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A GAUSS-BONNET FORMULA FOR DISCRETE ARITHMETICALLY DEFINED GROUPS

By G. HARDER,

Bonn (1).

—•—
*Dedicated to Ernst Witt
on the occasion
of his sixtieth birthday*

INTRODUCTION

Let G/\mathbf{R} be a semi-simple algebraic group over the field of real numbers. Let us denote the group of real points of G/\mathbf{R} by $G_{\mathbf{R}}$, i. e. $G_{\mathbf{R}} = G(\mathbf{R})$. If K is a maximal compact subgroup of $G_{\mathbf{R}}$ then $X = K \backslash G_{\mathbf{R}}$ is a symmetric space, we know that X is diffeomorphic to \mathbf{R}^d . Let $\Gamma \subset G_{\mathbf{R}}$ be a discrete subgroup without torsion, then X/Γ is a manifold and

$$H_i(\Gamma, \mathbf{R}) = H_i(X/\Gamma, \mathbf{R}) \quad \text{for } i \in \mathbf{N}.$$

If Γ is of finite cohomological dimension we define the Euler-Poincaré characteristic of Γ by

$$\chi(\Gamma) = \sum (-1)^i \dim H_i(\Gamma, \mathbf{R}).$$

It is well known from differential geometry that in the case of a *compact quotient* X/Γ there is a differential form ω_X on X such that

$$(\star) \quad \int_{X/\Gamma} \omega_X = \chi(\Gamma).$$

The form ω_X can be computed in terms of the Riemannian metric on X , and it will be called the *Euler-Poincaré form* on X . This formula (\star)

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is a generalisation of the classical formula of Gauss-Bonnet, and it is due to Allendoerfer and Weil (comp. [4]).

Now the question arises whether the formula (\star) holds also when the volume of X/Γ is finite. The goal of this paper is to show that for arithmetically defined groups Γ the formula (\star) is true without the assumption of compactness. This is an answer to a question posed by Ono in his paper [8].

If Γ_0 is the group of integral points of a simply connected Chevalley scheme over the ring of integers of a number field we will give an explicit formula for $\chi(\Gamma_0)$ (2.2). This will be done by using Langland's calculation of Tamagawa numbers for Chevalley groups. Similar calculations have been done by Ono in his paper [8].

The problem we are dealing with here has been solved by C. L. Siegel in the special case of an orthogonal group over \mathbf{Q} in [11]. Needless to say that the most important ideas of the present paper are already contained in Siegel's note.

The proof of (\star) rests on the reduction theory of Borel and Harish-Chandra; I recall this theory in 1.2 in the different form given in [5].

In the proof of (\star) we need a function

$$h: X/\Gamma \rightarrow (0, \infty)$$

- (i) $h^{-1}([\delta, \infty))$ is compact for $\delta > 0$;
- (ii) h has no small critical values.

A function h having these two properties has been constructed by Raghunathan in [9]. For the proof of (\star) we need some additional properties of h , for this reason we give an explicit construction of h in 1.3. The idea of the proof of (\star) is explained in the beginning of 1.3.

1.1. PRELIMINARIES ON $X = K \backslash G_\infty$. — Let G/\mathbf{R} be a semi-simple algebraic group $G_\infty = G(\mathbf{R})$ its group of real points. The space of maximal compact subgroups of G_∞ is denoted by X . If $x \in X$ we denote the corresponding maximal compact subgroup by K_x . The group G_∞ acts on X by conjugation

$$(K_x, g) \rightarrow g^{-1} K_x g.$$

Choosing $x \in X$ we get an identification

$$\begin{aligned} \varphi_x: K_x \backslash G_\infty &\xrightarrow{\sim} X, \\ \varphi_x: K_x g &\mapsto g^{-1} K_x g. \end{aligned}$$

The map φ_x is compatible with the right action of G_∞ on both sides.

The Lie algebra of G_x is denoted by \mathfrak{g}_x , if $x \in X$ we denote its corresponding Cartan decomposition by

$$\mathfrak{g}_x = \mathfrak{k}_x \oplus \mathfrak{p}_x$$

and the corresponding Cartan involution is denoted by Θ_x . It is well known that

$$\begin{aligned} \Theta_x(Y) &= Y & \text{for } Y \in \mathfrak{k}_x, \\ \Theta_x(Z) &= -Z & \text{for } Z \in \mathfrak{p}_x. \end{aligned}$$

Moreover it is well known that the Killing form B is negative definite on \mathfrak{k}_x and is positive definite on \mathfrak{p}_x . From this we get that the following quadratic form on \mathfrak{g}_x :

$$B_x(Y) = -B(Y, \Theta_x(Y)) \quad \text{for } Y \in \mathfrak{g}_x$$

is positive definite. It is obvious that B_x is invariant under the restriction of the adjoint action to K_x . By right translations we get a G_x -right and K_x -left invariant metric $d_x s^2$ on G_x . The space X is endowed with a G_x -invariant Riemannian metric. This metric is related to $d_x s^2$ as follows : The differential of the action of G_x on X yields a linear mapping

$$\lambda_x : \mathfrak{g}_x \rightarrow T_{X,x},$$

where $T_{X,x}$ is the tangent space of X in the point x . The kernel of λ_x is \mathfrak{k}_x and we get an isomorphism

$$\bar{\lambda}_x : \mathfrak{p}_x \xrightarrow{\sim} T_{X,x}.$$

By definition of the metric on X this map is an isometry if \mathfrak{p}_x carries the metric $B_x|_{\mathfrak{p}_x}$.

Let $P \subset G$ be a parabolic subgroup. The group $P_x = P(\mathbf{R})$ acts transitively on X . We now collect some facts concerning the restriction of the metric $d_x s^2$ to P_x . These facts will be important for the study of the fundamental domain X/Γ .

Before doing this I want to give some remarks on root systems. Let $R(P)$ [resp. $R_u(P) = U$] be the radical (resp. unipotent radical) of P . Let $S \subset R(P)$ be a maximal split torus. We consider the adjoint action of S on the Lie algebra \mathfrak{g} . Then we get a decomposition

$$\mathfrak{g} = \left(\bigoplus_{\alpha \in \Delta} \mathfrak{u}^{(\alpha)} \right) \oplus \mathfrak{Z}(S).$$

Here $\mathfrak{Z}(S)$ is the centralizer of S in \mathfrak{g} , the set $\Delta \subset \text{Hom}(S, G_m) = X(S)$ is the set of roots with respect to (S, P) and $\mathfrak{u}^{(\alpha)}$ is the root sub-space which belongs to α .

If we restrict the action of S to the Lie algebra \mathfrak{u} of U we get

$$\mathfrak{u} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{u}^{(\alpha)},$$

where $\Delta^+ \subset \Delta$ is the system of positive roots of S with respect to P . The restriction map

$$\text{Hom}(P, G_m) \rightarrow \text{Hom}(S, G_m)$$

is not surjective in general, but it becomes an isomorphism if we tensorise with \mathbf{Q} . If $\chi \in \text{Hom}(S, G_m)$ we get a homomorphism

$$\chi_\infty: S_\infty \rightarrow \mathbf{R}^*$$

and we define

$$|\chi|: S_\infty \rightarrow (\mathbf{R}^+)^*$$

by

$$|\chi|: s \mapsto |\chi_\infty(s)|.$$

It follows from the preceding remark that this homomorphism can be extended in a canonical way to a homomorphism of P_∞ to $(\mathbf{R}^+)^*$ which we also denote by

$$|\chi|: P_\infty \rightarrow (\mathbf{R}^+)^*.$$

The character

$$\gamma_P = \sum_{\alpha \in \Delta^+} (\dim \mathfrak{u}^{(\alpha)}) \cdot \alpha$$

is the restriction of a character of P which is also denoted by γ_P . This character is called *the sum of the positive roots of P* .

If $S_1 \subset R(P)$ is another maximal split torus then S and S_1 are conjugate by an inner automorphism of $R(P)$. From this we get a canonical identification between the systems of roots (positive roots) of S and S_1 .

After these remarks we come back to the investigation of the restriction of $d_x s^2$ to P_∞ . The transform P^{θ_x} of P by the Cartan involution is opposite to P and

$$L_{x,\infty} = P_\infty \cap P^{\theta_x}$$

is a Levi subgroup of P_∞ . Let S_x be the maximal split torus in the centre of $L_{x,\infty}$; we apply our previous considerations to S_x and get a decomposition

$$\mathfrak{u}_\infty = \bigoplus_{\alpha \in \Delta^+} \mathfrak{u}_{x,\infty}^{(\alpha)},$$

where \mathfrak{u}_∞ is of course the Lie algebra of $U_\infty = R_u(P)_\infty$. If \mathfrak{p}_∞ (resp. $\mathfrak{k}_{x,\infty}$) is the Lie algebra of P_∞ (resp. $L_{x,\infty}$), then the following proposition holds.

PROPOSITION 1.1.1. — *The decomposition*

$$\mathfrak{p}_\infty = \mathfrak{k}_{x,\infty} \oplus \left(\bigoplus_{\alpha \in \Delta^+} \mathfrak{u}_{x,\infty}^{(\alpha)} \right)$$

is orthogonal with respect to the restriction of B_x to \mathfrak{p}_∞ . If $\mathfrak{k}_x^p = \mathfrak{k}_x \cap \mathfrak{p}_\infty$ then the tangent space of X at x can be identified as a metric space with the orthogonal complement of \mathfrak{k}_x^p in \mathfrak{p}_∞ .

Proof. — The involution Θ_x leaves $\mathfrak{k}_{x,\infty}$ stable. It induces on S_x the mapping $y \mapsto y^{-1}$. From this it follows that Θ_x sends $\mathfrak{u}_{x,\infty}^{(\alpha)}$ onto $\mathfrak{u}_{x,\infty}^{(-\alpha)}$. Then the proposition follows from the definition of B_x , our previous remarks on the metric on X , and some well known properties of the Killing form.

We want to draw a simple consequence from this proposition. Let us consider an element $p \in P_\infty$. We are going to compare the restrictions of the metrics $d_x s^2$ and $d_{xp} s^2$ to the group P_∞ . The metric $d_{xp} s^2$ is defined by the maximal compact subgroup $p^{-1} K_x p$ and the corresponding Cartan involution is $\Theta_{xp} = \text{ad}(p^{-1}) \Theta_x \text{ad}(p)$. For $Y \in \mathfrak{p}_\infty$ we get

$$B_{xp}(Y) = -B(Y, \Theta_{xp}(Y)) = -B(\text{ad}(p)(Y), \Theta_x(\text{ad}(p)(Y))) = B_x(\text{ad}(p)(Y)).$$

So the restriction of $d_{xp} s^2$ to P_∞ is obtained by transforming the restriction of $d_x s^2$ by the inner automorphism $\text{ad}(p)$ of P_∞ . Especially for the volume element $d_x u$ on U_∞ which is defined by the restriction of $d_x s^2$ to U_∞ we get the formula

$$(1.1.2) \quad d_{xp} u = |\gamma_p(p)| d_x u.$$

1.2. PRELIMINARIES ON REDUCTION THEORY. — Let F be an algebraic number field. Its ring of integers is denoted by \mathfrak{O} . Let $\mathfrak{G}/\mathfrak{O}$ be a flat affine group scheme of finite type. Moreover we assume that its generic fiber $G = \mathfrak{G} \times_{\mathfrak{O}} F$ is semi-simple. Let $\Pi = \{ \alpha, \beta, \dots \}$ be the system of simple roots of G/F . Actually the set of simple roots is only defined with respect to pairs (P, S) , where P is a minimal parabolic subgroup of G/F and $S \subset P$ is a maximal split torus. Then the set Δ (resp. Δ^+ , resp. Π) of roots (resp. positive roots, resp. simple roots) is defined as a subset of $\text{Hom}(S, G_m)$ (comp. 1.1). But as before we have a canonical bijection between the corresponding sets if we have two pairs (P, S) and (P_1, S_1) . So we are allowed to speak of the set of roots (simple roots, positive roots) of G/F .

The parabolic subgroups of G/F containing a given minimal one correspond to the subsets of Π ([2], p. 86), the minimal subgroup itself corres-

ponds to the empty subset. The group $\mathfrak{G}(\mathfrak{O})$ of integral points of $\mathfrak{G}/\mathfrak{O}$ is denoted by Γ and we denote the group of real points by G_∞ , i. e. $G_\infty = G(F \otimes_{\mathfrak{Q}} \mathbf{R})$.

We consider a point $x \in X$ and a parabolic subgroup P of G/F . The unipotent radical of P is denoted by U . The quotient $U_\infty/U_\infty \cap \Gamma$ is compact. At the end of 1.1 I introduced the measure $d_x u$ on U_∞ . We put

$$p(x, P) = \int_{U_\infty/U_\infty \cap \Gamma} d_x u.$$

If P is a minimal parabolic subgroup of G/F and if $P^{(\alpha)}$ is the maximal parabolic subgroup of type $\Pi - \{\alpha\}$ containing P then we put

$$p_\alpha(x, P) = p(x, P^{(\alpha)}).$$

If $\alpha \in \Pi$ is a simple root, we denote by γ_α the corresponding fundamental dominant weight, i. e.

$$\frac{2 \langle \gamma_\alpha, \beta \rangle}{\langle \beta, \beta \rangle} = \delta_{\alpha, \beta} \quad (\alpha, \beta \in \Pi).$$

The sum of the positive roots of $P^{(\alpha)}$ is a positive integral multiple of the character γ_α

$$\gamma_{P^{(\alpha)}} = f_\alpha \gamma_\alpha, \quad f_\alpha > 0.$$

Therefore we can express the simple roots in terms of the characters $\gamma_{P^{(\alpha)}}$

$$\alpha = \sum_{\beta \in \Pi} c_{\alpha, \beta} \gamma_{P^{(\beta)}}, \quad c_{\alpha, \beta} \in \mathfrak{Q}.$$

We now introduce new numbers by

$$n_\alpha(x, P) = \prod_{\beta \in \Pi} p_\beta(x, P)^{c_{\alpha, \beta}}.$$

Let $S \subset P$ be a maximal split torus and let χ be a character on S . This character defines a homomorphism

$$\chi_\infty: S_\infty \rightarrow (F \otimes_{\mathfrak{Q}} \mathbf{R})^*.$$

Let

$$|\nu|: (F \otimes \mathbf{R})^* \rightarrow (\mathbf{R}^+)^*$$

be the absolute value of the norm mapping. The composite map $|\nu| \circ \chi_\infty$ can be extended in a canonical way to a homomorphism

$$|\chi|: P_\infty \rightarrow (\mathbf{R}^+)^*.$$

This follows again from the fact that $\text{Hom}(P, G_m) \otimes \mathfrak{Q} = \text{Hom}(S, G_m) \otimes \mathfrak{Q}$.

PROPOSITION 1.2.1. — *Let P be a minimal parabolic subgroup of G/F. If $x \in X$ and $p \in P_\infty$, then we have*

$$\begin{aligned} n_\alpha(xp, P) &= n_\alpha(x, P) |\alpha|(p), \\ p_\alpha(xp, P) &= p_\alpha(x, P) |\gamma_{P(\alpha)}|(p). \end{aligned}$$

This follows immediately from 1.1.2.

I now state the basic theorems of reduction theory. They are not formulated in the usual form, but I have shown in my paper ([5], § 2, p. 51-52) how to translate the present formulation into the language of Borel's book [1]. For the convenience of the reader I will give some indications about the relations between these two different points of view after having stated the main results.

THEOREM 1.2.2. — *There exists a constant C_1 , such that for every point $x \in X$ there is a minimal parabolic subgroup P of G/F, such that*

$$n_\alpha(x, P) < C_1 \quad \text{for all } \alpha \in \Pi.$$

Let us choose such a constant C_1 once for all. If $x \in X$ we call a minimal parabolic subgroup P of G/F *reduced with respect to x* or simply *x-reduced* if

$$n_\alpha(x, P) < C_1 \quad \text{for all } \alpha \in \Pi.$$

THEOREM 1.2.3. — *There is a constant $C_2 > 0$ having the following property: If $x \in X$ and if P is reduced with respect to x and if $n_{\alpha_0}(x, P) < C_2$ for some $\alpha_0 \in \Pi$, then every x-reduced minimal parabolic subgroup of G/F is contained in $P^{(\alpha_0)}$.*

Let us choose $C_2 > 0$ once for all.

