $P\backslash G$ are $(Y_i)_{red}$, where

$$Y_i = P\backslash G \cap \mathbf{P}^{2n-i}, \qquad i = 0, 1, \dots, n, \quad n+2, \dots, 2n$$

(the intersection being scheme-theoretic) where

$$\mathbf{P}^{2n-i} = \{(0, \dots, 0, \overset{i+1}{*}, *, \dots, *)\}, \qquad i = 0, 1, \dots, n, \quad n+2, \dots, 2n$$

are (all but one) Schubert varieties in $\mathbf{P}(V)$. In other words, we have

$$\begin{aligned} Y_0 &= \{Q_n = x_1 y_n + \dots + x_n y_1 + z^2 = 0 \text{ in } \mathbf{P} = \mathbf{P}(V)\}, \\ Y_1 &= \{Q_{n-1} = x_2 y_{n-1} + \dots + x_n y_1 + z^2 = 0 \text{ and } x_1 = 0 \text{ in } \mathbf{P}\}, \\ &\vdots \\ Y_i &= \{Q_{n-i} = x_{i+1} y_{n-i} + \dots + x_n y_1 + z^2 = 0 \text{ and } x_1 = \dots = x_i = 0 \text{ in } \mathbf{P}\}, \\ &\vdots \\ Y_{n-1} &= \{Q_1 = x_n y_1 + z^2 = 0 \text{ and } x_1 = \dots = x_{n-1} = 0 \text{ in } \mathbf{P}\}, \end{aligned}$$

$$\begin{aligned}
Y_n &= \{Q_0 = z^2 = 0 \text{ and } x_1 = \ldots = x_n = 0 \text{ in } \mathbf{P}\},\\
Y_{n+2} &= \{x_1 = \ldots = x_n = z = y_1 = 0 \text{ in } \mathbf{P}\},\\
&\vdots\\
Y_{n+i} &= \{x_1 = \ldots = x_n = z = y_1 = \ldots = y_{i-1} = 0 \text{ in } \mathbf{P}\},\\
&\vdots\\
Y_{2n} &= \{x_1 = \ldots = x_n = z = y_1 = \ldots = y_{n-1} = 0 \text{ in } \mathbf{P}\}.\\
&= \text{the point } \{(0, \ldots, 0, 1)\}.
\end{aligned}$$

Thus we find that

(i) Y_0 and Y_{n+i}, $2 \leqq i \leqq n$ are non-singular.

(ii) Y_i, $1 \leqq i \leqq n-1$, are (*generalised*) *cones* in $\mathbf{P}$ over the quadrics Q_{n-i} which lie in the lower dimensional projective spaces $\mathbf{P}^{2n-2i}$ with coordinates

$$(x_{i+1}, \ldots, x_n, z, y_1, \ldots, y_{n-i}).$$

In particular, the Y_i*'s* are Cohen-Macaulay (being locally complete intersections). Further, it is easy to see that the singular set of Y_i is Y_{2n-i+1} for $1 \leqq i \leqq n-1$, in particular non-singular in codimension 1 and hence normal.

(iii) Y_n is *not reduced* and $(Y_n)_{\text{red}}$ is the projective space $\mathbf{P}^{n-1}$ with coordinates $(0, \ldots, 0, y_1, \ldots, y_n)$ in $\mathbf{P}$.

7 (*a*). THE IDEAL SHEAF OF $X(w_i)_r$ IN $X(w_{i-1})_r$ $(i \neq n)$.

PROPOSITION B.7. — *The sheaf of ideals defining* $X(w_i)_r$ *in* $X(w_{i-1})_r$ *is precisely* $L(-\chi_{i-1})|_{X(w_{i-1})_r}$ *for* $i \neq n$. *i.e., the equalities in Propositions* B.4 *and* B.5 *are scheme-theoretic for* $i \neq n$.

Proof. — As in Proposition A. 7, this follows easily from the explicit description of the scheme-theoretic Schubert varieties in the quadric P\G and the smoothness of the morphisms $G \to P\backslash G$ and $G \to G/B$, etc.

REMARK B.8. — We know that the closed Bruhat cells $X(w_0), \ldots, X(w_n), \ldots, X(w_{2n-1})$ in G are the inverse images of $Y_0, \ldots, (Y_n)_{\text{red}}, \ldots, Y_{2n}$ in P\G. It follows therefore that $X(w_i)$, $i \neq n$, are reduced (in fact normal), etc. Further, it is easily seen that $X(w_n)$ [resp. $X(w_n)_r$] is *not* a Cartier divisor in $X(w_{n-1})$ [resp. $X(w_{n-1})_r$]. However, we see that $2X(w_n)$ [resp. $2X(w_n)_r$] is a Cartier divisor in $X(w_{n-1})$ [resp. $X(w_{n-1})_r$].

7 (*b*). THE IDEAL SHEAF OF $X(w_n)_r$ IN $X(w_{n-1})_r$. — Let Z (resp. Z_r, resp. Z_l) denote the subscheme of $X(w_{n-1})$ [resp. $X(w_{n-1})_r$, resp. Y_{n-1}] whose ideal sheaf is generated by the function f_{n-1} [resp. the section f_{n-1} of $L(\chi_{n-1})|_{X(w_{n-1})_r}$, resp. the equation $x_n = 0$], f_{n-1} as in Proposition B.5.

Let K_r be the $\mathcal{O}_{Z_r}$-module defined by the exact sequence

$$0 \to K_r \to \mathcal{O}_{Z_r} \to \mathcal{O}_{(Z_r)_{\text{red}}} \to 0$$

$[X(w_n)_r = (Z_r)_{red}]$. It will be easily seen that $K_r^2 = 0$ so that it can be canonically considered as an $\mathcal{O}_{X(w_n)_r}$-module. The *problem* is to compute K_r. For this purpose, we consider K_l, defined by the exact sequence

$$0 \to K_l \to \mathcal{O}_{Z_1} \to \mathcal{O}_{(Z_1)_{red}} \to 0.$$

From the explicit nature of cycles in P\G, it is easy to compute K_l. It requires some argument to compute K_r from knowing K_l and this is essentially achieved by Lemma B.10, below.

Let $\pi_r : G \to G/B$ (resp. $\pi_l : G \to P\backslash G$) be the canonical morphism. It is clear that

$$Z = \pi_r^{-1}(Z_r) = \pi_l^{-1}(Z_l)$$

as schemes and that $Z_{red} = X(w_n)$. Similary we find that $(Z_r)_{red} = X(w_n)_r$ and $Z_l = Y_n$. We denote by the same letter Z the Cartier divisor on $X(w_{n-1})$ defined by the ideal sheaf of Z. Let $J = \mathcal{O}_{X(w_{n-1})}(-Z)$. Notice that J can also be described as the sheaf of germs of regular functions on the normal variety $X(w_{n-1})$ vanishing up to order $\geqq 2$ on the codimension one subvariety $X(w_n)$ of $X(w_{n-1})$. We define similarly the $\mathcal{O}_{X(w_{n-1})_r}$-module J_r and the $\mathcal{O}_{(Y_n)_{red}}$-module J_l.

Denote by I the sheaf of ideals defining $X(w_n)$ in $X(w_{n-1})$. Similarly, define I_r and I_l. It is clear that $\pi_r^*(I_r) = I = \pi_l^*(I_l)$ and also $\pi_r^*(J_r) = J = \pi_l^*(J_l)$. We observe the following facts :

(*a*) $$I^2 \subset J \, (I_r^2 \subset J_r \text{ and } I_l^2 \subset J_l).$$

(*b*) We have natural commuting actions of P and B on $\mathcal{O}_{X(w_{n-1})}$ induced by the action of P on the left on $X(w_{n-1})$ and the action of B on the right on $X(w_{n-1})$. These actions leave stable the subscheaves I and J of $\mathcal{O}_{X(w_{n-1})}$; in particular, we have commuting actions of P and B on J (and I) compatible with their action on $X(w_{n-1})$. In particular, it follows that we have commuting actions of P and B on I/J compatible with their actions on $X(w_{n-1})$.

(*c*) It is clear that I/J is the sheaf of ideals defining $X(w_n) = Z_{red}$ in Z. In particular, by (*a*), I/J is an $\mathcal{O}_{X(w_n)}$-module. Similar statements hold for I_r/J_r and I_l/J_l. Write $K = I/J$ and similarly define K_r and K_l.

(*d*) We have $K_l \approx \mathcal{O}_{\mathbf{P}^{n-1}}(-1)$, or equivalently, $K \approx \pi_l^*(\mathcal{O}_{P\backslash G}(-1))|_{X(w_n)}$. [Recall that K_l is a sheaf on $(Z_l)_{red} = (Y_n)_{red} = \mathbf{P}^{n-1}$.]

