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TADAO ODA

The first de Rham cohomology group and Dieudonné modules

THE FIRST DE RHAM COHOMOLOGY GROUP AND DIEUDONNÉ MODULES (*)

By Tadao ODA.

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INTRODUCTION.

The purpose of this paper is to study the first De Rham cohomology group $H^1_{DR}(X)$ of a proper smooth scheme over a perfect field k of characteristic p.

It was shown by Grothendieck [15] that if k is the field of complex numbers, then $H^1_{DR}(X)$ is canonically isomorphic to $H^1(X_{class}, k)$. The spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_{X/k}) \Rightarrow H^n_{DR}(X)$$

(*cf.* Section 5) is degenerate when X is Kähler, giving an exact sequence

$$(\star) \qquad 0 \to H^0(X, \Omega^1_{X/k}) \to H^1_{DR}(X) \to H^1(X, \mathcal{O}_X) \to 0.$$

Moreover, the theory of harmonic forms gives a splitting of this exact sequence.

However, if $p \neq 0$, this spectral sequence does not degenerate in general. The first non-degeneracy comes in as d_1. For example $E_2^{1,0} = H^0(X, \Omega_{X/k}^1)_{d=0}$ and $E_2^{0,1} = H^1(X, \mathcal{O}_X)_{d=0}$ may be smaller than $E_1^{1,0}$ and $E_1^{0,1}$ in general (*cf.* Mumford [28]). Thus instead of $(\star)$ we get an exact sequence

$$0 \to H^0(X, \Omega_{X/k}^1)_{d=0} \to H^1_{DR}(X) \to H^1(X, \mathcal{O}_X)_{d=0} \xrightarrow{d_2} H^0(X, \Omega_{X/k}^2)_{d=0}.$$

But d_2 may not be zero.

Here we may ask whether $H^1_{DR}(X)$ is small enough, or, more precisely, whether $H^1_{DR}(X)$ is closely related to the Picard variety $\text{Pic}^0_{X/k, \text{red}}$ of X. The answer is no. The fact that the Picard scheme $\text{Pic}_{X/k}$ may not be reduced in characteristic p is one of the reasons (*cf.* Igusa [20], [21]], Serre [33] and Mumford [28], [29], lecture 27).

Then we may ask whether $H^1_{DR}(X)$ is closely related to the Picard scheme $\text{Pic}_{X/k}$. The answer is still no. $H^1_{DR}(X)$ may still be too big.

The Cartier operator (Cartier [6] and Seshadri [36]) can be used to define a canonical subspace of $H^1_{DR}(X)$, which is closely related to $\text{Pic}_{X/k}$ (*cf.* Corollary 5.12).

But before stating the result we have to clarify the term " closely related to $\text{Pic}_{X/k}$ ". This we can do using the Dieudonné modules. The subspace of $H^1_{DR}(X)$ we canonically obtain is isomorphic to the dual of the Dieudonné module of $_p\text{Pic}_{X/k}$, the k-group-scheme theoretic kernel of the endomorphism p on $\text{Pic}_{X/k}$.

The Dieudonné module is the major tool used to interpret finite commutative group schemes over a perfect field k of characteristic p. Let A be the ring $W(k)[F, V]$ defined by the relations $FV = VF = p$, $F\lambda = \lambda^\sigma F$ and $\lambda V = V\lambda^\sigma$ for all λ in the ring $W(k)$ of infinite Witt vectors with coefficients in k, where σ is the Frobenius endomorphism of the ring $W(k)$.

We generalize in Definition 3.12 the definition of the Dieudonné modules given in Gabriel [13] and Manin [24]. We obtain an anti-equivalence M from the category $\mathfrak{N}_p$ of finite commutative group schemes over k of p-primary rank over k to the category of left A-modules of $W(k)$-finite length. The key tool here is the theorem of Dieudonné-Cartier given in Sharma [37]. This functor M can easily be extended to one from the category $\text{Ind}(\mathfrak{N}_p)$ of inductive systems of objects in $\mathfrak{N}_p$ to the category of projective systems of left A-modules of $W(k)$-finite length.

We describe various fundamental properties satisfied by the Dieudonné modules in Section 3. Especially, we express in Theorem 3.19 the Cartier duals of group schemes in $\mathfrak{N}_p$ in terms of Dieudonné modules. One of the tools for this is the computation in Proposition 3.21 of the commutative group functors

$$\mathcal{H}om_{k-gr}(W, G_m) \quad \text{and} \quad \mathcal{H}om_{k-gr}(W_n, G_m).$$

The identification of the canonical subspace of $H^1_{DR}(X)$ as the dual of the Dieudonné module of $_p\mathrm{Pic}_{X/k}$ is derived from the more general Theorem 4.4, relating the dual of the Dieudonné module of the object, $_{p^\infty}\mathrm{Pic}_{X/k} = \varinjlim_n {}_{p^n}\mathrm{Pic}_{X/k}$, in $\mathrm{Ind}(\mathfrak{N}_p)$, to a left A-module I(X) of W(k)-cofinite type (cf. Definition 4.1), which is described entirely in terms of the cohomology of certain abelian sheaves on X in the Zariski topology. When k is algebraically closed, this module I(X) contains all the information about the classification of flat principal bundles over X with finite commutative k-group scheme in $\mathfrak{N}_p$ as the structure group. We dualize I(X) to get a left A-module of W(k)-finite type $H_1(X_{p\text{-flat}})$, which has properties similar to those satisfied by the first homology group with p-adic integral coefficients $H_1(X_{class}, \mathbf{Z}_p)$ when k is the field of complex numbers.

As a corollary to this we prove (cf. Theorem 4.12) the well known fact that a flat principal bundle over a k-abelian scheme is again a k-group scheme, if its structure group is a commutative algebraic k-group scheme killed by some power of p.

When X is an abelian scheme over a perfect field k of characteristic p, the spectral sequence for the De Rham cohomology is degenerate and we have the exact sequence (★), which may not split canonically in general. We can interpret this exact sequence in terms of the Dieudonné module of the kernel $_pX$ of the endomorphism p_X (cf. Corollary 5.11).

On the other hand, we can associate to X a p-divisible group $X(p) = {}_{p^\infty}X$, which is an object of $\mathrm{Ind}(\mathfrak{N}_p)$. The Dieudonné module M(X(p)) of X(p) is a left A-module which is W(k)-free and whose rank over W(k) is equal to $2\dim(X)$.

From the definition of the Dieudonné module it is not difficult to see that M(X(p)) is equal to Barsotti's module of canonical covectors for X (cf. Barsotti [2], [3], [4]).

Barsotti defines the Riemann form of a divisor on X on the module of canonical covectors. We can define the Riemann form of a divisor (or an invertible sheaf) on X in a different way (cf. Proposition 3.24), using the Cartier duality theorem (Corollary 1.3). This theorem states that the kernel of an isogeny from an abelian scheme to another is the Cartier dual of the kernel of the transpose of the isogeny. This was first proved over a field by Cartier [8]. Oort gave a proof in the general case over a prescheme (cf. [31]) (¹).

(¹) M. Nishi also proved this duality theorem independently. We should have called it the Cartier-Nishi-Oort duality theorem. *Cf.* M. NISHI, *The Frobenius theorem and the duality theorem on an abelian variety* (*Mem. Coll. Sci. Univ. Kyoto*, vol. 32, 1959, p. 333-350).

We give an entirely different proof using descent theory (FGA [16], exposé 190). In this method we can prove that the duality between the kernel of an isogeny and the kernel of its transpose is skew-symmetric in nature. This fact is essential to proving the skew-symmetry of the Riemann form of an invertible sheaf on X. We can prove as a corollary to the Cartier duality theorem that abelian schemes have no torsion in the Picard scheme. This fact was first proved by Barsotti.

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SECTION 1.

CARTIER DUALITY THEOREM.

For the definition and basic properties of the Picard schemes $\mathrm{Pic}_{X/S}$, $\mathrm{Pic}^{\tau}_{X/S}$ and $\mathrm{Pic}^{0}_{X/S}$ of an S-prescheme X we refer the reader to FGA [16], exposés 232 and 236, and Murre [30].

Let S be a prescheme. Let X and Y be abelian schemes over S (*cf.* Mumford [27], Chap. 6) and $\lambda : X \to Y$ be an S-isogeny. We get an S-homomorphism

$$\mathrm{Pic}_{\lambda/S} : \quad \mathrm{Pic}_{Y/S} \to \mathrm{Pic}_{X/S}.$$

For an S-prescheme S' we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}\,(S') & \longrightarrow & \mathrm{Pic}\,(X_{S'}) & \longrightarrow & \mathrm{Pic}_{X/S}\,(S') & \longrightarrow & 0 \\
 & & \| & & \lambda^{*}_{S'} \uparrow & & \mathrm{Pic}_{\lambda/S} \uparrow & & \\
0 & \longrightarrow & \mathrm{Pic}\,(S') & \longrightarrow & \mathrm{Pic}\,(Y_{S'}) & \longrightarrow & \mathrm{Pic}_{Y/S}\,(S') & \longrightarrow & 0
\end{array}
$$

whose rows are exact, where $X_{S'}$, $\lambda_{S'}$ and $Y_{S'}$ are the base extensions of X, λ and Y to S'. Hence we can easily see that

$$[\,\mathrm{ker}\,(\mathrm{Pic}_{\lambda/S})\,]\,(S') = \mathrm{ker}\Big[\mathrm{Pic}_{Y/S}\,(S') \xrightarrow{\mathrm{Pic}_{\lambda/S}} \mathrm{Pic}_{X/S}\,(S')\Big]$$
$$= \mathrm{ker}\Big[\mathrm{Pic}\,(Y_{S'}) \xrightarrow{\lambda^{*}} \mathrm{Pic}\,(X_{S'})\Big].$$

We now define the Cartier pairing

$$\langle\ ,\ \rangle_{\lambda} : \quad \mathrm{ker}\,(\lambda) \times_{S} \mathrm{ker}\,(\mathrm{Pic}_{\lambda/S}) \to \mathbf{G}_{m\,S}$$

as follows : let S' be an S-prescheme. Let x be in $\mathrm{ker}(\lambda)\,(S')$ and L be in $[\mathrm{ker}(\mathrm{Pic}_{\lambda/S})]\,(S')$. From what we have seen above we can think

of L as an invertible sheaf on $Y_{S'}$ such that there exists an isomorphism α of invertible sheaves on $X_{S'}$

$$\lambda_{S'}^\star L \xrightarrow[\sim]{\alpha} \mathcal{O}_{X_{S'}}.$$

If we denote by $T_x : X_{S'} \to X_{S'}$ the translation on $X_{S'}$ determined by the element x in $\ker(\lambda)(S')$, then the composition

$$\mathcal{O}_{X_{S'}} \xrightarrow[\sim]{\alpha^{-1}} \lambda_{S'}^\star L = T_x^\star \lambda_{S'}^\star L \xrightarrow{T_x^\star(\alpha)} T_x^\star \mathcal{O}_{X_{S'}} = \mathcal{O}_{X_{S'}}$$

gives an $\mathcal{O}_{X_{S'}}$-automorphism of $\mathcal{O}_{X_{S'}}$, which we identify with an element $\langle x, L \rangle_\lambda$ of

$$H^0(X_{S'}, \mathcal{O}_{X_{S'}}^\star) = H^0(S', \mathcal{O}_{S'}^\star) = G_{m\,S}(S').$$

It is obvious that $\langle x, L \rangle_\lambda$ does not depend on α.

For a group scheme N finite and flat over S we denote by $D_S(N)$ the Cartier dual

$$D_S(N) = \mathcal{H}om_{S-gr}(N, G_{m\,S})$$

(*cf.* Oort [31] and Gabriel [12]).

THEOREM 1.1. — *Let* X *and* Y *be abelian schemes over a prescheme* S *and let* $\lambda : X \to Y$ *be an* S-*isogeny. Then the Cartier pairing* $\langle \ , \ \rangle_\lambda$ *is a non-degenerate and biadditive pairing of group schemes finite and flat over* S, *i.e. it defines a canonical* S-*isomorphism*

$$\nu_\lambda : \ \ker(Pic_{\lambda/S}) \to D_S(\ker(\lambda)).$$

Moreover ν_λ *is functorial in* λ, *i.e. if*

$$
\begin{array}{ccc}
X & \xrightarrow{\ \lambda\ } & Y \\
\alpha \downarrow & & \beta \downarrow \\
X' & \xrightarrow{\ \lambda'\ } & Y'
\end{array}
$$

is a commutative diagram of S-*homomorphisms of abelian schemes such that* λ *and* λ' *are* S-*isogenies, then the following diagram is commutative :*

$$
\begin{array}{ccc}
\ker(Pic_{\lambda/S}) & \xrightarrow[\sim]{\nu_\lambda} & D_S(\ker(\lambda)) \\
{\scriptstyle Pic_{\beta/S}}\uparrow & & \uparrow{\scriptstyle D_S(\alpha)} \\
\ker(Pic_{\lambda'/S}) & \xrightarrow{\nu_{\lambda'}} & D_S(\ker(\lambda'))
\end{array}
$$

Proof. — We show that

$$\nu_\lambda : \ [\ker(Pic_{\lambda/S})](S') \to D_S(\ker(\lambda))(S')$$

is an isomorphism for all S-preschemes S'. We first note that everything is compatible with base extension. Hence we may assume $S' = S$.

The map

$$\ker\left[\operatorname{Pic}(Y) \xrightarrow{\lambda^*} \operatorname{Pic}(X)\right] \xrightarrow{\nu_\lambda} D_S(\ker(\lambda))(S)$$

is defined as follows : let L be an invertible sheaf on Y such that there exists an isomorphism

$$\alpha : \lambda^* L \xrightarrow{\sim} \mathcal{O}_X.$$

Then $\nu_\lambda(L)$ is an S-homomorphism from $\ker(\lambda)$ to G_{mS} such that for an S-prescheme S' and an element x in $\ker(\lambda)(S') = \ker\left[X(S') \xrightarrow{\lambda} Y(S')\right]$ we have

$$\nu_\lambda(L)(x) = T_x^*(\alpha_{S'}) \circ \alpha_{S'}^{-1}.$$

Since $\lambda_{S'} \circ T_x = T_{\lambda x} \circ \lambda_{S'} = \lambda_{S'}$, the right hand side is the $\mathcal{O}_{X_{S'}}$-isomorphism
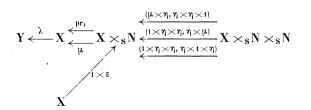
$$\mathcal{O}_{X_{S'}} \xrightarrow[\sim]{\alpha_{S'}^{-1}} \lambda_{S'}^* L_{S'} = T_x^* \circ \lambda_{S'}^* L_{S'} \xrightarrow[\sim]{T_x^*(\alpha_{S'})} \mathcal{O}_{X_{S'}}$$

which can be identified as an element of $H^0(X_{S'}, \mathcal{O}_{X_{S'}})^* = H^0(S', \mathcal{O}_{S'})^* = G_{mS}(S')$. Note that $\nu_\lambda(L)$ does not depend on the choice of α. It is easy to see that $\nu_\lambda(L)$ is a homomorphism. Thus $\nu_\lambda(L)$ is the canonical descent datum on $\mathcal{O}_X$ with respect to $\lambda : X \to Y$ induced by L on Y. But the descent theory (*cf.* FGA [16], exposé 190) tells us that

$$\ker\left[\operatorname{Pic}(Y) \xrightarrow{\lambda^*} \operatorname{Pic}(X)\right]$$

is isomorphis to the set of equivalence classes of descent data on $\mathcal{O}_X$ relative to $\lambda : X \to Y$ via the map sending L to $\lambda^* L \cong \mathcal{O}_X$ with its canonical descent datum. Hence it remains to show that the latter set is equal to $D_S(\ker(\lambda))(S)$.

For simplicity we write $N = \ker(\lambda)$. If we denote by $\eta : X \to S$ the structure morphism, $\mu : X \times X \to X$ the group law, and by $\varepsilon : S \to X$ the zero-section, we have the following diagram :

A descent datum on $\mathcal{O}_X$ is an $\mathcal{O}_{X \times_S N}$-endomorphism φ of $\mathcal{O}_{X \times_S N}$ such that

$$(1 \times \eta \times \eta, \eta \times \mu)^*(\varphi) = (1 \times \eta \times \eta, \eta \times 1 \times \eta)^*(\varphi) \circ (\mu \times \eta, \eta \times \eta \times 1)^*(\varphi)$$

and

$$(1 \times \varepsilon)^*(\varphi) = 1_{\mathcal{O}_X}.$$

But since

$$\text{End}_{\mathcal{O}_{X \times_s N}}(\mathcal{O}_{X \times_s N}) = H^0(N, \mathcal{O}_N)$$

and

$$\text{End}_{\mathcal{O}_{X \times_s N \times_s N}}(\mathcal{O}_{X \times_s N \times_s N}) = H^0(N \times_S N, \mathcal{O}_{N \times_s N})$$

φ is an element of $H^0(N, \mathcal{O}_N)$ such that

$$\mu^\star(\varphi) = (1 \times \eta)^\star(\varphi) \circ (\eta \times 1)^\star(\varphi)$$

and

$$\varepsilon^\star(\varphi) = 1.$$

Thus φ gives precisely an S-homomorphism from N to $G_{m\,\text{S}}$, i. e. an element of $D_S(N)(S)$.

To prove the functoriality of ν_λ in λ it is enough to show that if x is an S-valued point of $\ker(\lambda)$ and if L' is an invertible sheaf on Y' such that

$$\varphi : \quad \lambda'^\star L' \xrightarrow{\sim} \mathcal{O}_{X'}$$

then we get

$$\langle x, \beta^\star L' \rangle_\lambda = \langle \alpha x, L' \rangle_{\lambda'}.$$

The right hand side is equal to $T^\star_{\alpha x}(\varphi) \circ \varphi^{-1}$. On the other hand

$$\alpha^\star(\varphi) : \quad \lambda^\star \beta^\star L' = \alpha^\star \lambda'^\star L' \xrightarrow{\sim} \mathcal{O}_X.$$

Hence the left hand side is equal to

$$T^\star_x(\alpha^\star(\varphi)) \circ \alpha^\star(\varphi)^{-1} = \alpha^\star(T^\star_{\alpha x}(\varphi) \circ \varphi^{-1}) = T^\star_{\alpha x}(\varphi) \circ \varphi^{-1}$$

since $T^\star_{\alpha x}(\varphi) \circ \varphi^{-1}$ is in $H^0(X', \mathcal{O}_{X'})^\star = H^0(S, \mathcal{O}_S)^\star$.

<div align="right">Q. E. D.</div>

DEFINITION 1.2. — *Let* X *be an abelian scheme over a prescheme* S *such that* $\text{Pic}_{X/S}$ *is representable. We denote by* $X^{(t/S)}$ *or simply by* X^t *the dual abelian scheme* $\text{Pic}^0_{X/S}$. *For an S-homomorphism* $\lambda : X \to Y$ *of two abelian schemes such that* X^t *and* Y^t *are defined we denote by* λ^t *the induced S-homomorphism*

$$\text{Pic}^0_{\lambda/S} : \quad Y^t \to X^t.$$

For the definition of the canonical homomorphism $k_X : X \to X^{tt}$ we refer the reader to Lang ([22], p. 127).