We will need the *compactness criterion* in the following general formulation :

Let H/F be a connected affine algebraic group. Let $\chi_1, \chi_2, \dots, \chi_r$ be a basis of the character module $\text{Hom}(H, G_m)$. Let $C > 1$ be a real constant and

$$H_\infty(C) = \{ h \in H_\infty \mid C^{-1} < |\chi_i|(h) < C \}.$$

Let $\Gamma \subset H(F)$ be an arithmetically defined subgroup. Then the quotient $H_\infty(C)/\Gamma$ is compact if and only if the semi-simple part of H/F is anisotropic.

This is a slightly generalized version of Théorème 8.7 in [1]. The following fact is a consequence of the compactness criterion :

Let G/F be a semi-simple algebraic group and let $\Gamma \subset G(F)$ be an arithmetically defined subgroup. Then the set Σ of Γ -conjugacy classes of parabolic subgroups of G/F is finite.

This is proved in [1], 15.6.

Now I want to explain the relationship between the two theorems above and the corresponding theorems in Borel's book [1]. We start with the following trivial observation: Let us consider a point $x \in X$ and a minimal parabolic subgroup P of G/F which is reduced with respect to x . Then for any element $\gamma \in \Gamma$ the group $\gamma^{-1} P \gamma$ is reduced with respect to $x \gamma$. This follows from the obvious equality

$$n_x(x, P) = n_x(x \gamma, \gamma^{-1} P \gamma).$$

Let us choose representatives P_1, \dots, P_l for the Γ -conjugacy classes of minimal parabolic subgroups of G/F . We denote

$$X_i = \{x \in X \mid P_i \text{ is reduced with respect to } x\}, \quad 1 \leq i \leq l.$$

From the observation above we get

$$X = \left(\bigcup_{i=1}^l X_i \right) \Gamma.$$

We choose points $x_i \in X_i$ and we denote the corresponding maximal compact subgroups by K_{x_i} . The connected component of the identity of $P_{i,\infty}$ is denoted by $P_{i,\infty}^0$, it acts transitively on X . If $y \in X_i$ there is an element $p \in P_{i,\infty}^0$ such that $x_i p = y$. From proposition 1.2.1 follows

$$n_x(y, P_i) = n_x(x_i p, P_i) = n_x(x_i, P_i) |\alpha|(p).$$

The group P_i is reduced with respect to y and therefore

$$|\alpha|(p) \leq \frac{C_1}{n_x(x_i, P_i)} \leq C'_1.$$

If

$$P_{i,\infty}(C'_1) = \{p \in P_{i,\infty} \mid |\alpha|(p) \leq C'_1 \text{ for all } \alpha \in \Pi\},$$

then our arguments yield

$$X = \left(\bigcup_{i=1}^l x_i P_{i,\infty}^0(C'_1) \right) \Gamma$$

and this is equivalent to

$$G_\infty = \left(\bigcup_{i=1}^l K_{x_i} P_{i,\infty}^0(C'_1) \right) \Gamma.$$

By $\bar{P}_{i,\infty}$ we denote the intersection of the kernels of the homomorphisms $|\alpha|$, where $\alpha \in \Pi$. Then we have a decomposition

$$P_{i,\infty}^0 = A_i \bar{P}_{i,\infty}^0,$$

where A_i is the product of $|\Pi|$ copies of the multiplicative group of positive real numbers. We have $P_{i,\infty}^0 \cap \Gamma = \bar{P}_{i,\infty}^0 \cap \Gamma$ and it follows from the compactness criterion that the quotient

$$\bar{P}_{i,\infty}^0 / \bar{P}_{i,\infty}^0 \cap \Gamma$$

is compact. Let ω_i be a relatively compact open fundamental subset for this quotient, then

$$\bigcup_{i=1}^t K_{x_i} \cdot A_i (C'_1) \omega_i$$

is a fundamental set for the operation of Γ on G_∞ . The set $K_{x_i} \cdot A_i (C'_1) \omega_i$ is easily recognized as a Siegel domain (comp. [1], § 12) and this shows that theorem 1.2.2 is a translation of théorème 13.1 in [1].

Before I say some words about theorem 1.2.3, I want to state an important compactness criterion (comp. [5], Satz 2.2.2).

PROPOSITION 1.2.4. — *For any subset $\Omega \subset X$ the following two statements are equivalent :*

- (i) Ω is relatively compact mod Γ ;
- (ii) There is a constant $C > 0$, such that for any $x \in \Omega$, there is a minimal parabolic subgroup of G/F which is reduced with respect to x and fulfills

$$n_\alpha(x, P) > C \quad \text{for all } \alpha \in \Pi.$$

The implication (ii) \Rightarrow (i) follows quite easily from the general compactness criterion stated above and the previous considerations on the theorem 1.2.2. I want to mention that in the case of Chevalley groups the proposition 1.2.4 follows directly from theorem 1.2.2 and the finiteness of the class number, in this case one does not need the general compactness criterion.

Now I want to say some words on theorem 1.2.3 and at the end of these remarks I will indicate the proof of (i) \Rightarrow (ii) in the preceding proposition. We consider a point $x \in X$, we say that x is close to the boundary with respect to the root $\alpha \in \Pi$ if there is a minimal parabolic subgroup P of G/F which is reduced with respect to x , such that $n_\alpha(x, P) < C_2$.

Let us denote by X_α the set of points in X which are close to the boundary with respect to α .

The set X_α is obviously Γ -invariant. Theorem 1.2.3 tells us that a point $x \in X_\alpha$ determines a distinguished parabolic subgroup $P_x^{(\alpha)}$ of type $\Pi - \{\alpha\}$, this group contains all minimal parabolic subgroups of G/F which are reduced with respect to x . Let $Q \subset G/F$ be a parabolic subgroup of type $\Pi - \{\alpha\}$ we put

$$Y_\alpha^Q = \{x \in X_\alpha \mid P_x^{(\alpha)} = Q\}.$$

The following facts are obvious :

- (i) If $Q \neq Q_1$, then $Y_\alpha^Q \cap Y_\alpha^{Q_1} = \emptyset$;
- (ii) $Y_\alpha^Q \gamma = Y_\alpha^{\gamma^{-1}Q\gamma}$ for all $\gamma \in \Gamma$;
- (iii) An element $\gamma \in \Gamma$ leaves Y_α^Q stable if and only if $\gamma \in Q(F) \cap \Gamma$.

Let us consider an example. We take $G = \text{SL}(2)/\mathbf{Z}$ and $X = \mathbf{H}$ is the upper half plane. As a minimal parabolic subgroup we take

$$Q = \left\{ g = \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \mid g \in \text{SL}(2, \mathbf{R}) \right\}.$$

Then we may choose our constants C_1 and C_2 such that

$$Y_\alpha^Q = \{z \in \mathbf{H} \mid \text{Im}(z) > 2\}.$$

This follows from reduction theory and the fact that $p(z, Q) = \frac{1}{\text{Im}(z)}$ for all $z \in \mathbf{H}$. Let us have a look at the set X_α itself. The parabolic subgroups of $\text{SL}(2)$ correspond to the point ∞ and to the rational points p/q on the real axis. If $(p, q) = 1$ we denote by $D_{p,q} \subset \mathbf{H}$ the disc which has radius $(2q)^{-2}$, and is touching the real axis in the point p/q , then we have

$$X_\alpha = \left(\bigcup D_{p,q} \right) \cup Y_\alpha^Q.$$

Of course these discs are also sets of the type Y_α^Q .

Now we come back to the general case. The group Γ acts on X we denote by

$$f: X \rightarrow X/\Gamma = V$$

the natural projection of X onto its quotient under the action of Γ . The set X_α is Γ -invariant, therefore we get

$$X_\alpha/\Gamma = f(X_\alpha) = V_\alpha \quad \text{and} \quad X_\alpha = f^{-1}(V_\alpha).$$

We call V_α the set of points in V which are *close to the boundary with respect to the root α* .

Let us consider a point $v \in V_\alpha$ and two points x, y in the inverse image of v . There is an element $\gamma \in \Gamma$ such that $x\gamma = y$ and this implies $\gamma^{-1}P_x^{(\alpha)}\gamma = P_y^{(\alpha)}$. So we have seen that to any point $v \in V_\alpha$ corresponds a Γ -conjugacy class of parabolic subgroups of type $\Pi - \{\alpha\}$. We choose representatives Q_i, \dots, Q_{t_α} for the Γ -conjugacy classes of parabolic subgroups of type $\Pi - \{\alpha\}$. Let us denote the classes themselves by $[Q_i], \dots, [Q_{t_\alpha}]$. Then

$$V_\alpha^{[Q_i]} = \{v \in V_\alpha \mid \text{for } x \in f^{-1}(v) \text{ we have } P_x^{(\alpha)} \in [Q_i]\}$$

and V_α is the disjoint union of the sets $V_\alpha^{[Q_i]}$. Moreover the projection mapping

$$Y_\alpha^{[Q_i]} \rightarrow V_\alpha^{[Q_i]}$$

is surjective. If $x, x\gamma \in Y_\alpha^{[Q_i]}$ then it follows from the properties (i), (ii) and (iii) of the set $Y_\alpha^{[Q_i]}$ that $\gamma^{-1}Q_i\gamma = Q_i$ and this yields $\gamma \in Q_i(F) \cap \Gamma$. From this we get an isomorphism

$$Y_\alpha^{[Q_i]} / Q_i \cap \Gamma \xrightarrow{\sim} V_\alpha^{[Q_i]}.$$

At this point the relation of theorem 1.2.3 to proposition 17.9 in [4] and the lemma 2.1 in [9] becomes clear.

Now I want to define an important C^∞ -function which is defined on the set V_α . For a point $x \in X_\alpha$ we put

$$p_\alpha(x) = p(x, P_x^{(\alpha)}).$$

By definition, we have $x \in Y_\alpha^{P_x^{(\alpha)}}$ and this is an open set in X . This implies that $p_\alpha(x)$ is a C^∞ -function on X_α . Moreover this function p_α is obviously invariant under the action of Γ therefore it induces a C^∞ -function

$$\bar{p}_\alpha: V_\alpha \rightarrow \mathbf{R}^+ - \{0\}.$$

Remarks. — 1. These functions p_α are related to the functions of type (P, χ) in ([1], § 14). Actually in our case we have $\chi = \gamma_{P^{(\alpha)}}$ and the connection becomes clear from the relation

$$p_\alpha(xp) = p_\alpha(x) \mid \gamma_{P^{(\alpha)}} \mid (p)$$

(Prop. 1.2.4).

2. Using these functions p_α one can easily derive the implication (i) \Rightarrow (ii) in proposition 1.2.4. If $\Omega \subset X$ does not satisfy (ii) then at least one of the functions p_α tends to zero on $X_\alpha \cap \Omega$. For details, compare [5], p. 43.

3. The set of points in X which are not close to the boundary with respect to any simple root is relatively compact mod Γ . On the other hand a point which is close to the boundary with respect to some simple root determines a parabolic subgroup of G/F . This observation leads to a simple proof of the compactness criterion of Borel and Harish-Chandra (comp. [5], 2.2).

Let us consider a set π of simple roots. We denote

$$X_\pi = \bigcap_{\alpha \in \pi} X_\alpha \quad \text{and} \quad V_\pi = \bigcap_{\alpha \in \pi} V_\alpha.$$

It is clear that a point $x \in X_\pi$ determines a parabolic subgroup Q_x of type $\Pi - \pi$ of G/F and that a point $v \in V_\pi$ determines a Γ -conjugacy class of such subgroups. Let $Q \subset G/F$ be a parabolic subgroup of type $\Pi - \pi$, by $[Q]$ we denote the Γ -conjugacy class containing it. Then we put

$$Y_\pi^Q = \{ x \in X_\pi \mid Q_x = Q \},$$

$$V_\pi^{[Q]} = \{ v \in V_\pi \mid \text{for } x \in f^{-1}(v) \text{ we have } Q_x \in [Q] \}.$$

An obvious generalisation of our previous considerations shows that the restriction of f to Y_π^Q yields a surjective mapping

$$Y_\pi^Q \rightarrow V_\pi^{[Q]}$$

and from this map we get an isomorphism

$$\lambda^Q: Y_\pi^Q / Q_\infty \cap \Gamma \xrightarrow{\sim} V_\pi^{[Q]}.$$

This isomorphism will give us important informations about the structure of the sets $V_\pi^{[Q]}$. If Q_1, \dots, Q_{t_π} is a set of representatives for the Γ -conjugacy classes of subgroups of type $\Pi - \pi$, then V_π is the disjoint union of the sets $V_\pi^{[Q_i]}$.

Now I want to investigate the structure of the sets $V_\pi^{[Q]}$. As usual we denote the radical of Q by $R(Q)$ and we put $M = Q/R(Q) \cdot K$. If K is a maximal compact subgroup in G_∞ , then $Q_\infty \cap K = K_Q$ is a maximal compact subgroup in Q_∞ (this follows from the Iwasawa decomposition). By K_M we denote the unique maximal compact subgroup of M_∞ containing the image of K_Q . We have constructed a map

$$\psi_Q: X \rightarrow X_M,$$

where X_M is the space of maximal compact subgroups of M_∞ . This map factors through the action of Q_∞ . Therefore we get a map

$$\bar{\psi}_Q: X/\Gamma \cap Q_\infty \rightarrow X_M/\Gamma_M,$$

where Γ_M is the image of $\Gamma \subset Q_\infty$ in M_∞ . This is an arithmetically defined group ([1], 7.13).

Let $P \subset Q$ be a minimal parabolic subgroup, the image of P in M is denoted by \bar{P} and this is a minimal parabolic subgroup of M/F . The simple roots of M can be identified with the elements of $\Pi - \pi$. An equivalent of lemma 2.3.4 in [5] tells us

$$(1.2.5) \quad n_\alpha(x, P) \asymp n_\alpha(\psi_Q(x), \bar{P}) \quad \text{for all } \alpha \in \Pi - \pi.$$

Here \asymp means that the quotient of both sides is bounded away from zero and infinity by a constant not depending on x, P , and Q . Let $S \subset R(Q)$ be a maximal split torus. The roots $\alpha \in \Pi$ can be restricted to S , these restrictions will be denoted by α_π . We may express the roots $\alpha \in \pi$ in terms of the roots $\alpha \in \Pi - \pi$ and the fundamental dominant weights γ_α corresponding to the roots $\alpha \in \pi$. We get

$$(1.2.6) \quad \alpha = \sum_{\beta \in \pi} c_{\alpha, \beta}^\pi \gamma_\beta + \sum_{\beta \in \Pi - \pi} d_{\alpha, \beta} \beta \quad \text{for } \alpha \in \pi.$$

If we restrict α to S we obtain

$$(1.2.7) \quad \alpha_\pi = \sum_{\beta \in \pi} c_{\alpha, \beta}^\pi \gamma_\beta.$$

Now we put

$$(1.2.8) \quad n_\alpha^\pi(x, Q) = \prod_{\beta \in \pi} p_\beta(x, Q)^{c_{\alpha, \beta}^\pi}.$$

As in 1.2 the characters α_π yield homomorphisms

$$|\alpha_\pi|: Q_\infty \rightarrow (\mathbf{R}^+)^*$$

and for $q \in Q_\infty$ we get the formula

$$(1.2.9) \quad n_\alpha^\pi(xq, Q) = n_\alpha^\pi(x, Q) |\alpha_\pi|(q) \quad \text{for } \alpha \in \pi.$$

Now we define the map

$$\begin{aligned} r_Q: X &\rightarrow (\mathbf{R}^+)^* \times \dots \times (\mathbf{R}^+)^*, \\ r_Q: x &\mapsto (\dots, n_\alpha^\pi(x, Q) \dots). \end{aligned}$$

The number of factors is equal to $|\pi|$ = the number of elements in π . The map r_Q factors through the action of $\Gamma \cap Q_\infty$. This factorisation will be denoted by \bar{r}_Q .