To see this, recall that $Z_l = Y_n$ is the subscheme in $\mathbf{P}^n$ [with coordinates $(0, \ldots, 0, z, y_1, \ldots, y_n)$] defined by $z^2 = 0$. Now $z = 0$ defines $(Z_l)_{red}$ which is a hyperplane in $\mathbf{P}^n$ and hence its sheaf of ideals $\approx \mathcal{O}_{\mathbf{P}^n}(-1)$. But Z_l is defined by the square of this sheaf of ideals, i. e., by the sheaf $\mathcal{O}_{\mathbf{P}^n}(-2)$. We have

$$K_l = I_l/J_l \approx \operatorname{Ker}(\mathcal{O}_{Z_l} \to \mathcal{O}_{(Z_l)_{red}})$$
$$\approx \operatorname{Coker}(\mathcal{O}_{\mathbf{P}^n}(-2) \xrightarrow{\text{can. hom.}} \mathcal{O}_{\mathbf{P}^n}(-1)).$$

But we have the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^n}(-1) \to \mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_{(Z_l)_{red}} \to 0.$$

This gives the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^n}(-2) \to \mathcal{O}_{\mathbf{P}^n}(-1) \to \mathcal{O}_{(Z_l)_{\mathrm{red}}}(-1) \to 0.$$

But $(Z_l)_{\mathrm{red}} = \mathbf{P}^{n-1}$ and hence the assertion.

(*e*) We see that K is a locally principal $\mathcal{O}_{X(w_n)}$-module. This is clear from (*b*) and (*d*). Similar statements hold for K_r and K_l.

PROPOSITION B.9. — *The sheaf* K_r *on* $X(w_n)_r$ *is isomorphic to the sheaf of germs of sections of the line bundle* $L(-\chi_n)|_{X(w_n)_r}$. *In particular, we have an exact sequence*

$$0 \to \mathbf{L}(-\chi_n) \to \mathcal{O}_{Z_r} \to \mathcal{O}_{X(w_n)_r} \to 0$$

where $\mathbf{L}(-\chi_n)$ *denotes the sheaf associated to* $L(-\chi_n)$.

Proof. — We have seen that K is locally principal on $X(w_n)$ and that we have actions of P and B on K compatible with the actions on $X(w_n)$. Hence K defines a line bundle say M on $X(w_n)$ on which we are given commuting actions of P and B such that if $q: M \to X(w_n)$ is the canonical projection, then q is a P-B equivariant morphism. We denote by M_r and M_l the line bundles defined by K_r and K_l respectively.

Now the proposition is a consequence of the

CLAIM. — *There exists a non-zero* (P-B) *invariant rational section of* M *over* $X(w_n)$, *i. e., a rational section* $s: X(w_n) \to M$:

$$\begin{array}{c} M \\ {\scriptstyle q}\downarrow\uparrow{\scriptstyle s} \\ X(w_n) \end{array}$$

such that $s(pxb) = ps(x)b$ *for all* $p \in P$ *and* $b \in B$.

Suppose the claim is proved. Let s be such a section. Then s gives rise to non-zero rational sections s_r and s_l of M_r and M_l respectively. Further we have $\pi_r^*(s_r) = s = \pi_l^*(s_l)$, etc. We see that s cannot be non-zero everywhere, for otherwise, s_l would be an everywhere non-zero section of $\mathcal{O}_{\mathbf{P}^{n-1}}(-1)$ which is a contradiction.

Let D (resp. D_r, resp. D_l) be the union of the polar and zero sets of s (resp. s_r, resp. s_l). Then we have $\pi_r^*(D_r) = D = \pi_l^*(D_l)$. Now D (resp. D_r, resp. D_l) is of pure codimension 1 in $X(w_n)$ [resp. $X(w_n)_r$, resp. $X(w_n)_l = (Y_n)_{\mathrm{red}}$]. Now D is P-B stable and so D_l is a non-empty pure codimension 1 subset of $(Y_n)_{\mathrm{red}}$ which is also B-stable and hence D_l is a union of some codimension 1 Schubert varieties of $(Y_n)_{\mathrm{red}}$. But we know that Y_{n+2} is the only codimension 1 subvariety of $(Y_n)_{\mathrm{red}}$. Hence $D_l = Y_{n+2}$ (set-theoretically). We see also that s_l has a pole of order 1 along the subvariety $Y_{n+2} \approx \mathbf{P}^{n-2}$ since s_l represents a rational section of $\mathcal{O}_{\mathbf{P}^{n-1}}(-1)$. From this it follows that s_l has a pole of order 1 along Y_{n+2} and is non-zero on $(Y_n)_{\mathrm{red}} - Y_{n+2}$, consequently, the section s_r of M_r is non-zero at every point of $X(w_n)_r - X(w_{n+1})_r$ and it has a pole of order 1 along $X(w_{n+1})_r$. Now we know that the ideal sheaf of $X(w_{n+1})_r$ in $X(w_n)_r$ is $\approx L(-\chi_n)|_{X(w_n)_r}$ (*cf.* Proposition B.7). It follows therefore that $K_r \approx L(-\chi_n)|_{X(w_n)_r}$. Thus we have only to prove the existence of a non-zero P-B invariant section of M. But this is a general fact as assured by the following

LEMMA B.10. — *Let* G *be a semi-simple algebraic group,* B *a Borel subgroup and* $P \supset B$ *a parabolic subgroup. Let* $w \in W$ *be such that* $X = \overline{P w B} = \overline{B w B}$ *and let* M *be a line bundle on* X *on which* P *and* B *operate compatibly with their action on* X. *Then there exists a* P-B *invariant rational section* s *of* M *such that* $s(x) \neq 0$ *for all* $x \in Z = P w B$, *in particular there exists a non-zero* P-B *invariant section of* M.

Proof. — Let X_l (resp. X_r) denote the Schubert variety in P\G (resp. G/B) defined by X. The morphisms $X \to X_l$ and $X \to X_r$ are locally trivial principal fibrations with structure groups P and B respectively, and because of the hypothesis of P-B operation on M, M " goes down " to a line bundle M_r on X_r (resp. M_l on X_l). The (right) action of B on M induces a (right) action of B on M_l compatible with the action of B on X_l (similarly, we have a left action of P on M_r). The existence of a section s on X such that $s(x) \neq 0$ for all $x \in Z = P w B$ is easily seen to be equivalent to the existence of a B-invariant section s_l of M_l such that $s_l(x) \neq 0$ for all $x \in Z_l$ where Z_l is the *open* Schubert cell in X_l. We know that Z_l is the B^u-orbit through the point $w_P = P w$ in X_l. Let B_1 be the isotropy subgroup of B^u at w_P. Let M_0 be the fibre of the line bundle M_l over the point $w_P \in X_l$. We make the

CLAIM. — B_1 *operates trivially on* M_0.

This is so because B_1 operates through a character of B_1 identifying M_0 with the affine line, and B_1 being unipotent, every character of B_1 is trivial.

Recall that B_1 is the subgroup of G generated by H_α, $\alpha \in S_1$ for some subset S_1 of positive roots R^+. Let $S_2 = R^+ - S_1$ and let B_2 be the subgroup generated by H_α, $\alpha \in S_2$. Since any element $b \in B^u$ can be *uniquely* written as $b = b_1 b_2$ with $b_1 \in B_1$ and $b_2 \in B_2$, it follows that given $x \in Z_l$, there exists a unique element $b_2 \in B_2$ such that $x = w_P b_2$.

Now define a rational section s_l of M_l as follows. Take some $\theta \in M_0$, $\theta \neq 0$. Write $x \in Z_l$ as $x = w_P b_2$ as above. Then set $s_l(w_P) = \theta$ and $s_l(x) = \theta b_2$. It follows that s_l defines an everywhere non-zero section of $M_l|_{Z_l}$ and hence it is also a rational section of M_l.

Finally to conclude the proof of the lemma, it suffices to prove that s_l is B-invariant. For this, it suffices to show that $s_l|_{Z_l}$ is B^u-invariant. But this is equivalent to showing

$$(\bigstar) \qquad s_l(w_P b) = s_l(w_P) b \qquad \text{for} \quad b \in B.$$

This is true for $b \in B_2$ (by definition of s_l) and since $B = B_1 . B_2$, it suffices to check ($\bigstar$) for $b \in B_1$. We have $w_P b = w_P$ for $b \in B_1$. Further, we know that B_1 operates trivially on M_0 which implies that $s_l(w_P b) = s_l(w_P) = s_l(w_P) b$ for $b \in B_1$.

This completes the proof of the lemma and consequently the proposition.

8. VANISHING THEOREM.

THEOREM B.11. — *For every line bundle* $L(\chi)$ *on* G/B *belonging to the dominant chamber, we have*

$$H^p(X(w_i)_r, L(\chi)|_{X(w_i)_r}) = 0 \quad \textit{for all } p > 0$$

and $i = 0, \ldots, 2n-1$. *In particular for* $i = 0$, *we have*

$$H^p(G/B, L(\chi)) = 0 \quad \textit{for all} \quad p > 0.$$

Proof. — Write $X_i = X(w_i)_r$, $0 \leqq i \leqq 2n-1$.

Proceeding as in the proof of Theorem A.8, by increasing induction on the integer (χ, α^*_{n-j-1}) i.e., degree of $L(\chi)$ with respect to the $\mathbf{P}^1$-fibration $G/B \to G/P_{\alpha_{n-j-1}}$, where $i = n+j$, $0 \leqq j \leqq n-2$, we find that the theorem is true for X_i, $n \leqq i \leqq 2n-1$. Assuming that we have proved the theorem for X_{n-1}, again by the same procedure [now the induction being on (χ, α^*_{i+1}) we find that the theorem is also true for X_i, $0 \leqq i \leqq n-2$]. Thus we have only to prove the theorem for X_{n-1}, remembering that the theorem is true (for X_i, $i \geqq n$) in particular for X_n.