COROLLARY 1.3 (Cartier duality theorem). — *Let* S *be a prescheme. Then*

(i) *If* X *is an abelian scheme over* S *such that* $\text{Pic}_{X/S}$ *exists, then* X *has no torsion, i.e.*

$$\text{Pic}^\tau_{X/S} = \text{Pic}^0_{X/S}.$$

(ii) *If* $\lambda : X \to Y$ *is an S-isogeny of abelian schemes over* S *such that* X^t *and* Y^t *exist, then there is a canonical S-isomorphism*

$$\nu_\lambda : \quad \ker(\lambda^t) \xrightarrow{\sim} D_S(\ker(\lambda)).$$

ν_λ *is functorial in* λ*, i. e. if we have a commutative diagram*

$$\begin{array}{ccc}
X & \xrightarrow{\lambda} & Y \\
\alpha \downarrow & & \beta \downarrow \\
X' & \xrightarrow{\lambda'} & Y'
\end{array}$$

of S-*homomorphisms of abelian schemes such that* λ *and* λ' *are* S-*isogenies, then the following diagram is commutative :*

$$\begin{array}{ccc}
\ker(\lambda^t) & \xrightarrow[\sim]{\nu_\lambda} & D_S(\ker(\lambda)) \\
\beta^t \uparrow & & D_S(\alpha) \uparrow \\
\ker(\lambda'^t) & \xrightarrow[\sim]{\nu_{\lambda'}} & D_S(\ker(\lambda'))
\end{array}$$

Moreover the following diagram is commutative :

$$\begin{array}{ccc}
\ker(\lambda) & \xrightarrow[\sim]{\text{can.}} & D_S D_S(\ker(\lambda)) \\
k_X \downarrow \wr & & \wr \downarrow -D_S(\nu_\lambda) \\
\ker(\lambda''^t) & \xrightarrow[\sim]{\nu_{\lambda^t}} & D_S(\ker(\lambda^t))
\end{array}$$

Proof. — To prove (i) we may assume S to be the spectrum of an algebraically closed field. Since $\text{Pic}^0_{X/S}$ is an abelian scheme over S, we know that the multiplication by a non-zero integer in $\text{Pic}^0_{X/S}(S)$ is surjective. Hence $\text{Pic}^\tau_{X/S}(S)$ is generated by $\text{Pic}^0_{X/S}(S)$ and the elements of $\text{Pic}_{X/S}(S)$ of finite order. Thus it is enough to show that the elements of $\text{Pic}_{X/S}(S)$ of order n are contained in $\text{Pic}^0_{X/S}(S)$ for all n. Since by the Theorem of Square $\text{Pic}^\tau_{X/S}$ is contained in the kernel of

$$\text{Pic}_{\mu/S} - \text{Pic}_{p_1/S} - \text{Pic}_{p_2/S} : \text{Pic}_{X/S} \to \text{Pic}_{X \times_S X/S}$$

we know that the multiplication by n on $\text{Pic}_{X/S}$ coincides with $\text{Pic}_{n_X/S}$ on $\text{Pic}^\tau_{X/S}$. Hence (i) is the consequence of (ii) applied to $\lambda = n_X$.

To prove the first part of (ii) it suffices to show that

$$\ker(\text{Pic}_{\lambda/S}) \subset Y^t.$$

We may assume S is the spectrum of a field. From Theorem 1.1, it follows that

$$\deg(\lambda^t) \leq \deg(\text{Pic}_{\lambda/S}) = \text{rank}_S(D_S(\ker\lambda)) = \deg(\lambda).$$

It is well known (*cf.* Weil [40] and Lang [22]) that there is an S-isogeny $\lambda'.Y \to X$ such that $\lambda' \circ \lambda = m_X$ for a positive integer m. Hence $\lambda' \circ \lambda'' = m_{X'}$. Thus if we write $g = \dim X$, then

$$m^{2g} = \deg(\lambda' \circ \lambda'^t) = \deg(\lambda') . \deg(\lambda'^t) \leqq \deg(\lambda) . \deg(\lambda') = m^{2g}.$$

Thus $\deg(\lambda') = \deg(\lambda)$. The functoriality is obvious from Theorem 1.1. To prove the second half of (ii) we need the following :

Let P_Y be the Poincaré invertible sheaf on $Y \times_s Y^t$ normalized by $i_1^* P_Y \cong \mathcal{O}_Y$ and $i_2^* P_Y \cong \mathcal{O}_{Y^t}$, where $i_1 : Y \to Y \times_s Y^t$ and $i_2 : Y^t \to Y \times_s Y^t$ are the embedding into the first and the second factor respectively. Let $H[(\lambda \times 1)^* P_Y]$ be the kernel of the S-homomorphism

$$\Lambda[(\lambda \times 1)^* P_Y] : X \times_s Y^t \to (X \times_s Y^t)^t$$

which can be identified with

$$\begin{pmatrix} o & \lambda^t \\ \lambda & o \end{pmatrix} : \quad X \times_s Y^t \to X^t \times_s Y$$

and let $e^{[(\lambda \times 1)^* P_Y]}$ be the alternating biadditive pairing

$$H[(\lambda \times 1)^* P_Y] \times_s H[(\lambda \times 1)^* P_Y] \to G_{ms}$$

(*cf.* Mumford [26], Section 1). Let $\langle \ , \ \rangle_\lambda$ be the Cartier pairing given in Theorem 1.1.

We also denote by i_1 and i_2 the embedding of $\ker(\lambda)$ and $\ker(\lambda')$ into the first and the second factor of

$$H[(\lambda \times 1)^* P_Y] = \ker(\lambda) \times_s \ker(\lambda')$$

respectively.

LEMMA 1.4. — *The following diagram is commutative :*

$$
\begin{array}{ccc}
\ker(\lambda) \times_s \ker(\lambda') & \xrightarrow{\langle \ , \ \rangle_\lambda} & G_{ms} \\
\downarrow{\scriptstyle i_1 \times i_2} & & \parallel \\
H[(\lambda \times 1)^* P_Y] \times_s H[(\lambda \times 1)^* P_Y] & \xrightarrow{e^{[(\lambda \times 1)^* P_Y]}} & G_{ms}
\end{array}
$$

Proof of Lemma 1.4. — Since everything is compatible with base extension, it is enough to prove the commutativity of the diagram for S-valued points. Suppose that x is in $\ker(\lambda)(S)$ and v is in $\ker(\lambda')(S)$. Then $(1 \times v)^* P_Y$ is the invertible sheaf on Y corresponding to v. $\lambda'(v) = o$ implies that there is an isomorphism

$$\varphi : \lambda^*(1 \times v)^* P_Y \xrightarrow{\sim} \mathcal{O}_X.$$

Then by definition (*cf.* Theorem 1.1) we get

$$\langle x, v \rangle_\lambda = T_x(\varphi) \circ \varphi^{-1}.$$

For simplicity let us denote $L = (\lambda \times 1)^\star P_Y$. Then

$$L = (\lambda \times 1)^\star P_Y = T^\star_{(x, 0)} (\lambda \times 1)^\star P_Y = T^\star_{(x, 0)} L.$$

On the other hand since $(0, v)$ is also in $H[L]$, there is an isomorphism

$$\beta : \quad L \xrightarrow{\sim} T^\star_{(0, v)} L.$$

Then by definition the commutator of $[(x, 0), 1]$ and $[(0, v), \beta]$ in the group $G[L]$ (*cf.* Mumford, *ibid.*) is equal to

$$e^L((x, 0), (0, v)) = T^\star_{(0, v)}(1) \circ \beta \circ 1^{-1} \circ T^\star_{(x, 0)}(\beta^{-1})$$

which is the automorphism

$(\bigstar)$ $$T^\star_{(x, v)} L \xrightarrow{\sim} T^\star_{(x, 0)} L \xrightarrow{\sim} L \xrightarrow{\sim} T^\star_{(0, v)} L \xrightarrow{\sim} T^\star_{(x, v)} L.$$

Note that

$$(1 \times v)^\star L = (1 \times v)^\star (\lambda \times 1)^\star P_Y = \lambda^\star (1 \times v)^\star P_Y,$$

hence we have

$$\varphi : \quad (1 \times v)^\star L \xrightarrow{\sim} \mathcal{O}_X.$$

On the other hand since P_Y is normalized, we get

$$(1 \times \varepsilon)^\star L = \lambda^\star (1 \times \varepsilon)^\star P_Y \xrightarrow[\sim]{\psi} \mathcal{O}_X,$$

where ε is the zero section $S \to Y'$. Thus if we apply $(1 \times \varepsilon)^\star$ to the sequence $(\bigstar)$, we get a commutative diagram

$$
\begin{array}{ccc}
T^\star_x (1 \times v)^\star L & \xrightarrow[\sim]{T^\star_x(\varphi)} & \mathcal{O}_X \\
{\scriptstyle T^\star_x(1 \times \varepsilon)^\star (\beta^{-1})} \downarrow \wr & & \\
T^\star_x (1 \times \varepsilon)^\star L & \xrightarrow[\sim]{T^\star_x(\psi)} & \mathcal{O}_X \\
{\scriptstyle 1} \downarrow \wr & & \wr \uparrow {\scriptstyle \langle x, \varepsilon \rangle_\lambda = 1} \\
(1 \times \varepsilon)^\star L & \xrightarrow[\sim]{\psi} & \mathcal{O}_X \\
{\scriptstyle (1 \times \varepsilon)^\star (\beta)} \downarrow \wr & & \\
(1 \times v)^\star L & \xrightarrow[\sim]{\varphi} & \mathcal{O}_X \\
{\scriptstyle 1} \downarrow \wr & & \downarrow {\scriptstyle \langle x, v \rangle_\lambda} \\
T^\star_x (1 \times v)^\star L & \xrightarrow{T^\star_x(\varphi)} & \mathcal{O}_X
\end{array}
$$

We have

$$T^\star_x[\varphi \circ (1 \times \varepsilon)^\star (\beta) \circ \psi^{-1}] = \varphi \circ (1 \times \varepsilon)^\star (\beta) \circ \psi^{-1}$$

since the latter is the multiplication by an element of

$$H^0(X, \mathcal{O}_X)^\star = H^0(S, \mathcal{O}_S)^\star.$$

Hence

$$\langle x, v \rangle_\lambda = (1 \times \varepsilon)^\star e^L((x, 0), (0, v)) = e^L((x, 0), (0, v))$$

Q. E. D.

Proof of Corollary 1.3 *continued.* — It is enough to prove

$$\langle v, k_X(x) \rangle_{\lambda^t} = \langle x, v \rangle_\lambda^{-1}$$

for all x in $\ker(\lambda)$ (S) and v in $\ker(\lambda')$ (S). By Lemma 1.4 we get

$$\langle v, k_X(x) \rangle_{\lambda^t} = e^{[(\lambda^t \times 1)^\star P_X{}^t]} ((v, 0), (0, k_X(x)))$$

and

$$\langle x, v \rangle_\lambda = e^{[(\lambda \times 1)^\star P_Y]} ((x, 0), (0, v)).$$

By the definition of λ^t and the see-saw principle we get

$$(1 \times \lambda^t)^\star P_X = (\lambda \times 1)^\star P_Y.$$

Hence it is enough to prove

$$e^{[(\lambda^t \times 1)^\star P_X{}^t]} ((v, 0), (0, k_X(x))) = e^{[(1 \times \lambda^t)^\star P_X]} ((x, 0), (0, v))^{-1}.$$

We have the following commutative diagram :

$$
\begin{array}{ccc}
Y^t \times_S X^{tt} & \xrightarrow{\lambda^t \times 1} & X^t \times_S X^{tt} \\
\scriptstyle{1 \times k_X^{-1}} \downarrow \wr & & \wr \downarrow \scriptstyle{1 \times k_X^{-1}} \\
Y^t \times_S X & \xrightarrow{\lambda^t \times 1} & X^t \times_S X \\
\scriptstyle{s} \downarrow \wr & & \wr \downarrow \scriptstyle{s} \\
X \times_S Y^t & \xrightarrow{1 \times \lambda^t} & X \times_S X^t
\end{array}
$$

where s is the morphism which exchanges the factors.

By definition,

$$P_{X^t} = [s \circ (1 \times k_X^{-1})]^\star P_X.$$

Thus

$$e^{[(\lambda^t \times 1)^\star P_X{}^t]} ((v, 0), (0, k_X(x)))$$
$$= e^{[(1 \times k_X^{-1})^\star \circ s^\star \circ (1 \times \lambda^t)^\star P_X]} ((v, 0), (0, k_X(x)))$$
$$= e^{[(1 \times \lambda^t)^\star P_X]} (s \circ (1 \times k_X^{-1}) (v, 0), s \circ (1 \times k_X^{-1}) (0, k_X(x)))$$
$$= e^{[(1 \times \lambda^t)^\star P_X]} ((0, v), (x, 0)).$$

Hence the skew-symmetry of $e^{[(1 \times \lambda^t)^\star P_X]}$ gives the required result.

Q. E. D.

Let S be a prescheme and let $\mathfrak{N}_p$ be the category of commutative finite flat group schemes over S whose rank is a power of the prime number p. Ind $(\mathfrak{N}_p)$ is the category of inductive systems of objects in $\mathfrak{N}_p$.

DEFINITION 1.5. — *p-divisible group over* S *is an objet in* Ind$(\mathfrak{N}_p)$ *which can be given by*

$$G = \varinjlim_n \{ G_n, i_n \},$$

where $i_n : G_n \to G_{n+1}$ *is an* S-*homomorphism sending* G_n *isomorphically to the kernel of* p^n *in* G_{n+1} *such that there exists a positive integer* h *(called the corank of* G*) satisfying*

$$\operatorname{rank}_S (G_n) = p^{nh} \quad \text{for all } n.$$

DEFINITION 1.6. — *Let* $G = \varinjlim_n \{ G_n, i_n \}$ *be a* p-*divisible group over* S*.*

Let $j_n : G_{n+1} \to G_n$ *be the* S-*homomorphism induced by the multiplication by* p*. Then*

$$\varinjlim_n \{ D_S (G_n), D_S (j_n) \}$$

is again a p-*divisible group over* S*. We denote it by* G^t *and call it the Serre dual of* G*. To an* S-*homomorphism* $\lambda : G \to H$ *of* p-*divisible groups over* S *we associate an* S-*homomorphism of* p-*divisible groups* $\lambda^t : H^t \to G^t$ *in an obvious way.*

DEFINITION 1.7. — *Let* X *be an abelian scheme over* S*. We denote by* $_{p^n}X$ *the kernel of* $p_X^n : X \to X$*. We define a* p-*divisible group*

$$X(p) = \varinjlim_n {}_{p^n}X,$$

where i_n *is the canonical injection. To an* S-*homomorphism* $\lambda : X \to Y$ *we associate an* S-*homomorphism* $\lambda(p) : X(p) \to Y(p)$ *in an obvious way.*

For more details we refer the reader to Serre [35] and Tate [38].

PROPOSITION 1.8. — *Let* S *be a prescheme. Let* X *be an abelian scheme over* S *such that* X^t *exists. If* p *is a prime number, there is a canonical isomorphism of* p-*divisible groups over* S

$$\nu_X : \quad X^t(p) \xrightarrow{\sim} X(p)^t,$$

where $X(p)^t$ *is the Serre dual of* $X(p)$*.*

ν_X *is functorial in* X*, i.e. if* $\lambda : X \to Y$ *is an* S-*homomorphism, then the following diagram is commutative* :

$$
\begin{array}{ccc}
X^t(p) & \xrightarrow[\sim]{\nu_X} & X(p)^t \\
{\scriptstyle \lambda^t(p)} \uparrow & & \uparrow {\scriptstyle \lambda(p)^t} \\
Y^t(p) & \xrightarrow[\sim]{\nu_Y} & Y(p)^t
\end{array}
$$

Moreover the following diagram is commutative :

$$
\begin{array}{ccc}
X(p) & \xrightarrow[\sim]{k_X} & X^{tt}(p) \\
{\scriptstyle \text{can.}} \downarrow \wr & & \wr \downarrow {\scriptstyle \nu_{X^t}} \\
X(p)^{tt} & \xrightarrow[\sim]{-(\nu_X)^t} & X^t(p)^t
\end{array}
$$

Proof. — By definition $X(p) = \varinjlim_n p^n X$ and $X^t(p) = \varinjlim_n p^n X^t$. Applying the functoriality in Corollary 1.3 (ii) to the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{p^{n+1}} & X \\
\alpha = p \downarrow & & \downarrow \beta = 1 \\
X & \xrightarrow{p^n} & X
\end{array}
$$

we get a commutative diagram

$$
\begin{array}{ccc}
p^{n+1}X^t & \xrightarrow[\sim]{\nu_{p^{n+1}}} & D_S(p^{n+1}X) \\
1_x \uparrow & & \uparrow D_s(p) \\
p^n X^t & \xrightarrow[\sim]{\nu_{p^n}} & D_S(p^n X)
\end{array}
$$

We put $\nu_X = \varinjlim_n \{\nu_{p^n_x}\}$. The functoriality of ν_X and the commutativity of the diagram in the proposition are obvious from Corollary 1.3 (ii).

<div align="right">Q. E. D</div>

Definition 1.9. — *Let $\lambda : X \to X^t$ be an S-homomorphism of abelian schemes. We define an S-homomorphism ρ_λ of p-divisible groups over S by the composite*

$$
\rho_\lambda = \nu_X \circ \lambda(p) : \quad X(p) \xrightarrow{\lambda(p)} X^t(p) \xrightarrow[\sim]{\nu_X} X(p)^t.
$$

Proposition 1.10. — *ρ_λ is additive in λ and compatible with base extension. $\rho_\lambda = 0$ if and only if $\lambda = 0$. Moreover if $\alpha : X \to Y$ is an S-homomorphism of abelian schemes and $\lambda : Y \to Y^t$ is an S-homomorphism, then we get*

$$
\rho[\alpha^t \circ \lambda \circ \alpha] = \alpha(p)^t \circ \rho_\lambda \circ \alpha(p).
$$

Proof. — First two assertions are obvious. It is clear that $\rho_\lambda = 0$ if and only if $\lambda(p) = 0$. But by Weil [40] $\lambda(p) = 0$ if and only if $\lambda = 0$. By Proposition 1.8 we have a commutative diagram

$$
\begin{array}{ccccc}
X(p) & \xrightarrow{(\alpha^t \circ \lambda \circ \alpha)(p)} & X^t(p) & \xrightarrow[\sim]{\nu_X} & X(p)^t \\
\alpha(p) \downarrow & & \alpha^t(p) \uparrow & & \alpha(p)^t \uparrow \\
Y(p) & \xrightarrow{\lambda(p)} & Y^t(p) & \xrightarrow[\sim]{\nu_Y} & Y(p)^t
\end{array}
$$

composite of the first row and the second row being equal to the left hand side and the right hand side of the equality respectively.

<div align="right">Q. E. D.</div>

DEFINITION 1.11. — *We call an S-homomorphism* $\lambda : X \to X^t$ *symmetric if the following diagram is commutative :*

$$
\begin{array}{ccc}
X & \xrightarrow{\ \lambda\ } & X^t \\
{\scriptstyle k_X}\Big\downarrow\wr & \nearrow{\scriptstyle \lambda^t} & \\
X^{tt} & &
\end{array}
$$

PROPOSITION 1.12. — *Let* X *be an abelian scheme over a prescheme* S *and let* $\lambda : X \to X^t$ *be a symmetric S-homomorphism. Then the S-homomorphism* ρ_λ *of p-divisible groups is skew-symmetric, i. e.*

$$(\rho_\lambda)^t = - \rho_\lambda$$

if we identify $X(p)$ *and* $X(p)^{tt}$ *by the canonical map.*

Proof. — By Proposition 1.8 we have a commutative diagram

$$
\begin{array}{ccc}
X(p)^t \xleftarrow{\ \lambda(p)^t\ } X^t(p)^t \xleftarrow[\sim]{\ (\nu_X)^t\ } X(p)^{tt} \\
{\scriptstyle -\nu_X}\Big\uparrow\wr \qquad {\scriptstyle -\nu_{X^t}}\Big\uparrow\wr \qquad \Big\Vert \\
X^t(p) \xleftarrow{\ \lambda^t(p)\ } X^{tt}(p) \xleftarrow[\sim]{\ k_X(p)\ } X(p)
\end{array}
$$

The composite of the first row is equal to $(\rho_\lambda)^t$ while the composite of the second row is equal to $\lambda(p)$ by assumption.

Q. E. D.

An invertible sheaf L on X gives a symmetric S-homomorphism $\Lambda(L) : X \to X^t$ defined in Mumford [27], Chapter 6.

DEFINITION 1.13. — *We define a skew-symmetric S-homomorphism of p-divisible groups* $\rho(L)$ *by*

$$\rho(L) = \rho_{\Lambda(L)} : X(p) \to X(p)^t$$

and we call it the Riemann homomorphism defined by L.

It is clear from Proposition 1.10 that $\rho(L) = 0$ if and only if $\Lambda(L) = 0$.