Let us denote

$$A = A_\pi = (\mathbf{R}^+)^* \times \dots \times (\mathbf{R}^+)^*, \quad |\pi| \text{ factors,}$$

then we get the following important diagram :

$$(1.2.10) \quad \begin{array}{ccc} r_Q \times \psi_Q : & X & \longrightarrow A \times X_M \\ & \downarrow & \downarrow \\ \bar{r}_Q \times \bar{\psi}_Q : & X/\Gamma \cap Q_\infty & \longrightarrow A \times X_M/\Gamma_M \end{array}$$

The fibers of the map $r_Q \times \psi_Q$ are the orbits of the group

$$R(Q)_\infty(1) = \{ p \in R(Q)_\infty \mid |\alpha|(p) = 1 \text{ for all } \alpha \in \pi \}.$$

The fibers of $\bar{r}_Q \times \bar{\psi}_Q$ are equal to these orbits divided by the action of $R(Q)_\infty \cap \Gamma$, and they are compact.

Let me say a few words about the functorial properties of the diagram 1.2.10. Suppose Q_1 is another parabolic subgroup of type $\Pi - \pi$ which is conjugate to Q under Γ . If $Q_1 = \gamma Q \gamma^{-1}$ then this transformation yields an identification

$$\rho : A \times X_M/\Gamma_M \xrightarrow{\sim} A \times X_{M_1}/\Gamma_{M_1}$$

which does not depend on the choice of γ . The isomorphisms

$$\begin{aligned} \lambda^Q : Y_\pi^Q/\Gamma \cap Q_\infty &\xrightarrow{\sim} V_\pi^{[Q]} \\ \lambda^{Q_1} : Y_\pi^{Q_1}/\Gamma \cap Q_{1\infty} &\xrightarrow{\sim} V_\pi^{[Q_1]} \end{aligned}$$

yield the following commutative diagram :

$$(1.2.11) \quad \begin{array}{ccc} & Y_\pi^Q/\Gamma \cap Q_\infty & \xrightarrow{\bar{r}_Q \times \bar{\psi}_Q} A \times X_M/\Gamma_M \\ (\lambda^Q)^{-1} \nearrow \sim & & \downarrow \rho \\ V_\pi^{[Q]} & & \\ (\lambda^{Q_1})^{-1} \searrow \sim & Y_\pi^{Q_1}/\Gamma \cap Q_{1\infty} & \xrightarrow{\bar{r}_{Q_1} \times \bar{\psi}_{Q_1}} A \times X_{M_1}/\Gamma_{M_1} \end{array}$$

I am going to study the infinitesimal properties of the map

$$r_Q \times \psi_Q : X \rightarrow A \times X_M.$$

Let us denote the tangent bundle of X by T , the tangent bundle of A (resp. X_M) by T_A (resp. T_M), and the bundles induced on X by T_A (resp. T_M) by T_A^* (resp. T_M^*). The Riemannian metric gives an orthogonal decomposition

$$T = T_F \oplus T_M^* \oplus T_A^*.$$

The bundle T_F is the bundle of tangent vectors along the fibers.

The group Q_∞ acts on the spaces X and $A \times X_M$. It acts on A by translations. The map $r_Q \times \psi_Q$ is compatible with this action. The tangent space T_x at the point $x \in X$ can be identified with a quotient of the Lie algebra \mathfrak{Q}_∞ of Q_∞ , we have

$$\mathfrak{Q}_\infty / \mathfrak{k}_x \cap \mathfrak{Q}_\infty \rightarrow T_x.$$

We consider a refinement of the decomposition in proposition 1.1.1

$$\mathfrak{Q}_\infty = \mathfrak{f}_{x,\infty} \oplus \left(\bigoplus_{\alpha \in \Delta_Q^+} \mathfrak{u}_{x,\infty}^{(\alpha)} \right) = \mathfrak{f}'_{x,\infty} \oplus \mathfrak{h}_{x,\infty} \oplus \left(\bigoplus_{\alpha \in \Delta_Q^+} \mathfrak{u}_{x,\infty}^{(\alpha)} \right),$$

where $\mathfrak{h}_{x,\infty}$ is the Lie algebra of the centre $H_{x,\infty}$ of $L_{x,\infty} = Q_\infty \cap Q_\infty^{\theta_x}$. This decomposition is orthogonal with respect to B_x .

Now we consider the torus $H = R(Q)/R_u(Q)$. Let us denote its Lie algebra by \mathfrak{h} , the Lie algebra of H_∞ is \mathfrak{h}_∞ . The injection of $H_{x,\infty}$ to $R(Q)_\infty$ yields an isomorphism

$$\mu_x : \mathfrak{h}_{x,\infty} \xrightarrow{\sim} \mathfrak{h}_\infty.$$

To any character $\chi : H \rightarrow G_m$ we associated a homomorphism

$$|\chi| : H_\infty \rightarrow (\mathbf{R}^+)^*$$

and $H_\infty(1)$ is the intersection of the kernels of these homomorphisms. The Lie algebra of $H_\infty(1)$ is denoted by \mathfrak{h}_∞^0 and its orthogonal complement with respect to the Killing form is denoted by \mathfrak{a} . Then we have the following decompositions

$$\begin{aligned} \mathfrak{h}_\infty &= \mathfrak{h}_\infty^0 \oplus \mathfrak{a}, \\ \mathfrak{h}_{x,\infty} &= \mathfrak{h}_{x,\infty}^0 \oplus \mathfrak{a}_x, \end{aligned}$$

where the second decomposition is induced by μ_x . From the mapping from \mathfrak{Q}_∞ to T_x we get following isometries

$$\begin{aligned} \mathfrak{f}_{x,\infty} / \mathfrak{f}'_{x,\infty} \cap \mathfrak{k}_x &\xrightarrow{\sim} T_{M,x}^*, \\ \mathfrak{a}_x &\xrightarrow{\sim} T_{A,x}^*, \\ \mathfrak{h}_{x,\infty}^0 / \mathfrak{h}_{x,\infty} \cap \mathfrak{k}_x \oplus \left(\bigoplus_{\alpha \in \Delta_Q^+} \mathfrak{u}_{x,\infty}^{(\alpha)} \right) &\xrightarrow{\sim} T_{E,x}. \end{aligned}$$

By means of the isomorphisms μ_x we get a natural trivialisation of T_A^*

$$T_A^* = X \times \mathfrak{a}$$

and this trivialisation is compatible with the action of Q_∞ . We may associate to any vector $Z \in \mathfrak{a}$ a Q_∞ -invariant vector field $\tilde{Z} \in \Gamma(X, T_A^*)$.

Then we get from (1.2.9) the formula

$$(1.2.12) \quad \tilde{Z} n_{\alpha}^{\pi}|_x = \alpha'(Z) n_{\alpha}^{\pi}(x, Q),$$

where $\alpha' : \mathfrak{a} \rightarrow \mathbf{R}$ is the differential of $|\alpha|$ for $\alpha \in \pi$.

Let Q_1 be a parabolic subgroup containing Q and let $\Pi - \pi_1$ be its type, we know $\pi_1 \subset \pi$. We have a natural inclusion $H_1 = R(Q_1)/R_u(Q_1) \rightarrow H$ and this yields an imbedding $\mathfrak{a}_1 \hookrightarrow \mathfrak{a}$. Using our result above we get

$$\begin{array}{ccc} T_{\Lambda_1}^* & \xrightarrow{\sim} & X \times \mathfrak{a}_1 \\ \downarrow & & \downarrow \\ T_{\Lambda}^* & \xrightarrow{\sim} & X \times \mathfrak{a} \end{array}$$

Remark. — In our situation we have an obvious mapping $X_{M_1} \rightarrow X_M$ and we get a commutative diagram

$$\begin{array}{ccc} \bar{\Psi}_Q : X/\Gamma \cap Q_{\infty} & \rightarrow & X_M/\Gamma_M \\ \downarrow & & \downarrow \\ \bar{\Psi}_{Q_1} : X/\Gamma \cap Q_{1,\infty} & \rightarrow & X_{M_1}/\Gamma_{M_1} \end{array}$$

but there is no commutative diagram including also the map r_Q . This is due to the fact that r_Q is defined by “neglecting” the roots in $\Pi - \pi$.

1.3. A DISSECTION OF X/Γ AND THE CONSTRUCTION OF h . — At this point I want to explain briefly the idea of the proof of the Gauss-Bonnet formula. I will construct a function

$$h : X/\Gamma = V \rightarrow (\mathbf{R}^+)^*$$

which can be written

$$h(v) = \prod_{\alpha \in \Pi} \bar{p}_{\alpha}(v)^{\sigma_{\alpha}(v)},$$

where σ_{α} is a bounded positive C^{∞} -function having support in V_{α} . The set $h^{-1}([\delta, \infty])$ will be compact for any $\delta > 0$ and h will not have small critical values. We consider the set

$$V(\delta) = \{v \mid h(v) \geq \delta\}.$$

If ω_V is the Euler-Poincaré form on V (we assume from now on that Γ has no torsion), then (comp. [4])

$$\int_{V(\delta)} \omega_V = \chi(V(\delta)) + \int_{\partial V(\delta)} \Pi_{\delta},$$

where Π_δ is a form of highest degree on $\partial V(\delta)$. The only thing which remains to be shown is

$$(\star\star) \quad \lim_{\delta \rightarrow 0} \int_{\partial V(\delta)} \Pi_\delta = 0.$$

This will be done in 2.1.

Here we are going to construct the functions σ_α . This will be done by covering V by subsets $V_\pi^{[Q]}(c'_Q, \Omega'_Q)$ of $V_\pi^{[Q]}$, constructing these σ_α separately on these subsets, and adding up. The decomposition of V by the sets $V_\pi^{[Q]}(c'_Q, \Omega'_Q)$ may be of independent interest (Th. 1.3.2). We look at the following diagram :

$$(1.3.0) \quad \begin{array}{ccccccc} Y_\pi^Q & \xleftarrow{i} & X & \xrightarrow{\bar{r}_Q \times \bar{\psi}_Q} & A \times X_M & \xrightarrow{p_1} & X_M \\ \downarrow p_Y & & \downarrow p_X & & \downarrow p_M & & \downarrow \bar{p}_M \\ Y_\pi^Q/\Gamma \cap Q_\infty & \hookrightarrow & X/\Gamma \cap Q_\infty & \xrightarrow{\bar{r}_Q \times \bar{\psi}_Q} & A \times X_M/\Gamma_M & \xrightarrow{\bar{p}_1} & X_M/\Gamma_M \\ \uparrow (i, \vartheta)^{-1} & & & & & & \\ V_\pi^{[Q]} & & & & & & \end{array}$$

Let $\Omega \subset X_M/\Gamma_M$ be a relatively compact set. Then $X(\Omega)$, $Y_\pi^Q(\Omega)$ and $V_\pi^{[Q]}(\Omega)$ are the inverse images of Ω in X , and $V_\pi^{[Q]}$. Let $c > 0$ be a positive real number, and $\Omega \subset X_M/\Gamma_M$ be relatively compact. Then

$$A(c) = \{ (x_1, \dots, x_s) \in A \mid 0 < x_i < c \}$$

and

$$X(c, \Omega) = \{ x \in X \mid (\bar{r}_Q \times \bar{\psi}_Q)(x) \in A(c) \times \Omega \}.$$

LEMMA 1.3.1. — *Let $\Omega \subset X_M/\Gamma_M$ be relatively compact. Then there exists a constant $c > 0$, such that*

$$X(c, \Omega) \subset Y_\pi^Q.$$

Proof. — It follows from the relative compactness of Ω that there are constants $0 < c_\Omega < c'_\Omega$, such that for all $x \in X(\Omega)$ there is a minimal parabolic subgroup $\bar{P} \subset M$ for which

$$c_\Omega < n_\alpha(\psi_Q(x), \bar{P}) < c'_\Omega \quad \text{for all } \alpha \in \Pi - \pi.$$

Let $P \subset Q$ be the preimage of \bar{P} then we get from (1.2.5)

$$c_\Omega < n_\alpha(x, P) < c'_\Omega \quad \text{for all } \alpha \in \Pi - \pi,$$

where c_Ω, c'_Ω are constants depending only on Ω , and where x varies in $X(\Omega)$. We get from (1.2.6) for $\alpha \in \pi$

$$(1.3.1.1) \quad n_\alpha(x, P) = \left(\prod_{\beta \in \pi} p_\beta(x, P)^{c_{\alpha, \beta}^\pi} \right) \left(\prod_{\beta \in \pi} n_\beta(x, P)^{d_{\alpha, \beta}} \right).$$

Now we have by definition $p_\beta(x, P) = p_\beta(x, Q)$ for $\beta \in \pi$ and we see that the first factor on the right hand side is equal to $n_x^\pi(x, Q)$ [comp. (1.2.8)] The second factor on the right hand side is bounded away from zero and infinity by constants depending only on Ω . So there is a constant $C > 1$ which depends only on Ω , such that

$$(1.3.1.2) \quad C^{-1} n_x^\pi(x, Q) < n_\alpha(x, P) < C n_x^\pi(x, Q) \quad \text{for } \alpha \in \pi.$$

If $n_x^\pi(x, Q)$ is very small then the number $n_\alpha(x, P)$ is very small too. It follows from theorem 1.2.2 and 1.2.3 that any minimal parabolic subgroup P' which is reduced with respect to x must be contained in Q if $n_x^\pi(x, Q)$ is sufficiently small. This is clear because for $\alpha \in \Pi - \pi$ the numbers $n_\alpha(x, P)$ are bounded from above, and for $\alpha \in \pi$ the numbers $n_\alpha(x, P)$ are very small. Now if $P' \subset Q$ is reduced with respect to x it follows from the compactness criterion (prop. 1.2.4) applied to X_M/Γ_M and (1.2.5) that

$$C_\Omega^{-1} < n_\alpha(x, P') < C_\Omega \quad \text{for all } \alpha \in \Pi - \pi,$$

where $C_\Omega > 1$ depends only on Ω . Reversing our previous argument we see that for $\alpha \in \pi$ the numbers $n_\alpha(x, P')$ will be very small if $n_x^\pi(x, Q)$ is very small. But if for an x -reduced subgroup $P' \subset Q$ the numbers $n_\alpha(x, P')$ are small for $\alpha \in \pi$, then we have by definition $x \in Y_\pi^Q$. This proves the lemma.

Remark. — The difficulty in the proof of the lemma arises from the fact that we do not know *a priori* that Q contains a minimal parabolic subgroup which is reduced with respect to x .

If we have chosen Ω and $c > 0$, such that the lemma 1.3.1 holds we put

$$V_\pi^{(Q)}(c, \Omega) = \lambda^Q(X(c, \Omega))/\Gamma \cap Q^\infty.$$

It is easily seen that the map

$$X(c, \Omega) \rightarrow A(c) \times \tilde{\Omega},$$

where $\tilde{\Omega} = \bar{p}_M^{-1}(\Omega)$ is surjective. The fibers of this map have been described already in 1.2. The fibers of the map

$$V_\pi^{(Q)}(c, \Omega) \rightarrow A(c) \times \Omega$$

can be identified with $R(Q)_\infty(1)/\Gamma \cap R(Q)_\infty$.

During the proof of lemma 1.3.1 we have seen that the sets $V_\pi^{(Q)}(c, \Omega)$ have the following property: If $v \in V_\pi^{(Q)}(c, \Omega)$, and if $x \in X$ is in the preimage

of v , then for any minimal parabolic subgroup P which is reduced with respect to x the numbers $n_\alpha(x, P)$ for $\alpha \in \Pi - \pi$ are bounded away from zero by a constant which only depends on Ω .