Proof of the theorem for X_{n-1}. — Since no confusion is likely, we simply write [by abuse of notation *see* 7 (*b*) above] Z for Z_r, I for I_r, etc. Recall that Z is the (closed) subscheme of X_{n-1} ($Z_{\text{red}} = X_n$) whose sheaf of ideals (J) is generated by the section of the line bundle $L(\chi_{n-1})|_{X_{n-1}}$ vanishing along X_n and that K, which is isomorphic to the sheaf of germs of sections of the line bundle $L(-\chi_n)|_{X_n}$ (*cf.* Proposition B.9), is the sheaf of ideals defining X_n in Z. Notice that K is also canonically a sheaf of $\mathcal{O}_{X_n}$-modules. We have the exact sequence

$$0 \to L(-\chi_n)|_{X_n} \to \mathcal{O}_Z \to \mathcal{O}_{X_n} \to 0.$$

Tensoring this by the given line bundle $L(\chi)$ on G/B we get the exact sequence

$$0 \to L(\chi-\chi_n)|_{X_n} \to L(\chi)|_Z \to L(\chi)|_{X_n} \to 0.$$

Recall that $\chi_n = \tilde{\omega}_{n-1} - 2\,\tilde{\omega}_n$ (*cf.* Proposition B.6). Write $\chi' = \chi - \chi_n = \chi - \tilde{\omega}_{n-1} + 2\,\tilde{\omega}_n$. We have the cohomology exact sequence

$$\to H^p(X_n, L(\chi')) \to H^p(Z, L(\chi)) \to H^p(X_n, L(\chi)) \to .$$

Now we prove

$$\text{(I)} \qquad H^p(Z, L(\chi)) = 0 \quad \text{for all} \quad p > 0.$$

To see this, observe that $H^p(X_n, L(\chi)) = 0$ for $p > 0$ (the theorem being true for X_n by induction). Hence we have only to prove that $H^p(X_n, L(\chi')) = 0$ for $p > 0$. We prove this by induction on (χ', α^*_{n-1}). Notice that

$$\text{(i)} \qquad (\chi', \alpha^*_{n-1}) = (\chi, \alpha^*_{n-1}) - 1 \geqq -1$$

and

$$\text{(ii)} \qquad (\chi', \alpha^*_j) \geqq (\chi, \alpha^*_j) \geqq 0 \quad \text{for} \quad j \neq n-1.$$

Suppose $(\chi', \alpha^*_{n-1}) = -1$. Then by Remark B.3 (ii) and Corollary 1.15, we find that the claim is true. Assume $(\chi', \alpha^*_{n-1}) \geqq 0$. Now by (ii) above, $\chi' \geqq 0$ and so (I) follows since the theorem is true for X_n.

We prove the theorem for X_{n-1} by induction on (χ, α_n^*). Let us verify the result when $(\chi, \alpha_n^*) = 0$.

(II) $$(\chi, \alpha_n^*) = 0 \Rightarrow H^p(X_{n-1}, L(\chi)) = 0 \quad \text{for} \quad p > 0.$$

To see this, let I denote the sheaf of ideals defining the closed subscheme X_n in X_{n-1}. We have an exact sequence

$$0 \to I \to \mathcal{O}_{X_{n-1}} \to \mathcal{O}_{X_n} \to 0.$$

Tensoring with $L(\chi)$, we get the exact sequence

$$0 \to I \otimes L(\chi)|_{X_{n-1}} \to L(\chi)|_{X_{n-1}} \to L(\chi)|_{X_n} \to 0.$$

This gives the exact sequence

$$\to H^p(X_{n-1}, I \otimes L(\chi)) \to H^p(X_{n-1}, L(\chi)) \to H^p(X_n, L(\chi)) \to.$$

By induction

$$H^p(X_n, L(\chi)) = 0 \quad \text{for} \quad p > 0$$

and so we have only to prove that

$$H^p(X_{n-1}, I \otimes L(\chi)) = 0 \quad \text{for} \quad p > 0.$$

To see this, recall that I is the sheaf associated to the Weil divisor X_n (note $2X_n = Z$) i. e., the sheaf of germs of sections of $L(-\chi_{n-1})$, etc. It follows that the degree of I with respect to the $\mathbf{P}^1$-fibration $X_{n-1} \to X_{n-1}/P_{\alpha_n}$ [*cf.* Remark B.3 (ii)] is precisely -1. Hence the degree of $I \otimes L(\chi)$ with respect to this $\mathbf{P}^1$-fibration is -1 [since $(\chi, \alpha_n^*) = 0$]. Hence by Proposition 1.14, (since X_{n-1} is normal) we get that

$$H^p(X_{n-1}, I \otimes L(\chi)) = 0 \quad \text{for} \quad p \geqq 0.$$

This proves (II). Finally

(III) Assume $(\chi, \alpha_n^*) \geqq 1$ and the induction hypothesis [for all $\chi' \geqq 0$ we have

$$(\chi', \alpha_n^*) < (\chi, \alpha_n^*) \Rightarrow H^p(X_{n-1}, L(\chi')) = 0 \quad \text{for all } p > 0].$$

We have the exact sequence

$$0 \to L(-\chi_{n-1})|_{X_{n-1}} \to \mathcal{O}_{X_{n-1}} \to \mathcal{O}_Z \to 0$$

which gives the exact sequence

$$0 \to L(\chi - \chi_{n-1})|_{X_{n-1}} \to L(\chi)|_{X_{n-1}} \to L(\chi)|_Z \to 0.$$

Recall that $\chi_{n-1} = 2\tilde{\omega}_n - \tilde{\omega}_{n-1}$ (*cf.* Proposition B.6). Write

$$\chi' = \chi - \chi_{n-1} = \chi + \tilde{\omega}_{n-1} - 2\tilde{\omega}_n.$$

We have the cohomology exact sequence

$$\to H^p(X_{n-1}, L(\chi')) \to H^p(X_{n-1}, L(\chi)) \to H^p(Z, L(\chi)) \to$$

This reduces [by (I)] to the exact sequences

$$(\bigstar) \qquad H^p(X_{n-1}, L(\chi')) \to H^p(X_{n-1}, L(\chi)) \to 0$$

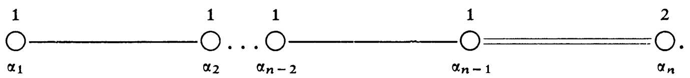
for all $p \geqq 1$. Notice that we have $(\chi', \alpha_n^*) = (\chi, \alpha_n^*) - 2$ and $(\chi', \alpha_j^*) \geqq (\chi, \alpha_j^*)$ for $j \neq n$. In particular $\chi' \geqq 0$ if $(\chi, \alpha_n^*) \geqq 2$. Now by Remark B.3 (ii) and Corollary 1.15, ($\bigstar$) and the induction hypothesis, we get the result whenever (χ, α_n^*) is odd. Similarly, by (II), ($\bigstar$) and the induction hypothesis, we get the result whenever (χ, α_n^*) is even.

This completes the proof of the theorem.

C. Type $\mathbf{C}_n$ ($n \geqq 2$)

1. Numerical data. — Recall the following facts for a group G of type $\mathbf{C}_n$.

Dynkin diagram:



The Cartan numbers

$$n_{ij} = (\alpha_i, \alpha_j^*) = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$$

are given by

$$n_{ij} = \begin{cases} 2 & \text{if } \ i = j, \\ -1 & \text{if } \ i = j \pm 1 \ \text{ and } \ i, j \leqq n-1, \\ -1 & \text{if } \ (i, j) = (n-1, n), \\ -2 & \text{if } \ (i, j) = (n, n-1), \\ 0 & \text{otherwise.} \end{cases}$$

The number of roots $= 2n^2$.
The order of the Weyl group W ($=$ W ($\mathbf{C}_n$)) of G $= 2^n.n!$

2. A reduced expression for w_0.

Proposition C.1. — *A reduced expression for the element $w_0 \in$ W (of largest length) is given by*

$$w_0 = s_n(s_{n-1}s_n s_{n-1}) \dots (s_i s_{i+1} \dots s_n s_{n-1} \dots s_i) \dots (s_1 \dots s_n s_{n-1} \dots s_1).$$

Proof. — Similar to that of Proposition A.1. In this case we define the sequences $\{v_i\}$ and $\{u_i\}$ as follows:

$$\begin{aligned} \{v_i\}: &\quad v_n = s_n, \quad v_i = s_i v_{i+1} s_i, \qquad 1 \leqq i < n; \\ \{u_i\}: &\quad u_n = v_n, \quad u_i = u_{i+1} v_i, \qquad 1 \leqq i < n. \end{aligned}$$

The following assertion for u_1 together with Proposition 1.1 (iii) proves the assertion.

ASSERTION I.

$$u_i(\alpha_j) = \begin{cases} -\alpha_j & \textit{for} \quad i \leqq j \leqq n, \\ (\alpha_{i-1} + 2\alpha_i + \ldots + 2\alpha_{n-1} + \alpha_n) & \textit{for} \quad j = i-1, \\ \alpha_j & \textit{for} \quad j \leqq i-2. \end{cases}$$

(*In particular* $u_1 = -\mathrm{Id}$.)