DEFINITION 1.14. — *An invertible sheaf* L *on* X *defines a compatible system of skew-symmetric pairings*

$$
{}_{p^n}X \times_{S} {}_{p^n}X \xrightarrow{\ e^{[L \otimes p^n]}\ } G_{mS}
$$

(*cf.* Mumford [26], Section 1), *which gives a skew-symmetric S-homomorphism of p-divisible groups*

$$e(L) : \quad X(p) \to X(p)^t.$$

We now show that these two ways of defining a skew-symmetric S-homomorphism are the same.

PROPOSITION 1.15. — *If* L *is an invertible sheaf on an abelian scheme* X, *then* [2]

$$\rho(L) = e(L).$$

Proof. — Since both are compatible with base extension, it is enough to show that they coincide on the S-valued points of $_{p^n}X$, i. e.

$$\langle x, \Lambda(L)\, y \rangle_{p_X^n} = e^{[L^{\otimes p^n}]}(x, y)$$

for all x and y in $_{p^n}X(S)$.   But by Lemma 1.4, we know

$$\langle x, \Lambda(L)\, y \rangle_{p_X^n} = e^{[(p_X^n \times 1)^* P_X]}((x, o), (o, \Lambda(L)\, y)) = e^{[P_X^{\otimes p^n}]}((x, o), (o, \Lambda(L)\, y)).$$

Here we used the fact $P_X^{\otimes m} = (m_X \times 1)^* P_X$ which can be proved easily using the see-saw principle. But the last term can also be written as

$$e^{[(1 \times \Lambda(L))^* P_X^{\otimes p^n}]}((x, o), (o, y)).$$

Since by definition

$$(1 \times \Lambda(L))^* P_X = \mu^* L \otimes \mathrm{pr}_1^* L^{-1} \otimes \mathrm{pr}_2^* L^{-1}$$

---

[2] As a consequence of Proposition 1.15, the descent theory of invertible sheaves on abelian schemes developed in Mumford [26] can be expressed in terms of the Riemann homomorphism as follows :

PROPOSITION. — *Let* L *be an invertible sheaf on an abelian scheme* X/S.   *Then there exists an invertible sheaf* L' *on* X *such that*

$$L^{\otimes p^n} \cong p_X^n \star L'$$

if and only if the Riemann homomorphism

$$\rho(L) : \quad X(p) \to X(p)^t$$

is trivial on $_{p^n}X$.

A. Weil [40] obtained this result when S is a field and p is different from the characteristic.

Once we know this proposition, we can prove the following theorem which M. Nishi obtained independently using the purely inseparable descent. The author thanks Nishi for pointing out to him the possibility of this application of the Riemann homomorphism.

THEOREM. — *Let* $\lambda : X \to X^t$ *be a homomorphism of abelian schemes.   Then* λ *is symmetric if and only if there exists an invertible sheaf* L' *on* X *such that* $\lambda = \Lambda(L')$. -

The sufficiency is obvious.   For the necessity, consider the invertible sheaf $L = (l, \lambda) \star P_X$ on X, where P_X is the normalized Poincaré invertible sheaf on $X \times_S X^t$ and $(l, \lambda) : X \to X \times_S X^t$ sending x in X to $(x, \lambda(x))$.   Then we can easily prove that when λ is symmetric, we have $\Lambda(L) = 2\lambda$.   Then taking $p = 2$, we get

$$2\rho_\lambda = \rho_{2\lambda} = \rho_{\Lambda(L)} = \rho(L).$$

Hence $\rho(L)$ is trivial on $_2X$.   By the previous proposition we get $L^{\otimes 2} = 2_X^* L'$ for an invertible sheaf L' on X.   Then

$$4\lambda = 2\Lambda(L) = \Lambda(L^{\otimes 2}) = \Lambda(2_X^* L') = 4\Lambda(L').$$

Hence $\lambda = \Lambda(L')$.

we can easily see that this is equal to $e^{(L \otimes p^n)}(x, y)$. Here we used the following fact several times : if $\alpha : X \to Y$ is an S-homomorphism of abelian schemes and if L is an invertible sheaf on Y, then $\alpha^{-1} \ker(\Lambda(L))$ is contained in $\ker \Lambda(\alpha^\star L)$ and for S-valued points x and y in $\alpha^{-1} \ker(\Lambda(L))$ we have

$$e^{\alpha^\star L}(x, y) = e^{L}(\alpha x, \alpha y).$$

<div align="right">Q. E. D.</div>

SECTION 2.

SOME AUXILIARY RESULTS IN CHARACTERISTIC p.

Let S be a prescheme of characteristic p. Let X be a contravariant functor from the category of S-preschemes to the category of sets (for simplicity we call such X an S-functor). Then even if X is not representable, we can define an S-functor $X^{(p/S)}$ and a canonical morphism of S-functors

$$F : \quad X \to \quad X^{(p/S)}$$

as follows : for an S-prescheme T, there is a morphism of preschemes $\pi_T : T \to T$, which is the identity map as a topological space and $\pi_T^\star$ takes an element in the structure sheaf to its p-th power. π_T is functorial in T, i. e. for an S-morphism $u : T_1 \to T_2$ the following diagram is commutative :

$$
\begin{array}{ccc}
T_1 & \xrightarrow{u} & T_2 \\
\pi_{T_1} \downarrow & & \downarrow \pi_T \\
T_1 & \xrightarrow{u} & T_2
\end{array}
$$

We write $\pi = \pi_S$. For an S-prescheme T we denote by (T, π) the S-prescheme whose structure morphism is the composite of the structure morphism of T with π. The functoriality of π_T implies that π_T is an S-morphism from (T, π) to T. We define an S-functor $X^{(p/S)}$ by

$$X^{(p/S)}(T) = X((T, \pi))$$

and the morphism $F : X \to X^{(p/S)}$ by

$$X(\pi_T) : \quad X(T) \to X((T, \pi)).$$

It is easy to see that if X is representable, then $X^{(p/S)}$ is represented by the S-prescheme $\mathrm{pr}_2 : X \times_S (S, \pi) \to S$ and $F : X \to X^{(p/S)}$ is given by $(\pi_X, \eta) : X \to X \times_S (S, \pi)$, where $\eta : X \to S$ is the structure morphism of X.

PROPOSITION 2.1. — *Let* S *be a prescheme of characteristic p and let* X *be an S-prescheme with a section* $\varepsilon : S \to X$ *such that* $\eta_* \mathcal{O}_X = \mathcal{O}_S$, *where* η *is the structure morphism* $X \to S$. *Then there is a canonical isomorphism* α *such that the following diagram is commutative :*

$$
\begin{array}{ccc}
\mathrm{Pic}_{X/S} & \xrightarrow{\ F\ } & [\mathrm{Pic}_{X/S}]^{(p/S)} \\
{\scriptstyle p}\downarrow & & {\scriptstyle \alpha}\downarrow{\scriptstyle \wr} \\
\mathrm{Pic}_{X/S} & \xleftarrow[\mathrm{Pic}_{F/S}]{} & \mathrm{Pic}_{[X^{(p/S)}/S]}
\end{array}
$$

Especially if $\mathrm{Pic}_{X/S}$ *is representable and flat over* S, *and* $F : \mathrm{Pic}_{X/S} \to [\mathrm{Pic}_{X/S}]^{(p/S)}$ *is an epimorphism then* (*if we identify* $[\mathrm{Pic}_{X/S}]^{(p/S)}$ *and* $\mathrm{Pic}_{[X^{(p/S)}/S]}$ *by* α) *we get*

$$ V_{[\mathrm{Pic}_{X/S}]} = \mathrm{Pic}_{F/S}. $$

(For the definition of V for flat commutative group schemes we refer the reader to Gabriel [12], SGAD 1963-1964, exposé 7 A, Section 4.3.)

Proof. — Since everything is compatible with base extension, it is enough to prove the commutativity of the diagram for S-valued points. But by definition

$$ \mathrm{Pic}_{X/S}(S) = \mathrm{Pic}(X/S), $$
$$ [\mathrm{Pic}_{X/S}]^{(p/S)}(S) = \mathrm{Pic}_{X/S}((S, \pi)) = \mathrm{Pic}(X \times_S (S, \pi)/S) $$

and

$$ \mathrm{Pic}_{[X^{(p/S)}/S]}(S) = \mathrm{Pic}(X^{(p/S)}/S) = \mathrm{Pic}(X \times_S (S, \pi)/S). $$

Moreover it is easy to see that by the identification above F is equal to

$$ (\mathrm{pr}_1/\pi)^* : \quad \mathrm{Pic}(X/S) \to \mathrm{Pic}(X \times_S (S, \pi)/S). $$

On the other hand $\mathrm{Pic}_{F/S}$ is equal to

$$ (F/1_S)^* : \quad \mathrm{Pic}(X^{(p/S)}/S) \to \mathrm{Pic}(X/S). $$

We get $(\mathrm{pr}_1/\pi) \circ (F/1_S) = (\pi_X/\pi)$. Thus

$$ (\mathrm{Pic}_{F/S}) . \alpha . F = (F/1_S)^* . (\mathrm{pr}_1/\pi)^* = (\pi_X/\pi)^*. $$

Since π_X^* and π^* are equal to the multiplication by p in the Picard groups $\mathrm{Pic}(X)$ and $\mathrm{Pic}(S)$, we are done. If $\mathrm{Pic}_{X/S}$ is representable and flat over S, then $V = V_{[\mathrm{Pic}_{X/S}]}$ is defined and $V \circ F = p$. On the other hand, we have $(\mathrm{Pic}_{F/S}) \circ F = p$.

<div align="right">Q. E. D.</div>

COROLLARY 2.2. — *Let* S *be a prescheme of characteristic p and let* X *be an abelian scheme over* S *such that* X^t *exists. Then we have*

$$ V_{(X^t)} = (F_X)^t \qquad \text{and} \qquad F_{(X^t)} = (V_X)^t. $$

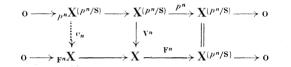*Remark.* — This corollary is stated in Matsumura-Miyanishi [25], Lemma 3. The proof there is incomplete.

PROPOSITION 2.3. — *Let* S *be a prescheme of characteristic p. If* X *is an abelian scheme over* S, *then there are exact sequences of finite flat* S-*group schemes*

(i) $0 \to {}_{V^n}X^{(p^n/S)} \xrightarrow{i} {}_{p^n}X^{(p^n/S)} \xrightarrow{v_n} {}_{F^n}X \to 0$;

(ii) $0 \to {}_{F^n}X \xrightarrow{j} {}_{p^n}X \xrightarrow{f_n} {}_{V^n}X^{(p^n/S)} \to 0$

such that $i \circ f_n = F^n$ *and* $j \circ v_n = V^n$ *for all positive integers n.*

Proof. — We have a commutative diagram whose rows are exact

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & {}_{p^n}X^{(p^n/S)} & \longrightarrow & X^{(p^n/S)} & \xrightarrow{p^n} & X^{(p^n/S)} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle v_n} & & \Big\downarrow{\scriptstyle V^n} & & \Big\| & & \\
0 & \longrightarrow & {}_{F^n}X & \longrightarrow & X & \xrightarrow{F^n} & X^{(p^n/S)} & \longrightarrow & 0
\end{array}
$$

Since V^n is an epimorphism the snake lemma implies that v_n is an epimorphism and $\ker(v_n) = \ker(V^n) = {}_{V^n}X^{(p^n/S)}$. Thus we get (i). (ii) is similarly proved.

Q. E. D.

SECTION 3.

DIEUDONNÉ MODULES.

From now on we let k be a perfect field of characteristic p. We write $X(B)$ instead of $X(\mathrm{Spec}(B))$ for a k-prescheme X and a (commutative) k-algebra B.

We denote by W the k-ring scheme of infinite Witt vectors. Then $W(k)$ is a discrete valuation ring. The Frobenius endomorphism F of W induces a ring endomorphism σ of $W(k)$.

DEFINITION 3.1. — *We denote by* A *the non-commutative ring* $W(k)$ [F, V] *defined by the relations*

$$FV = VF = p,$$
$$F\lambda = \lambda^\sigma F,$$
$$\lambda V = V\lambda^\sigma$$

for all λ *in* $W(k)$.

The scheme W has a canonical structure of left A-module scheme over k. Hence for a positive integer n the k-scheme W_n of Witt vectors of length n has a canonical structure of left A-module.

DEFINITION 3.2. — *Let* M *be a left* A-*module, and let* n *be an integer. We denote by*

$$(W(k), \sigma^n) \otimes_{W(k)} M$$

the left A-*module defined as follows : for* λ *and* λ' *in* $W(k)$ *and* x *in* M

$$\lambda' \lambda^{\sigma^n} \otimes x = \lambda' \otimes \lambda x,$$
$$\lambda(\lambda' \otimes x) = \lambda\lambda' \otimes x,$$
$$F(\lambda' \otimes x) = \lambda'^{\sigma} \otimes Fx,$$
$$V(\lambda' \otimes x) = \lambda'^{\sigma^{-1}} \otimes Vx.$$

Note that the map from M to $(W(k), \sigma^n) \otimes_{W(k)} M$ sending x to $1 \otimes x$ is σ^n-linear.

DEFINITION 3.3. — *For a positive integer* n *we denote by* C_{-n} *the left* A-*module scheme over* k *defined by*

$$C_{-n} = (W(k), \sigma^n) \otimes_{W(k)} W_n,$$

i. e. for a k-*algebra* B

$$C_{-n}(B) = (W(k)\,\sigma^n) \otimes_{W(k)} W_n(B).$$

The k-group homomorphism $\nu : W_n \to W_{n+1}$ sending

$$x = (x_0, x_1, \ldots, x_{n-1}) \quad \text{in } W_n(B)$$

to

$$\nu(x) = (0, x_0, \ldots, x_{n-1}) \quad \text{in } W_{n+1}(B)$$

is not a $W(k)$-module homomorphism, but it induces a left A-module homomorphism

$$i_n : \quad C_{-n} \to C_{-(n+1)}.$$

This defines an inductive system of left A-module functors

$$\{ C_{-n}, i_n \}.$$

DEFINITION 3.4. — *We denote by* C *the left* A-*module functor over* k

$$C = \varinjlim_n C_{-n}$$

defined by

$$C(B) = \varinjlim_n C_{-n}(B)$$

for a k-*algebra* B. *After Barsotti we call* C *the left* A-*module functor over* k *of Witt covectors.*

Note that the $W(k)$-module homomorphism $R : W_{n+1} \to W_n$ sending $x = (x_0, \ldots, x_n)$ in $W_{n+1}(B)$ to $Rx = (x_0, \ldots, x_{n-1})$ in $W_n(B)$ induces

a σ^{-1}-homomorphism of $W(k)$-modules $C_{-(n+1)} \to C_{-n}$. The induced σ^{-1}-endomorphism of $W(k)$-module on C coincides with the multiplication by V in the left A-module structure of C.

We can define C directly as follows : for a k-algebra B, $C(B)$ is the set of all sequences

$$x = (\ldots, x_{-n}, \ldots, x_{-2}, x_{-1}),$$

where x_{-n} is in B and all but a finite number of x_{-n} are zero. We define the sum $x + y$ of x and y in $C(B)$ using phantom components by

$$(x + y)^{(m)} = x^{(m)} + y^{(m)}$$

for all negative integers m, the phantom component $x^{(m)}$ of x being defined by

$$x^{(m)} = \sum_{i \geq 0} (1/p^i)\, x_{m-i}^{p^i}.$$

The action of $W(k)$ is defined by

$$(\{ a \} x)_m = a^{p^m} x_m$$

for all a in k and x in $C(B)$, where $\{ a \}$ is the element of $W(k)$ with a as the o-th component and with o as the rest of the components. The action of F and V is defined by

$$(Fx)_m = x_m^p$$
$$(Vx) = x_{m-1}$$

for all negative integers m.

It can be easily seen that the set of k-valued points $C(k)$ is canonically isomorphic as a $W(k)$-module to $B(k)/W(k)$, where $B(k)$ is the quotient field of $W(k)$ (Barsotti's bivectors with values in k). Moreover the Frobenius endomorphism $F : C \to C$ defines a σ-endomorphism of $W(k)$-modules on $C(k)$ which we also denote by σ.

DEFINITION 3.5. — *Let* M *be a left* A-*module. We define a left* A-*module* $D(M)$ *as the set of* $W(k)$-*module homomorphisms from* M *to* $C(k)$, *i. e.*

$$D(M) = \mathrm{Hom}_{W(k)}(M, C(k)),$$

where elements of $W(k)$ *operate on* $D(M)$ *in the usual way and* F *and* V *operate as follows : for* α *in* $D(M)$ *and* x *in* M

$$(F\alpha)(x) = \alpha(Vx)^\sigma,$$
$$(V\alpha)(x) = \alpha(Fx)^{\sigma^{-1}}.$$

DEFINITION 3.6. — *Let* M *be a left* A-*module.* *We define a left* A-*module* M^t *by*

$$M^t = \text{Hom}_{W(k)}(M, W(k)),$$

where elements of $W(k)$ *operate in the usual way and* F *and* V *operate as follows : for* α *in* M^t *and* x *in* M

$$(F\alpha)(x) = \alpha(Vx)^\sigma,$$
$$(V\alpha)(x) = \alpha(Fx)^{\sigma^{-1}}.$$

DEFINITION 3.7. — *Let* M *be a left* A-*module.* *We define a left* A-*module structure on*

$$C(k) \otimes_{W(k)} M \quad [resp. \; B(k) \otimes_{W(k)} M]$$

by operating with $W(k)$ *in the usual way and operating with* F *and* V *in the following way : for* c *in* $C(k)$ [*resp. in* $B(k)$]

$$F(c \otimes x) = c^\sigma \otimes Fx,$$
$$V(c \otimes x) = c^{\sigma^{-1}} \otimes Vx.$$

DEFINITION 3.8. — *Let* M *be a left* A-*module.* *We define a left* A-*module structure on the* p-*adic Tate module*

$$T_p(M) = \text{Hom}_{W(k)}(C(k), M)$$

by operating with elements of $W(k)$ *in the usual way and by operating with* F *and* V *as follows : for* c *in* $C(k)$ *and* α *in* $T_p(M)$

$$(F\alpha)(c) = F(\alpha(c^{\sigma^{-1}})),$$
$$(V\alpha)(c) = V(\alpha(c^\sigma)).$$

PROPOSITION 3.9.

(i) D *induces an anti-equivalence between the category of left* A-*modules of* $W(k)$-*finite type and the category of left* A-*modules of* $W(k)$-*cofinite type* [*i. e. isomorphic to a sub-*$W(k)$-*module of a finite direct sum of* $C(k)$]. *This induces an anti-equivalence between the category of left* A-*modules* $W(k)$-*free of* $W(k)$-*finite rank and the category of left* A-*modules* $W(k)$-*divisible of* $W(k)$-*finite corank* [*i. e. isomorphic as* $W(k)$-*module to a finite direct sum of* $C(k)$].

(ii) D *induces a dualizing functor from the category of left* A-*modules of* $W(k)$-*finite length into itself, i. e.* D *is a contravariant functor from the category into itself such that* DD *is isomorphic to the identity functor.*

(iii) t *induces a dualizing functor from the category of left* A-*modules* $W(k)$-*free of* $W(k)$-*finite rank into itself.*

(iv) *For a left A-module* M *there is a canonical homomorphism of left A-modules*

$$C(k) \otimes_{W(k)} T_p(M) \to M$$

which is an isomorphism when M *is* $W(k)$-*divisible of* $W(k)$-*finite corank.*

(v) *For a left A-module* M *there is a canonical homomorphism of left A-modules*

$$M \to T_p(C(k) \otimes_{W(k)} M)$$

which is an isomorphism when M *is* $W(k)$-*free of* $W(k)$-*finite rank.*

(vi) *For a left A-module* M *of* $W(k)$-*finite type there are canonical isomorphisms of left A-modules*

$$M^t \xrightarrow{\sim} D(C(k) \otimes_{W(k)} M) \xrightarrow{\sim} T_p D(M).$$

(vii) *For a left A-module* M *of* $W(k)$-*cofinite type there exists a canonical isomorphism of left A-modules*

$$(DM)^t \xrightarrow{\sim} T_p M.$$

Proof. — Obvious.