A subset $\Omega \subset X_M/\Gamma_M$ is called a subset of *special type* if it can be described in the following way : There is a real number $t_0 > 0$, such that a point $\bar{v} \in X_M/\Gamma_M$ is in Ω if and only if there is a point $y \in Y_\pi^0$ which is in the preimage of \bar{v} under the map

$$\bar{p}_M \circ p_1 \circ (r_Q \times \psi_Q) \circ i : Y_\pi^0 \rightarrow X_M/\Gamma_M,$$

and a minimal parabolic subgroup P which is reduced with respect to y , such that

$$n_\alpha(x, P) > t_0 \quad \text{for all } \alpha \in \Pi - \pi.$$

Then we denote $\Omega = \Omega_{t_0}$. It follows from theorem 1.2.3 that $P \subset Q$.

Remark. — I claim that these sets of special type are relatively compact. We know from (1.2.5) that the numbers $n_\alpha(\psi_Q(y), \bar{P})$ are bounded away from zero and infinity for $\alpha \in \Pi - \pi$ so we apply the proposition 1.2.4 for $\Omega_{t_0} \subset X_M/\Gamma_M$. Moreover we have

$$\bigcup_{t_0 > 0} \Omega_{t_0} = X_M/\Gamma_M.$$

This is an immediate consequence of lemma 1.3.1

We have introduced the sets Ω_{t_0} for technical reasons which will become clear in the proof of theorem 1.3.2 (iv).

If $\pi \subset \Pi$ we denote by Σ_π the set of Γ -conjugacy classes of parabolic subgroups of G/F which are of type $\Pi - \pi$. We put $\Sigma = \bigcup_{\pi \subset \Pi} \Sigma_\pi$. The

elements of these sets will be denoted as before by $[Q], [Q_1], \dots$. We assume that we have chosen a group $Q \in [Q]$ for all elements of Σ . If $\pi_1 \subset \pi$, and $[Q] \in \Sigma_\pi, [Q_1] \in \Sigma_{\pi_1}$ we say that $[Q_1]$ *dominates* $[Q]$ if there is an element $\gamma \in \Gamma$, such that $\gamma Q \gamma^{-1} \subset Q_1$.

In the formulation of the next theorem we consider G/F itself as a parabolic subgroup which is of course of type Π . I hope that the reader does not take offence at the fact that this group is “more maximal” than the maximal one’s which are of type $\Pi - \{\alpha\}$.

THEOREM 1.3.2. — *We can choose for all $[Q] \in \Sigma$ relatively compact open sets $\Omega_Q \subset \mathcal{C} \subset \Omega'_Q \subset X_M/\Gamma_M$, and constants $0 < c_Q < c'_Q$ such that the follo-*

wing statements are true :

(i) Lemma 1.3.1 holds for all pairs (c'_0, Ω'_0) ;

(ii) We have

$$X/\Gamma = V = \bigcup_{\pi \subset \Pi} \left(\bigcup_{[Q] \in \Sigma_\pi} V_\pi^{[Q]}(c_0, \Omega_0) \right);$$

(iii) If $|\pi| \geq |\pi_1|$, and if

$$V_\pi^{[Q]}(c'_0, \Omega'_0) \cap V_{\pi_1}^{[Q_1]}(c'_0, \Omega'_0) \neq \emptyset,$$

then $\pi_1 \subset \pi$ and $[Q_1]$ dominates $[Q]$;

(iv) If $\pi_1 \subset \pi$, and if

$$v \in V_\pi^{[Q]}(c'_0, \Omega'_0) \cap V_{\pi_1}^{[Q_1]}(c'_0, \Omega'_0),$$

then the fiber of the map

$$V_\pi^{[Q]}(c'_0, \Omega'_0) \rightarrow A \times \Omega'_0,$$

which passes through v is contained in $V_{\pi_1}^{[Q_1]}(c'_0, \Omega'_0)$.

Proof. — We will choose the constants c_0, c'_0 and the sets $\Omega_0 \subset \subset \Omega'_0$ by decreasing induction on $|\pi|$ where $[Q] \in \Sigma_\pi$.

If $\pi = \Pi$ we put for all $[Q] \in \Sigma_\Pi$

$$\Omega_0 = \Omega'_0 = X_M/\Gamma_M.$$

Then we choose $0 < c_0 < c'_0$ such that the condition (i) is fulfilled. It follows from theorem 1.2.3 and the following considerations that for $[Q] \neq [Q_1]$ we have

$$V_\Pi^{[Q]}(c'_0, \Omega'_0) \cap V_\Pi^{[Q_1]}(c'_0, \Omega'_0) = \emptyset,$$

and this tells us that the conditions (iii) and (iv) are satisfied.

Let us assume that we have chosen for all $[Q] \in \Sigma_\pi$ with $|\pi| > s$ the constants c_0, c'_0 and the sets $\Omega_0 \subset \subset \Omega'_0$ such that the conditions (i), (iii) and (iv) are fulfilled. Moreover we require the following condition to be satisfied :

(ii)_s : There exists a constant $t_s > 0$ such that for any point

$$v \notin \bigcup_{\pi, |\pi| > s} \left(\bigcup_{[Q] \in \Sigma_\pi} V_\pi^{[Q]}(c_0, \Omega_0) \right)$$

there exists a point $x \in X$ in the preimage of v , a minimal parabolic subgroup P which is reduced with respect to x , and a subset $\pi_1 \subset \Pi$ with $|\pi_1| = s$, such that

$$n_\alpha(x, P) > t_s \text{ for all } \alpha \in \Pi - \pi_1.$$

This rather technical condition guarantees that eventually the condition (ii) will be fulfilled too because we have $(ii)_{-1} \Leftrightarrow (ii)$. It is clear that $(ii)_{|\pi| - 1}$ is satisfied by our initial choice.

Now we choose the sets Ω_0 for all $[Q] \in \Sigma_\pi$ with $|\pi| = s$: for Ω_0 we take the sets of special type

$$\Omega_0 = \Omega_{t_s} = \Omega_{0, t_s}.$$

Then we choose for Ω'_0 relatively compact open sets of special type such that $\Omega_0 \subset \subset \Omega'_0$. Now we have to choose our constants. Before doing this I will show that the condition $(ii)_{s-1}$ is fulfilled automatically if the constants $c_0 > 0$ are chosen sufficiently small. To see this let us choose the constants $c_0 > 0$ provisorily. We consider a point

$$v \notin \bigcup_{\pi, |\pi| > s} \left(\bigcup_{[Q] \in \Sigma_\pi} V_\pi^{[Q]}(c_0, \Omega_0) \right).$$

The condition $(ii)_s$ is satisfied, we can find a point x in the preimage of v , a minimal parabolic subgroup which is reduced with respect to x , and a subset $\pi_1 \subset \Pi$ consisting of s elements such that

$$n_\alpha(x, P) > t_s \text{ for all } \alpha \in \Pi - \pi_1.$$

Let $Q_1 \supset P$ be the parabolic subgroup of type $\Pi - \pi$ containing P , we may assume that Q_1 is the representative in its Γ -conjugacy class which we have chosen before. By assumption we have

$$v \notin V_{\pi_1}^{[Q_1]}(c_0, \Omega_0).$$

I claim that there is a root $\beta \in \pi_1$ such that $n_{\beta^{-1}}(x, Q_1) \geq c_0$. To see this we compare the numbers $n_{\beta^{-1}}(x, Q_1)$ and $n_\beta(x, P)$ by using the formula (1.3.1.1) in the proof of lemma 1.1.1. The numbers $n_\alpha(x, P)$ for $\alpha \in \Pi - \pi$ are bounded between two non zero constants which depend only on our previous choices (actually these constants are $t_s < C_1$). Then it follows from formula (1.3.1.1) that there is a constant $C_{Q_1} > 1$ which depends only on Ω_0 , such that

$$(1.3.1.2) \quad C_{Q_1}^{-1} n_\beta(x, P) < n_{\beta^{-1}}(x, Q_1) < C_{Q_1} n_\beta(x, P) \quad \text{for } \beta \in \pi_1.$$

Now it follows from $n_{\beta^1}(x, Q_1) < c_{Q_1}$ that $n_{\beta}(x, P) < C_2$ if c_{Q_1} is chosen sufficiently small. Here C_2 is the constant in theorem 1.2.3 and therefore way can conclude that $x \in Y_{\pi_1}^{Q_1}$. If we denote by abuse of language ⁽²⁾ $\bar{\Psi}_{Q_1} = \bar{p}_{M_1} \circ p_1 \circ (r_{Q_1} \times \psi_{Q_1}) \circ i$ in the diagram (1.3.0) then this last assertion implies immediately that $\bar{\Psi}_{Q_1}(x) \in \Omega_{Q_1} = \Omega_{Q_1, t}$. But then it follows from $n_{\beta^1}(x, Q_1) < c_{Q_1}$ for all $\beta \in \pi_1$ that $v \in V_{\pi_1}^{[Q_1]}(c_{Q_1}, \Omega_{Q_1})$ and this contradicts our assumption. To fulfill (ii)_{s-1} we only have to chose $t_{s-1} < C_{Q_1}^{-1} c_{Q_1}$ for all $[Q_1] \in \Sigma_{\pi_1}$ with $|\pi_1| = s$.

Now I claim that the conditions (i), (iii) and (iv) are satisfied if we choose the constants c'_{Q_1} sufficiently small. If this is shown then the proof of our theorem will be finished. This is clear because we have a description of the sets $\Omega_{Q_1} \subset \subset \Omega'_{Q_1}$ for all $[Q_1] \in \Sigma_{\pi_1}$ where $|\pi_1| = s$, then we choose the constants c'_{Q_1} such that (i), (iii) and (iv) are satisfied, and then we choose arbitrary constants $0 < c_{Q_1} < c'_{Q_1}$. The condition (i)_{s-1} is fulfilled automatically as we have seen before.

There is no trouble with condition (i) because of lemma 1.3.1. We consider condition (iii). Let π_1, π_2 be sets of simple roots, we assume $|\pi_1| \geq |\pi_2| = s$. We take a class $[Q_1] \in \Sigma_{\pi_1}$, and a class $[Q_2] \in \Sigma_{\pi_2}$ and we assume

$$v \in V_{\pi_1}^{[Q_1]}(c'_{Q_1}, \Omega'_{Q_1}) \cap V_{\pi_2}^{[Q_2]}(c'_{Q_2}, \Omega'_{Q_2}).$$

We choose a point $x \in Y_{\pi_1}^{Q_1}$ in the preimage of v . If we replace Q_2 by another representative in its Γ -conjugacy class we may also assume that $x \in Y_{\pi_2}^{Q_2}$. Let P be a minimal parabolic subgroup which is reduced with respect to x . It follows from our definitions that $P \subset Q_1 \cap Q_2$. Now let us assume that there is a root $\alpha \in \pi_1$ which is not contained in π_2 . Roughly speaking we will get a contradiction because $\alpha \notin \pi_2$ yields that $n_{\alpha}(x, P)$ is bounded away from zero, and $\alpha \in \pi_1$ yields that $n_{\alpha}(x, P)$ is very close to zero. Now I will give the precise argument. In the first step I show that $\alpha \notin \pi_2$ implies that $n_{\alpha}(x, P)$ is bounded away from zero by a constant not depending on v, x and P . We have by definition $\bar{\Psi}_{Q_2}(x) \in \Omega'_{Q_2}$. Moreover there are constants $c_1, x_2 < 0$ such that

$$c_1 n_{\alpha}(x, P) \leq n_{\alpha}(\bar{\Psi}_{Q_2}(x), \bar{P}) \leq c_2 n_{\alpha}(x, P).$$

This is the relation (1.2.5). The group P is reduced with respect to x and we obtain the inequality $n_{\alpha}(\bar{\Psi}_{Q_2}(x), \bar{P}) \leq c_2 C_1$ where C_1 is the constant in theorem 1.2.2. The compactness criterion (prop. 1.2.4)

⁽²⁾ This map is essentially the same map as the map denoted by $\bar{\Psi}_{Q_1}$ before. These two maps differ only by the projection p_Y and therefore I denoted them by the same letter.

applied to X_{M_2}/Γ_{M_2} shows that the number $n_\alpha(\bar{\Psi}_{Q_2}(x), \bar{P})$ is bounded away from zero by a constant not depending on x, v and P . The inequality above yields that the same is true for $n_\alpha(x, P)$.

In the second step we make use of the fact that $\alpha \in \pi_1$. This shows that $n_{\bar{\alpha}'}(x, Q_1) < c'_{Q_1}$. But we know from the proof of lemma 1.3.1 that [comp. (1.3.1.2)]

$$C_{Q_1}^{-1} n_\alpha(x, P) \leq n_{\bar{\alpha}'}(x, Q_1),$$

where $C_{Q_1} > 0$ depends only on Ω'_{Q_1} . If $c'_{Q_1} > 0$ is sufficiently small we get a contradiction, and therefore we have $\pi_1 \subset \pi_2$.

Now I will prove that (iv) is fulfilled if the numbers c'_0 are chosen sufficiently small. We keep over the notations of our previous considerations. Let v be a point in the intersection

$$V_{\pi_1}^{[Q_1]}(c'_{Q_1}, \Omega'_{Q_1}) \cap V_{\pi_2}^{[Q_2]}(c'_{Q_2}, \Omega'_{Q_2}).$$

We know already that $\pi_1 \subset \pi_2$, and we may assume that $Q_2 \subset Q_1$. Let $y \in Y_{\pi_2}^{Q_2}$ be a point in the preimage of v . Because we have obviously $Y_{\pi_2}^{Q_2} \subset Y_{\pi_1}^{Q_1}$ it follows that $y \in X(c'_{Q_1}, \Omega'_{Q_1}) \subset Y_{\pi_1}^{Q_1}$. We put

$$\bar{\Psi}_{Q_1}(y) = \bar{v} \in \Omega'_{Q_1}.$$

The sets Ω'_{Q_i} are of special type, say $\Omega'_{Q_i} = \Omega'_{Q'_i}$. By definition there is a point $y_1 \in Y_{\pi_1}^{Q_1}$ in the preimage of \bar{v} , and a minimal parabolic subgroup P which is reduced with respect to y_1 such that

$$n_\alpha(y_1, P) > t_s \quad \text{for all } \alpha \in \Pi - \pi_1.$$

From the fact that y and y_1 have the same image in X_{M_1}/Γ_{M_1} we get $yq = y_1\gamma$ where $q \in R(Q_1)_\infty$ and $\gamma \in Q_{1,\infty} \cap \Gamma$. We may replace P by $\gamma^{-1}P\gamma$, and y_1 by $y_1\gamma$, therefore we may assume that $\gamma = e$. Now I claim that P is also reduced with respect to y . We get from proposition 1.2.1,

$$n_\alpha(y, P) = n_\alpha(yq, P) = n_\alpha(y_1, P) \quad \text{for all } \alpha \in \Pi - \pi_1.$$

Now it follows from $n_{\bar{\alpha}'}(x, Q_1) \leq c'_{Q_1}$ for $\alpha \in \pi_1$ that $n_\alpha(y, P) \leq C_2$ for $\alpha \in \pi_1$ if c'_{Q_1} is chosen sufficiently small. (This follows from a standard argument which has been used already several times.) So we have seen that P is reduced with respect to y . Then we get from our theorem 1.2.3 that $P \subset Q_2$. We want to show that the fiber of the map which passes through v

$$V_{\pi_2}^{[Q_2]}(c'_{Q_2}, \Omega'_{Q_2}) \rightarrow A(c'_{Q_2}) \times \Omega'_{Q_2},$$

is contained in $V_{\pi_1}^{(Q_2)}(c'_{Q_1}, \Omega'_{Q_1})$. But this fiber is the image of the set $\{yq \mid q \in R(Q_2)_\infty(1)\}$ under the projection map. Now $P \subset Q_2$ implies that $R(Q_2) \subset P$. Now we apply proposition 1.2.1 again and we get

$$n_\alpha(yq, P) = n_\alpha(y, P) \quad \text{for all } \alpha \in \Pi - \pi_1.$$

This tells us that $\bar{\Psi}_{Q_1}(yq) \in \Omega'_{Q_1} = \Omega'_{Q_1}$. On the other hand we have $r_{Q_1}(yq) = r_{Q_1}(y)$, and both facts together yield

$$(\bar{r}_{Q_1} \times \bar{\Psi}_{Q_1})(yq) \in A(c_{Q_1}) \times \Omega'_{Q_1} \quad \text{for all } q \in R(Q_2)_\infty(1)$$

and this proves that (iv) is fulfilled if the c'_0 are chosen sufficiently small. This finishes the proof of the theorem.