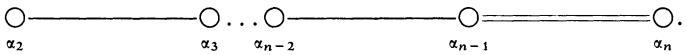
We prove this by decreasing induction on i using the fact that $u_i = u_{i+1} v_i$ and the values $v_i(\alpha_j)$ from the following

ASSERTION II. — *We have*

$$v_i(\alpha_j) = \begin{cases} \alpha_j & \textit{for} \quad j \neq i-1 \quad \text{and} \quad i, \\ -(\alpha_i + 2\alpha_{i+1} + \ldots + 2\alpha_{n-1} + \alpha_n) & \textit{for} \quad j = i, \\ (\alpha_{i-1} + 2\alpha_i + \ldots + 2\alpha_{n-1} + \alpha_n) & \textit{for} \quad j = i-1. \end{cases}$$

We prove this again by decreasing induction i noting that $v_i = s_i v_{i+1} s_i$.

3. THE PARABOLIC SUBGROUP $P(= P_{\hat{\alpha}_1})$. — $P = P_{\hat{\alpha}_1}$ admits a similar description as before, namely, the "semi-simple part" of P is of type C_{n-1} (or A_1 if $n = 2$) and its Dynkin diagram can be taken to be



In particular, the element w'_0 in the Weyl group $W_P (= W(C_{n-1}))$ of P is given by

$$w'_0 = s_n (s_{n-1} s_n s_{n-1}) \ldots (s_i \ldots s_n \ldots s_i) \ldots (s_2 \ldots s_n \ldots s_2).$$

Thus we have $w_0 = w'_0 (s_1 \ldots s_n \ldots s_1)$. The number of Schubert varieties in

$$P \backslash G = [W : W_P] = 2n.$$

4. THE FAMILY OF CLOSED BRUHAT CELLS $\{ X(w_i) \}$. — Define the sequences $\{ r_i \}$, $\{ \tau_i \}$ and $\{ w_i \}$ of elements in W as follows :

$$\{ r_i \} : \quad r_0 = \mathrm{Id}, \quad r_1 = s_1, \ldots, r_n = s_n,$$
$$r_{n+1} = s_{n-1}, \ldots, r_{n+i} = s_{n-i}, \ldots, r_{2n-1} = s_1;$$
$$\{ \tau_i \} : \quad \tau_0 = \mathrm{Id}, \quad \tau_i = r_{2n-1} \ldots r_{2n-i}, \qquad 0 \leqq i \leqq 2n-1,$$
$$\{ w_i \} : \quad w_i = w_0 \tau_i = w'_0 r_1 \ldots r_{2n-1-i}, \qquad 0 \leqq i \leqq 2n-1.$$

PROPOSITION C.2.

(i) $$G = X(w_0) \supset X(w_1) \supset \ldots \supset X(w_{2n-1}) = P.$$

(ii) *Each* $X(w_i)$ *is of codimension* 1 *in* $X(w_{i-1})$.

(iii) $X(w_i)$ *are the inverse images* (*under the natural morphism* $G \to P \backslash G$) *of the Schubert varieties in* $P \backslash G$.

Proof. — Similar to that of Proposition A.2.

REMARK C.3. — We have

(i) $$X(w_i) = P\,w_i\,B \cup X(w_{i+1}), \qquad 0 \leqq i \leqq 2n-2$$

the union being set-theoretic and disjoint

(ii) $X(w_i)$ is stable for multiplication on the right by the (minimal) parabolic subgroup $P_{\alpha_{i+1}}$ if $0 \leqq i \leqq n-1$ or $P_{\alpha_{n-j-1}}$ if $i = n+j$, $0 \leqq j \leqq n-2$.

Proof. — Similar to that of Remark A.3.

PROPOSITION C.4. — *We have (set-theoretically) :*

$$X(w_i) \cap X(w_1)\,\tau_i = X(w_{i+1}), \qquad 0 \leqq i \leqq 2n-2$$

(where $\tau_i = w_0\,w_i$*).*

Proof. — Similar to that of Proposition A.4.

PROPOSITION C.5. — *Let* $\chi_i = \tau_i^{-1}(\tilde{\omega}_1)$, $0 \leqq i \leqq 2n-2$. *For each i, there exists an element* $f_i \in H^0(X(w_i)_r, L(\chi_i)|_{X(w_i)_r})$ *such that (set-theoretically) the set of zeros of* f_i *in* $X(w_i)_r$ *is* $X(w_{i+1})_r$.

Proof. — Same (word by word) as in Proposition B.5.

5. THE FAMILY OF CHARACTERS $\{\chi_i\}$.

PROPOSITION C.6. — *The characters* $\chi_i\,(=\tau_i^{-1}(\tilde{\omega}_1))$, $0 \leqq i \leqq 2n-2$, *are given by* $\chi_0 = \tilde{\omega}_1$ *and*

(*a*) $$\chi_i = \tilde{\omega}_{i+1} - \tilde{\omega}_i \quad \textit{for} \quad 1 \leqq i \leqq n-1.$$

(*b*) $$\chi_n = \tilde{\omega}_{n-1} - \tilde{\omega}_n$$

(*c*) $$\chi_{n+j} = \tilde{\omega}_{n-j-1} - \tilde{\omega}_{n-j} \quad \textit{for} \quad 1 \leqq j \leqq n-2.$$

Proof. — Similar to that of Proposition B.6. In this case the recurrence relations are (same as in type $\mathbf{B}_n$) :

$$\tau_i = s_1 \ldots s_i = \tau_{i-1}\,s_i \quad \text{for } 1 \leqq i \leqq n \qquad \text{and} \qquad \tau_{n+j} = \tau_{n+j-1}\,s_{n-j} \quad \text{for } 1 \leqq j \leqq n-1.$$

6. STRUCTURE OF SCHUBERT VARIETIES IN P\G. — Here we take $G = Sp_{2n}$, the *Symplectic group* (*cf.* [2], [7], [23] or [25]). Recall that Sp_{2n} is realised as a subgroup of $GL(2n)$ as follows : let

$$M = \begin{pmatrix} & \mathbf{0} & & & 1 \\ & & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & -1 & & \mathbf{0} & \\ -1 & & & & \end{pmatrix}$$

be the skew-symmetric matrix, i. e., with 1 or -1 at the $(i, 2n-i+1)$-th place according as $i \leqq n$ or $\geqq n+1$ and 0 elsewhere. Then

$$\mathrm{Sp}_{2n} = \{ \mathrm{A} \in \mathrm{GL}(2n) / \mathrm{A M A}^t = \mathrm{M} \}.$$

For this imbedding $\mathrm{Sp}_{2n} \subsetneq \mathrm{GL}(2n)$, the subgroup T (resp. B) of Sp_{2n} consisting of the diagonal (resp. upper triangular) matrices is a maximal torus (resp. Borel subgroup). Fixing these, it is easy to check that $\mathrm{P} = \mathrm{P}_{\hat{\alpha}_1}$ is precisely the subgroup of Sp_{2n} consisting of matrices of the form $(g_{ij}) \in \mathrm{Sp}_{2n}$ such that $g_{i1} = 0$ for $i \geqq 2$ and $g_{2nj} = 0$ for $j \leqq 2n-1$, i. e., of the form

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ \cdot & * & & * \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

and its " semi-simple part " is the set of matrices of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & \boxed{\mathrm{X}} & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \approx \mathrm{Sp}_{2n-2}.$$

Let P′ be the (maximal) parabolic subgroup of GL $(2n)$ consisting of the matrices of the form

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{pmatrix}.$$

It is easy to check that $\mathrm{P} = \mathrm{Sp}_{2n} \cap \mathrm{P}'$ (scheme-theoretically).

CLAIM. — *The canonic al closed immersion* $\mathrm{P}\backslash\mathrm{G} \subsetneq \mathrm{P}'\backslash\mathrm{GL}(2n)$ *is an isomorphism i. e.,* $\mathrm{P}\backslash\mathrm{G} \approx \mathbf{P}(\mathrm{V})$ *where* V *is a vector space of dimension* $2n$.

This is immediate from the consideration of dimension, etc.

As in the case of type $\mathbf{A}_n$, fixing a coordinate system $(x_1, \ldots, x_{2n})$ on V, we see that the Schubert varieties in P\G are obtained by taking

$$x_1 = 0; \qquad x_1 = x_2 = 0; \qquad \ldots; \qquad x_1 = \ldots = x_{2n-1} = 0.$$

In particular (these are non-singular and) each is obtained from the previous one with the (scheme-theoretic) intersection of a hyper-plane in **P** (V).

7. THE IDEAL SHEAF OF $\mathrm{X}(w_i)_r$ IN $\mathrm{X}(w_{i-1})_r$.

PROPOSITION C.7. — *The sheaf of ideals defining* $\mathrm{X}(w_i)_r$ *in* $\mathrm{X}(w_{i-1})_r$ *is precisely* $\mathrm{L}(-\chi_i)|_{\mathrm{X}(w_i)_r}$ (*i. e., the equalities in Proposition* C.4 *and* C.5 *are scheme-theoretic*).

Proof. — Similar to that of Proposition A.7.

8. Vanishing Theorem.

Theorem C.8. — *For every line bundle* L (χ) *on* G/B *belonging to the dominant chamber, we have*

$$H^p(X(w_i)_r, L(\chi)|_{X(w_i)_r}) = 0 \quad \textit{for all} \ p > 0$$

and $i = 0, \ldots, 2n-1$. *In particular for* $i = 0$ *we have* $H^p(G/B, L(\chi)) = 0$ *for* $p > 0$.