DEFINITION 3.10. — *We denote by* $\mathfrak{U}$ *the category of commutative affine k-group schemes of finite type over k which are killed by some power of* V *(i. e. commutative unipotent algebraic group schemes over k).*

DEFINITION 3.11. — *We denote by* $\mathfrak{N}$ *the category of commutative finite k-group schemes. The Cartier dualizing functor* D *from* $\mathfrak{N}$ *into itself is defined by*

$$D(G) = \mathcal{H}om_{k\text{-group} \atop \text{schemes}}(G, G_{mk}),$$

i. e. for a k-algebra B

$$D(G)(B) = \mathcal{H}om_{B\text{-group} \atop \text{schemes}}(G_B, G_{mB}).$$

It is well known (*cf.* Gabriel [13], Manin [24] and Oort [31]) that $\mathfrak{N}$ can be decomposed as a product

$$\mathfrak{N} = \mathfrak{N}_{rr} \times \mathfrak{N}_{rl} \times \mathfrak{N}_{ll} \times \mathfrak{N}_{lr},$$

where $\mathfrak{N}_{rr}$ is the subcategory of commutative reduced finite k-group schemes whose Cartier dual is also reduced, $\mathfrak{N}_{rl}$ is the subcategory of commutative reduced finite k-group schemes whose Cartier dual is local (i. e. the spectrum of a local ring), $\mathfrak{N}_{ll}$ is the subcategory of commutative local finite k-group schemes whose Cartier dual is also local, and finally $\mathfrak{N}_{lr}$ is the subcategory of commutative local finite k-group schemes whose Cartier dual is reduced.

$\mathfrak{N}_{rl}$ and $\mathfrak{N}_{ll}$ are subcategories of $\mathfrak{U}$.

As defined in Section 1, $\mathfrak{N}_p$ is the subcategory

$$\mathfrak{N}_p = \mathfrak{N}_{rl} \times \mathfrak{N}_{ll} \times \mathfrak{N}_{lr}$$

consisting of commutative finite k-group schemes whose rank over k is a power of the characteristic p.

D preserves $\mathfrak{N}_{rr}$, $\mathfrak{N}_{ll}$ and $\mathfrak{N}_p$, and interchanges $\mathfrak{N}_{rl}$ and $\mathfrak{N}_{lr}$.

Finally we denote by $\mathscr{P}$ the category

$$\mathscr{P} = \mathfrak{U} \times \mathfrak{N}_{lr}$$

of commutative k-group schemes of finite type over k killed by some power of p.

For k-group schemes G and G' we denote by

$$\mathrm{Hom}_{k-gr}(G, G')$$

the set of homomorphisms of k-group schemes from G to G'. We also denote by

$$\mathrm{Hom}_{k-gr}(G, C)$$

the inductive limit

$$\varinjlim_{n} \mathrm{Hom}_{k-gr}(G, C_{-n}),$$

where C is the covector functor in Definition 3.4 considered as an ind-object of commutative k-group schemes.

DEFINITION 3.12. — *For a k-group scheme G we define a left A-module* M(G) *by*

$$M(G) = \mathrm{Hom}_{k-gr}(G, C) \oplus \left\{ W(\overline{k}) \otimes_{\mathbf{Z}} \mathrm{Hom}_{\overline{k}-gr}(G_{\overline{k}}, G_{m\overline{k}}) \right\}^{\mathrm{Gal}(\overline{k}/k)},$$

where $\overline{k}$ is an algebraic closure of k and $\mathrm{Gal}(\overline{k}/k)$ is the Galois group. We give the structure of left A-module on M(G) as follows : on the first factor we give the left A-module structure induced from the structure of left A-module on the functor C in Definition 3.4. As for the second factor let λ be in W(k), λ' in $W(\overline{k})$ and x in

$$\mathrm{Hom}_{\overline{k}-gr}(G_{\overline{k}}, G_{m\overline{k}}).$$

Then

$$\lambda(\lambda' \otimes x) = \lambda\lambda' \otimes x,$$
$$F(\lambda' \otimes x) = \lambda'^\sigma \otimes px,$$
$$V(\lambda' \otimes x) = \lambda'^{\sigma^{-1}} \otimes x.$$

These induce a left A-module structure on the $\mathrm{Gal}(\overline{k}/k)$-invariants. We call M(G) *the Dieudonné module of* G.

Note that if G is a unipotent k-group scheme the second factor in $M(G)$ is zero while if G is semi-simple k-group scheme the first factor in $M(G)$ is zero.

PROPOSITION 3.13. — *Let* G *be a* k-*group scheme.   Then there is a canonical isomorphism of left* A-*modules*

$$M(G^{(p)}) \xrightarrow{\sim} (W(k), \sigma) \otimes_{W(k)} M(G).$$

If we identify those modules by this isomorphism, then the k-*homomorphisms*

$$G \xrightarrow{F} G^{(p)} \xrightarrow{V} G$$

become

$$M(G) \xleftarrow{M(F)} (W(k), \sigma) \otimes_{W(k)} M(G) \xrightarrow{M(V)} M(G),$$

where $M(F)$ *sends* $a \otimes x$ *to* aFx *for* a *in* $W(k)$ *and* x *in* $M(G)$, *while* $M(V)$ *sends* x *in* $M(G)$ *to* $1 \otimes Vx$.

Proof. — As remarked in Section 2, $G^{(p)}$ is the base extension of G by (k, σ) over k.   Hence there is a σ-isomorphism of $W(k)$-modules

$$\mathrm{Hom}_{k-gr}(G, C) \xrightarrow{(p)} \mathrm{Hom}_{k-gr}(G^{(p)}, C^{(p)}).$$

But we know that $C^{(p)} = C$ and that

$$x^{(p)} \circ F = F \circ x,$$
$$x \circ V = V \circ x^{(p)}$$

for all x in $\mathrm{Hom}_{k-gr}(G, C)$.   The same is true for the second factor of M. Thus we get a canonical σ-isomorphism of $W(k)$-modules

$$M(G) \xrightarrow{(p)} M(G^{(p)})$$

such that

$$M(F) \, x^{(p)} = Fx,$$
$$M(V) \, x = V x^{(p)}$$

for all x in $M(G)$.   The rest of the proof is obvious.

Q. E. D.

Let K/k be a perfect extension field.   We denote by M^K and A^K the corresponding notions over K.   Then there is a canonical homomorphism of A^K-modules

$(\star)$ $W(K) \otimes_{W(k)} M(G) \to M^K(G_K)$

for a k-group scheme G.   We say the Dieudonné module M is compatible with perfect base field extensions if $(\star)$ is an isomorphism for all K/k.

THEOREM 3.14 (Dieudonné-Cartier).

(i) M *induces an anti-equivalence from the category* $\mathcal{U}$ *to the category of left A-modules of A-finite type killed by some power of* V. M *is compatible with perfect base field extensions for* $\mathcal{U}$.

(ii) M *induces an anti-equivalence from the category* $\mathcal{N}_{rl}$ *to the category of left A-modules of* W(k)-*finite length killed by some power of* V *and on which* F *acts bijectively.* M *is compatible with perfect base field extensions for* $\mathcal{N}_{rl}$. *Moreover we have*

$$(\bigstar\bigstar) \qquad\qquad \mathrm{rank}_k(G) = p^{[\mathrm{length}_{W(k)}(M(G))]}$$

for G *in* $\mathcal{N}_{rl}$.

(iii) M *induces an anti-equivalence from the category* $\mathcal{N}_{ll}$ *to the category of left A-modules of* W(k)-*finite length killed by some power of* F *and* V. M *is compatible with perfect base extensions for* $\mathcal{N}_{ll}$. *Moreover the formula* $(\bigstar\bigstar)$ *holds for* G *in* $\mathcal{N}_{ll}$.

Proof. — The first part of (i) is proved in Sharma [37], exposé 11, Theorem 8.4. Actually the Dieudonné-Cartier Theorem proved there is much more general, i. e. we can omit the finite type assumption both from $\mathcal{U}$ and from the category of left A-modules. By devissage it is enough to prove $(\bigstar)$ when G is killed by V. Then, by definition,

$$M(G) = (W(k), \sigma) \otimes_{W(k)} \mathrm{Hom}_{k-gr}(G, G_a).$$

Hence it is enough to prove that the canonical K-homomorphism

$$K \otimes_k \mathrm{Hom}_{k-gr}(G, G_a) \to \mathrm{Hom}_{K-gr}(G_K, G_{aK})$$

is an isomorphism. As remarked by F. Oort this can be proved as follows : $\mathrm{Hom}_{k-gr}(G, G_a)$ is equal to the kernel of the k-linear homomorphism

$$H^0(G, \mathcal{O}_G) \xrightarrow{\mu^\star - p_1^\star - p_2^\star} H^0(G, \mathcal{O}_G) \otimes_k H^0(G, \mathcal{O}_G),$$

where μ is the group law of G and p_1 and p_2 are projections. If we apply $K \otimes_k$ to this homomorphism we get the corresponding K-linear homomorphism for G_K. Since K/k is flat we are done.

Except for $(\bigstar\bigstar)$, the assertions (ii) and (iii) follow immediately from (i). To prove $(\bigstar\bigstar)$ in (ii) and (iii) we may assume by $(\bigstar)$ that k is algebraically closed and that G is a simple object. But $(\bigstar\bigstar)$ can be easily checked for unique simple object $Z/(p)$ and α_p in $\mathcal{N}_{rl}$ and $\mathcal{N}_{ll}$ respectively.

Q. E. D,

COROLLARY 3.15. — *For G in $\mathfrak{N}_{lr}$ we get a canonical isomorphism of left A-modules*

$$M(G) \cong DMD(G),$$

where D on the right hand side is the Cartier dualizing functor in Definition 3.11, while D on the left hand side is the one in Definition 3.5. In particular M induces an anti-equivalence from the category $\mathfrak{N}_{lr}$ to the category of left A-modules of $W(k)$-finite length killed by some power of F and on which V acts bijectively. M is compatible with perfect base field extensions for $\mathfrak{N}_{lr}$. Moreover the formula ($\bigstar\bigstar$) holds for G in $\mathfrak{N}_{lr}$.

Proof. — By definition we get

$$\mathrm{Hom}_{\bar{k}-gr}\left(G_{\bar{k}}, G_{m\bar{k}}\right) = D(G)(\bar{k}) = D(G)_{\bar{k}}(\bar{k}).$$

When G is in $\mathfrak{N}_{lr}$, $D(G)$ is in $\mathfrak{N}_{rl}$. Hence by definition and Theorem 3.14 (i) we get

$$W(\bar{k}) \otimes_{W(k)} M(DG) \cong M^{\bar{k}}\left(D(G)_{\bar{k}}\right)$$

$$= \mathrm{Hom}_{\substack{\text{abelian} \\ \text{groups}}}\left(D(G)_{\bar{k}}(\bar{k}), C(\bar{k})\right)$$

$$= \mathrm{Hom}_{W(\bar{k})}\left(W(\bar{k}) \otimes_{\mathbf{Z}} D(G)(\bar{k}), C(\bar{k})\right)$$

$$= D^{\bar{k}}\left(W(\bar{k}) \otimes_{\mathbf{Z}} D(G)(\bar{k})\right).$$

By canonicalness we thus get an isomorphism of left A-Gal$\left(\bar{k}/k\right)$-modules

$$W(\bar{k}) \otimes_{W(k)} DMD(G) \cong D^{\bar{k}}\left(W(\bar{k}) \otimes_{W(k)} M(DG)\right)$$

$$\cong W(\bar{k}) \otimes_{\mathbf{Z}} D(G)(\bar{k}).$$

Taking Gal$\left(\bar{k}/k\right)$-invariants on both sides we get a canonical isomorphism

$$DMD(G) = M(G).$$

Compatibility with base field extensions and the formula ($\bigstar\bigstar$) follows easily from this and Theorem 3.14 (ii).

Q. E. D.

COROLLARY 3.16. — *M induces an anti-equivalence from the category $\mathfrak{N}_p$ to the category of left A-modules of $W(k)$-finite length. M is compatible with perfect base field extensions for $\mathfrak{N}_p$. The formula ($\bigstar\bigstar$) holds for G in $\mathfrak{N}_p$.*

Proof. — Immediate from Theorem 3.14, (ii), (iii) and Corollary 3.15.

DEFINITION 3.17. — *M induces an anti-equivalence from the category* Ind$(\mathfrak{N}_p)$ *to the category* Pro [*left A-modules of $W(k)$-finite length*] *as follows : for* $G = \varinjlim_n G_n$ *in* Ind$(\mathfrak{N}_p)$ *we put*

$$M(G) = \varprojlim_n M(G_n).$$

We also call M(G) *the Dieudonné module of* G.

It is obvious that M is compatible with perfect base field extensions for G in $\mathrm{Ind}(\mathfrak{N}_p)$ and that Proposition 3.13 also holds for G in $\mathrm{Ind}(\mathfrak{N}_p)$.

PROPOSITION 3.18. — *Let* $\mathrm{G} = \varinjlim_{n} \mathrm{G}_n$ *be a p-divisible group over* k

(*cf. Definition* 1.5) *of corank* h. *Then* M(G) *is a left* A-*module* W(k)-*free of rank* h *with the usual p-adic topology.*

Proof. — Obvious.

Remark. — On the subcategory $\mathrm{Ind}(\mathfrak{N}_{ll})$ our Dieudonné module M(G) gives rise to the Dieudonné module over W(k) [[F, V]] defined in Gabriel [13] The definition of the Dieudonné module on $\mathrm{Ind}(\mathfrak{N}_p)$ is given in Manin [24] when k is algebraically closed.

THEOREM 3.19. — *Let* G *be in* $\mathfrak{N}_p$. *Then there exists a canonical isomorphism of left* A-*modules*

$$d(\mathrm{G}) : \quad \mathrm{MD}(\mathrm{G}) \xrightarrow{\sim} \mathrm{DM}(\mathrm{G}),$$

where D *on the left hand side is the Cartier dualizing functor in Definition* 3.11, *while* D *on the right hand side is the one in Definition* 3.5. *Moreover the following diagram is commutative:*

$$
\begin{array}{ccc}
\mathrm{M}(\mathrm{G}) & \xleftarrow{\mathrm{M}(\mathrm{can.})}_{\sim} & \mathrm{MDD}(\mathrm{G}) \\
\text{can.} \downarrow \wr & & \wr \downarrow d(\mathrm{D}(\mathrm{G})) \\
\mathrm{DDM}(\mathrm{G}) & \xrightarrow[\sim]{\mathrm{D}(d(\mathrm{G}))} & \mathrm{DMD}(\mathrm{G})
\end{array}
$$

where can. *denotes the obvious canonical isomorphism.*

Proof. — It is enough to prove the Theorem in the following three cases separately : (i) G is in $\mathfrak{N}_{rl}$; (ii) G is in $\mathfrak{N}_{ll}$, and (iii) G is in $\mathfrak{N}_{lr}$. The cases (i) and (iii) follow easily from Corollary 3.15. It remains to prove in case (ii).

We define the dualizing functor T from the category of left A-modules of W(k)-finite length into itself by

$$\mathrm{T}(\mathrm{M}(\mathrm{G})) = \mathrm{M}(\mathrm{D}(\mathrm{G}))$$

for all G in $\mathfrak{N}_p$.

LEMMA 3.20. — *Let* A *and* B *be left Nœtherian rings.*

(i) *Let* T *be a contravariant additive functor from the category of left* A-*modules of finite type to that of left* B-*modules. Then there is a morphism of functors*

$$\beta : \quad \mathrm{T} \to \mathrm{T}' = \mathrm{Hom}_{\mathrm{A}}(?, \mathrm{T}(\mathrm{A})),$$

where T(A) *is a left* A-*module via* T(R_a) *for* a *in* A (R_a *is the right multiplication by* a) *and the right hand side is given the structure of left* B-*module via that of* T(A). T *is left exact if and only if* β *is an isomorphism.*

(ii) *Let* T *be a contravariant additive functor from the category of left* A-*modules of finite type to that of left* B-*modules of finite type, and let* S *be a contravariant additive functor from the category of left* B-*modules of finite type to that of left* A-*modules of finite type such that there is a morphism of functors*

$$\rho : \quad I \to ST$$

(I *is the identity functor on the category of left* A-*modules of finite type*). *If we denote by* $\alpha : S \to S'$ *and* $\beta : T \to T'$ *the morphisms of functors we get from* (i), *then the following diagram is commutative :*

$$
\begin{array}{ccc}
I & \xrightarrow{\ \rho\ } & ST \\
\gamma \downarrow & & \downarrow \alpha T \\
S'T' & \xrightarrow{\ S'\beta\ } & S'T
\end{array}
$$

where γ *is defined as follows : let* φ *be the* B-*homomorphism* $T(A) \to S(B)$ *which is the image of* 1_A *by*

$$A \xrightarrow{\ \rho\ } ST(A) \xrightarrow{\alpha(T(A))} S'T(A) = \mathrm{Hom}_B(T(A), S(B)).$$

Then for a left A-*module* M *of finite type* γ *is the* A-*homomorphism*

$$M \to S'T'(M) = \mathrm{Hom}_B(\mathrm{Hom}_A(M, T(A)), S(B))$$

such that for x *in* M *and* f *in* $T'(M) = \mathrm{Hom}_A(M, T(A))$

$$\gamma(x)(f) = \varphi(f(x)).$$

(iii) *Suppose* $\{m_i\}$ *is a decreasing sequence of two-sided ideals in* A. *Let* T *be a contravariant additive functor from the category of left* A-*modules of finite type annihilated by one of the two-sided ideals to the category of left* B-*modules. Then there is a morphism of functors*
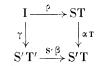
$$\beta : \quad T \to T' = \mathrm{Hom}_A\left(?, \varinjlim_i T(A/m_i)\right),$$

where as in (i), $\varinjlim_i T(A/m_i)$ *is a left* A-*module via* $T(R_a)$ *for* a *in* A *and the right hand side is given the left* B-*module structure via that of* $\varinjlim_i (T(A/m_i))$. T *is left exact if and only if* β *is an isomorphism.*

(iv) *Suppose* $\{m_i\}$ *and* $\{n_i\}$ *are decreasing sequences of two-sided ideals in* A *and* B *respectively. Let* T *be a contravariant additive functor from the category of left* A-*modules of finite type annihilated by one of the two-sided ideals to the category of left* B-*modules of finite type annihilated by one of the two-sided ideals. Let* S *be a contravariant additive functor from the latter category to the former category such that there is a morphism of functors*

$$\rho : 1 \to ST.$$

If we denote by $\alpha : S \to S'$ *and* $\beta : T \to T'$ *the morphisms of functors we get from* (iii), *then the following diagram is commutative :*

$$
\begin{array}{ccc}
I & \xrightarrow{\ \rho\ } & ST \\
\gamma \downarrow & & \downarrow \alpha T \\
S'T' & \xrightarrow{\ S'\beta\ } & S'T
\end{array}
$$

where γ *is defined as follows : let* φ_i *be the* B*-module homomorphism*

$$
T(A/m_i) \to \varinjlim_j S(B/n_j)
$$

which is the image of 1 *by*

$$
A/m_i \xrightarrow{\rho} ST(A/m_i) \xrightarrow{\alpha T} S'T(A/m_i).
$$

Then $\{\varphi_i\}$ *are compatible and definies a* B*-homomorphism*

$$
\varphi : \ \varinjlim_i T(A/m_i) \to \varinjlim_j S(B/n_j).
$$

Then for a left A*-module* M *of finite type annihilated by one of the two-sided ideals* γ *is the* A*-homomorphism*

$$
M \to S'T'(M)
$$

such that for x *in* M *and* f *in* $T'(M)$

$$
\gamma(x)(f) = \varphi(f(x)).
$$

Proof of Lemma 3.20. — This is just a generalization of the result in Grothendieck [18], Section 4 to the non-commutative case. Let M be a left A-module. Let h be a homomorphism from M to $\mathrm{Hom}_A(A, M)$ defined by $h_x(a) = ax$. Then we get a homomorphism

$$
M \xrightarrow{Th} \mathrm{Hom}_B(T(M), T(A))
$$

sending x to $T(h_x)$. Hence we have a homomorphism of B-modules

$$
T(M) \xrightarrow{\beta} \mathrm{Hom}(M, T(A))
$$

defined by $\beta(u)(x) = T(h_x)(u)$ for u in $T(M)$ and x in M. Since $R_{aa'} = R_{a'} \circ R_a$ and $h_x \circ R_a = h_{ax}$ for a and a' in A, u in $T(M)$ and x in M, we get $\beta(u)(ax) = T(R_a)\beta(u)(x)$. Thus if we give $T(A)$ a left A-module structure by $T(R_a)$, we see that the image of β is in $\mathrm{Hom}_A(M, T(A))$. The rest of (i) is straightforward.