Now we are able to construct the function

$$h: V \rightarrow \mathbf{R}^+ - \{0\},$$

$$h(v) = \prod_{\alpha \in \Pi} \bar{p}_\alpha(v)^{\sigma_\alpha(v)},$$

where σ_α will be a suitably chosen bounded positive C^∞ -function having its support in V_α . If we have constructed these functions σ_α , then it is clear that the expression for h is well defined, because σ_α vanishes outside of V_α . The construction of σ_α will be done by constructing functions

$$\sigma_\alpha^0: V_{\pi}^{(Q)}(c'_Q, \Omega'_Q) \rightarrow [0, 1]$$

and then we put

$$\sigma_\alpha = \sum_{(Q) \in \Sigma} \sigma_\alpha^0.$$

Roughly speaking σ_α^0 will have the following shape: If Q is of type $\Pi - \pi$ and $\alpha \in \pi$, then σ_α^0 will be equal to one on $V_{\pi}^{(Q)}(c'_Q, \Omega'_Q)$ and its support will be a compact subset of $V_{\pi}^{(Q)}(c'_Q, \Omega'_Q)$. If $\alpha \notin \pi$ the σ_α^0 will be identically zero. To be more precise we choose a C^∞ -function

$$\Phi: \Omega'_Q \rightarrow [0, 1]$$

which is equal to one on Ω_Q and has compact support in Ω'_Q . Then we choose a function

$$\Psi: (0, c'_Q) \rightarrow [0, 1]$$

which is C^∞ , equal to one on $(0, c_Q)$, and has support in $(0, c'_Q - \varepsilon)$. Then we put for $a = (\cdot, a_\alpha, \cdot)_{\alpha \in \pi} \in A$

$$\bar{\Psi}(a) = \prod_{\alpha \in \pi} \Psi(a_\alpha).$$

We define for $v \in V_{\pi}^{(Q)}(c_Q, \Omega_Q)$

$$\sigma_{\alpha}^0(v) = \begin{cases} 0 & \text{if } \alpha \notin \pi, \\ \bar{\Psi}(\bar{r}_Q((\lambda^Q)^{-1}(v))) \Phi(\bar{\psi}_Q((\lambda^Q)^{-1}(v))) & \text{if } \alpha \in \pi. \end{cases}$$

This function will be extended to a C^{∞} -function on V by zero. The extended function will also be denoted by σ_{α}^0 . We put

$$\sigma_{\alpha} = \sum_{(Q) \in \Sigma} \sigma_{\alpha}^0.$$

Now we may define h by the expression above

THEOREM 1.3.3. — *The function h is C^{∞} on V . If $\delta > 0$, then the set $h^{-1}([\delta, \infty])$ is compact. The function h has no small critical values, i. e. there is a constant $\delta_0 > 0$ such that $h(v) < \delta_0$ implies $dh|_v \neq 0$.*

Proof. — The first assertion is clear. For the second assertion we restrict h to one of the sets $V_{\pi}^{(Q)}(c_Q, \Omega_Q)$. On this set we consider the functions

$$\begin{aligned} n_{\alpha}^{\pi} &: V_{\pi}^{(Q)}(c_Q, \Omega_Q) \rightarrow (0, c_Q) & \text{for } \alpha \in \pi, \\ n_{\alpha}^{\pi} &: v \rightarrow n_{\alpha}^{\pi}(y, Q), \end{aligned}$$

where y is a point in the preimage of v . Inverting the relation (1.2.7) we may express the characters γ_{α} in terms of the characters β_{π}

$$\gamma_{\alpha} = \sum_{\beta \in \pi} b_{\alpha, \beta}^{\pi} \beta_{\pi}.$$

It is well known that in this expression $b_{\alpha, \beta}^{\pi} \geq 0$, and $b_{\alpha, \alpha}^{\pi} > 0$. From this formula we get an expression for the functions \bar{p}_{α}

$$\bar{p}_{\alpha}(v) = \prod_{\beta \in \pi} n_{\beta}^{\pi}(v)^{b_{\alpha, \beta}^{\pi}} \quad \text{for } \alpha \in \pi.$$

As we have seen in the proof of lemma 1.3.1 there is a constant $C > 1$ depending only on Ω_Q such that for all minimal parabolic subgroups P which are reduced with respect to a point y in the preimage of v we have

$$\begin{aligned} C^{-1} n_{\alpha}^{\pi}(v) &\leq n_{\alpha}(y, P) \leq C n_{\alpha}^{\pi}(v) & \text{for } \alpha \in \pi, \\ C^{-1} &\leq n_{\alpha}(y, P) \leq C & \text{for } \alpha \notin \pi. \end{aligned}$$

For the proof of the second statement of our theorem we have to show that h takes arbitrarily small values on a subset $D \subset V_{\pi}^{(Q)}(c_Q, \Omega_Q)$ which is not relatively compact. It follows from proposition 1.2.4 (the com-

pactness criterion), and the inequalities above that there is a root $\alpha_0 \in \pi$ such that $n_{\alpha_0}^\pi$ takes arbitrarily small values on D . It follows from our definitions that $\sigma_{\alpha_0}(v) \geq 1$ for $v \in D$. Now we look at the expression for $\bar{p}_\alpha(v)$ in terms of the n_β^π . The values $n_\beta^\pi(v)$ are obviously bounded from above, so we see that the values $\bar{p}_\alpha(v)$ are bounded from above on D . But from $b_{\alpha_0, \alpha_0}^\pi > 0$ we get that $p_{\alpha_0}(v)^{\sigma_{\alpha_0}(v)}$ takes arbitrarily small values on D . This altogether shows that h takes arbitrary small values on D .

For the proof of the last assertion we replace the function h by the function $f(v) = \log(h(v))$. We shall see that the length of the derivative $df|_v$ is bounded away from zero by a strictly positive constant if $h(v)$ is sufficiently small. The length is of course taken with respect to the given riemannian metric on X .

Let me make a small remark before the proof starts. We consider two Γ -conjugacy classes $[Q_1]$ and $[Q_2]$, and we suppose

$$V_{\pi_1}^{[Q_1]}(c'_{Q_1}, \Omega'_{Q_1}) \cap V_{\pi_2}^{[Q_2]}(c'_{Q_2}, \Omega'_{Q_2}) \neq \emptyset.$$

By theorem 1.3.2 we may assume that $\pi_1 \subset \pi_2$. Let us consider a root $\alpha \in \pi_2 - \pi_1$. The function $n_\alpha^{\pi_2}$ is defined on the intersection, and it follows easily from the arguments we have used in the proof of theorem 1.3.2 that the values of this function on the given intersection are bounded away from zero by a strictly positive constant which depends only on the choice of the sets Ω'_i . If $\beta \in \pi_1$, then we may consider the functions $n_\beta^{\pi_1}$, and $n_\beta^{\pi_2}$ on this intersection. It follows from the same kind of arguments that the quotient of these two functions is bounded away from zero and infinity on this intersection.

Now let us consider a point $v \in V$ where the value $h(v)$ is very small. We consider all Γ -conjugacy classes $[Q_1], \dots, [Q_t]$ for which

$$v \in V_{\pi_\nu}^{[Q_\nu]}(c'_{Q_\nu}, \Omega_{Q_\nu}) \quad (\nu = 1, \dots, t).$$

As we have seen before there is a class, say $[Q_1]$, and a root $\alpha_0 \in \pi_1$ such that $n_{\alpha_0}^{\pi_1}(v)$ is very small. Our preceding remark shows that we have $\alpha_0 \in \pi_\nu$ for all $1 \leq \nu \leq t$, and that $n_{\alpha_0}^{\pi_\nu}(v)$ is small for all π_ν . From the definition of the function σ_α we get

$$f = \sum_{\nu=1}^t f_\nu,$$

where

$$f_\nu = \sum_{\alpha \in \Pi} \sigma_\alpha^{\alpha_0}(v) \log(\bar{p}_\alpha(v)).$$

At the end of 1.2 we have defined a certain Lie algebra

$$\mathfrak{a}_v \subset \text{Lie}(\mathbb{R}(Q_v)/\mathbb{R}_u(Q_v))_\infty,$$

and to any element $H \in \mathfrak{a}_v$ we associated a vector field \tilde{H} on X which is invariant under the action of $Q_{v,\infty}$. If we restrict \tilde{H} to $Y_{\pi_v}^{Q_v}$ the projection yields a vectorfield \tilde{H} on $V_{\pi_v}^{[Q_v]}$. Let $H_{\alpha_0} \in \mathfrak{a}_v$ be the element defined by

$$\alpha'(H_{\alpha_0}) = \delta_{\alpha, \alpha_0} \text{ for all } \alpha \in \pi_v.$$

(Here α' is the derivative of $|\alpha|$.) It follows from the diagram (1.2.13) that the vectorfield \tilde{H}_{α_0} restricted to

$$\bigcap_{\mu=1}^l V_{\pi_\mu}^{[Q_\mu]}(c'_{Q_\mu}, \Omega'_{Q_\mu})$$

does not depend on v . The last assertion in our theorem will be proved if we have shown that in our situation

$$\tilde{H}_{\alpha_0}(f)|_v = \sum_{v=1}^l \tilde{H}_{\alpha_0}(f_v)|_v \geq M > 0,$$

where M is a constant not depending on v . Because the length of the vectors in our vector field \tilde{H}_{α_0} is constant this proves also that the length of $df|_v$ is bounded away from zero if $h(v)$ is sufficiently small. Let us recall the definition of the functions f_v

$$f_v(v) = \log(h_v(v)) = \sum_{\alpha \in \Pi} \sigma_\alpha^{Q_v}(v) \log(\bar{p}_\alpha(v)).$$

The functions $\sigma_\alpha^{Q_v}$ have been defined as follows : Let **be**

$$\bar{r}_v \times \bar{\psi}_v = (\bar{r}_{Q_v} \times \bar{\psi}_{Q_v}) \circ (\lambda^{Q_v})^{-1} : V_{\pi_v}^{[Q_v]}(c'_{Q_v}, \Omega'_{Q_v}) \rightarrow A(c'_{Q_v}) \times \Omega'_{Q_v},$$

then for $(a, \bar{v}) = (\bar{r}_v \times \bar{\psi}_v)(v)$ we have

$$\sigma_\alpha^{Q_v}(v) = \bar{\psi}_v(a) \Phi_v(\bar{v}).$$

The function \bar{p}_α is given by $\bar{p}_\alpha = p_\alpha \circ (\bar{r}_v \times \bar{\psi}_v)$ where we have for $a = (\dots, a_\alpha, \dots)_{\alpha \in \pi_v} \in A$,

$$p_\alpha(a, \bar{v}) = \bar{p}_\alpha(a) = \prod_{\beta \in \pi_v} a_\beta^{b_{\alpha, \beta}^{v, 3}}$$

[comp. (1.3.4)], and if we write $f_v = f_v \circ (\bar{r}_v \times \bar{\psi}_v)$ we obtain

$$\bar{f}_v(a, \bar{v}) = \bar{\psi}(a) \Phi(\bar{v}) \left(\sum_{\alpha \in \pi_v} \log(\bar{p}_\alpha(a)) \right) = \bar{\psi}_v(a) \Phi_v(\bar{v}) \left(\sum_{\alpha, \beta \in \pi_v} b_{\alpha, \beta}^v \log(a_\beta) \right).$$

It is clear from the definition of \tilde{H}_{α_0} that

$$\tilde{H}_{\alpha_0}(f_v)|_v = t_{\alpha_0} \frac{\partial}{\partial t_{\alpha_0}} (\bar{f}_v)|_{(a, \bar{v})}$$

i. e. essentially we have to take the partial derivative with respect to the α_0 -th variable in $A_v = \prod_{\alpha \in \pi_v} (\mathbf{R}^+)^*$. We assumed that $a_{\alpha_0} = n_{\alpha_0}^{\pi_v}(v)$ is very small, and this implies that $\bar{\psi}_v$ does not depend on a_{α_0} . Now we get

$$\tilde{H}_{\alpha_0}(f_v)|_v = \bar{\psi}_v(a) \Phi_v(\bar{v}) \left(\sum_{\alpha \in \pi_v} b_{\alpha, \alpha_0}^v \right).$$

We know that the sum in the bracket is strictly positive. It follows from theorem 1.3.2 (ii), that there is an index ν_0 such that $\bar{\psi}_{\nu_0}(a) \Phi_{\nu_0}(\bar{v}) = 1$. This shows that $\tilde{H}_{\alpha_0}(f)|_v$ is bounded away from zero, and the theorem is proved.

2.1. THE PROOF OF THE GAUSS-BONNET FORMULA. — I am going to prove the limit formula

$$(\star \star) \quad \lim_{\delta \rightarrow 0} \int_{\partial V(\delta)} \Pi_\delta = 0.$$

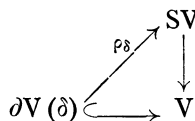
We always assume that $\delta > 0$ is very small so $\partial V(\delta)$ is a hypersurface. The first step in the proof consists in comparing the differential form Π_δ with the volume element ω_δ on the hypersurface $\partial V(\delta)$. The form ω_δ is nowhere zero, and therefore we may consider the ratio between these two forms. I claim

$$(2.1.1) \quad \frac{\Pi_\delta}{\omega_\delta} = O((-\log \delta)^M),$$

where the O -constant and M are independent of δ .