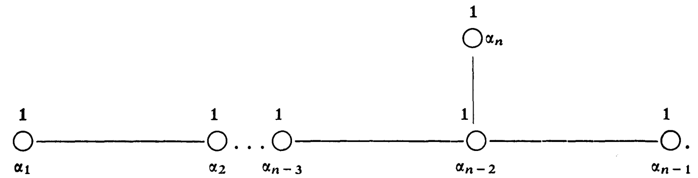
Proof. — Similar to that of Theorem A.8. In this case, the proof is by increasing induction on (χ, α_{i+1}^*) or (χ, α_{n-j-1}^*) according as $0 \leqq i \leqq n-1$ or $i = n+j$, $0 \leqq j \leqq n-2$.

D. Type $\mathbf{D}_n$ ($n \geqq 3$)

Since $\mathbf{D}_3 = \mathbf{A}_3$, we can assume that $n \geqq 4$.

1. Numerical data. — Recall the following facts for a group G of type $\mathbf{D}_n$.

Dynkin diagram :



The Cartan numbers

$$n_{ij} = (\alpha_i, \alpha_j^*) = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$$

are given by

$$n_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \ i, j \leqq n-1, \\ -1 & \text{if } (i, j) = (n-2, n) \ \text{ or } (n, n-2), \\ 0 & \text{otherwise.} \end{cases}$$

The number of roots $= 2n(n-1)$.

The order of the Weyl group W (= W ($\mathbf{D}_n$)) of G = $2^{n-1}.n!$

2. A reduced expression for w_0.

Proposition D. 1. — *A reduced expression for the element* $w_0 \in W$ (*of largest length*) *is given by*

$$\begin{aligned} w_0 &= (s_{n-1} s_n)(s_{n-2} s_{n-1} s_n s_{n-2}) \ldots (s_1 \ldots s_{n-2} s_{n-1} s_n s_{n-2} \ldots s_1) \\ &= (s_n s_{n-1})(s_{n-2} s_n s_{n-1} s_{n-2}) \ldots (s_1 \ldots s_{n-2} s_n s_{n-1} s_{n-2} \ldots s_1). \end{aligned}$$

Proof. — Similar to that of Proposition A. 1. In this case, we define the sequences $\{v_i\}$ and $\{u_i\}$ as follows :

$$\{v_i\} : \quad v_{n-1} = s_{n-1} s_n \quad \text{and} \quad v_i = s_i v_{i+1} s_i, \qquad 1 \leqq i < n-1;$$
$$\{u_i\} : \quad u_{n-1} = v_{n-1} \quad \text{and} \quad u_i = u_{i+1} v_i, \qquad 1 \leqq i < n-1.$$

The following assertion for u_1 together with Proposition 1.1 (iii) proves the proposition.

ASSERTION I. — *We have*

$$u_i(\alpha_j) = \begin{cases} -\alpha_n \quad \text{or} \quad -\alpha_{n-1} \quad \textit{according as } n-i \textit{ is odd or even, if } j=n \quad (\textit{for all } i), \\ -\alpha_{n-1} \quad \text{or} \quad -\alpha_n \quad \textit{according as } n-i \textit{ is odd or even, if } j=n-1 \quad (\textit{for all } i), \\ -\alpha_j \quad \textit{for} \quad i \leqq j \leqq n-2, \\ (\alpha_{i-1}+2\alpha_i+\ldots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n) \quad \textit{for} \quad j=i-1, \\ \alpha_j \quad \textit{for} \quad j \leqq i-2. \end{cases}$$

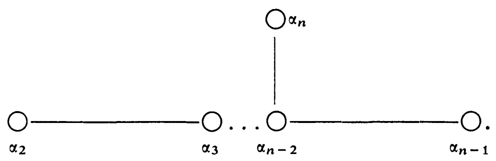
We prove this by decreasing induction on i using $u_i = u_{i+1} v_i$ and the values of $v_i(\alpha_j)$ (for $i \leqq n-2$) from the following

ASSERTION II. — *We have (for* $i \leqq n-2$) :

$$v_i(\alpha_j) = \begin{cases} \alpha_j \quad \textit{for} \quad j \neq i-1, \ i, \ n-1 \ \textit{and} \ n, \\ \alpha_{n-1} \quad \textit{for} \quad j=n, \\ \alpha_n \quad \textit{for} \quad j=n-1, \\ -(\alpha_i+2\alpha_{i+1}+\ldots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n) \quad \textit{for} \quad j=i, \\ (\alpha_{i-1}+2\alpha_i+\ldots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n) \quad \textit{for} \quad j=i-1. \end{cases}$$

We prove this again by decreasing induction on i noting that $v_i = s_i v_{i+1} s_i$.

3. THE PARABOLIC SUBGROUP P ($= P_{\hat{\alpha}_1}$). — It is easy to see that the "semi-simple part" of P is of type $\mathbf{D}_{n-1}$ and its Dynkin diagram can be taken to be



In particular, the element w'_0 in the Weyl group W_P ($= W(D_{n-1})$) of P is given by

$$w'_0 = (s_{n-1} s_n)(s_{n-2} s_{n-1} s_n s_{n-2}) \ldots (s_2 \ldots s_n s_{n-2} \ldots s_2).$$

Thus we have

$$w_0 = w'_0 (s_1 \ldots s_n s_{n-2} \ldots s_1) = w'_0 (s_1 \ldots s_{n-1} s_n s_{n-1} \ldots s_1).$$

The number of Schubert varieties in P\G is $[W : W_P] = 2n$.

4. THE FAMILY OF CLOSED BRUHAT CELLS $\{ X(w_i), X(w'_{n-1}) \}$. — Define the sequences $\{ \tau_i, \tau'_{n-1} \}$ and $\{ w_i, w'_{n-1} \}$ of elements in W as follows :

$$\{ \tau_i \, \tau'_{n-1} \} : \quad \tau_0 = \text{Id}, \quad \tau_i = \tau_{i-1} s_i \quad \text{for} \quad 1 \leqq i \leqq n-2,$$

$$\tau_{n-1} = \tau_{n-2} s_n, \quad \tau'_{n-1} = \tau_{n-2} s_{n-1},$$

$$\tau_n = \tau_{n-1} s_{n-1} = \tau'_{n-1} s_n$$

and

$$\tau_{n+j} = \tau_{n+j-1} s_{n-j-1}, \qquad 1 \leqq j \leqq n-2,$$

$$\{w_i, w'_{n-1}\}: \quad w_i = w_0 \tau_i \quad \text{for} \quad 0 \leqq i \leqq 2n-2 \quad \text{and} \quad w'_{n-1} = w_0 \tau'_{n-1}.$$

PROPOSITION D. 2.

(i) $$G = X(w_0) \supset X(w_1) \supset \ldots \supset X(w_{n-2}) \begin{smallmatrix} \nearrow X(w_{n-1}) \searrow \\ \searrow X(w'_{n-1}) \nearrow \end{smallmatrix} X(w_n) \supset \ldots \supset X(w_{2n-2}) = P.$$

(ii) *Each* $X(w_i)$ *is of codimension* 1 *in* $X(w_{i-1})$ *and also* $X(w_n)$ [*resp.* $X(w'_{n-1})$] *is of codimension* 1 *in* $X(w'_{n-1})$ [*resp.* $X(w_{n-2})$].

(iii) $X(w_i)$ *and* $X(w'_{n-1})$ *are the inverse images (under the natural morphism* $G \to P\backslash G$) *of the Schubert varieties in* $P\backslash G$.

Proof. — Similar to that of Proposition A. 2.

REMARK D.3. — We have

(i) $$X(w_i) = P w_i B \cup X(w_{i+1}) \quad \text{for} \quad 0 \leqq i \leqq n-3 \quad \text{and} \quad n-1 \leqq i \leqq 2n-3,$$

$$X(w'_{n-1}) = P w'_{n-1} B \cup X(w_n),$$

$$X(w_{n-2}) = P w_{n-2} B \cup (X(w_{n-1}) \cup X(w'_{n-1})),$$

the unions being set-theoretic and disjoint.

(ii)

Bruhat cell	Stable on the right by the (minimal) parabolic subgroup
$X(w_i)\ldots\ldots$	$P_{\alpha_{i+1}}$ for $0 \leqq i \leqq n-3$ $P_{\alpha_{n-1}}$ if $i = n-1$ $P_{\alpha_{n-j-2}}$ for $i = n+j$, $0 \leqq j \leqq n-3$
$X(w_{n-2})\ldots$	P_{α_n} and $P_{\alpha_{n-1}}$
$X(w'_{n-1})\ldots$	P_{α_n}

Proof. — Similar to that of Remark A. 3.

PROPOSITION D. 4. — *We have (set-theoretically)*:

(i) $X(w_i) \cap X(w_1)\tau_i = X(w_{i+1})$ *for* $0 \leqq i \leqq n-3$ *or* $n-1 \leqq i \leqq 2n-3$.

(ii) $$X(w'_{n-1}) \cap X(w_1)\tau'_{n-1} = X(w_n).$$

(iii) $$X(w_{n-2}) \cap X(w_1)\tau_{n-2} = X(w_{n-1}) \cup X(w'_{n-1}).$$

(iv) $$X(w_{n-1}) \cap X(w'_{n-1}) = X(w_n).$$

Proof. — Similar to that of Proposition A. 4 (with the obvious modifications).