As for (ii) the diagram is commutative by definition for free A-modules of finite rank. Hence it is also commutative for any A-module of finite type, since an A-module of finite type is a quotient of a free A-module of finite rank.

(iii) Let M be a left A-module of finite type annihilated by m_n. Then (i) implies that we have a homomorphism of B-modules

$$\beta_i : \quad T(M) \to \mathrm{Hom}_A(M, T(A/m_i))$$

for i greater than n. It is easy to see that β_i are compatible with the inductive system $\{T(A/m_i)\}$. Thus we have a homomorphism of B-modules

$$\beta : \quad T(M) \to \varinjlim_i \mathrm{Hom}_A(M, T(A/m_i)).$$

The right hand side is equal to $\mathrm{Hom}_A\left(M, \varinjlim_i T(A/m_i)\right)$, since M is an A-module of finite type. The rest of (iii) and (iv) is clear.

<div align="right">Q. E. D.</div>

To apply Lemma 3.20 to the proof of Theorem 3.19 we need the following: as before let W be the k-group scheme of infinite Witt vectors. We denote by W' the subfunctor of W (considered as a contravariant functor from the category of k-algebras to the category of sets) defined by

$$W'(B) = \left\{ b = (b_0, b_1, \ldots) \,\middle|\, \begin{array}{l} b_n = 0 \text{ for all but a finite number of } n, \\ b_n \quad \text{nilpotent element in B for all } n \end{array} \right\}$$

for a k-algebra B. W'(B) is an ideal in W(B). In fact the addition and multiplication in W(B) is defined by

$$(x + y)_n = s_n(x_0, \ldots, x_n; y_0, \ldots, y_n)$$
$$(xy)_n = m_n(x_0, \ldots, x_n : y_0, \ldots, y_n),$$

where if we define the weight of x_i and y_i to be both equal to p^i, s_n is isobaric of weight p^n in $\{x_0, \ldots, x_n, y_0, \ldots, y_n\}$ and m_n is isobaric of weight p^n both in $\{x_0, \ldots, x_n\}$ and in $\{y_0, \ldots, y_n\}$. The Artin-Hasse exponential series E defines a homomorphism of k-group functors

$$E : \quad W' \to G_m$$

as follows : let $e(z)$ be the formal power series in one variable z defined by

$$e(z) = \exp\left(-\sum_{i \geq 0} (1/p^i) z^{p^i} \right).$$

Then it is well known (*cf.* Serre [32]) that

$$e(z) = \prod_{(m, p) = 1} (1 - z^m)^{\mu(m)/m},$$

where $\mu(m)$ is the Möbius function. Thus $e(z)$ is a formal power series with coefficients in $\mathbf{Z}_p$. For a k-algebra B, $e(z)$ defines a map from the set of nilpotent elements in B to $G_m(B) = B^\star$.

The Artin-Hasse exponential series is the series

$$E(x) = \prod_{n \geq 0} e(x_n)$$

in an infinite number of variables $x = (x_0, x_1, \dots)$. It is easy to see that

$$E(x) = \exp\left(-\sum_{m \geq 0} (1/p^m) x^{(m)}\right),$$

where

$$x^{(m)} = \sum_{0 \leq i \leq m} p^i x_i^{p^{m-i}}$$

for a non-negative integer m. Thus $E(x)$ defines a homomorphism from the additive group $W'(B)$ to the multiplicative group $G_m(B)$.

From the definition it is easy to see that $E(Vx) = E(x)$.

We now define a biadditive pairing

$$W' \times W \to G_m$$

by sending (u, x) in $W'(B) \otimes W(B)$ to $E(ux)$ in $G_m(B)$. Here we use the fact that $W'(B)$ is an ideal in $W(B)$. We can easily see that

$$E((Vu)x) = E(V(uFx)) = E(u(Fx)),$$
$$E(uVx) = E(V([Fu]x)) = E((Fu)x).$$

Hence we get a homomorphism of k-group functors

$$\xi : W' \to \mathcal{H}om_{k-gr}(W, G_m).$$

PROPOSITION 3.21.

(i) *The homomorphism ξ of k-group functors is an isomorphism.*

(ii) *ξ induces an isomorphism of k-group functors*

$$\xi_n : {}_{F^n}W' \xrightarrow{\sim} \mathcal{H}om_{k-gr}(W_n, G_m).$$

(iii) *Let $W_{n,m}$ be the kernel ${}_{F^m}W_n$. Then the biadditive pairing*

$$W_{m,n} \times W_{n,m} \to G_m$$

sending (u, x) in $W_{m,n}(B) \times W_{n,m}(B)$ to $E(g_m(u) g_n(x))$ in $G_m(B)$ for a k-algebra B is non-degenerate, i. e. defines an isomorphism

$$\xi_{n,m} : W_{m,n} \xrightarrow{\sim} D(W_{n,m}),$$

where g_n is the section from W_n to W sending

$$x = (x_0, \dots, x_{n-1}) \quad in \ W_n(B) \qquad to \qquad g_n(x) = (x_0, \dots, x_{n-1}, 0, 0, \dots) \quad in \ W(B).$$

Moreover if we denote by v and f the homomorphisms (for $n \leq n'$ and $m \leq m'$)

$$v : \quad W_{n,m} \to W_{n',m'},$$
$$f : \quad W_{m',n'} \to W_{m,n}$$

sending $x = (x_0, \ldots, x_{n-1})$ *in* $W_{n,m}(B)$ *to* $vx = (0, 0, \ldots, 0, x_0, \ldots, x_{n-1})$ *in* $W_{n',m'}(B)$ *and*

$u = (u_0, \ldots, u_{m'-1})$ *in* $W_{m',n'}(B)$ *to* $fu = (u_0^{p^{n'-n}}, \ldots, u_{m-1}^{p^{n'-n}})$ *in* $W_{m,n}(B)$,

then the following diagram is commutative :

$$
\begin{array}{ccc}
W_{m,n} & \xleftarrow{\quad f \quad} & W_{m',n'} \\
\xi_{n,m} \downarrow & & \downarrow \xi_{n',m'} \\
D(W_{n,m}) & \xleftarrow{\ D(v)\ } & D(W_{n',m'})
\end{array}
$$

Proof of Proposition 3.21. — ξ is injective. In fact if $x = (t, 0, 0, \ldots)$ then $ux = (u_0 t, u_1 t^p, u_2 t^{p^2}, \ldots)$. Hence $E(ux) = \prod_n e(u_n t^{p^n})$. If u_i is the first non-zero component of u, then

$$E(ux) \equiv 1 - u_i t^{p^i} \bmod t^{p^{i+1}}.$$

ξ is surjective. In fact for a k-algebra B an element L of

$$\mathcal{H}om_{k-gr}(W, G_m)(B) = \mathrm{Hom}_{k-gr}(W_B, G_{m\,B})$$

is given by a polynomial $L(X)$ in $B[X_0, X_1, \ldots]$ such that

$$L(X + Y) = L(X) L(Y)$$

and $L(0) = 1$ (here $X + Y$ is the Witt addition). We have a commutative diagram

$$
\begin{array}{ccc}
W_B & \xrightarrow{\quad L \quad} & G_{m\,B} \\
V \uparrow & & \uparrow V = 1 \\
W_B = W_B^{(p/B)} & \xrightarrow[L^{(p/B)}]{} & G_{m\,B}^{(p/B)} = G_{m\,B}
\end{array}
$$

where $L^{(p/B)}$ is given by the polynomial $L^p(X)$ whose coefficients are p-th power of the corresponding coefficients of $L(X)$. Thus we get $L(VX) = L^p(X)$. Since $L(X)$ contains only a finite number of variables, there exists an integer n such that $L(V^n X) = 1$. Thus
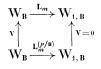
$$L^{p^n}(X) = 1,$$

i. e. the coefficients of the non-constant terms are all nilpotent in B. Let us write

$$L(X) = 1 + L_m(X) + L'(X),$$

where $L_m(X)$ is the sum of terms of the least positive weight m and $L'(X)$ is the sum of terms of higher weight (where as before weight of X_n is p^n). Then we get

$$L_m(X+Y) = L_m(X) + L_m(Y),$$

i. e. L_m defines a homomorphism $L_m : W_B \to W_{1,B}$. By the commutative diagram,

$$
\begin{array}{ccc}
W_B & \xrightarrow{\ L_m\ } & W_{1,B} \\
{\scriptstyle V}\uparrow & & \uparrow{\scriptstyle V=0} \\
W_B & \xrightarrow[\ L_m^{(p/B)}\ ]{} & W_{1,B}
\end{array}
$$

we conclude that $L_m(VX) = 0$. Hence L_m factors through the canonical projection $R : W_B \to W_{1,B}$, i. e. $L_m(X)$ is a polynomial only in X_0 which is additive and of weight m. Hence there exists a positive integer h and a nilpotent element a in B such that

$$L_m(X) = aX_0^{p^h}.$$

Then since

$$E((V^h\{a\})\,x) = E(\{a\}\,F^h x) \equiv 1 - ax_0^{p^h} \bmod \text{weight} > p^h$$

we get

$$L(x) \equiv E((V^h\{-a\})\,x) \quad \bmod \text{weight} > p^h.$$

If we proceed in this fashion, we finally get

$$L(x) = E(ux)$$

for some u in $W(B)$.

We still have to show that u is actually in $W'(B)$. But first of all since $L(x)$ only involves a finite number of variables, we get

$$0 = L(V^n x) = E(uV^n x) = E((F^n u)\,x)$$

for some positive integer n. Thus $F^n u = 0$, hence the components of u are all nilpotent. We now show that all but a finite number of components of u are actually zero.

Since $x = \displaystyle\sum_{r=0}^{\infty} V^r\{x_r\}$, we have

$$E(ux) = \prod_{r=0}^{\infty} E((F^r u)\,\{x_r\}).$$

Therefore it is enough to show that given u in $W(B)$ such that $F^n u = 0$ for some positive integer n, then the formal power series $E(u\{t\})$ in one variable t is a polynomial in t, if and only if all but a finite number of components of u are zero.

Let $A(m)$ be the coefficient of t^m in $E(u\{t\})$, and let the weight of $A(m)$ be m. As before the weight of u_r is p^r. Then $A(m)$ is a polynomial in $\{u_0, u_1, \ldots\}$ isobaric of weight m. Conversely u_r is a polynomial in $\{A(0), A(1) \ldots\}$ isobaric of weight p^r (cf. Serre [32], chap. V, No. 17, and Bergman [29], lecture 26, Section E).

Our assertion follows immediately from this fact.

We now prove (ii). The exact sequence

$$0 \to W \xrightarrow{V^n} W \to W_n \to 0$$

and $E((Fu)x) = E(u(Vx))$ give a commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{H}om_{k-gr}(W_n, G_m) & \longrightarrow & \mathcal{H}om_{k-gr}(W, G_m) & \xrightarrow{V^{n*}} & \mathcal{H}om_{k-gr}(W, G_m) \\
& & \big\uparrow{\xi_n} & & \wr\big\uparrow{\xi} & & \wr\big\uparrow{\xi} \\
0 & \longrightarrow & {}_{F^n}W' & \longrightarrow & W' & \xrightarrow{F^n} & W'
\end{array}
$$

whose rows are exact and whose two vertical arrows on the right hand side are isomorphisms. Hence ξ_n is an isomorphism.

As for (iii) we note that
$$W_{m,n} = {}_{F^n}W'_m$$
and
$$D(W_{n,m}) = \mathcal{H}om_{k-gr}({}_{F^m}W_n, G_m).$$

Thus we have a commutative diagram

$$
\begin{array}{ccccccc}
\mathcal{H}om_{k-gr}(W_n, G_m) & \xrightarrow{F^{m*}} & \mathcal{H}om_{k-gr}(W_n, G_m) & \longrightarrow & D(W_{n,m}) \\
\wr\big\uparrow{\xi_n} & & \wr\big\uparrow{\xi_n} & & \big\uparrow{\xi_{n,m}} \\
{}_{F^n}W' & \xrightarrow{V^m} & {}_{F^n}W' & \longrightarrow & W_{m,n} \longrightarrow 0
\end{array}
$$

whose rows are exact and whose vertical arrows on the left are isomorphisms by (ii). Hence $\xi_{n,m}$ is an injection. But $W_{m,n}$ and $D(W_{n,m})$ are finite k-group schemes of the same rank p^{nm}, hence $\xi_{n,m}$ is an isomorphism.

The second half of (iii) follows immediately from

$$g_{n'}(v(x)) = V^{n'-n}g_n(x),$$
$$g_m(f(u)) \equiv F^{n'-n}g_{m'}(u) \mod V^m W'.$$

Q. E. D.

Remark. — Dieudonné [11] proved Proposition 3.21 (i) in the formal group case. The proof given here is a slight modification. Cartier [7] proved Proposition 3.21 (ii) when $n = 1$. He also announced (iii) without proof in the same paper. *See* also a recent paper of Cartier [41].

Proof of Theorem 3.19 *continued.* — We apply Lemma 3.20 (iii) and (iv) to $A = B = W(k) [F, V]$ and $m_i = n_i = A(F^i, V^i)$, since, for G in $\mathfrak{N}_{ll}$, $M(G)$ is annihilated by some m_i. Thus there is an isomorphism of functors
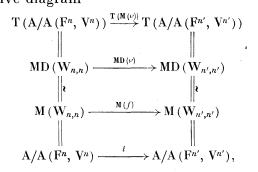
$$\beta : \quad T \xrightarrow{\sim} T' = \mathrm{Hom}_A(?, L),$$

where $L = \varinjlim_n T(A/A(F^n, V^n))$. But we know that $A/A(F^n, V^n) = M(W_{n,n})$ and the canonical surjection $A/A(F^{n'}, V^{n'}) \to A/A(F^n, V^n)$ for $n \leq n'$ is equal to $M(\nu)$, where $\nu : W_{n,n} \to W_{n',n'}$ is as defined in Proposition 3.21 (iii). Hence by Proposition 3.21 (iii) we get

$$T(A/A(F^n, V^n)) = MD(W_{n,n}) \cong M(W_{n,n}) = A/A(F^n, V^n)$$

and the commutative diagram

$$
\begin{array}{ccc}
T(A/A(F^n, V^n)) & \xrightarrow{\ T(M(\nu))\ } & T(A/A(F^{n'}, V^{n'})) \\
\| & & \| \\
MD(W_{n,n}) & \xrightarrow{\ MD(\nu)\ } & MD(W_{n',n'}) \\
\| \wr & & \| \wr \\
M(W_{n,n}) & \xrightarrow{\ M(f)\ } & M(W_{n',n'}) \\
\| & & \| \\
A/A(F^n, V^n) & \xrightarrow{\quad i \quad} & A/A(F^{n'}, V^{n'}),
\end{array}
$$
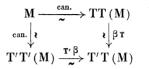
where for $n \leq n'$, $f : W_{n',n'} \to W_{n,n}$ is as defined in Proposition 3.21 (iii).

It is easy to see that i is the A-module homomorphism which sends the coset of $1 \bmod A(F^n, V^n)$ to the coset of $p^{n'-n} \bmod A(F^{n'}, V^{n'})$. $L = \varinjlim_n A/A(F^n, V^n)$ has two structures of left A-modules; (i) usual left A-module structure and this is used to define a left A-module structure on the functor $T' = \mathrm{Hom}_A(?, L)$; (ii) new left A-module structure defined by $T(R_a)$ for a in A. This is the structure by which we take Hom_A. Since $R_F = M(F)$, we get $T(R_F) = MD(F) = M(V) = R_V$. Similarly we get $T(R_V) = R_F$ and $T(R_\lambda) = R_\lambda$ for λ in $W(k)$. Hence by the second left A-module structure the multiplications by F, V and λ are the usual right multiplications by V, F and λ respectively. Thus we finally get an isomorphism of functors

$$
T(M) \xrightarrow[\sim]{\beta} T'(M) = \left\{ u : M \to L \;\middle|\; \begin{array}{l} u \text{ is additive} \\ u(Fx) = u(x) V \\ u(Vx) = u(x) F \\ u(\lambda x) = u(x) \lambda \\ \text{for } x \text{ in } M \\ \text{and } \lambda \text{ in } W(k) \end{array} \right\},
$$

the left A-module structure on the right hand side being given by

$$(Fu)(x) = F u(x), \qquad (Vu)(x) = V u(x) \qquad \text{and} \qquad (\lambda u)(x) = \lambda u(x).$$

It is also clear that $\varphi : L \to L$ defined in Lemma 3.20 (iv) is equal to the identity map, hence $\gamma : M \to T'T'(M) = \mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(M, L), L)$ is the canonical isomorphism of functors defined by $\gamma(x)(f) = f(x)$ for x in M and f in $\mathrm{Hom}_\Lambda(M, L)$. Thus we have a commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow[\sim]{\text{can.}} & TT(M) \\
{\scriptstyle\text{can.}}\downarrow{\wr} & & {\wr}\downarrow{\scriptstyle\beta T} \\
T'T'(M) & \xrightarrow[\sim]{T'\beta} & T'T(M)
\end{array}
$$

We can express the module L more explicitly as follows : let $\{(V/p)^i ; i \geqq 0\}$ and $\{(F/p)^j ; j > 0\}$ be just abstract symbols. Then we have an isomorphism of two-sided A-modules

$$
L = \Big\{ \bigoplus_{i \geqq 1} (V/p)^i C(k) \Big\} \oplus C(k) \oplus \Big\{ \bigoplus_{j \geqq 1} (F/p)^j C(k) \Big\},
$$

where we give two-sided A-module structure on the right hand side as follows :

$$
\begin{aligned}
i \geqq 0 : \quad & V[(V/p)^i c] = (V/p)^{i+1} pc = [(V/p)^i c^{\sigma^{-1}}] V, \\
j \geqq 1 : \quad & V[(F/p)^j c] = (F/p)^{j-1} c = [(F/p)^j c^{\sigma^{-1}}] V, \\
i \geqq 1 : \quad & F[(V/p)^i c] = (V/p)^{i-1} c = [(V/p)^i c^{\sigma}] F, \\
j \geqq 0 : \quad & F[(F/p)^j c] = (F/p)^{j+1} pc = [(F/p)^j c^{\sigma}] F, \\
i \geqq 0 : \quad & \lambda[(V/p)^i c] = (V/p)^i \lambda^{\sigma^i} c = [(V/p)^i c] \lambda^{\sigma^i}, \\
j \geqq 0 : \quad & \lambda[(F/p)^j c] = (F/p)^j \lambda^{\sigma^{-j}} c = [(F/p)^j c] \lambda^{\sigma^{-j}}
\end{aligned}
$$

for c in $C(k)$ and λ in $W(k)$. Under this isomorphism the submodule $A/A(F^n, V^n)$ is mapped isomorphically onto the submodule

$$
\Big\{ \bigoplus_{n > i > 0} (V/p)^i C_{-n+i}(k) \Big\} \oplus C_{-n}(k) \oplus \Big\{ \bigoplus_{0 < j < n} (F/p)^j C_{-n+j}(k) \Big\}
$$

of L.

If we write an additive map $u : M \to L$ as

$$
u(x) = \sum_{i \geqq 1} (V/p)^i u_{-i}(x) + u_0(x) + \sum_{j \geqq 1} (F/p)^j u_j(x),
$$

then it is easy to see that $u(Fx) = u(x) V$, $u(Vx) = u(x) F$ and $u(\lambda x) = u(x) \lambda$ for all x in M and λ in $W(k)$ if and only if u_0 is a homomorphism of $W(k)$-modules from M to $C(k)$ and

$$
\begin{aligned}
u_j(x) &= u_0(F^j x)^{\sigma^{-j}} \quad \text{for } j \geqq 0, \\
u_{-i}(x) &= u_0(V^i x)^{\sigma^i} \quad \text{for } i \geqq 0.
\end{aligned}
$$

Moreover

$$
(Fu)_0(x) = [u_0(Vx)]^{\sigma}, \qquad (Vu)_0(x) = [u_0(Fx)]^{\sigma^{-1}}
$$

and $(\lambda u)_0(x) = \lambda u_0(x)$ for all x in M and λ in $W(k)$.