The proof of (2.1.1) consists in some rather technical estimations which we shall carry through now. Let us denote the sphere bundle of unit tangent vectors at V by SV . In the following considerations we identify the tangent and the cotangent bundle on V by means of the riemannian metric. If Y_1 and Y_2 are tangent vectors at the point $v \in V$ we denote by $\langle Y_1, Y_2 \rangle|_v$ their scalar product, and the length of Y_1

is denoted by $\|Y_1\|_v$. If $v \in \partial V(\delta)$ the vector $N_v = \frac{dh|_v}{\|dh\|_v}$ is a unit normal vector to $\partial V(\delta)$. The vector field $N = \frac{dh}{\|dh\|}$ is defined outside of a compact set (theorem 1.3.3). This vector field of normal vectors defines a section



In his paper [4] Chern has constructed a differential form α of degree $\dim V - 1$ on SV such that $\rho_\delta^*(\alpha) = \Pi_\delta$. On SV we have a canonical riemannian metric is defined by means of the canonical symmetric connection on V . The length of the differential form α is bounded with respect to this metric. This is clear because α can be calculated from local data on V , and these local data are determined by the metric. The boundedness then follows from the fact that V is the quotient of a symmetric space. Let us denote by ω_δ^* the volume element of the manifold $\rho_\delta(\partial V(\delta))$. Our previous remark shows

$$(2.1.2) \quad \frac{\alpha|_{\rho_\delta(\partial V(\delta))}}{\omega_\delta^*} = 0 \quad (1).$$

Now we identify the manifolds $\rho_\delta(\partial V(\delta))$ and $\partial V(\delta)$, and we compare the forms ω_δ and ω_δ^* . To do this we look at the differential of the map ρ_δ . For $v \in \partial V(\delta)$ we consider

$$d\rho_\delta: T_v \rightarrow T_{\rho_\delta(v)},$$

where T_v (resp. $T_{\rho_\delta(v)}$) is the tangent space at v [resp. $\rho_\delta(v)$]. It is clear that

$$\frac{\omega_\delta^*}{\omega_\delta} \leq \|d\rho_\delta\|^{\dim V - 1},$$

where $\|d\rho_\delta\|$ is the norm of the linear map $d\rho_\delta$. From well known formulas in differential geometry we get for a tangent vector $Y \in T_v$

$$\|d\rho_\delta(Y)\|_v^2 = \|Y\|_v^2 + \|\nabla_Y(N)\|_v^2$$

I claim that from this expression we get an estimate

$$(2.1.3) \quad \|d\rho_\delta\|_v = O((-\log \delta)^M).$$

This together with (2.1.1) yields (2.1.1). To prove (2.1.3) we consider the function $f = \log(h)$ it is clear that $\frac{dh}{\|dh\|} = \frac{df}{\|df\|}$. If Y is a tangent

vector at a point $v \in \partial V(\delta)$, then we have

$$\nabla_Y \left(\frac{df}{\|df\|} \right) \Big|_v = \frac{(-Y \|df\|)_v df|_v + \|df\|_v \nabla_Y df|_v}{\|df\|_v^2}.$$

We have seen in the proof of theorem 1.3.3 that the denominator in this expression is bounded away from zero. We shall give an estimate for the numerator for all tangent vectors Y which are of length ≤ 1 . As in

the proof of theorem 1.3.3 let us write $f = \sum_{\nu=1}^t f_\nu$ where

$$f_\nu = \sum_{\alpha \in \Pi} \sigma_\alpha^{\mathcal{Q}_\nu} \log(\bar{p}_\alpha).$$

Then we have

$$df_\nu = \sum_{\alpha \in \Pi} \left(\sigma_\alpha^{\mathcal{Q}_\nu} \frac{d\bar{p}_\alpha}{\bar{p}_\alpha} + \log(\bar{p}_\alpha) d\sigma_\alpha^{\mathcal{Q}_\nu} \right).$$

At first we will estimate the length of the forms df_ν , then we will estimate the length of the covariant derivatives $\nabla_Y df_\nu$, and from there we will easily get the desired result. The functions $\sigma_\alpha^{\mathcal{Q}_\nu}$ are bounded, and also the differential forms $d\sigma_\alpha^{\mathcal{Q}_\nu}$ are of bounded length. The vectors in the vector field $d \log(\bar{p}_\alpha)$ are of constant length. To see this we consider the function

$$p_\alpha^\nu: x \mapsto p_\alpha(x, Q_\nu)$$

on X . This function is invariant under the action of $Q_{\nu, \infty} \cap \Gamma$ and it induces on $Y_{\pi_\nu}^{\mathcal{Q}_\nu} / \Gamma \cap Q_{\nu, \infty} = V_{\pi_\nu}^{(\mathcal{Q}_\nu)}$ the function \bar{p}_α . Now it follows from proposition 1.2.1 that $d \log(p_\alpha^\nu)$ is a form on X which is invariant under the action of $Q_{\nu, \infty}$. But this form induces $d \log(\bar{p}_\alpha)$ on $V_{\pi_\nu}^{(\mathcal{Q}_\nu)}$, and therefore the latter vector field is even parallel, this fact will be needed later.

The only non bounded terms in our expression for df_ν are the functions $\log(\bar{p}_\alpha(v))$. To estimate these terms we prove the following.

LEMMA 2.1.5. — *There exist constants $M > 0$ and $t > 0$ such that for all $v \in V_\alpha$*

$$\bar{p}_\alpha(v) \geq M h(v)^t.$$

Proof. — Let us distinguish two cases. In the first case we assume that v is contained in a set $V_\pi^{(\mathcal{Q})}(c_\mathcal{Q}, \Omega_\mathcal{Q})$, and that $\alpha \in \pi$. Then we have by construction $\sigma_\alpha(v) \geq 1$. But then the assertion is clear because of the definition of h , and the fact that the functions $\bar{p}_\beta(v)^{\sigma_\beta(v)}$ are bounded from above on V . In the second case we assume $v \in V_\pi^{(\mathcal{Q})}(c_\mathcal{Q}, \Omega_\mathcal{Q})$, and

$\alpha \notin \pi$. In this case the argument is more complicated. Let x be a point in the preimage of v , and let us choose a minimal parabolic subgroup P which is reduced with respect to x . Now we express the character γ_α in terms of the roots $\alpha \in \Pi - \pi$, and the fundamental dominant weights $\gamma_{\beta'}$, with $\beta' \in \pi$. This linear expression yields as usual a multiplicative relation

$$\bar{p}_\alpha(v) = \left(\prod_{\beta' \in \pi} \bar{p}_{\beta'}(v)^{n_{\beta'}} \right) \left(\prod_{\beta \in \Pi - \pi} n_\beta(x, P)^{m_\beta} \right).$$

We know that the second factor is bounded away from zero and infinity on $V_\pi^{(0)}(c_0, \Omega_0)$. This reduces the second case to the first case. The lemma tells us that for $v \in \partial V(\delta)$

$$\|df_v\|_v = 0 \quad (-\log \delta).$$

Now we consider the covariant derivatives $\nabla_Y df_v$ for tangent vectors of length ≤ 1 . The covariant derivatives $\nabla_Y(d\sigma_x^{Q_v})$ and the derivatives $Y(\sigma_x^{Q_v})$ are bounded. The covariant derivatives $\nabla_Y(d \log(\bar{p}_\alpha))$ do vanish because $d \log(\bar{p}_\alpha)$ is parallel as we have just seen. The derivatives $Y(\log(\bar{p}_\alpha)) = \langle Y, d \log(\bar{p}_\alpha) \rangle$ are bounded too. Again we have that the only non bounded term in our expression for $\nabla_Y df_v$ are the terms $\log(\bar{p}_\alpha(v))$. If we apply lemma 2.1.5 a second time we get

$$\|\nabla_Y df_v\|_v = 0 \quad (-\log \delta).$$

If we add up we get the same estimates for $\|df\|_v$ and $\|\nabla_Y df\|_v$. Now we come back to the estimation of the numerator of our expression for $\nabla_Y \frac{df}{\|df\|}$. The term $Y \|df\|_v$ is estimated as follows : We have

$$Y \|df\| = Y(\langle df, df \rangle)^{1/2} = \frac{\langle \nabla_Y df, df \rangle}{\|df\|}$$

and this together with our previous estimates yields

$$Y \|df\|_v = 0 \quad ((-\log \delta)^2) \quad \text{for } v \in \partial V(\delta).$$

One glance at the numerator in question shows that it can be estimated by $0 \quad ((-\log \delta)^3)$ if $v \in \partial V(\delta)$ and $\|Y\|_v \leq 1$. This yields the desired estimate (2.1.3).

Now we come to the last step in the proof of our limit formula, this is the estimation of the volume of $\partial V(\delta)$. Actually we will prove that there are constant $M_1 \geq 0$ and $\eta > 0$ such that

$$(2.1.6) \quad \int_{\partial V(\delta)} \omega_\delta = 0 \quad (\delta^\eta (-\log \delta)^{M_1}).$$

Once this is proved, then our limit formula follows easily from our previous estimate (2.1.1).

To any point $v \in V$ we may associate a set $\pi(v)$ of simple roots, and a class $[Q_v] \in \Sigma_{\pi(v)}$ such that

- (i) $v \in V_{\pi(v)}^{[Q_v]}(c'_{Q_v}, \Omega'_{Q_v})$;
- (ii) If $v \in V_{\pi}^{[Q]}(c'_Q, \Omega'_Q)$, then $\pi(v) \subset \pi$.

It follows easily from theorem 1.3.2 that this can be done. Now we introduce the sets

$$\tilde{V}_{\pi}^{[Q]}(c'_Q, \Omega'_Q) = \{v \mid v \in V_{\pi}^{[Q]}(c'_Q, \Omega'_Q) \text{ and } \pi(v) = \pi\}.$$

We have an alternative description of these sets

$$\tilde{V}_{\pi}^{[Q]}(c'_Q, \Omega'_Q) = V_{\pi}^{[Q]}(c'_Q, \Omega'_Q) - \left(\bigcup_{\substack{[Q'] \in \Sigma_{\pi'} \\ |\pi'| < |\pi|}} V_{\pi'}^{[Q']}(c'_{Q'}, \Omega'_{Q'}) \right).$$

This shows that $\tilde{V}_{\pi}^{[Q]}(c'_Q, \Omega'_Q)$ is closed in $V_{\pi}^{[Q]}(c'_Q, \Omega'_Q)$. It is clear that V is the disjoint union of the sets $\tilde{V}_{\pi}^{[Q]}(c'_Q, \Omega'_Q)$. Therefore it suffices to prove (2.1.6) on the pieces

$$\partial V(\delta) \cap \tilde{V}_{\pi}^{[Q]}(c'_Q, \Omega'_Q).$$

Let us simplify the notations. We put

$$V_Q = \tilde{V}_{\pi}^{[Q]}(c'_Q, \Omega'_Q) \quad \text{and} \quad V_Q(\delta) = \partial \tilde{V}_{\pi}^{[Q]} \cap (c'_Q, \Omega'_Q).$$

Let

$$F_Q: V_Q \rightarrow A(c'_Q) \times \Omega'_Q$$

be the natural projection from $V_{\pi}^{[Q]}(c'_Q, \Omega'_Q)$ to $A(c'_Q) \times \Omega'_Q$ restricted to V_Q .

LEMMA 2.1.7. — *A fiber of F_Q is either empty or it is equal to the corresponding fiber of the projection*

$$V_{\pi}^{[Q]}(c'_Q, \Omega'_Q) \rightarrow A(c'_Q) \times \Omega'_Q.$$

The function h restricted to V_Q is constant on the fibers of F_Q . If $U = F_Q^{-1}(V_Q)$ and $h = \bar{h} \circ F_Q$, then $\bar{h}: U \rightarrow \mathbf{R}$ is a C^∞ -function on U , i. e. it is the restriction of a C^∞ -function defined in an open neighbourhood of U .

Proof. — The first assertion is obviously a consequence of theorem 1.3.2 (iv). To prove the second assertion let us write as in the proof of theorem 1.3.3,

$$h(v) = \prod_{\nu=1}^l h_\nu(v),$$

where

$$h_\nu(v) = \prod_{\alpha \in \Pi} p_\alpha(v)^{\sigma_\alpha^{\Omega_\nu}(v)}.$$

The function h_ν is by construction constant on the fibers of the projection

$$V_{\pi_\nu}^{(\Omega_\nu)}(c'_{\Omega_\nu}, \Omega'_{\Omega_\nu}) \rightarrow A(c'_{\Omega_\nu}) \times \Omega'_{\Omega_\nu}$$

and since we know $\pi \subset \pi_\nu$ these fibers contain the fibers of F_Q .

The last assertion follows from the simple observation that we can find an open neighbourhood $V_Q^* \supset V_Q$ such that for the map

$$F_Q: V_Q^* \rightarrow A(c'_Q) \times \Omega'_Q$$

the first two assertions of the present lemma remain true. The existence of such a set V_Q^* follows from the fact that the functions

$$\sigma_\alpha^{\Omega'_Q}: V_{\pi'}^{(\Omega'_Q)}(c'_{\Omega'_Q}, \Omega'_{\Omega'_Q}) \rightarrow [0, 1]$$

are compactly supported.

At this point it seems to be reasonable to give a rough idea of what follows. Let us put $U(\delta) = F_Q(V_Q(\delta))$. Then we shall see that

$$\int_{V_Q(\delta)} \omega_\delta = \int_{U(\delta)} g \omega_\delta^*,$$

where ω_δ^* is the volume element on $U(\delta)$ (this will be specified later), and where $g(u)$ is the volume of the fiber $F_Q^{-1}(u)$. The point of the whole story is that these fibers “shrink” very rapidly if δ tends to zero (Lemma 2.1.8). The reason for this “shrinking of the fibers” is that their volume is essentially equal to the number $\bar{p}(v) = p(y, Q)$ and we will see that this tends to zero as fast as δ^η .

The next step consists in the proof of the formula

$$\int_{V_Q(\delta)} \omega_\delta = \int_{U(\delta)} g \omega_\delta^*.$$

For this purpose I have to recall the infinitesimal properties of the map

$$r_Q \times \psi_Q: X \rightarrow A \times X_M$$

which have been studied in 1.2. For a point $x \in X$ we gave an orthogonal decomposition of the tangent space

$$T_{X,x} = T_{F,x} \oplus T_{A,x}^* \oplus T_{M,x}^*$$

and this decomposition is invariant under the action of Q_∞ . If $r_Q \times \psi_Q(x) = (a, \bar{x})$, then we get from the definition of an induced bundle

$$T_{A,x}^* \oplus T_{F,x}^* \xrightarrow{\sim} T_{A,a} \oplus T_{M,\bar{x}}$$

where $T_{A,a}$ (resp. $T_{M,\bar{x}}$) is the tangent space of A (resp. X_M) in the point a (resp. \bar{x}). The restriction of the metric on $T_{X,x}$ to $T_{A,a}^* \oplus T_{M,\bar{x}}^*$ yields a well defined metric on $T_{A,a} \oplus T_{M,\bar{x}}$. This metric is well defined because the metric on the tangent bundle T_X is invariant under the action of Q_∞ . Therefore we have constructed a riemannian metric on $A \times X_M$ which is invariant under the action of Q_∞ , and which is „nicely adapted” to the metric on X .

Let us choose an open neighbourhood V_Q^* of V_Q in $V_\pi^{(Q)}(c'_Q, \Omega'_Q)$ such that the assertions of lemma 2.1.7 are still true for the extended map $F_Q: V_Q^* \rightarrow A(c'_Q) \times \Omega'_Q$. Let us denote $F_Q(V_Q^*) = U^*$ and let $V_Q^*(\delta)$ [resp. $U^*(\delta)$] be the hypersurfaces defined by $h(v) = \delta$ [resp. $\bar{h}(u) = \delta$] in V_Q^* (resp. U^*). Let ω_δ^* be the volume element on $U^*(\delta)$ which is defined by the metric we have constructed above. From the decomposition of the tangent bundle of V_Q^* which is induced by the decomposition of T_X we obtain

$$\omega_\delta = \omega_F \wedge F_Q^*(\omega_\delta^*),$$

where ω_F is the volume element on the fibers. This implies the formula we wanted to prove.

LEMMA 2.1.8. — *There exist constants $C \geq 0$ and $\eta > 0$ such that*

$$\int_{V_Q(\delta)} \omega_\delta \leq C \delta^\eta \int_{U(\delta)} \omega_\delta^*.$$

Proof. — Let us consider a point $u \in U(\delta)$ and a point $v \in V_Q(\delta)$. Let $y \in Y_\pi^0$ be a point in the preimage of v . Then the natural map

$$y R(Q)_\infty(1) / R(Q)_\infty(1) \cap \Gamma \xrightarrow{\sim} F_Q^{-1}(u)$$

is an isometry [comp. diagram (1.2.10) and the considerations following it]. Let $d_y q$ be the measure on $R(Q)_\infty(1)$ which is obtained from the metric $d_y s$ restricted to $R(Q)_\infty(1)$. Then we have

$$\text{vol}(F_Q^{-1}(u)) = k \int_W d_y q,$$

where $W = R(Q)_\infty(1)/R(Q)_\infty(1) \cap \Gamma$, and where k is a constant : it is the volume of the compact torus $K_y \cap R(Q)_\infty$. The integral on right hand side is equal to

$$\text{vol}(H_\infty(1)/\Gamma_H) p(y, Q),$$

where $H = R(Q)/R_u(Q)$, and Γ_H is the image of $\Gamma \cap R(Q)_\infty$ in H_∞ . The first factor is a constant, and therefore the lemma is proved if we have shown $p(y, Q) = 0(\delta^\eta)$ with $\eta > 0$. To see this we express the function $p(y, Q) = \bar{p}(v)$ in terms of the functions $n_\alpha^\pi(v)$ on $V_\pi^{(Q)}(c'_Q, \Omega'_Q)$. We have multiplicative relation

$$\bar{p}(v) = \prod_{\alpha \in \Pi} n_\alpha^\pi(v)^{n_\alpha},$$

where the exponents n_α are strictly positive (the function \bar{p} corresponds to the sum of the positive roots of Q).