PROPOSITION D. 5. — *Let* $\chi_i = \tau_i^{-1}(\tilde{\omega}_1)$, $0 \leqq i \leqq 2n-3$ *and* $\chi'_{n-1} = \tau'^{-1}_{n-1}(\tilde{\omega}_1)$. *Then there exist sections* $f_i \in H^0(X(w_i)_r, L(\chi_i)|_{X(w_i)_r})$ and $f'_{n-1} \in H^0(X(w'_{n-1})_r, L(\chi'_{n-1})|_{X(w'_{n-1})_r})$ *such that* (*set-theoretically*) *the set of zeros of*

(i) f_i *in* $X(w_i)_r$ *is* $X(w_{i+1})_r$ *if* $0 \leqq i \leqq n-3$ *or* $n-1 \leqq i \leqq 2n-3$.

(ii) f_{n-2} *in* $X(w_{n-2})_r$ *is* $X(w_{n-1})_r \cup X(w'_{n-1})_r$.

(iii) f'_{n-1} *in* $X(w'_{n-1})_r$ *is* $X(w_n)_r$.

Proof. — Similar to that of Proposition B. 5.

5. THE FAMILY OF CHARACTERS $\{\chi_i, \chi'_{n-1}\}$.

PROPOSITION D. 6. — *The characters*

$$\chi_i (= \tau_i^{-1}(\tilde{\omega}_1)), \quad 0 \leqq i \leqq 2n-3 \quad \textit{and} \quad \chi'_{n-1} (= \tau'^{-1}_{n-1}(\tilde{\omega}_1))$$

are given by $\chi_0 = \tilde{\omega}_1$ *and*

(*a*) $$\chi_i = \tilde{\omega}_{i+1} - \tilde{\omega}_i \quad \textit{for} \quad 1 \leqq i \leqq n-3;$$

(*b*) $$\chi_{n-2} = -\chi_n = \tilde{\omega}_n + \tilde{\omega}_{n-1} - \tilde{\omega}_{n-2};$$

(*c*) $$\chi_{n-1} = -\chi'_{n-1} = \tilde{\omega}_{n-1} - \tilde{\omega}_n.$$

(*d*) $$\chi_{n+j} = \tilde{\omega}_{n-j-2} - \tilde{\omega}_{n-j-1} \quad \textit{for} \quad 1 \leqq j \leqq n-3.$$

Proof. — Similar to that of Proposition of B. 6. In this case, recall that the recurrence relations are as follows :

$$\tau_i = \tau_{i-1} s_i \quad \text{for} \quad 1 \leqq i \leqq n-2 \quad (\tau_0 = \mathrm{Id}),$$
$$\tau_{n-1} = \tau_{n-2} s_n \quad \text{and} \quad \tau'_{n-1} = \tau_{n-2} s_{n-1},$$
$$\tau_n = \tau_{n-1} s_{n-1} = \tau'_{n-1} s_n$$

and

$$\tau_{n+j} = \tau_{n+j-1} s_{n-j-1} \quad \text{for} \quad 1 \leqq j \leqq n-3.$$

6. STRUCTURE OF SCHUBERT VARIETIES IN P\G. — For the purpose of this section, we take G = SO (2 n), called the (*even*) *Orthogonal group in* 2 n *variables* (*cf.* [2], [23] or [25]). Recall that SO (2 n) is realised as a subgroup of GL (2 n) as follows : let V be a vector space of dimension 2 n over the ground field k. With respect to a basis $e_1, \ldots, e_{2n}$ of V, write any point $v \in V$ as $v = (x_1, \ldots, x_n, y_1, \ldots, y_n)$. Let $Q = Q_n$ be the quadratic form on V defined by $Q(v) = x_1 y_n + \ldots + x_n y_n$.

Let O (2 n) be the subgroup of GL (2 n) which leaves the quadratic form Q invariant i.e.

$$O(2n) = \{ A \in GL(2n) / Q(Av) = Q(v) \text{ for all } v \in V \}.$$

Then SO (2 n) is the connected component through the identity element of O (2 n). Recall (*cf.* [25]) that we have

$$SO(2n) = \begin{cases} O(2n) \cap SL(2n) & \text{if char } k \neq 2, \\ \text{ker of Dick} : O(2n) \to \mathbf{Z}/2\mathbf{Z}, & \text{otherwise.} \end{cases}$$

For this imbedding SO (2 n) $\subsetneq$ GL (2 n), the subgroup T (resp. B) of SO (2 n) consisting of the diagonal (resp. upper triangular) matrices is a maximal torus (resp. Borel subgroup). Fixing these, it is easy to check that $P = P_{\hat{\alpha}_1}$ is precisely the subgroup of SO (2 n) consisting of the set of matrices of the form

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \cdot & \cdot & \cdots & \cdot \\ 0 & * & \cdots & * \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let P′ be the (maximal) parabolic subgroup of GL (2 n) consisting of the matrices of the form

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \cdot & & \cdots & \cdot \\ 0 & * & \cdots & * \end{pmatrix}.$$

It is seen easily that P = SO (2 n) $\cap$ P′ (as schemes).

As usual identifying P′\GL (2 n) canonically with **P** (V), we see easily that the canonical closed immersion P\G $\subsetneq$ P′\GL (2 n) = **P** (V) identifies P\G with the quadratic Q = 0. Notice that the points on P\G fixed by the maximal torus T are the points $(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)$, 1 at the i-th place, $i = 1, \ldots, 2n$.

Let $Y_0, \ldots, Y_{n-1}, Y'_{n-1}, Y_n, \ldots, Y_{2n-2}$ denote the Schubert varieties inP\G. Then in a manner similar to the case of type $\mathbf{B}_n$, we find easily the following :

$$Y_0 = \{ Q_n = x_1 y_n + \ldots + x_n y_1 = 0 \text{ in } \mathbf{P} = \mathbf{P}(V) \},$$

$$Y_1 = \{ Q_{n-1} = x_2 y_{n-1} + \ldots + x_n y_1 = 0 \text{ and } x_1 = 0 \text{ in } \mathbf{P} \},$$

$$\vdots$$

$$Y_{n-2} = \{ Q_2 = x_{n-1} y_2 + x_n y_1 = 0 \text{ and } x_1 = \ldots = x_{n-2} = 0 \text{ in } \mathbf{P} \},$$

$$Y_{n-1} \cup Y'_{n-1} = \{ Q_1 = x_n y_1 = 0 \text{ and } x_1 = \ldots = x_{n-1} = 0 \text{ in } \mathbf{P} \} = \{ x_1 = \ldots = x_n = 0 \} \cup \{ x_1 = \ldots = x_{n-1} = y_1 = 0 \},$$

$$Y_n = Y_{n-1} \cap Y'_{n-1} = \{ x_1 = \ldots = x_n = y_1 = 0 \},$$

$$Y_{2n-2} = \{ x_1 = \ldots = x_n = y_1 = \ldots = y_{n-1} = 0 \} = \text{the } pt \{ (0, \ldots, 0, 1) \}.$$

Thus we find (as in the case of type $\mathbf{B}_n$) that

(i) Y_0, Y'_{n-1} and Y_{n+j}, $-1 \leqq j \leqq n-2$, are non-singular.

(ii) Y_i, $1 \leqq i \leqq n-2$ are (*generalised*) *cones* in **P** over the quadrics Q_{n-i} which lie in the lower dimensional projective spaces $\mathbf{P}^{2n-1-2i}$ with coordinates $(x_{i+1}, \ldots, x_n, y_n, \ldots, y_{n-i})$, and are in particular normal.

(iii) Y_{n-1} (resp. Y'_{n-1}) is *not* a Cartier divisor in Y_{n-2}, however, $Y_{n-1} \cup Y'_{n-1}$ is a Cartier divisor in Y_{n-2}.

(iv) The union $Y_{n-1} \cup Y'_{n-1}$ and the intersection $Y_{n-1} \cap Y'_{n-1}$ are scheme-theoretic.

7. THE IDEAL SHEAF $X(w_i)_r$ IN $X(w_{i-1})_r$.

PROPOSITION D. 7. — *The sheaf of ideals defining*

(i) $X(w_i)_r$ *in* $X(w_{i-1})_r$ *is* $L(-\chi_{i-1})|_{X(w_{i-1})_r}$ *for* $1 \leqq i \leqq n-2$ *or* $n \leqq i \leqq 2n-2$.

(ii) $X(w_n)_r$ *in* $X(w'_{n-1})_r$ *is* $L(-\chi'_{n-1})|_{X(w'_{n-1})_r}$.

(iii) $X(w_{n-1})_r \cup X(w'_{n-1})_r$ *in* $X(w_{n-2})_r$ *is* $L(-\chi_{n-2})|_{X(w_{n-2})_r}$

(*i.e., the equalities in Propositions* D. 4 *and* D. 5 *are scheme-theoretic*).

Proof. — Similar to that of Proposition B. 7.