Thus we get a canonical isomorphism of left A-modules

$$T'(M) \xrightarrow{\sim} D(M)$$

sending u to u_0. The commutativity of the diagram

$$
\begin{array}{ccc}
M & \xrightarrow[\sim]{\text{can.}} & TT(M) \\
\text{can.}\downarrow\wr & & \wr\downarrow \\
DD(M) & \xrightarrow{\sim} & DT(M)
\end{array}
$$

is obvious from the corresponding diagram for T and T'.

$$\text{Q. E. D.}$$

PROPOSITION 3.22. — *Let* $G = (G_n, i_n, j_n)$ *be a p-divisible group over a perfect field k of characteristic p. Let $G^t = (D(G_n), D(j_n), D(i_n))$ be its Serre dual (cf. Definition 1.6). Then there is a canonical isomorphism of left A-modules*

$$d(G) : \quad M(G^t) \xrightarrow{\sim} M(G)^t$$

such that the following diagram is commutative :

$$
\begin{array}{ccc}
M(G) & \xrightarrow[\sim]{M(\text{can.})} & M(G^{tt}) \\
\text{can.}\downarrow\wr & & \wr\downarrow d(G^t) \\
M(G)^{tt} & \xrightarrow[\sim]{[d(G)]^t} & M(G^t)^t
\end{array}
$$

Proof. — By definition,

$$M(G^t) = \varprojlim_n \{ M(D(G_n)), M(D(j_n)) \}.$$

But by Theorem 3.19 there is a canonical isomorphism

$$M(D(G_n)) \cong D(M(G_n)).$$

Since $i_n \circ j_n = p_{G_{n+1}}$, we have the following commutative diagram :

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M(G) & \xrightarrow{p^n} & M(G) & \longrightarrow & M(G_n) & \longrightarrow & 0 \\
& & \| & & \downarrow p & & \downarrow M(j_n) & & \\
0 & \longrightarrow & M(G) & \xrightarrow{p^{n+1}} & M(G) & \longrightarrow & M(G_{n+1}) & \longrightarrow & 0
\end{array}
$$

Thus we get a canonical isomorphism

$$\varinjlim_n \{ M(G_n), M(j_n) \} \cong C(k) \otimes_{W(k)} M(G),$$

where $M(G_n)$ corresponds to the submodule $C_{-n}(k) \otimes_{W(k)} M(G)$ and $M(j_n)$ corresponds to the inclusion. Thus we get from Proposition 3.9 (vi)

$$M(G^t) = D(C(k) \otimes_{W(k)} M(G)) = M(G)^t.$$

The commutativity of the diagram in the proposition is obvious.

$$\text{Q. E. D.}$$

We now interpret Proposition 1.8 in terms of the Dieudonné modules.

PROPOSITION 3.23. — *If* X *is an abelian scheme over a perfect field* k *of characteristic* p. *Then there is a canonical isomorphism of left* A-*modules*

$$M(\nu_X) \circ d(X(p))^{-1} : \quad M(X(p))^t \xrightarrow{\sim} M(X^t(p))$$

such that the following diagram is commutative :

$$
\begin{array}{ccc}
M(X(p)) & \xleftarrow{\ \ M(k_X(p))\ \ } & M(X^{tt}(p)) \\
\text{can.} \downarrow \wr & & \wr \downarrow M(\nu_{X^t}) \circ d(X^t(p)) \\
M(X(p))^{tt} & \xrightarrow[\sim]{\ -[M(\nu_X) \circ d(X(p))^{-1}]^t\ } & M(X^t(p))^t
\end{array}
$$

Proof. — Obvious from Proposition 3.22 and Proposition 1.8.

We now interpret the Riemann homomorphism defined at the end of Section 1 in terms of the Dieudonné modules.

We first note that for a $W(k)$-module M free and of finite type we get canonical isomorphism

$$\mathrm{Hom}_{W(k)}(M^t, M) = M \otimes_{W(k)} M = \mathrm{Hom}_{W(k)}(M^t \otimes_{W(k)} M^t, W(k)).$$

If M is a left A-module $W(k)$-free of finite rank, then a homomorphism of A-modules $r : M^t \to M$ corresponds by the above isomorphisms to an element c in $M \otimes_{W(k)} M$ and to a $W(k)$-bilinear form b on M^t with values in $W(k)$ such that

$$(V \otimes 1)(c) = (1 \otimes F)(c^{(p)}) \qquad \text{and} \qquad (1 \otimes V)(c) = (F \otimes 1)(c^{(p)})$$

in the diagram

$$
\begin{array}{ccc}
M \otimes_{W(k)} M & \xrightarrow{\ \ V \otimes 1\ \ } & M^{(p)} \otimes_{W(k)} M \\
1 \otimes V \downarrow & & \uparrow 1 \otimes F \\
M \otimes_{W(k)} M^{(p)} & \xleftarrow{\ F \otimes 1\ } M^{(p)} \otimes_{W(k)} M^{(p)} = & (M \otimes_{W(k)} M)^{(p)}
\end{array}
$$

and such that

$$b(Fx, y) = b(x, Vy)^\sigma,$$
$$b(x, Fy) = b(Vx, y)^\sigma$$

for all x and y in M^t. It is obvious that if r is a skew-symmetric homomorphism, then c is a skew-symmetric tensor, and b is a skew-symmetric bilinear form.

PROPOSITION 3.24. — *Let* X *be an abelian scheme over a perfect field* k *of characteristic* p. *Then :*

(i) *If* $\lambda : X \to X^t$ *is a* k-*homomorphism, there is a left* A-*module homomorphism*

$$r_\lambda = M(\rho_\lambda) \circ d(X(p))^{-1} : \quad M(X(p))^t \to M(X(p))$$

an element c_λ in $M(X(p)) \otimes_{W(k)} M(X(p))$ and a $W(k)$-bilinear from b_λ on $M(X(p))^t$ with values in $W(k)$ such that

$$(V \otimes 1)(c_\lambda) = (1 \otimes F)(c_\lambda^{(p)})),$$
$$(1 \otimes V)(c_\lambda) = (F \otimes 1)(c_\lambda^{(p)}))$$

and

$$b_\lambda(Fx, y) = b_\lambda(x, Vy)^\sigma,$$
$$b_\lambda(x, Fy) = b_\lambda(Vx, y)^\sigma$$

for all x and y in $M(X(p))^t$.

r_λ, c_λ and b_λ are additive in λ.   They are zero if and only if λ is zero.

(ii) *If λ is a symmetric k-homomorphism, then r_λ, c_λ and b_λ are skew-symmetric.*

Proof. — It follows immediately from Proposition 1.10 and Proposition 1.12.

<div align="right">Q. E. D.</div>

In particular, if L is an invertible sheaf on X, we have a skew-symmetric homomorphism of left A-modules

$$r(L) = r_{\Lambda(L)} : \quad M(X(p))^t \to M(X(p))$$

a skew-symmetric element

$$c(L) = c_{\Lambda(L)} \quad \text{in } M(X(p)) \otimes_{W(k)} M(X(p))$$

such that

$$(V \otimes 1)(c(L)) = (1 \otimes F)(c(L)^{(p)})$$

and a skew-symmetric $W(k)$-bilinear form

$$b(L) = b_{\Lambda(L)} \quad \text{on } M(X(p))^t \text{ with values in } W(k)$$

such that

$$b(L)(Fx, y) = b(L)(x, Vy)^\sigma$$

for all x and y in $M(X(p))^t$.

$r(L)$, $c(L)$ and $b(L)$ are additive in L.   Moreover they are zero if and only if L is contained in $\mathrm{Pic}_{X/k}^\tau(k)$.

DEFINITION 3.25. — *We call $r(L)$ the Riemann homomorphism of the Dieudonné modules, $c(L)$ the fundamental class of L, and $b(L)$ the Riemann form of L.*

We can also interpret Proposition 2.3 in terms of the Dieudonné modules.

PROPOSITION 3.26. — *Let* X *be an abelian scheme over a perfect field* k *of characteristic* p. *Then we have*

$$\mathrm{v}^n \mathrm{M} \left({}_{p^n}\mathrm{X} \right) = \mathrm{F}^n \mathrm{M} \left({}_{p^n}\mathrm{X} \right),$$
$$\mathrm{F}^n \mathrm{M} \left({}_{p^n}\mathrm{X} \right) = \mathrm{V}^n \mathrm{M} \left({}_{p^n}\mathrm{X} \right)$$

for all positive integer n.

Proof. — Immediate from Proposition 2.3.

Remark. — The Dieudonné module is closely related to the " module of canonical covectors " in the sense of Barsotti [3], Chap. I and Chap. III. In fact if we define a " formal group " cov by using Φ in [3], Th. 1.11, then cov is a left A-module in a natural way ([3], I, Section 6). We can easily show that the " module of canonical covectors " (*ibid.*, Section 5) $\mathrm{Hom}_{k-gr}(\mathrm{G}, \mathrm{cov})$ is isomorphic to $\mathrm{M}(\mathrm{G})$ for G in Ind $(\mathscr{L})$. In [3], Chap. VI and Chap. VII Barsotti defines in a method different from ours, the Riemann homomorphism and the Riemann form for a divisor on an abelian scheme over an algebraically closed field of characteristic p.

SECTION 4.

PICARD SCHEMES AND DIEUDONNÉ MODULES.

Let k be a perfect field and let X be a k-prescheme. The left A-module functor C of covectors defines a sheaf of left A-modules $\mathrm{C_X}$ on X in the Zariski topology. The multiplicative group scheme G_m defines a sheaf of abelian groups $\mathrm{G}_{m\mathrm{X}} = \mathcal{O}_\mathrm{X}^\ast$ on X. We denote by $\overline{\mathrm{X}}$ the base extension $\mathrm{X}_{\overline{k}}$ to an algebraic closure $\overline{k}$ of k.

DEFINITION 4.1. — *Let* X *be a prescheme over a perfect field* k *of characteristic* p. *Let* $\mathrm{I}(\mathrm{X})$ *be the left A-module defined by*

$$\mathrm{I}(\mathrm{X}) = \mathrm{H}^1(\mathrm{X}, \mathrm{C_X}) \oplus \left\{ \mathrm{W}(\overline{k}) \otimes_{\mathbf{Z}_{p^\infty}} \mathrm{H}^1 \left(\overline{\mathrm{X}}, \mathrm{G}_{m\overline{\mathrm{X}}} \right) \right\}^{\mathrm{Gal}(\overline{k}/k)},$$

where $\mathrm{H}^1(\mathrm{X}, \mathrm{C_X})$ *is given the left A-module structure induced by that of* $\mathrm{C_X}$, *while the second factor is given the left A-module structure by*

$$\lambda(\lambda' \otimes x) = \lambda\lambda' \otimes x,$$
$$\mathrm{F}(\lambda' \otimes x) = \lambda'^\sigma \otimes px,$$
$$\mathrm{V}(\lambda' \otimes x) = \lambda'^{\sigma^{-1}} \otimes x$$

for λ *in* $\mathrm{W}(k)$, λ' *in* $\mathrm{W}(\overline{k})$ *and* x *in* $_{p^\infty}\mathrm{H}^1\left(\overline{\mathrm{X}}, \mathrm{G}_{m\overline{\mathrm{X}}}\right)$. *We define left A-modules by*

$$\mathrm{H}_1(\mathrm{X}_{p-\mathrm{flat}}) = \mathrm{D}(\mathrm{I}(\mathrm{X})),$$
$$\mathrm{H}^1(\mathrm{X}_{p-\mathrm{flat}}) = \mathrm{T}_p(\mathrm{I}(\mathrm{X})).$$

Note that the second factor can also be expressed as

$$\{ W(\overline{k}) \otimes_{\mathbf{Z}} {}_{p^z}\mathrm{Pic}\,(\overline{X}) \}^{\mathrm{Gal}\,(\overline{k}/k)}.$$

I defines a contravariant functor from the category of k-preschemes to the category of p-torsion left A-modules.

Note also that $H^1(X_{p\text{-flat}}) = H_1(X_{p\text{-flat}})^t$, if $I(X)$ is of $W(k)$-cofinite type. The p-adic flat homology and cohomology notation is used here just to indicate the properties which these A-modules have and which we prove below (*cf.* Prop. 4.2, Th. 4.4, Prop. 4.9, Prop. 4.12, and Cor. 4.13)

PROPOSITION 4.2. — *Let* X *be an abelian scheme over a perfect field* k *of characteristic* p. *Then there are canonical isomorphisms of left* A-*modules*

$$M(_{p^n}X) \cong {}_{p^n}I(X),$$
$$I(X) \cong C(k) \otimes_{W(k)} M(X(p)),$$
$$M(X(p)) \cong T_p(I(X)) = H^1(X_{p-\text{flat}}).$$

Proof. — We have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & {}_{p^{n+1}}X & \longrightarrow & X & \xrightarrow{p^{n+1}} & X & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle i_n} & & \| & & \uparrow{\scriptstyle p} & & \\
0 & \longrightarrow & {}_{p^n}X & \longrightarrow & X & \xrightarrow{p^n} & X & \longrightarrow & 0
\end{array}
$$

whose rows are exact. Since $\mathrm{Hom}_{k-gr}(X, C_{-N}) = 0$ and $\mathrm{Hom}_{\overline{k}-gr}(\overline{X}, G_{m\overline{k}}) = 0$ for all positive integers N, we get commutative diagrams

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}\,(_{p^{n+1}}X, C_{-N}) & \xrightarrow{\partial} & \mathrm{Ext}^1(X, C_{-N}) & \xrightarrow{p^{n+1}} & \mathrm{Ext}^1(X, C_{-N}) \\
 & & \downarrow{\scriptstyle i_n^\star} & & \downarrow{\scriptstyle p} & & \| \\
0 & \longrightarrow & \mathrm{Hom}\,(_{p^n}X, C_{-N}) & \xrightarrow{\partial} & \mathrm{Ext}^1(X, C_{-N}) & \xrightarrow{p^n} & \mathrm{Ext}^1(X, C_{-N})
\end{array}
$$

and

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}\,(_{p^{n+1}}\overline{X}, G_{m\overline{k}}) & \xrightarrow{\partial} & \mathrm{Ext}^1(\overline{X}, G_{m\overline{k}}) & \xrightarrow{p^{n+1}} & \mathrm{Ext}^1(\overline{X}, G_{m\overline{k}}) \\
 & & \downarrow{\scriptstyle i_n^\star} & & \downarrow{\scriptstyle p} & & \| \\
0 & \longrightarrow & \mathrm{Hom}\,(_{p^n}\overline{X}, G_{m\overline{k}}) & \xrightarrow{\partial} & \mathrm{Ext}^1(\overline{X}, G_{m\overline{k}}) & \xrightarrow{p^n} & \mathrm{Ext}^1(\overline{X}, G_{u\overline{k}})
\end{array}
$$

But by Serre ([32], VII, No. 18, Th. 8) we know that

$$\mathrm{Ext}^1(X, C_{-N}) = H^1(X, C_{-N, x}).$$

Thus we get

$$\mathrm{Hom}_{k-gr}(_{p^n}X, C) \cong {}_{p^n}H^1(X, C_X).$$

On the other hand Serre ([32], VII, No. 16, Th. 6) shows that

$$\mathrm{Ext}^1(\overline{X}, G_{m\overline{k}}) = \ker\left[\,\mathrm{Pic}\,(\overline{X}) \xrightarrow{\mu^\star - p_1^\star - p_2^\star} \mathrm{Pic}\,(\overline{X} \times \overline{X})\,\right].$$

But Lang ([22], IV, Section 2, Cor. 3 to Th. 4) and the Theorem of Square shows that the right hand side is equal to $\mathrm{Pic}^\tau(\overline{X})$. This in turn is equal to $\mathrm{Pic}^0(\overline{X})$ by Corollary 1.3 (Cartier duality theorem). Thus we get

$$\mathrm{Hom}_{k-gr}\left({}_{p^n}\overline{X},\ G_{m\overline{k}}\right) \doteq {}_{p^n}\mathrm{Pic}^0(\overline{X}) = {}_{p^n}\mathrm{Pic}(\overline{X}).$$

Thus by the definition of the Dieudonné module we get a commutative diagram

$$
\begin{array}{ccc}
\mathrm{M}\left({}_{p^{n+1}}X\right) & \cong & {}_{p^{n+1}}\mathrm{I}(X) \\
{\scriptstyle \mathrm{M}(i_n)}\downarrow & & \downarrow{\scriptstyle p} \\
\mathrm{M}\left({}_{p^n}X\right) & \cong & {}_{p^n}\mathrm{I}(X)
\end{array}
$$

Q. E. D.

For the sake of completeness we now interpret the left A-module $H^1(X, W_X)$ studied by Serre ([33], [34]) in terms of the Dieudonné modules. *See* also Barsotti ([3], Chap. VI).

PROPOSITION 4.3. — *Let* X *be an abelian scheme over a perfect field* k *of characteristic* p. *Then for every positive integer* n *there is a commutative diagram of left* A-*modules whose rows are isomorphisms*

$$
\begin{array}{ccc}
\mathrm{M}\left({}_{V^{n+1}}X\right) & \xrightarrow{\ \sim\ } & H^1(X, W_{n+1, X}) \\
{\scriptstyle \mathrm{M}(i_n)}\downarrow & & \downarrow{\scriptstyle R} \\
\mathrm{M}\left({}_{V^n}X\right) & \xrightarrow{\ \sim\ } & H^1(X, W_{n, X})
\end{array}
$$

Thus there is a canonical left A-*module isomorphism*

$$\mathrm{M}\left(\varinjlim_n {}_{V^n}X\right) \cong H^1(X, W_X).$$

Proof. — The commutative diagram for a positive integer n

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & {}_{V^{n+1}}X & \longrightarrow & X & \xrightarrow{V^{n+1}} & X^{(p^{-n-1})} & \longrightarrow & 0 \\
& & {\scriptstyle i_n}\uparrow & & \| & & \uparrow{\scriptstyle V} & & \\
0 & \longrightarrow & {}_{V^n}X & \longrightarrow & X & \xrightarrow{V^n} & X^{(p^{-n})} & \longrightarrow & 0
\end{array}
$$

gives a commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}\left({}_{V^{n+1}}X, C\right) & \xrightarrow{\ \hat{\partial}\ } & \mathrm{Ext}^1\left(X^{(p^{-n-1})}, C\right) & \xrightarrow{V^{n+1}{}^\star} & \mathrm{Ext}^1(X, C) \\
& & {\scriptstyle i_n^\star}\downarrow & & \downarrow{\scriptstyle V^\star} & & \| \\
0 & \longrightarrow & \mathrm{Hom}\left({}_{V^n}X, C\right) & \xrightarrow{\ \hat{\partial}\ } & \mathrm{Ext}^1\left(X^{(p^{-n})}, C\right) & \xrightarrow{V^n{}^\star} & \mathrm{Ext}^1(X, C)
\end{array}
$$

However there is a canonical isomorphism

$$\mathrm{Ext}^1(A, B) \xrightarrow{\ (p)\ } \mathrm{Ext}^1(A^{(p)}, B^{(p)})$$

for commutative k-group schemes A and B such that $(V_A)^\star = (V_B)_\star \circ (p)$ (*cf.* Matsumura-Miyanishi [25], Lemma 2). On the other hand the

isomorphism of k-group schemes $W_n \to C_{-n}$ is a σ^n-isomorphism as left A-module functors. Since $\mathrm{Ext}^1(X, C) = H^1(X, C_X)$ as in the proof of Proposition 4.2 we get the following commutative diagram whose rows are exact :

$$
\begin{array}{ccccccc}
0 \longrightarrow & H^1(W_{n+1, X}) & \longrightarrow & \mathrm{Ext}^1(X^{(p^{-n-1})}, C) & \xrightarrow{\;V^{n+1*}\;} & \mathrm{Ext}^1(X, C) \\
& \searrow{\scriptstyle R} & & \searrow{\scriptstyle V^*} & & \diagdown \\
0 \longrightarrow & H^1(W_{n, X}) & \longrightarrow & \mathrm{Ext}^1(X^{(p^{-n})}, C) & \xrightarrow{\;V^{n*}\;} & \mathrm{Ext}^1(X, C) \\
{\scriptstyle (p^{n+1})}\downarrow & {\scriptstyle (p^n)} & {\scriptstyle (p^{n+1})} & {\scriptstyle (p^n)} & & \\
0 \longrightarrow & H^1(C_{-n-1, X}) & \longrightarrow & H^1(C_X) & \xrightarrow{\;V^{n+1}\;} & H^1(C_X) \\
& \searrow{\scriptstyle V} & & \searrow{\scriptstyle V} & & \diagdown \\
0 \longrightarrow & H^1(C_{-n, X}) & \longrightarrow & H^1(C_X) & \xrightarrow{\;V^n\;} & H^1(C_X)
\end{array}
$$

Thus finally we get a commutative diagram

$$
\begin{array}{ccc}
M(_{V^{n+1}}X) & \xrightarrow{\;\sim\;} & H^1(X, W_{n+1, X}) \\
\downarrow{\scriptstyle M(i_n)} & & \downarrow{\scriptstyle R} \\
M(_{V^n}X) & \xrightarrow{\;\sim\;} & H^1(X, W_{n, X})
\end{array}
$$

<div align="right">Q. E. D.</div>

For a proper scheme X over a perfect field k of characteristic p, such that $\mathrm{Pic}^\tau_{X/k}$ is proper over k we define objects in $\mathrm{Ind}(\mathfrak{N}_p)$ by

$$
_{p^\infty}\mathrm{Pic}_{X/k} = \varinjlim_n {}_{p^n}\mathrm{Pic}_{X/k},
$$

$$
\widehat{\mathrm{Pic}}_{X/k} = \varinjlim_n {}_{F^n}\mathrm{Pic}_{X/k},
$$

$$
\widehat{\mathrm{Pic}}_{X/k, \mathrm{red}} = \varinjlim_n {}_{F^n}(\mathrm{Pic}_{X/k, \mathrm{red}}).
$$

The last two objects are the formal group obtained by the usual completion at the origin of the corresponding group schemes.