Let $P \subset Q$ be a minimal parabolic subgroup which is reduced with respect to y . If we express the numbers $p_\alpha(y, P)$ in terms of the numbers $n_\alpha(x, P)$ we get

$$p_\alpha(y, P) = \prod_{\beta \in \Pi} n_\beta(y, P)^{a_{\alpha, \beta}},$$

where the $a_{\alpha, \beta}$ are positive integers. Then we get

$$h(v) = \prod_{\alpha \in \Pi} \bar{p}_\alpha(v)^{\sigma_\alpha(v)} = \prod_{\alpha \in \Pi} n_\alpha(y, P)^{\lambda_\alpha(v)}.$$

The functions λ_α are positive and bounded. The numbers $n_\alpha(y, P)$ for $\alpha \in \Pi - \pi$ are bounded away from zero and infinity, since $v \in V_\pi^{(Q)}(c'_Q, \Omega'_Q)$. If we take into account (1.3.1.2) we see that $p(y, Q) = 0(\delta^\eta)$ is satisfied if we choose $\eta > 0$ such that

$$\eta \lambda_\alpha(v) < n_\alpha.$$

We have seen that for proof of (2.1.6) it suffices to show

$$\int_{U(\hat{\sigma})} \omega_{\hat{\sigma}}^* = 0 ((-\log \delta)^{M_1}).$$

Before I prove this I have to add some remarks on our function

$$\begin{array}{ccc} V_Q & & \\ \downarrow & \searrow h & \\ U & \xrightarrow{\bar{h}} & \mathbf{R} \end{array}$$

Let us consider the function

$$h_Q : V_\pi^{(Q)}(c'_Q, \Omega'_Q) \rightarrow \mathbf{R}$$

which is defined as follows

$$h_Q(v) = \prod_{\substack{[Q'] \in \Sigma_{\pi'} \\ |\pi'| \geq |\pi|}} h_{v'}(v),$$

where

$$h_{v'}(v) = \prod_{\alpha \in \Pi} \bar{p}_\alpha(v)^{\sigma_\alpha^{v'}(v)}.$$

The restrictions of h_Q and h to V_Q coincide. Outside of V_Q these two functions may be different, since the product defining h_Q is only a part of the product defining h . We have neglected the factors

$$\prod_{\alpha \in \Pi} \bar{p}_\alpha(v)^{\sigma_\alpha^{Q'}(v)},$$

where $[Q'] \in \Sigma_\pi$ and $\|\pi'\| < \|\pi\|$. It is clear that the function h_Q is constant on the fibers of the projection map

$$V_\pi^{(Q)}(c'_Q, \Omega'_Q) \rightarrow A(c'_Q) \times \Omega'_Q$$

(see proof of lemma 2.1.7) and therefore we obtain an extension of \bar{h}

$$\begin{array}{ccc} \bar{h} : & U & \longrightarrow \mathbf{R} \\ & \downarrow & \parallel \\ h_Q : & A(c'_Q) \times \Omega'_Q & \longrightarrow \mathbf{R} \end{array}$$

We denote $\bar{f} = \log(\bar{h})$ and $\bar{f}_Q = \log(\bar{h}_Q)$. Now let us consider a point $(a, \bar{v}) = (\dots, a_{\alpha_0}, \dots, \bar{v}) \in A(c'_Q) \times \Omega'_Q$. We assume that the coordinate a_{α_0} is very small. Then I claim

$$t_{\alpha_0} \frac{\partial \bar{f}_Q}{\partial t_{\alpha_0}} \Big|_{(a, \bar{v})} \geq 0$$

and if $(a, \bar{v}) \in U$,

$$t_{\alpha_0} \frac{\partial \bar{f}_Q}{\partial t_{\alpha_0}} \Big|_{(a, \bar{v})} \geq C > 0.$$

where C is a constant not depending on $(a, \bar{v}) \in U$ provided a_{α_0} is small enough. To see this we go back into the proof of theorem 1.3.3. There

we have written $f = \sum_{\nu=1}^t f_\nu$ and we have seen

$$H_{\alpha_0}(f_\nu)|_v = t_{\alpha_0} \frac{\partial \bar{f}_\nu}{\partial t_{\alpha_0}} \Big|_{(a, \bar{v})} \geq 0.$$

The sum defining \bar{f}_0 is a part of the sum defining \bar{f} , and this proves the first statement. The second statement has already been stated in the proof of theorem 1.3.3.

For the estimation of the volume we divide the sets U and $U(\delta)$ into pieces

$$U_{\alpha_0} = \{ u \in U \mid u = (\dots, a_\alpha, \dots, \bar{v}), a_{\alpha_0} \leq a_\alpha \text{ for all } \alpha \in \pi \},$$

$$U_{\alpha_0}(\delta) = U(\delta) \cap U_{\alpha_0}.$$

We put

$$A_{\alpha_0} = \{ (\dots, a_\alpha, \dots)_{\substack{\alpha \in \pi \\ \alpha \neq \alpha_0}} \mid a_\alpha \in (\mathbf{R})^* \},$$

i. e. we drop the α_0 -th coordinate. There is an obvious diagram

$$\begin{array}{ccc} U_{\alpha_0} & \longrightarrow & A_{\alpha_0}(c_0) \times \Omega'_0 \\ \uparrow \cup & & \parallel \\ \varphi_{\alpha_0}: U_{\alpha_0}(\delta) & \longrightarrow & A_{\alpha_0}(c_0) \times \Omega_0 \end{array}$$

I claim that φ_{α_0} is injective provided $\delta > 0$ is small. Otherwise we would have points $(a, \bar{v}), (a', \bar{v}) \in U_{\alpha_0}(\delta)$ such that

$$a_\alpha = a'_\alpha \text{ for all } \alpha \neq \alpha_0$$

and

$$\bar{f}(a, \bar{v}) = \bar{f}(a', \bar{v}) = \log \delta.$$

Because δ is very small at least one of the coordinates a_α (resp. a'_α) is very small. From the definition of U_{α_0} it follows that a_{α_0} (resp. a'_{α_0}) must be very small. We know that the derivative of \bar{f} with respect to the α_0 -th variable is strictly positive in (a, \bar{v}) and (a', \bar{v}) . The derivative of the extended function \bar{f}_0 is positive on the path

$$t \rightarrow (\dots, a'_{\alpha_0} a_{\alpha_0}^{t-1}, \dots, \bar{v}) \quad (0 \leq t \leq 1)$$

which joins these two points. Then we necessarily have $(a, \bar{v}) = (a', \bar{v})$.

Now we identify $U_{\alpha_0}(\delta)$ with its image $\varphi_{\alpha_0}(U_{\alpha_0}(\delta)) = U_{\alpha_0}^*(\delta)$, and we compare the volume element ω_δ^* with the volume element ω_δ^{**} on $U_{\alpha_0}^*(\delta)$.

The vector field $t_{x_0} \frac{\partial}{\partial t_{x_0}}$ on A is invariant under translations, and therefore of constant length. We have seen that for points $(a, \bar{v}) \in U_{x_0}(\delta)$

$$t_{x_0} \frac{\partial}{\partial t_{x_0}} \bar{f}|_{(a, \bar{v})} \geq C > 0.$$

Therefore we can find a constant C' such that

$$\frac{\omega_{\delta}^*}{\omega_{\delta}^{**}} \leq C' \|d\bar{f}\|.$$

We have already seen in the beginning of this section that

$$\|d\bar{f}|_{(a, \bar{v})} = 0 \text{ (} -\log \delta \text{)}.$$

Now there exists a constant $\xi > 0$ such that $(a, v) \in U$, and $\bar{h}(a, \bar{v}) = \delta$ imply that $a_x \geq \delta^{\xi}$. This follows immediately from lemma 2.1.5. This yields

$$\int_{U_{x_0}^*(\delta)} \omega_{\delta}^{**} \leq \text{vol}(\Omega_0) \left(\iint_{\delta^{\xi}}^{c'_0} \dots \int \frac{dt_x}{t_x} \right) = 0 \text{ (} (-\log \delta)^{|\pi_1-1|} \text{)}$$

and this implies the limit formula ($\star \star$).

2.2. EXPLICIT CALCULATIONS FOR CHEVALLEY GROUPS. — Let F be an algebraic number field, the ring of integers of F will be denoted by \mathfrak{O} . We consider a simple, simply connected Chevalley scheme G/\mathfrak{O} and we denote by Γ_0 the group of its integral points. Again we denote the group of its real points $G(L \otimes_{\mathfrak{Q}} \mathfrak{R})$ by G_{∞} and the variety of maximal compact subgroups of G_{∞} by X . Let $\Gamma \subset \Gamma_0$ be a subgroup of finite index operating freely on X . Then it follows from 2.1

$$\int_{X/L} \omega_X = \chi(\Gamma).$$

We now put $\chi(\Gamma_0) = [\Gamma_0 : \Gamma]^{-1} \chi(\Gamma)$ and get

$$\int_{X/\Gamma_0} \omega_X = \chi(\Gamma_0).$$

I want to give an explicit expression for this integral. For this purpose we may assume that F is totally real, otherwise the form ω_X is identically zero and we only get $\chi(\Gamma_0) = 0$. Let us take a left invariant differential form ω on G/\mathfrak{O} of highest degree whose reduction mod \mathfrak{p} is not zero for

all finite primes \mathfrak{p} . This form yields a measure on the group $G(\mathbb{A})$ of adèles (comp. [12])

$$\omega'_\mathbb{A} = \prod_{\mathfrak{p}} \omega_{\mathfrak{p}}.$$

Here \mathfrak{p} is running over the set of all primes and $\omega_{\mathfrak{p}}$ is the measure on $G(K_{\mathfrak{p}})$ induced by ω . The Tamagawa measure on $G(\mathbb{A})$ is defined by

$$\omega_\mathbb{A} = |d_F|^{-\frac{\dim G}{2}} \omega'_\mathbb{A},$$

where d_F is the discriminant of our field F . In [7] Langlands has shown

$$\int_{G(\mathbb{A})/G(F)} \omega_\mathbb{A} = 1.$$

The strong approximation theorem for simply connected split groups yields

$$G(\mathbb{A})/G(F) = G_\infty/\Gamma_0 \times \prod_{\mathfrak{p} \text{ finite}} G(\mathfrak{O}_{\mathfrak{p}})$$

and it follows from [12] and [13]

$$1 = \int_{G(\mathbb{A})/G(F)} \omega_\mathbb{A} = \int_{G_\infty/\Gamma_0} \omega_\infty \prod_{\mathfrak{p} \text{ finite}} \text{vol}_{\omega_{\mathfrak{p}}}(G(\mathfrak{O}_{\mathfrak{p}})) = \int_{G_\infty/\Gamma_0} \omega_\infty \prod_{i=1}^r \zeta_F(m_i)^{-1}.$$

Here ζ_F is the Dedekind ζ -function of F , and the numbers m_i are the degrees of the invariant polynomials of G/F (comp. [3], chap. V, § 5-6).

Let us denote by $\mathfrak{g}/\mathfrak{O}$ the Lie algebra of the scheme G/\mathfrak{O} , and by $\mathfrak{g}(\mathfrak{O})$ we denote the algebra of its points in \mathfrak{O} . Then $\mathfrak{g}(\mathfrak{O})$ is a lattice in $\mathfrak{g}_\infty = \mathfrak{g}(\mathfrak{O}) \otimes \mathbf{R}$. There exists a volume element $\tilde{\omega}_\infty$ on \mathfrak{g}_∞ which corresponds to the volume element ω_∞ on G_∞ and by definition of the Tamagawa measure we have

$$\text{vol}_{\omega_\infty}(\mathfrak{g}_\infty/\mathfrak{g}(\mathfrak{O})) = 1.$$

Let $\mathfrak{g}_0/\mathbf{Z}$ be the Lie algebra of the corresponding Chevalley scheme over \mathbf{Z} . Let $\tilde{\omega}_0$ be the measure on $\mathfrak{g}(\mathbf{R})$ such that

$$\text{vol}_{\tilde{\omega}_0}(\mathfrak{g}_0(\mathbf{R})/\mathfrak{g}_0(\mathbf{Z})) = 1.$$

We have $\mathfrak{g}_\infty = \mathfrak{g}_0(\mathbf{R}) \otimes_{\mathbf{Z}} \mathbf{L}$, and $\mathfrak{g}(\mathfrak{O}) = \mathfrak{g}_0(\mathbf{Z}) \otimes_{\mathbf{Z}} \mathfrak{O}$. The measure $\tilde{\omega}_0$ induces a measure $\tilde{\omega}'_0$ on \mathfrak{g}_∞ and it is well known from number theory that we have

$$\text{vol}_{\tilde{\omega}'_0}(\mathfrak{g}_\infty/\mathfrak{g}(\mathfrak{O})) = |d_F|^{\frac{\dim G}{2}}.$$

We are going to compare the measure $\tilde{\omega}_\infty$ on \mathfrak{g}_∞ with the measure $\tilde{\omega}_k$ which is defined by means of the Killing form. This will be done by doing this first on $\mathfrak{g}_0(\mathbf{R})$. Then we observe that the measure defined by the Killing form behaves nicely under base extensions.

Let Δ be the system of roots of \mathfrak{g}_0 . Let us choose split Cartan algebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$. We may find a set of root vectors $e_\alpha \in \mathfrak{g}_0(\mathbf{Z})$, such that

- (1) $[e_\alpha, e_\beta] = r_{\alpha, \beta} e_{\alpha+\beta}$ with $r_{\alpha, \beta} \in \mathbf{N}$;
- (2) $[e_\alpha, e_{-\alpha}] = h_\alpha$ with $\alpha(h_\alpha) = 2$;
- (3) The elements h_α where α is running over the set Π of simple roots, and the element e_α , where $\alpha \in \Delta$ form a basis of $\mathfrak{g}_0(\mathbf{Z})$.

If we denote by $(\ , \)$ the Killing form on \mathfrak{g}_0 we get for the volume of $\mathfrak{g}_0(\mathbf{R})/\mathfrak{g}_0(\mathbf{Z})$ with respect to $\tilde{\omega}_{0, k}$

$$\text{vol}_{\tilde{\omega}_{0, k}}(\mathfrak{g}_0(\mathbf{R})/\mathfrak{g}_0(\mathbf{Z})) = \sqrt{\det((\langle h_\alpha, h_\beta \rangle)_{\alpha, \beta \in \Pi})} \prod_{\alpha \in \Delta^+} (e_\alpha, e_{-\alpha}) = C_\Delta.$$

Then it is clear that we have

$$\tilde{\omega}_\infty = |d_F|^{-\frac{\dim G}{2}} \cdot C_\Delta^n \cdot \tilde{\omega}_k.$$

This together with our previous formula yields

$$(2.2.1) \quad \int_{G_\infty/\Gamma_0} \omega_k = \prod_{i=1}^r \zeta_F(m_i) \cdot C_\Delta^n \cdot |d_F|^{-\frac{\dim G}{2}}.$$

Now we have to compare this integral with the integral

$$\int_{X/\Gamma_0} \omega_X.$$

Let K be a maximal compact subgroup of G_∞ . We assume that K is of maximal rank, i. e. there is a maximal torus $T \subset K$ having rank equal to the rank of G_∞ . Otherwise the form ω_X is zero and we only get $\chi(\Gamma_0) = 0$.

We identify $X = K \backslash G$ and put $Y = T \backslash G$.