8. VANISHING THEOREM.

THEOREM D. 8. — *For every line bundle* $L(\chi)$ *on* G/B *belonging to the dominant chamber, we have*

$$H^p(X(w_i)_r, L(\chi)|_{X(w_i)_r}) = 0 \quad \textit{for all } p > 0$$

and $i = 0, \ldots, 2n-2$ [*a similar assertion for* $X(w'_{n-1})_r$]. *In particular for* $i = 0$, *we have*

$$H^p(G/B, L(\chi)) = 0 \quad \textit{for all } p > 0.$$

Proof. — Write $X_i = X(w_i)_r$, $0 \leqq i \leqq 2n-2$ and $X'_{n-1} = X(w'_{n-1})_r$. Proceeding as in the proof of Theorem A. 8, by increasing induction on the integer (χ, α^*_{n-j-2}) for $i = n+j$, $0 \leqq j \leqq n-3$, we find that the theorem is true for X_i, $n \leqq i \leqq 2n-2$; similarly, by increasing induction on (χ, α^*_{n-1}) [resp. (χ, α^*_n)], we find that the theorem is true for X_{n-1} (resp. X'_{n-1}). Assuming that we have proved the theorem for X_{n-2}, again by the same procedure [now the induction being on (χ, α^*_{i+1}) we find that the theorem is true for X_i, $0 \leqq i \leqq n-3$]. Thus we have only to prove the theorem for X_{n-2}, remembering that the theorem is true in particular for X_n, X_{n-1} and X'_{n-1}.

Proof of the theorem for X_{n-2}. — Let $Z = X_{n-1} \cup X'_{n-1}$. By Proposition D. 7, we know that Z is the closed subscheme of X_{n-2} obtained as the scheme-theoretic union of X_{n-1} and X'_{n-1} patched along the subscheme $X_n = X_{n-1} \cap X'_{n-1}$. Now we prove

(I) $$H^p(Z, L(\chi)|_Z) = 0 \quad \text{for all } p > 0 \quad (\text{and } \chi \geqq 0).$$

Consider the exact sequence

$$0 \to \mathcal{O}_Z \to \mathcal{O}_{X_{n-1}} \oplus \mathcal{O}_{X'_{n-1}} \to \mathcal{O}_{X_n} \to 0.$$

This gives the exact sequence

$$0 \to L(\chi)|_Z \to L(\chi)|_{X_{n-1}} \oplus L(\chi)|_{X'_{n-1}} \to L(\chi)|_{X_n} \to 0$$

and hence we get the exact sequence

$$(\bigstar)\qquad \begin{aligned} &0 \to H^0(Z, L(\chi)) \to H^0(X_{n-1}, L(\chi)) \oplus H^0(X'_{n-1}, L(\chi)) \xrightarrow{\lambda} H^0(X_n, L(\chi)) \\ &\to H^p(Z, L(\chi)) \to H^p(X_{n-1}, L(\chi)) \oplus H^p(X'_{n-1}, L(\chi)) \to H^p(X_n, L(\chi)) \to . \end{aligned}$$

Since the theorem is true for X_{n-1}, X'_{n-1} and X_n, ($\bigstar$) implies that $H^p(Z, L(\chi)) = 0$ for all $p \geqq 2$. To prove $H^1(Z, L(\chi)) = 0$, it suffices to prove the

CLAIM. — *λ is surjective.*

Recall that we have the exact sequence [given by the divisor X_n in X_{n-1} whose ideal sheaf is isomorphic to $L(-\chi_{n-1})|_{X_{n-1}}$, *cf.* Proposition D. 7] :

$$0 \to L(-\chi_{n-1})|_{X_{n-1}} \to \mathcal{O}_{X_{n-1}} \to \mathcal{O}_{X_n} \to 0.$$

Tensoring by $L(\chi)$, we get the exact sequence

$$0 \to L(\chi - \chi_{n-1})|_{X_{n-1}} \to L(\chi)|_{X_{n-1}} \to L(\chi)|_{X_n} \to 0$$

and hence the exact sequence

$$H^0(X_{n-1}, L(\chi)) \xrightarrow{\lambda_1} H^0(X_n, L(\chi)) \to H^1(X_{n-1}, L(\chi')),$$

where

$$\chi' = \chi - \chi_{n-1} = \chi - \tilde{\omega}_{n-1} + \tilde{\omega}_n \quad (cf.\ \text{Proposition D.6}).$$

Notice that

$$(\chi', \alpha^*_{n-1}) = (\chi, \alpha^*_{n-1}) - 1$$

and

$$(\chi', \alpha^*_j) \geqq (\chi, \alpha^*_j) \qquad \text{for} \quad j \neq n-1.$$

Hence by Remark D. 3 (ii) and Corollary 1.15, and the theorem being true for X_{n-1}, we find that $H^1(X_{n-1}, L(\chi')) = 0$. Hence λ_1 is surjective. Similarly, replacing X_{n-} by X'_{n-1} in the above argument, we get that

$$H^0(X'_{n-1}, L(\chi)) \xrightarrow{\lambda'_1} H^0(X_n, L(\chi))$$

is surjective. But now the claim is obvious since $\lambda = \lambda_1 - \lambda'_1$.

(II) We now conclude the proof of the theorem as follows :

By Proposition D. 7, the ideal sheaf of Z in X_{n-2} is $\approx L(-\chi_{n-2})|_{X_{n-2}}$ and so we have the exact sequence

$$0 \to L(-\chi_{n-2})|_{X_{n-2}} \to \mathcal{O}_{X_{n-2}} \to \mathcal{O}_Z \to 0.$$

Tensoring by $L(\chi)$, we get the exact sequence

$$0 \to L(\chi - \chi_{n-2})|_{X_{n-2}} \to L(\chi)|_{X_{n-2}} \to L(\chi)|_Z \to 0.$$

This gives the exact sequence

$$\to H^p(X_{n-2}, L(\chi')) \to H^p(X_{n-2}, L(\chi)) \to H^p(Z, L(\chi)) \to,$$

where

$$\chi' = \chi - \chi_{n-2} = \chi + \tilde{\omega}_{n-2} - \tilde{\omega}_{n-1} - \tilde{\omega}_n.$$

By (I), this reduces to the exact sequences

$$H^p(X_{n-2}, L(\chi')) \to H^p(X_{n-2}, L(\chi)) \to 0$$

for all $p \geqq 1$. We prove the result by induction on (χ, α_{n-1}^*) or (χ, α_n^*). First notice that

(i) $$(\chi', \alpha_{n-1}^*) = (\chi, \alpha_{n-1}^*) - 1,$$
$$(\chi', \alpha_n^*) = (\chi, \alpha_n^*) - 1$$

and

(ii) $$(\chi', \alpha_j^*) \geqq (\chi, \alpha_j^*) \quad \text{for} \quad j \neq n-1 \quad \text{or} \quad n.$$

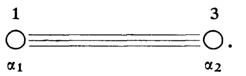
Suppose one of (χ, α_{n-1}^*) or (χ, α_n^*) is 0, say $(\chi, \alpha_{n-1}^*) = 0$. Then by Remark D. 3 (ii) and Corollary 1.15, $H^p(X_{n-2}, L(\chi')) = 0$ for all $p \geqq 0$ and so $H^p(X_{n-2}, L(\chi)) = 0$ as required. So we can assume $(\chi, \alpha_{n-1}^*) > 0$ and $(\chi, \alpha_n^*) > 0$. Hence $\chi' \geqq 0$. Now proceeding by induction on either of (χ, α_{n-1}^*) and (χ, α_n^*), we get the result.

This completes the proof of the theorem.

G. Type $\mathbf{G}_2$

1. Numerical data. — Recall the following facts for a group G of type $\mathbf{G}_2$.

Dynkin diagram :



The Cartan numbers

$$n_{ij} = (\alpha_i, \alpha_j^*) = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$$

are given by the matrix

$$(n_{ij}) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

The number of roots = 12.

The order of the Weyl group W (= W ($\mathbf{G}_2$)) of G = 12.

2. A reduced expression for w_0.

Proposition G. 1. — *A reduced expression for the element* $w_0 \in W$ (*of largest length*) *is given by*

$$w_0 = s_2 s_1 s_2 s_1 s_2 s_1.$$

Proof. — Write $u = s_2 s_1 s_2 s_1 s_2 s_1$. We have $l(u) \leqq 6$. It is trivial to check that $u(\alpha_1) = -\alpha_1$ and $u(\alpha_2) = -\alpha_2$ (i.e., $u = -\mathrm{Id}$). Hence by Proposition 1.1 (iii), we get that $u (= w_0)$ is the element of largest length in W. But $l(w_0) = 6 = 1/2$ number of roots and hence the result.

3. THE PARABOLIC SUBGROUP $P (= P_{\hat{\alpha}_1})$. — Since the rank of G is 2, it is clear that $P = P_{\hat{\alpha}_1} = P_{\alpha_2}$ (i.e., minimal parabolic subgroups P_{α_1} and P_{α_2} of G are also the maximal ones). We have $W_P = \{1, s_2\}$ and the number of Schubert varieties in P\G is $[W : W_P] = 6$.

4. THE FAMILY OF CLOSED BRUHAT CELLS $\{X(w_i)\}$. — Define the sequences $\{\tau_i\}$ and $\{w_i\}$ of elements in W as follows :

$$\{\tau_i\} : \quad \tau_0 = \text{Id}, \quad \tau_1 = s_1, \quad \tau_2 = s_1 s_2, \quad \tau_3 = s_1 s_2 s_1,$$
$$\tau_4 = s_1 s_2 s_1 s_2 \quad \text{and} \quad \tau_5 = s_1 s_2 s_1 s_2 s_1,$$
$$\{w_i\} : \quad w_i = w_0 \tau_i, \qquad 0 \leqq i \leqq 5,$$

PROPOSITION G. 2.