THEOREM 4.4. — *Let k be a perfect field of characteristic p. Let X be a proper k-scheme such that $X(k)$ is non-empty, $H^0(X, \mathcal{O}_X) = k$ and $\mathrm{Pic}^\tau_{X/k}$ is proper over k (e. g. X is proper normal k-scheme, cf.* Murre [30] *and* Chevalley [9] *([3])). Then there are following canonical isomorphisms of left A-modules :*

(1) $$\mathrm{DM}\left(\widehat{\mathrm{Pic}_{X/k}}\right) \cong H^1(X, C_X).$$

([3]) It is also an easy consequence of the valuative criterion of properness. *Cf.* EGA [19], chap. II, 7.3.

In particular, $H^1(X, C_X)$ *is of* $W(k)$*-cofinite type,*

$$(2) \qquad DM(_{p^\infty}Pic_{X/k}) \cong I(X), \qquad M(_{p^\infty}Pic_{X/k}) \cong H_1(X_{p\text{-flat}}).$$

In particular, $I(X)$ *is of* $W(k)$*-cofinite type,*

$$(3) \qquad DM\left(\widehat{Pic_{X/k, \text{red}}}\right) \cong \bigcap_n V^n H^1(X, C_X) \cong C(k) \otimes_{W(k)} H^1(X, W_X)$$

and

$$M\left(\widehat{Pic_{X/k, \text{red}}}\right) \cong H^1(X, W_X)',$$

$$(4) \qquad DM(_F(Pic_{X/k, \text{red}})) = (k, \sigma) \otimes_k [H^1(X, W_X)/V H^1(X, W_X)].$$

In particular the irregularity of X *(i. e. the dimension of the Picard variety* $Pic^0_{X/k, \text{red}}$*) is equal to* $\dim_k H^1(X, W_X)/V H^1(X, W_X)$.

(5) $DM(_{p^\infty}(Pic^0_{X/k, \text{red}}))$ *is isomorphic as an* A*-module to the submodule of* p*-divisible elements of* $I(X)$.

Remark. — If X is an abelian scheme over k, then (2) follows from Proposition 4.2 and Corollary 3.23 noting that

$$_{p^\infty}Pic_{X/k} = X^t(p).$$

(4) recovers the computation of irregularity in Mumford [29], lecture 27.

Proof. — For simplicity we write $P = Pic_{X/k}$. Then (1) implies (2). In fact since k is perfect, there is a p-primary etale object G in $Ind(\mathfrak{N}_{rl})$ such that

$$_{p^\infty}P = G \times \hat{P}$$

and

$$_{p^\infty}P(\bar{k}) = G(\bar{k}).$$

Then $M(_{p^\infty}P) = M(\hat{P}) \oplus M(G)$. But since by the proof of Corollary 3.15 we get

$$M(G) = \varprojlim_n Hom_{k-gr}(_{p^n}G, C) = \underset{gr}{Hom_{\text{abel.}}}(G(\bar{k}), C(\bar{k}))^{Gal(\bar{k}/k)}$$

$$= \{ D^{\bar{k}}(W(\bar{k}) \otimes_{\mathbf{Z}} G(\bar{k})) \}^{Gal(\bar{k}/k)}$$

$$= D\left(\{ W(\bar{k}) \otimes_{\mathbf{Z}} G(\bar{k}) \}^{Gal(\bar{k}/k)} \right)$$

as a left A-module, we obtain an isomorphism of left A-modules

$$DM(G) \cong \{ W(\bar{k}) \otimes_{\mathbf{Z}} {}_{p^\infty}P(\bar{k}) \}^{Gal(\bar{k}/k)}.$$

The rest of (2) is clear.

(1) implies (3). In fact there is a finite group scheme N such that

$$0 \to \hat{P}_{\text{red}} \to \hat{P} \to N \to 0$$

is exact. N is killed by some power of F. Hence we get an exact sequence of A-modules

$$0 \leftarrow M(\hat{P}_{red}) \leftarrow M(\hat{P}) \leftarrow M(N) \leftarrow 0.$$

Since $\hat{P}_{red}$ is a direct factor of the p-divisible group $P_{red}^0(p) = {}_{p^\infty}(P_{red}^0)$, $\hat{P}_{red}$ itself is a p-divisible group. Hence $M(\hat{P}_{red})$ is $W(k)$-free of finite rank. Thus $M(N)$ can be identified as the submodule of $M(\hat{P})$ of F-torsion elements (i. e. killed by some power of F). Hence

$$M(\hat{P}_{red}) = \varinjlim_n M(\hat{P})/_{F^n}M(\hat{P})$$
$$= \varinjlim_n \text{Coimage}\left[M(\hat{P}) \xrightarrow{F^n} (W(k), \sigma^{-n}) \otimes_{W(k)} M(\hat{P})\right].$$

On the other hand by (1) we know that

$$DM(\hat{P}) \cong H^1(X, C_X).$$

Hence

$$DM(\hat{P}_{red}) = D\left(\varinjlim_n \text{Coimage}\left[M(\hat{P}) \xrightarrow{F^n} (W(k), \sigma^{-n}) \otimes_{W(k)} M(\hat{P})\right]\right)$$
$$= \varprojlim_n \text{Image}\left[(W(k), \sigma^{-n}) \otimes_{W(k)} DM(\hat{P}) \xrightarrow{V^n} DM(\hat{P})\right]$$
$$= \varprojlim_n V^n DM(\hat{P}) \cong \bigcap_n V^n H^1(X, C_X).$$

Thus we get the first isomorphism of (3). On the other hand we have

$$\bigcap_n V^n H^1(C_X) = \varinjlim_i \bigcap_n V^n H^1(C_{-i-n, X}).$$

But we have

$$\bigcap_n V^n H^1(C_{-i-n, X}) = \text{Image}\left[\varprojlim_n \{ (W(k), \sigma^{-n}) \otimes_{W(k)} H^1(C_{-i-n, X}), V \} \to H^1(C_{-i, X})\right],$$

where $\{ (W(k), \sigma^{-n}) \otimes_{W(k)} H^1(C_{-i-n, X}), V \}$ is the projective system

$$\xleftarrow{V} (W(k), \sigma^{-n}) \otimes_{W(k)} H^1(C_{-i-n, X}) \xleftarrow{V} (W(k), \sigma^{-n-1}) \otimes_{W(k)} H^1(C_{-i-n-1, X}).$$

Since by definition $C_{-i-n} = (W(k), \sigma^{i+n}) \otimes_{W(k)} W_{i+n}$, this projective system is equal to

$$\xleftarrow{R} (W(k), \sigma^i) \otimes_{W(k)} H^1(W_{i+n, X}) \xleftarrow{R} (W(k), \sigma^i) \otimes_{W(k)} H^1(W_{i+n+1, X}) \leftarrow$$

where R is the canonical projection $R : W_{i+n+1} \to W_{i+n}$. Thus we get

$$\bigcap_n V^n H^1(C_{-i-n, X}) = (W(k), \sigma^i) \otimes_{W(k)} [H^1(W_X)/V^i H^1(W_X)].$$

Since the injection $C_{-i} \to C_{-i-1}$ is equal to

$$(W(k), \sigma^i) \otimes_{W(k)} W_i \xrightarrow{V} (W(k), \sigma^{i+1}) \otimes_{W(k)} W_{i+1}$$

we get

$$\bigcap_n V^n H^1(C_X) = \varinjlim_i \{ (W(k), \sigma^i) \otimes_{W(k)} [H^1(W_X)/V^i H^1(W_X)], V \}.$$

LEMMA 4.5. — *Let* X *be as in Theorem 4.4. Then* $H^1(X, W_X)$ *is* $W(k)$-*free of finite rank.*

Proof of Lemma 4.5. — This is a slight generalization of the result of Serre ([33], p. 20, Prop. 4). First of all $H^1(W_X)$ is V-torsion free (i. e. no non-zero element is killed by any power of V), since $H^0(W_X) \to H^0(W_{1,X})$ is surjective. By Serre ([33], p. 15, Prop. 3), $H^1(W_X)$ is $W(k)$-finite type if and only if $H^1(W_X)/FH^1(W_X)$ is of $W(k)$-finite length. This latter condition is satisfied. In face we have

$$H^1(W_X)/FH^1(W_X) = \varprojlim_n [H^1(W_{n,X})/FH^1(W_{n,X})]$$

and an exact sequence

$$0 \to H^0(W_{n,X}/FW_{n,X}) \xrightarrow{\partial} H^1(W_{n,X}) \xrightarrow{F} H^1(W_{n,X}),$$

where $H^1(W_{n,X})$ is of $W(k)$-finite length. Hence we get

$$\text{length}_{W(k)}[H^1(W_{n,X})/FH^1(W_{n,X})] = \text{length}_{W(k)}[H^0(W_{n,X}/FW_{n,X})].$$

But by definition

$$H^0(W_{n,X}/FW_{n,X}) = (W(k), \sigma^{-n}) \otimes_{W(k)} H^0(C_{-n,X}/FC_{-n,X})$$

and we know that $H^0(C_{-n,X}/FC_{-n,X})$ is imbedded canonically in $H^0(C_X/FC_X) = {}_F H^1(C_X)$. From (1) we know that this is isomorphic to $DM(_V\hat{P})$, which is of $W(k)$-finite length. Hence the $W(k)$-length of $H^1(W_{n,X})/FH^1(W_{n,X})$ is bounded in n. Hence we are done. We know from Serre ([33], p. 13, Prop. 2) that for a left A-module of $W(k)$-finite type on which V acts topologically nilpotently in the p-adic topology, the set of V-torsion elements is equal to the set of p-torsion elements.

                                                        Q. E. D.

LEMMA 4.6. — *Suppose* M *is a left* A-*module which is* $W(k)$-*free of finite rank and on which* V *acts topologically nilpotently in the* p-*adic topology. Then there is a canonical isomorphism of left* A-*modules*

$$\varinjlim_i \{ (W(k), \sigma^i) \otimes_{W(k)} M/V^i M, V \} \xrightarrow[\sim]{\alpha} C(k) \otimes_{W(k)} M.$$

Proof of Lemma 4.6. — We denote by ρ_i the projection $\rho_i : M \to M/V^i M$. Then for x in M we define

$$\alpha_i(1 \otimes \rho_i(x)) = (1/p^i) \otimes F^i x.$$

Then α_i is an A-module homomorphism from $(W(k), \sigma^i) \otimes_{W(k)} M/V^i M$ to $C(k) \otimes_{W(k)} M$. In fact

$$\alpha_i(\lambda \otimes \rho_i(x)) = \alpha_i(1 \otimes \rho_i(\lambda^{\sigma^{-i}} x)) = (1/p^i) \otimes F^i(\lambda^{\sigma^{-i}} x)$$
$$= (1/p^i) \otimes \lambda F^i x = \lambda \alpha_i(1 \otimes \rho_i(x))$$

for λ in $W(k)$, and α_i obviously commutes with F and V. We now show that α_i is well defined and injective. In fact $\alpha_i(1 \otimes \rho_i(x)) = 0$ if and only if $F^i x$ is in $p^i M$. Since M is $W(k)$-free, there is no element in M killed by F^i. Hence $\alpha_i(1 \otimes \rho_i(x)) = 0$ if and only if x is in $V^i M$, i.e. $1 \otimes \rho_i(x) = 0$.

$\{\alpha_i\}$ are compatible, since

$$\alpha_{i+1}(1 \otimes V\rho_i(x)) = \alpha_{i+1}(1 \otimes \rho_{i+1}(Vx)) = (1/p^{i+1}) \otimes F^{i+1}(Vx)$$
$$= (1/p^i) \otimes F^i x = \alpha_i(1 \otimes \rho_i(x)).$$

Thus we get an injective A-module homomorphism α. Now we show that α is surjective. Let $(1/p^m) \otimes y$ be an element of $C(k) \otimes_{W(k)} M$. Since V acts topologically nilpotently on M, there is an integer i sufficiently large and an element z in M such that

$$V^i y = p^m z.$$

Multiplying F^i on both sides we obtain $p^m(p^{i-m} y - F^i z) = 0$. Since M is $W(k)$-free we get $p^{i-m} y = F^i z$. Then

$$\alpha_i(1 \otimes \rho_i(z)) = (1/p^i) \otimes F^i z = (1/p^i) \otimes p^{i-m} y = (1/p^m) \otimes y.$$

Q. E. D.

Proof of Theorem 4.4 continued. — We now apply Lemma 4.5 and Lemma 4.6 to $M = H^1(W_X)$ and get the second isomorphism in (3). The rest is trivial.

(4) follows from (3). In fact we have

$$\mathrm{DM}(_F(P_{red})) = D\left(\mathrm{coker}\left[(W(k), \sigma) \otimes_{W(k)} M(\hat{P}_{red}) \xrightarrow{F} M(\hat{P}_{red})\right]\right)$$
$$= \ker\left[\mathrm{DM}(\hat{P}_{red}) \xrightarrow{V} (W(k), \sigma) \otimes_{W(k)} \mathrm{DM}(\hat{P}_{red})\right]$$
$$\cong \ker\left[\bigcap_n V^n H^1(C_X) \xrightarrow{V} (W(k), \sigma) \otimes_{W(k)} \bigcap_n V^n H^1(C_X)\right]$$
$$= \mathrm{Image}\left[\varprojlim_n \{(W(k), \sigma^{-n}) \otimes_{W(k)} H^1(C_{-1-n,X}), V\} \to H^1(C_{-1,X})\right]$$
$$= (k, \sigma) \otimes_k [H^1(W_X)/VH^1(W_X)].$$

It is not difficult to see that we have a homomorphism of left A-modules

$$M\left(_{F}(P_{red})\right) = (k, \sigma) \otimes_k \omega^1 (P_{red}^0/k),$$

where $\omega^1(P_{red}^0/k)$ denotes the vector space of invariant differential forms of the abelian scheme P_{red}^0. F acts trivially and V acts through the Cartier operator (*cf.* Section 5 and Seshadri [36]). Hence we get an isomorphism of left A-modules

$$DM\left(_{F}(P_{red})\right) = (k, \sigma) \otimes_k Lie\,(P_{red}^0/k),$$

where V acts trivially on the right hand side and F acts through p-operation of the p-Lie algebra structure.

(5) follows from (2) and the exact sequence

$$0 \to {}_{p^\infty}(P_{red}^0) \to {}_{p^\infty}P \to N' \to 0,$$

where N' is a finite k-group scheme killed by some power of p.

We now prove (1) of Theorem 4.4. Since $\hat{P} = \varinjlim_{n} {}_{F^n}P$ we get

$$DM(\hat{P}) = D\left(\varprojlim_{n} M\left(_{F^n}P\right)\right) = \varinjlim_{n} DM\left(_{F^n}P\right) \cong \varinjlim_{n} M\left(D\left(_{F^n}P\right)\right)$$

by Theorem 3.19. But we can write

$$D\left(_{F^n}P\right) = Spec\,(H_n(P)),$$

where $H_n(P)$ is the dual vector space of $\mathcal{O}_{P,0}/\mathcal{O}_{P,0}\,F^n(\mathfrak{M}_{P,0})$, $\mathcal{O}_{P,0}$ and $\mathfrak{M}_{P,0}$ being the local ring of P and the maximal ideal respectively at the origin. $H_n(P)$ can be identified with a sub-bialgebra of the bialgebra of invariant differential operators

$$H(P) = \bigcup_{n} H_n(P)$$

·of P (i. e. the hyperalgebra of P). If we denote by i_n the imbedding $_{F^n}P \to {}_{F^{n+1}}P$, then the epimorphism $D(i_n): D\left(_{F^{n+1}}P\right) \to D\left(_{F^n}P\right)$ corresponds to the canonical inclusion $H_n(P) \to H_{n+1}(P)$. Hence (1) becomes

(1') $$H^1(X, C_X) \cong \varinjlim_{n} Hom_{k-gr}\,(Spec\,(H_n(P)), C).$$

To prove (1') it is enough to prove that for each positive integer N

$$H^1(C_{-N,X}) \cong \varinjlim_{n} Hom_{k-gr}\,(Spec\,(H_n(P)), C_{-N})$$

$$= Hom_{k-gr}\,(Spec\,(H(P)), C_{-N}).$$

Since $C_{-N} = (W(k), \sigma^N) \otimes_{W(k)} W_N$ by definition, it is enough to prove

(1'') $$H^1(X, W_{N,X}) \cong Hom_{k-gr}\,(Spec\,(H(P)), W_N)$$

as left A-modules, where the structure of left A-module on the left and right hand sides of $(1'')$ is induced by that of W_N.

Let R be an augmented Artinian local k-algebra. We denote by M and π the maximal ideal and the augmentation $R \to k$ respectively. Let E be the dual k-vector space of M. Then the dual k-vector space $\mathrm{Hom}_k(R, k) = k\pi \oplus E$ has the structure of augmented unitary commutative k-coalgebra. Hence the symmetric algebra $S(E)$ which canonically contains $k\pi \oplus E$ has the structure of commutative, cocommutative, augmented and unitary k-bialgebra.

LEMMA 4.7. — $\mathrm{Spec}(S(E))$ is a commutative group scheme over k, whose B-valued points for a k-algebra B are given by $\mathrm{Spec}(S(E))(B) = 1 + B \otimes_k M$, where the right hand side is the multiplicative group which is the kernel of the homomorphism

$$(B \otimes_k R)^\star \xrightarrow{1 \otimes \pi} (B \otimes_k k)^\star = B^\star.$$

Proof of Lemma 4.7. — If we call a vector space B over k unitary when B is given a homomorphism $k \to B$, then a k-algebra B is a unitary k-vector space with the canonical injection of k in B and $k\pi \oplus E = \mathrm{Hom}_k(R, k)$ is also a unitary k-vector space with the map sending 1 to π. Thus we get

$$\mathrm{Spec}(S(E))(B) = \mathrm{Hom}_{\substack{\text{unitary} \\ k\text{-algebra}}}(S(E), B)$$
$$= \mathrm{Hom}_{\substack{\text{unitary} \\ k\text{-vector space}}}(k\pi \oplus E, B)$$
$$= \mathrm{Hom}_{\substack{\text{unitary} \\ k\text{-vector space}}}(\mathrm{Hom}_k(R, k), B)$$
$$= (1 \otimes \pi)^{-1}(1) = 1 + B \otimes_k M$$

Q. E. D.

$\mathrm{Spec}(S(E))$ thus gives the abelian sheaf

$$\mathrm{Spec}(S(E))_X = 1 + \mathcal{O}_X \otimes_k M$$

on X in the Zariski topology.