Let $\mathfrak{h} \subset \mathfrak{g}_\infty$ be the Lie algebra of T , then we have

$$\mathfrak{g}_\infty = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_n^+} \mathfrak{C}_\alpha \right) = \mathfrak{h} \oplus \mathfrak{C},$$

where \mathfrak{C}_α is a two dimensional real vector space corresponding to the roots $\alpha, -\alpha$ and where Δ_n^+ is disjoint union of $n = [F : \mathbf{Q}]$ copies of the set of positive roots of G/\mathbf{Q} . The Killing form restricted to \mathfrak{C}_α is posi-

tive definite if α is not a root of K and is negative definitive if α is root of K . On the space \mathfrak{E}_α , where α is a root of K we replace the Killing form by its negative and the sum of these forms on the \mathfrak{E}_α yields a positive definite form \tilde{B} on \mathfrak{E} which is invariant under the operation of $\text{ad } (T)$. We identify the space \mathfrak{E} with the tangent space at Y in the point $T e$. Then \tilde{B} yields a positive definite G_α -invariant metric on Y . Let us denote by ω_Y the corresponding Euler-Poincaré form on Y . Then we have

$$(2.2.2) \quad \int_{Y/\Gamma_0} \omega_Y = \chi(T \setminus K) \int_{X/\Gamma_0} \omega_X = \chi(T \setminus K) \chi(\Gamma_0).$$

Now we transform the integral

$$\int_{G_\alpha/\Gamma_0} \omega_k$$

into an integral over Y/Γ_0 . For this purpose we consider the Lie algebra $\mathfrak{p}_\infty \otimes \mathbf{C} \subset \mathfrak{g}_\infty \otimes \mathbf{C}$. These algebras are direct sums of $n = [F : \mathbf{Q}]$ copies of $\mathfrak{h}_0(\mathbf{C}) \subset \mathfrak{g}_0(\mathbf{C})$, i. e.

$$\begin{aligned} \mathfrak{h}_0(\mathbf{C}) \oplus \dots \oplus \mathfrak{h}_0(\mathbf{C}) &= \mathfrak{h} \oplus_{\mathbf{R}} \mathbf{C}, \\ \mathfrak{g}_0(\mathbf{C}) \oplus \dots \oplus \mathfrak{g}_0(\mathbf{C}) &= \mathfrak{g}_\infty \otimes_{\mathbf{R}} \mathbf{C} \end{aligned}$$

n -summands

The basis $\{h_\alpha\}_{\alpha \in \Pi}$ of $\mathfrak{h}_0(\mathbf{Z})$ yields a basis of $\mathfrak{h}_0(\mathbf{C}) \oplus \dots \oplus \mathfrak{h}_0(\mathbf{C})$ if we repeat it n -times. Now the elements $2\pi\sqrt{-1}h_\alpha$ are contained in \mathfrak{h} and the kernel of the map

$$\text{exp} : \mathfrak{h} \rightarrow T$$

is equal to the lattice generated by these elements. If we consider on T the volume element ω_k^T defined by means of the Killing form the previous considerations show

$$\int_T \omega_k^T = (2\pi)^{r^n} (\sqrt{\det(((h_\alpha, h_\beta))_{\alpha, \beta \in \Pi})})^n$$

and we get

$$(2.2.3) \quad \int_{G_\alpha/\Gamma_0} \omega_k = (2\pi)^{r^n} (\sqrt{\det(((h_\alpha, h_\beta))_{\alpha, \beta \in \Pi})})^n \int_{Y/\Gamma_0} \omega_k^Y,$$

where ω_k^Y is the volume element defined by the metric on Y . Our problem is now reduced to the comparison of ω_k^Y , and the Euler-Poincaré form ω_Y . The form ω_Y is calculated from a connection on Y . Usually one take the connection without torsion, but here we take the canonical connection an Y as defined ([6], vol. II, p. 192).

The curvature tensor of the canonical connection at the point $T e \in Y$ is given by

$$R(U, V)W = -[[U, V]_h, W]$$

([6], chap. X, Th. 2.6).

In each of the vectorspaces \mathfrak{C}_α we choose an orthonormal basis U_α, V_α with respect to the metric \tilde{B} . It is obvious from the definition of $R(U, V)$ that the endomorphism $R(U, V)$ leaves the subspaces \mathfrak{C}_α invariant. Moreover one checks easily that

$$(R(U, V)U_\alpha, U_\alpha)_{\tilde{B}} = 0,$$

$$(R(U, V)V_\alpha, V_\alpha)_{\tilde{B}} = 0.$$

With respect to the basis formed by the U_α, V_α the curvature tensor looks as follows

$$R(U, V) = \begin{pmatrix} 0 & (R(U, V)U_\alpha, V_\alpha)_{\tilde{B}} & 0 \\ (R(U, V)V_\alpha, U_\alpha)_{\tilde{B}} & 0 & \\ 0 & & \boxed{} \end{pmatrix}.$$

Now the map

$$(U, V) \rightarrow (R(U, V)U_\alpha)_{\tilde{B}}$$

is a skew symmetric bilinear map from $\mathfrak{C} \times \mathfrak{C}$ to \mathfrak{R} and so it may be considered as an element $T_\alpha \in \check{\mathfrak{C}} \wedge \check{\mathfrak{C}}$. If we consider $\mathfrak{C} \otimes \mathfrak{C} = \mathfrak{C}e_\alpha + \mathfrak{C}e_{-\alpha}$ and express the elements U_α, V_α in terms of the $e_\alpha, e_{-\alpha}$ an easy calculation shows

$$T_\alpha = \sum_{\beta \in \Delta_n^+} \frac{2(e_\alpha, e_{-\alpha})}{(e_\beta, e_{-\beta})^2} (-1)^{\varepsilon(\alpha)} \alpha(h_\beta) U_\beta \wedge V_\beta,$$

where $\varepsilon(\alpha) = 1$ if α is a root of K and $\varepsilon(\alpha) = 0$ if α is not a root of K . Now the form ω_Y in the point $T e$ is given by (comp. [6], chap. XII, Th. 5.4),

$$(-1)^{d'} \frac{1}{2^{2d'} \pi^{d'}} \bigwedge_{\alpha \in \Delta_n^+} T_\alpha,$$

where $d' = \frac{1}{2} \dim Y$ and this is the number of elements in Δ_n^+ .

The number of roots α with $\varepsilon(\alpha) = +1$ is equal to the number n_k of positive roots of K . We put $a' = \frac{1}{2} \dim X = d' - n_k$ and get

$$\begin{aligned} \omega_Y &= (-1)^{a'} \frac{1}{\pi^{d'} \cdot 2^{2d'}} \bigwedge_{\beta \in \Delta_n^+} \left(\sum_{\beta \in \Delta_n^+} \frac{2(e_\alpha, e_{-\alpha})}{(e_\beta, e_{-\beta})^2} \alpha(h_\beta) U_\beta \wedge V_\beta \right) \\ &= (-1)^{a'} \frac{1}{(2\pi)^{d'}} \left(\prod_{\beta \in \Delta_n^+} \frac{1}{(e_\beta, e_{-\beta})^n} \right) \left(\sum_{\varphi} \prod_{\alpha \in \Delta_n^+} \alpha(h_{\varphi(\alpha)}) \right) \left(\bigwedge_{\alpha \in \Delta_n^+} U_\alpha \wedge V_\alpha \right) \end{aligned}$$

and φ is running over the set of permutations of the set Δ_n^+ .

The expression

$$\sum_{\varphi} \prod_{\alpha \in \Delta_n^+} \alpha(h_{\varphi(\alpha)})$$

has been calculated by Steinberg in a letter to Tits. This letter is copied (up to notation) in the appendix with the kind permission of Steinberg. His result is

$$\sum_{\varphi} \prod_{\alpha \in \Delta_n^+} \alpha(h_{\varphi(\alpha)}) = \left(\prod_{i=1}^r m_i! \right)^n,$$

where the m_i are the degrees of the invariant polynomials of G/\mathbb{O} . The set Δ_n^+ is the disjoint union of n copies of the set Δ of simple roots of G/\mathbb{O} , we have $d' = nd$, where d is the number of positive roots of G/\mathbb{O} . We get

$$\omega_Y = (-1)^{a'} \frac{1}{2^{nd} \pi^{nd}} \left(\prod_{\alpha \in \Delta_n^+} \frac{1}{(e_\alpha, e_{-\alpha})^n} \right) \left(\prod_{i=1}^r m_i! \right)^n \left(\bigwedge_{\alpha \in \Delta_n^+} U_\alpha \wedge V_\alpha \right).$$

The measure on Y defined by the metric is

$$\omega_k^Y = \bigwedge_{\alpha \in \Delta_n^+} U_\alpha \wedge V_\alpha.$$

We obtain the formula

$$(2.2.4) \quad \omega_Y = (-1)^{a'} \frac{1}{2^{nd} \pi^{nd}} \prod_{\alpha \in \Delta_n^+} \frac{1}{(e_\alpha, e_{-\alpha})^n} \left(\prod_{i=1}^r (m_i!) \right)^n \omega_k^Y.$$

Combining (2.2.2) and (2.2.4) we obtain

$$\chi_r(\Gamma_0) \chi_r(\Gamma \setminus K) = \int_{Y/\Gamma_0} \omega_Y = (-1)^{a'} \frac{1}{2^{nd} \pi^{nd}} \prod_{\alpha \in \Delta_n^+} \frac{1}{(e_\alpha, e_{-\alpha})^n} \left(\prod_{i=1}^r (m_i!) \right)^n \int_{Y/\Gamma_0} \omega_k^Y.$$

Now formula (2.2.3) yields for this last expression

$$(-1)^{a'} \frac{1}{(2\pi)^{nd}} \prod_{\alpha \in \Delta_+^*} \left(\frac{1}{(e_\alpha, e_{-\alpha})^n} \right) \left(\prod_{i=1}^r (m_i!) \right)^n.$$

Using (2.2.4) we get

$$= (-1)^{a'} \frac{1}{(2\pi)^{n(d+r)}} \left(\prod_{i=1}^r m_i! \right)^n |d_F|^{\frac{\dim G}{2}} \prod_{i=1}^r \zeta_F(m_i).$$

Let G_0/\mathbf{Z} be the Chevalley scheme of the same type as G/\mathfrak{O} . Then we have

$$G_\infty = \prod_{i=1}^r G_0(\mathbf{R}).$$

Let be K_0 a maximal compact subgroup of $G_0(\mathbf{R})$ then

$$K = \prod_{i=1}^r K_0.$$

We put $a = \frac{1}{2} \dim K_0 \backslash G_0(\mathbf{R})$ then $a' = a.n$. If $T \subset K$ is a maximal compact torus, then

$$T \backslash K = \prod_{i=1}^r T_0 \backslash K_0$$

and for the Euler-Poincaré characteristic we get

$$\chi(T \backslash K) = (\chi(T_0 \backslash K_0))^n.$$

It is well known (Theorem of Hopf-Samelson) that

$$\chi(T_0 \backslash K_0) = |W_{K_0}|,$$

where W_{K_0} is the Weyl group of K_0 . This gives the formula

$$\chi(\Gamma_0) = (-1)^{an} \frac{\left(\prod_{i=1}^r m_i! \right)^n |d_F|^{\frac{\dim G}{2}}}{(2\pi)^{n(d+r)} |W_{K_0}|^n} \prod_{i=1}^r \zeta_F(m_i).$$

We are going to use the functional equation of the Dedekind ζ -function to bring this expression into a much prettier form. The functional equa-

tion tells us (comp. [14], chap. VII, th. 3)

$$\zeta_{\mathbb{F}}(m_i) = \zeta_{\mathbb{F}}(1 - m_i) \circ \Gamma\left(\frac{m_i}{2}\right)^{-n} \Gamma\left(\frac{1 - m_i}{2}\right)^n \cdot \pi^{n(m_i - \frac{1}{2})} \cdot |d_{\mathbb{F}}|^{\frac{1}{2} - m_i}.$$

The following relations are well known ([3], chap. V, p. 122)

$$\sum_{i=1}^r m_i = d + r,$$

$$\prod_{i=1}^r m_i = |W_G|,$$

where W_G is the Weyl group of G . Using these facts a simple calculation yields

$$\chi(\Gamma_0) = \frac{|W_G|^n}{2^{rn} |W_K|^n} \prod_{i=1}^r \zeta_{\mathbb{F}}(1 - m_i).$$

APPENDIX.

In the last section I used the formula

$$\prod_{\alpha \in \Delta^+} \alpha(h_{\varphi(\alpha)}) = \prod_{i=1}^r m_i !.$$

This formula has been proved by Steinberg in a letter to Tits. Steinberg considers the real vector space V generated by the roots and evaluates

$$h = \sum_{\varphi} \prod_{\alpha \in \Delta^+} (\alpha, \varphi(\alpha)),$$

where $(\ , \)$ denotes an inner product on V which is invariant under the operation of the Weyl group W . Now I am going to copy Steinberg's argument with his kind permission :

The given inner product on V extends to one the symmetric algebra of V by the formula

$$(\alpha_1 \dots \alpha_n, \beta_1 \dots \beta_n) = \sum_{\varphi \text{ permutations}} (\alpha_i, \beta_{\varphi(i)}) \alpha_i, \beta_i \in V.$$

Thus if $P = \prod_{\alpha \in \Delta^+} \alpha$, the product of positive roots, then $h = (P, P)$.

We may evaluate as follows. Let $d = |\Delta^+| = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. From Weyl's identity (of formal power series) comp. [3], chap. VI, n° 3.3)

$$\sum_{w \in W} \varepsilon(w) e^{w\rho} = \prod_{\alpha \in \Delta^+} \left(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}} \right) \quad (\varepsilon(w) = \text{sgn } w)$$

it follows that (compare terms of degree d)

$$\sum \varepsilon(w) (w\rho)^d = (d!) P.$$

Thus

$$\begin{aligned} h = (P, P) &= \frac{1}{d!} \left(\sum_{w \in W} \varepsilon(w) (w\rho)^d, \prod_{\alpha \in \Delta^+} \alpha \right) \\ &= \frac{1}{d!} |W| \cdot \left(\rho^d, \prod_{\alpha \in \Delta^+} \alpha \right) \quad \text{because } \prod_{\alpha \in \Delta^+} \alpha \text{ is skew} \\ &= \frac{1}{d!} |W| \cdot d! (\rho, \alpha) \\ &= |W| \prod_{\alpha \in \Delta^+} \frac{(\alpha, \alpha)}{2} \prod_{\alpha \in \Delta^+} \frac{2(\rho, \alpha)}{(\alpha, \alpha)}. \end{aligned}$$

Now for any root α , $2(\rho, \alpha)/(\alpha, \alpha) = ht \alpha$ (the height of α) and it is known that if $m_1 \leq m_2 \leq \dots \leq m_r$ are the degrees of the basic invariants the number of roots of height j minus the number of roots of height $j + 1$ is just the number of m_i 's equal to $j + 1$.

Thus

$$\prod_{\alpha \in \Delta^+} \frac{2(\rho, \alpha)}{(\alpha, \alpha)} = \prod_{i=1}^r (m_i - 1).$$

Since also $|W| = \prod_{i=1}^r m_i!$ the earlier expression becomes

$$h = 2^{-d} \prod_{i=1}^r m_i! \prod_{\alpha \in \Delta^+} (\alpha, \alpha).$$

This is Steinberg's argument. If we identify V with its dual V and consider the h_β as elements of V , we have

$$h_\beta = a_\beta \beta$$

and get

$$\begin{aligned} \sum_{\varphi} \prod_{\alpha \in \Delta^+} \alpha(h_{\varphi(\alpha)}) &= \sum_{\varphi} \prod_{\alpha \in \Delta^+} (\alpha, \varphi(\alpha)) a_{\varphi(\alpha)}^{-1} = h \left(\prod_{\alpha \in \Delta^+} a_{\alpha} \right)^{-1} \\ &= \left(\prod_{\alpha \in \Delta^+} a_{\alpha} \right)^{-1} \frac{1}{2^l} \prod_{i=1}^r m_i! \prod_{\alpha \in \Delta^+} \alpha(h_{\alpha}). \end{aligned}$$

Now $\alpha(h_{\alpha}) = 2$ by definition. This yields the desired formula.

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