(i) $G = X(w_0) \supset X(w_1) \supset \ldots \supset X(w_5) = P$.

(ii) *Each* $X(w_i)$ *is of codimension* 1 *in* $X(w_{i-1})$.

(iii) $X(w_i)$ *are the inverse images (under the natural morphism* $G \to P\backslash G$) *of the Schubert varieties in* P\G.

Proof. – Similar to that of Proposition A. 2.

REMARK G.3. — We are

(i) $$X(w_i) = P w_i B \cup X(w_{i+1}), \qquad 0 \leqq i \leqq 4$$

the union being set-theoretic and disjoint.

(ii) $X(w_i)$ is stable for multiplication on the right by the parabolic subgroups P_{α_1} or P_{α_2} according as i is even or odd, $0 \leqq i \leqq 4$.

Proof. – Similar to that of Remark A. 3.

PROPOSITION G. 4. – *We have (set-theoretically) :*

$$X(w_i) \cap X(w_1) \tau_i = X(w_{i+1}), \qquad 0 \leqq i \leqq 4$$

(*where* $\tau_i = w_0 w_i$).

Proof. – Similar to that of Proposition A. 4.

PROPOSITION G. 5. – *Let* $\chi_i = \tau_i^{-1}(\tilde{\omega}_1)$, $0 \leqq i \leqq 4$. *For each i there exists an element* $f_i \in H^0(X(w_i)_r, L(\chi_i)|_{X(w_i)_r})$ *such that (set-theoretically) the set of zeros of* f_i *in* $X(w_i)_r$ *is* $X(w_{i+1})_r$.

Proof. – Same as in Proposition B. 5.

5. THE FAMILY OF CHARACTERS $\{\chi_i\}$.

PROPOSITION G. 6. – *The characters* $\chi_i (= \tau_i^{-1}(\tilde{\omega}_1))$, $0 \leqq i \leqq 4$, *are given by* $\chi_0 = \tilde{\omega}_1$ *and*

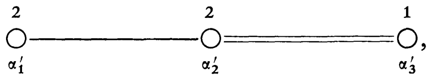
$$\chi_1 = -\chi_4 = \tilde{\omega}_2 - \tilde{\omega}_1 \qquad \text{and} \qquad \chi_2 = -\chi_3 = 2\tilde{\omega}_1 - \tilde{\omega}_2.$$

Proof. — Straight forward verification.

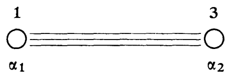
6. STRUCTURE OF SCHUBERT VARIETIES IN P\G. — Recall that there exists only one connected semi-simple linear algebraic group G upto isomorphism which is of type $\mathbf{G}_2$ and that we have a faithful representation of G in GL (7) which factors through G′ = SO (7) (*cf.* [7], [9], [15], etc.). We can choose a maximal torus T′ (resp. a Borel subgroup B′ ⊃ T′) in G′ such that the following facts hold, namely

(i) T = G ∩ T′ (resp. B = G ∩ B′) (as schemes) is a maximal torus (resp. a Borel subgroup) in G.

(ii) If α'_1, α'_2 and α'_3 are the simple roots of G′ relative to T′ and B′ with Dynkin diagram



then α_1 and α_2 where $\alpha_1 = \alpha'_1|_T = \alpha'_3|_T$ and $\alpha_2 = \alpha'_2|_T$ are the simple roots of G relative to T and B with Dynkin diagram



and further.

(iii) P = G ∩ P′ (as schemes) where $P' = P_{\hat{\alpha}'_1}$ in G′ and $P = P_{\hat{\alpha}_1}$ in G.

Consequently, we have a closed immersion P\G ⊊ P′\G′. But this is an isomorphism because dim P\G = 5 = dim P′\G′, in other words, P\G is a *five dimensional quadric* and hence from the section 6 of Type B_n for $n = 3$, we get to know the structure of Schubert varieties in P\G.

7. The ideal sheaves of $X(w_i)_r$ in $X(w_{i-1})_r$ are determined exactly as in the case of Type B_3.

8. VANISHING THEOREM.

THEOREM G.11. — *For every line bundle* L (χ) *on* G/B *belonging to the dominant chamber, we have*

$$H^p(X(w_i)_r, L(\chi)|_{X(w_i)_r}) = 0 \qquad for \quad p > 0$$

and $i = 0, \ldots, 5$. *In particular for* $i = 0$, *we have*

$$H^p(G/B, L(\chi)) = 0 \quad for \ all \ p > 0.$$

Proof. — Exactly same as that of Theorem B. 11 for $n = 3$. However, in carrying out the details in this case one must use the values of the characters χ_i, $0 \leqq i \leqq 4$, as given by Proposition G. 6 (but *not* as in Proposition B. 6 for $n = 3$). While proving the theorem for $X_2 = X(w_2)_r$, the proof of (I) goes verbatum and the proof is completed by proceeding by increasing induction on (χ, α_1^*).

RÉFÉRENCES

[1] A. BOREL, *Linear Algebraic Groups*, W. A. Benjamin, New York, 1969.

[2] A. BOREL, *Introduction aux groupes arithmétiques*, *Publications de l'Institut Mathématique de l'Université de Strasbourg*, vol. 15, Hermann, Paris, 1969.

[3] A. BOREL et J. TITS, *Groupes réductifs* (*Publ. Math. I. H. E. S.*, vol. 27, 1965, p. 55-150).

[4] A. BOREL et J. TITS, *Compléments à l'article « Groups réductifs »* (*Publ. Math. I. H. E. S.*, vol. 41, 1972, p. 253-276).

[5] R. BOTT, *Homogeneous vector bundles* (*Ann. of Math.*, series (2), vol. 66, 1957, p. 203-248).

[6] N. BOURBAKI, *Groupes et algèbres de Lie*, chapitres 4, 5 et 6, Hermann, Paris, 1968.

[7] C. C. CHEVALLEY, *Classification de groupes de Lie algébriques* (*Séminaire*, 1956-1958, Secrétariat mathématique, 11, rue Pierre-Curie, Paris, 1958).

[8] C. C. CHEVALLEY, *Sur les décompositions cellulaires des espaces* G/B (manuscrit non publié), Circa, 1958.

[9] M. DEMAZURE, *The case of the group* G_2, *Appendix to « Adeles and Algebraic Groups »* (*Lectures* by ANDRÉ WEIL, The Institute for Advanced Study, Princeton, New Jersey, 1961).

[10] M. DEMAZURE, *Une démonstration algébrique d'un théorème de Bott* (*Inventiones Math.*, vol. 5, 1968, p. 349-356).

[11] M. DEMAZURE, *Sur la formule des caractères de H. Weyl* (*Inventiones Math.*, vol. 9, 1970, p. 249-252.)

[12] M. DEMAZURE, *Désingularisation des variétés de Schubert généralisées* (*Ann. Sc. Éc. Norm. Sup.*, *t.* 7, 1974, p. 53-88).

[13] M. DEMAZURE et A. GROTHENDIECK, *Schémas en Groupes III* (SGA 3), *Lecture Notes in Mathematics*, n° 153, Springer-Verlag, 1970.

[14] M. HOCHSTER, *Grassmannians and their Schubert varieties are arithmetically Cohen-Macaulay* (*J. Algebra*, vol. 25, 1973, p. 40-57).

[15] J.-I. IGUSA, *On certain representations of semi-simple algebraic groups and the arithmatic of the corresponding invariants* (1) (*Inventiones Math.*, vol. 12, 1971, p. 62-94).

[16] G. KEMPF, *Schubert methods with an application to algebraic curves* (Stichting mathematisch centrum, Amsterdam, 1971).

[17] B. KOSTANT, *Lie algebra cohomology and the generalised Borel-Weil theorem* (*Ann. of Math.*, vol. 74, 1961, p. 329-387).

[18] D. LAKSOV, *The arithmetic Cohen-Macaulay character of Schubert schemes* (*Acta Mathematica*, vol. 129, 1972, p. 1-9).

[19] D. MUMFORD, *Abelian Varieties*, Oxford University Press, Bombay, 1970.

[20] C. MUSILI, *Postulation formula for Schubert varieties* (*J. Indian Math. Soc.*, vol. 36, 1972, p. 143-171).

[21] J.-P. SERRE, *Algèbres de Lie semi-simples complexes*, W. A. Benjamin, New York, 1966.

[22] C. S. SESHADRI, *Quotient spaces modulo reductive groups* (*Ann. of Math.*, vol. 95, 1972, p. 511-556).

[23] R. STEINBERG, *Lectures on Chevalley groups*, *Lecture Notes in Mathematics*, Yale University, New Haven, Conn., 1967.

[24] J. TITS, *Classification of semi-simple algebraic groups*, *Proceeding Symposia in Pure Math.*, vol. 9, 1966, p. 33-62.

[25] *Groupes algébriques linéaires*, Exposé 5 (*Séminaire Heidelberg*, Strasbourg, 1965-1966, Publication I. R. M. A., Strasbourg, n° 2, 1967).

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(Manuscrit reçu le 19 novembre 1973.)