LEMMA 4.8. — *There are canonical isomorphisms*

$$H^1(X, \mathrm{Spec}(S(E))_X) \cong \ker\left[P(R) \xrightarrow{\pi_\star} P(k)\right]$$
$$\cong \mathrm{Hom}_{k\text{-}gr}(\mathrm{Spec}(H(P)), \mathrm{Spec}(S(E))).$$

Proof of Lemma 4.8. — Since $X(k)$ is non-empty and $H^0(X, \mathcal{O}_X) = k$ by assumption, we get

$$\ker\left[P(R) \xrightarrow{\pi_\star} P(k)\right] = \ker\left[H^1(X_R, \mathcal{O}_{X_R}^\star) \xrightarrow{\pi_\star} H^1(X, \mathcal{O}_X^\star)\right]$$
$$= \ker\left[H^1(X, (\mathcal{O}_X \otimes_k R)^\star) \xrightarrow{\pi} H^1(X, \mathcal{O}_X^\star)\right]$$
$$= H^1(X, 1 + \mathcal{O}_X \otimes_k M) = H^1(X, \mathrm{Spec}(S(E))_X).$$

Thus we get the first isomorphism.   On the other hand if we write $\mathcal{O} = \mathcal{O}_{P,0}$ then

$$\ker\left[P(R) \xrightarrow{\pi_*} P(k) \right] = \mathrm{Hom}_{\substack{\text{augmented} \\ \text{unitary} \\ k\text{-algebra}}} (\mathcal{O}, R)$$

$$= \mathrm{Hom}_{\substack{\text{augmented} \\ \text{unitary} \\ k\text{-coalgebra}}} (\mathrm{Hom}_k(R, k), H(P))$$

$$= \mathrm{Hom}_{\substack{\text{augmented} \\ \text{unitary} \\ k\text{-bialgebra}}} (S(E), H(P))$$

$$= \mathrm{Hom}_{k-gr} (\mathrm{Spec}(H(P)), \mathrm{Spec}(S(E))).$$

Thus the second isomorphism is obtained.

<div align="right">Q. E. D.</div>

Proof of Theorem 4.4 (1) *continued.* — We now apply Lemma 4.7 and Lemma 4.8 to the special case where

$$R = k[t]/(t^{p^N}).$$

Then $S(E) = k[A_1, A_2, \ldots, A_{p^N-1}]$, where if we put $A_0 = 1$, then $\{A_i\}$ is the dual basis of $\{t^i\}$.   The comultiplication is given by

$$\delta A_i = \sum_{j+k=i} A_j \otimes A_k.$$

Thus $\mathrm{Spec}(S(E))$ is the group scheme over k of truncated power series of length p^N and with 1 as the constant term, i. e. B-valued point is an element of the form

$$1 + b_1 t + b_2 t + \ldots + b_{p^N-1} t^{p^N-1}.$$

But according to Serre [32], V, No. 17, and Bergman [29], lecture 26, there is a canonical decomposition

$$\mathrm{Spec}(S(E)) = \prod_{\substack{0 < i < p^N \\ (i,p)=1}} W_{r_i} = W_N \times \prod_{\substack{1 < i < p^N \\ (i,p)=1}} W_{r_i},$$

where $r_i = \min[r; p^r \geq p^N/i]$.   Thus from Lemma 4.8 we get (1'').

<div align="right">Q. E. D.</div>

Let X be as in Theorem 4.4.   Then the Albanese variety $\mathrm{Alb}(X)$ of X is defined by (*cf.* Chevalley [9])

$$\mathrm{Alb}(X) = (\mathrm{Pic}^0_{X/k, \text{red}})^t.$$

There is a canonical k-morphism $i : X \to \mathrm{Alb}(X)$ unique up to the translation by k-valued points of $\mathrm{Alb}(X)$.   This defines a canonical homomorphism of left A-modules

$$i^* : \quad I(\mathrm{Alb}(X)) \to I(X)$$

which does not depend on the translation of i by k-valued points of $\mathrm{Alb}(X)$. In fact by the results of Serre [32] quoted in the proof of Proposition 4.2 the homomorphism

$$\mu^\star - p_1^\star - p_2^\star : \quad \mathrm{I}(\mathrm{Alb}(X)) \to \mathrm{I}(\mathrm{Alb}(X) \times \mathrm{Alb}(X))$$

is equal to zero. Thus the translation of $\mathrm{Alb}(X)$ by a k-valued point induces a trivial action on $\mathrm{I}(\mathrm{Alb}(X))$. From this we also see that the canonical homomorphism of left A-modules

$$i_\star : \quad \mathrm{H}_1(X_{p\text{-flat}}) \to \mathrm{H}_1(\mathrm{Alb}(X)_{p\text{-flat}})$$

does not depend on the translations (*cf.* Definition 4.1).

PROPOSITION 4.9. — *Let* X *be as in Theorem* 4.4. *Then :*

(i) *the canonical homomorphism of left* A-*modules*

$$i^\star : \quad \mathrm{I}(\mathrm{Alb}(X)) \to \mathrm{I}(X)$$

is injective and the image is equal to the submodule of p-divisible elements of $\mathrm{I}(X)$;

(ii) *the canonical homomorphism of left* A-*modules*

$$i_\star : \quad \mathrm{H}_1(X_{p\text{-flat}}) \to \mathrm{H}_1(\mathrm{Alb}(X)_{p\text{-flat}})$$

is surjective and the kernel is equal to the submodule of p-torsion elements of $\mathrm{H}_1(X_{p\text{-flat}})$.

Proof. — (ii) follows trivially from (i). By (2) of Theorem 4.4 we have a commutative diagram of left A-modules

$$
\begin{array}{ccc}
\mathrm{DM}\left(_{p^*}\mathrm{Pic}_{X/k}\right) & \xrightarrow{\ \sim\ } & \mathrm{I}(X) \\
\uparrow & & \uparrow{\scriptstyle i^\star} \\
\mathrm{DM}\left(_{p^*}(\mathrm{Pic}^0_{X/k,\,\mathrm{red}})\right) = \mathrm{DM}\left(_{p^*}\mathrm{Pic}_{\mathrm{Alb}(X)/k}\right) & \xrightarrow{\ \sim\ } & \mathrm{I}(\mathrm{Alb}(X))
\end{array}
$$

By (5) of Theorem 4.4 the left vertical arrow is injective and the image is equal to the submodule of p-divisible elements in $\mathrm{DM}(_{p^*}\mathrm{Pic}_{X/k})$. Hence $i^\star$ satisfies the same properties.

Q. E. D.

DEFINITION 4.10. — *Let* G *be an object of* $\mathscr{X} = \mathfrak{U} \times \mathfrak{N}_{lr}$, *i.e. commutative k-group scheme of finite type killed by some power of* p. *As in* SGA [17], 1960-1961, *exposé* XI, *we define*

$$\mathrm{H}^1(X_{\mathrm{flat}}, G_X)$$

to be the commutative group of equivalence classes of principal homogeneous spaces over X *with group* G_X.

This is a covariant functor from $\mathcal{R}$ to the category of abelian groups, and is a part of the flat cohomology theory of X. If G is smooth and connected over k, then we get

$$H^1(X_{\text{flat}}, G_X) = H^1(X, G_X),$$

where the right hand side is in the Zariski topology by SGA [17], *ibid.*, Prop. 5.1 [4].

We recall that for G in $\mathcal{R}$

$$M(G) = \text{Hom}_{k-gr}(G, C) \oplus \left\{ W(\bar{k}) \otimes_{\mathbf{Z}} \text{Hom}_{\bar{k}-gr}(G_{\bar{k}}, G_{m\bar{k}}) \right\}^{\text{Gal}(\bar{k}/k)}.$$

We also remark that

$$I(X) = H^1(X, C_X) \oplus \left\{ W(\bar{k}) \otimes_{\mathbf{Z}_{p^a}} H^1(\bar{X}, G_{m\bar{X}}) \right\}^{\text{Gal}(\bar{k}/k)}$$

$$= H^1(X_{\text{flat}}, C_X) \oplus \left\{ W(\bar{k}) \otimes_{\mathbf{Z}_{p^a}} H^1(\bar{X}_{\text{flat}}, G_{m\bar{X}}) \right\}^{\text{Gal}(\bar{k}/k)}$$

by the remark above and by the fact that $C = \varinjlim_n C_{-n}$ and that C_{-n} and G_m are smooth and connected.

DEFINITION 4.11. — *Let* G *be in* $\mathcal{R}$. *Then we define a homomorphism of commutative groups*

$$u: \quad H^1(X_{\text{flat}}, G_X) \to \text{Hom}_A(M(G), I(X))$$

as follows : let x *be an element of* $H^1(X_{\text{flat}}, G_X)$. *Then the homomorphism* $u(x)$ *from* $M(G)$ *to* $I(X)$ *is defined in the following way : an element* f *in* $\text{Hom}_{k-gr}(G, C)$ *induces a homomorphism*

$$f_*: \quad H^1(X_{\text{flat}}, G_X) \to H^1(X_{\text{flat}}, C_X) \subseteq I(X).$$

Then we put $u(x)(f) = f_*(x)$. *On the other hand an element* f *in* $\text{Hom}_{\bar{k}-gr}(G_{\bar{k}}, G_{m\bar{k}})$ *induces a homomorphism*

$$f_*: \quad H^1(\bar{X}_{\text{flat}}, G_{\bar{X}}) \to {}_{p^a}H^1(\bar{X}_{\text{flat}}, G_{m\bar{X}})$$

since G *is killed by some power of* p. *Hence if* $\bar{x}$ *is the image of* x *by* $H^1(X_{\text{flat}}, G_X) \to H^1(\bar{X}_{\text{flat}}, G_{m\bar{X}})$, *then we have a homomorphism*

$$\bar{u}(\bar{x}): \quad W(\bar{k}) \otimes_{\mathbf{Z}} \text{Hom}_{\bar{k}-gr}(G_{\bar{k}}, G_{m\bar{k}}) \to W(\bar{k}) \otimes_{\mathbf{Z}_{p^a}} H^1(\bar{X}_{\text{flat}}, G_{m\bar{X}})$$

by $\bar{u}(\bar{x})(\lambda \otimes f) = \lambda \otimes f_*(\bar{x})$. *We put* $u(x)$ *to be the restriction of* $\bar{u}(\bar{x})$ *to the* $\text{Gal}(\bar{k}/k)$-*invariants.*

---

[4] *See* also A. GROTHENDIECK, *Le groupe de Brauer III* : Exemples et complements, Appendice (Notes miméographiées, I. H. E. S., mars 1966). Cf. *Ad. St. in pure Maths.*, vol. 3, 1968, p. 88-188, North-Holland Pub.

It is easy to see that $u(x)$ is a homomorphism of left A-modules. We remark that if $I(X)$ is of $W(k)$-cofinite type and G is in $\mathfrak{N}_p$, i.e. finite k-group scheme whose rank over k is a power of p, then

$$\mathrm{Hom}_A(M(G), I(X)) = \mathrm{Hom}_A(H_1(X_{p\text{-flat}}), DM(G)).$$

Thus the following Theorem justifies the notation $H^1(X_{p\text{-flat}})$. A slightly different formulation of the same statement is in SGA [17], exposé XI, Section 6, and Milne [42], Section 2 ([5]).

THEOREM 4.12. — *Let k be an algebraically closed field of characteristic p. Let X be a k-prescheme such that $H^0(X, \mathcal{O}_X) = k$, and let G be a commutative group scheme of finite type killed by some power of p. Then the canonical homomorphism of commutative groups*

$$u: \quad H^1(X_{\text{flat}}, G_X) \to \mathrm{Hom}_A(M(G), I(X))$$

is an isomorphism.

Proof. — Since k is algebraically closed and $H^0(X, \mathcal{O}_X) = k$, the flat cohomology theory assures us that $H^1(X_{\text{flat}}, G_X)$ is left exact in G. On the other hand since M is exact, $\mathrm{Hom}_A(M(G), I(X))$ is left exact in G. Since G is in $\mathfrak{T}$ and k is algebraically closed G can be written as the kernel of a homomorphism

$$\prod_i C_{-n_i} \times \prod_j \mu_{p^{m_j}} \to \prod_{i'} C_{-n'_{i'}} \times \prod_{j'} \mu_{p^{m'_{j'}}},$$

where the products are finite in number. Hence it is enough to prove the theorem when : (i) $G = C_{-n}$ or (ii) $G = \mu_{p^n}$ for all n. In case (i), we have $M(G) = A/AV^n$ and the residue class of $1 \bmod AV^n$ corresponds to the canonical embedding $C_{-n} \to C$. This defines an isomorphism from $H^1(X, C_{-n,X})$ to the submodule of $H^1(X, C_X)$ killed by V^n. Hence we are done in case (i). In case (ii), $M(G) = A/A(V-1, F^n)$ and the residue class of $1 \bmod A(V-1, F^n)$ corresponds to the canonical embedding $\mu_{p^n} \to G_m$. This defines an isomorphism from $H^1(X_{\text{flat}}, \mu_{p^n})$ to the subgroup of $W(k) \otimes_{\mathbf{Z}_{p^n}} H^1(X, G_{m,X})$ killed by $V-1$ and F^n, i.e. the subgroup $1 \otimes_{p^n} H^1(X, G_{m,X})$.

Q. E. D.

COROLLARY 4.13. — *Let X_k be an abelian scheme over an algebraically closed field of characteristic p. If G is a commutative group scheme of finite type over k killed by some power of p, then the canonical homomorphism*

$$\mathrm{Ext}^1(X, G) \to H^1(X_{\text{flat}}, G_X)$$

is an isomorphism.

([5]) *See* also M. MIYANISHI, *Quelques remarques sur la première cohomologie d'un préschéma affine en groupes commutatifs* (to appear in *J. Math. Kyoto Univ.*).

Remark. — This was first proved by Lang-Serre [23], for a finite etale G, and by Miyanishi [43] for a finite G.   *See* also Milne [42], Section 2.

Proof. — Since by Proposition 4.2 we have

$$I(X) = C(k) \otimes_{W(k)} M(X(p)),$$

Theorem 4.12 gives an isomorphism

$$H^1(X_{\text{flat}}, G_X) \cong \text{Hom}_A(M(G), C(k) \otimes_{W(k)} M(X(p))).$$

On the other hand since $M(X(p))$ is $W(k)$-free, we have an exact sequence of left A-modules

$$0 \to M(X(p)) \to B(k) \otimes_{W(k)} M(X(p)) \to C(k) \otimes_{W(k)} M(X(p)) \to 0.$$

Since $M(G)$ is killed by some power of p, the long exact sequence gives an isomorphism

$$\text{Hom}_A(M(G), C(k) \otimes_{W(k)} M(X(p))) \xrightarrow[\sim]{\delta} \text{Ext}_A^1(M(G), M(X(p))).$$

The right hand side is equal to $\text{Ext}^1_{\text{Ind}(\mathfrak{L})}(X(p), G)$.   We now show that the canonical homomorphism

$$\text{Ext}^1(X, G) \to \text{Ext}^1_{\text{Ind}(\mathfrak{L})}(X(p), G)$$

is an isomorphism.   The following argument is due to F. Oort.

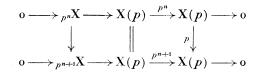Suppose G is killed by p^m.   Then for $n \geq m$ the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & {}_{p^n}X & \longrightarrow & X & \xrightarrow{p^n} & X & \longrightarrow & 0 \\
& & \downarrow & & \| & & \downarrow{\scriptstyle p} & & \\
0 & \longrightarrow & {}_{p^{n+1}}X & \longrightarrow & X & \xrightarrow{p^{n+1}} & X & \longrightarrow & 0
\end{array}
$$

gives a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Ext}^1(X, G) & \longrightarrow & \text{Ext}^1({}_{p^n}X, G) & \xrightarrow{\delta} & \text{Ext}^2(X, G) & \longrightarrow & 0 \\
& & \| & & \uparrow & & \uparrow{\scriptstyle p} & & \\
0 & \longrightarrow & \text{Ext}^1(X, G) & \longrightarrow & \text{Ext}^1({}_{p^{n+1}}X, G) & \xrightarrow{\delta} & \text{Ext}^2(X, G) & \longrightarrow & 0
\end{array}
$$

Taking the projective limit for increasing n we get an isomorphism

$$\text{Ext}^1(X, G) \xrightarrow{\sim} \varprojlim_{n} \text{Ext}^1({}_{p^n}X, G)$$

since $\text{Ext}^2(X, G)$ is killed by p^m.   Similarly from the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & {}_{p^n}X & \longrightarrow & X(p) & \xrightarrow{p^n} & X(p) & \longrightarrow & 0 \\
& & \downarrow & & \| & & {\scriptstyle p}\downarrow & & \\
0 & \longrightarrow & {}_{p^{n+1}}X & \longrightarrow & X(p) & \xrightarrow{p^{n+1}} & X(p) & \longrightarrow & 0
\end{array}
$$

we get an isomorphism

$$\mathrm{Ext}^1_{\mathrm{Ind}(\mathcal{Q})}(X(p),\ G) \xrightarrow{\sim} \varprojlim_n \mathrm{Ext}^1_{\mathrm{Ind}(\mathcal{Q})}({}_{p^n}X,\ G).$$

The right hand side is equal to $\varprojlim_n \mathrm{Ext}^1({}_{p^n}X,\ G)$.

<div style="text-align:right">Q. E. D.</div>

SECTION 5.

DE RHAM COHOMOLOGY AND DIEUDONNÉ MODULES.

Let X be a scheme over a ring k. Let $(\Omega^{\bullet}_{X/k},\ d)$ be the complex of sheaves of Kähler differential forms on X. Then the *De Rham cohomology of* X is defined by

$$H^n_{DR}(X) = \mathbf{H}^n(X,\ (\Omega^{\bullet}_{X/k},\ d)),$$

where the right hand side is the hypercohomology of the functor $H^0(X,\quad)$ with respect to the complex of abelian sheaves $(\Omega^{\bullet}_{X/k},\ d)$ (*cf.* EGA, $\mathrm{o_{III}}$, Section 11.4).

There is a spectral sequence

$$E^{p,q}_1 = H^q(X,\ \Omega^p_{X/k}) \Rightarrow H^n_{DR}(X).$$

Let $\mathfrak{U} = \{U(i)\}$ be an affine open covering of X. Then we have a first quadrant double complex

$$\{C^{\bullet}(\mathfrak{U},\ \Omega^{\bullet}_{X/k}),\ d,\ \delta\},$$

where $C^q(\mathfrak{U},\ \Omega^p_{X/k})$ is the set of all q-cochains $\alpha = \{\alpha(i_0,\ i_1,\ \ldots,\ i_q)\}$, where $(i_0, i_1,\ \ldots, i_q)$ runs through q-nerves of the covering $\mathfrak{U}$, and $\alpha(i_0, ..., i_q)$ is in $H^0(U(i_0,\ \ldots,\ i_q),\ \Omega^p_{X/k})$. The coboundary operators are

$$d:\quad C^q(\mathfrak{U},\ \Omega^p_{X/k}) \to C^q(\mathfrak{U},\ \Omega^{p+1}_{X/k})$$

sending α to $d\alpha$ defined by

$$(d\alpha)(i_0,\ i_1,\ \ldots,\ i_q) = d(\alpha(i_0,\ i_1,\ \ldots,\ i_q))$$

and

$$\delta:\quad C^q(\mathfrak{U},\ \Omega^p_{X/k}) \to C^{q+1}(\mathfrak{U},\ \Omega^p_{X/k})$$

sending α to $\delta\alpha$ defined by

$$(\delta\alpha)(i_0,\ i_1,\ \ldots,\ i_{q+1}) = \sum_{0 \leq r \leq q+1} (-1)^{p+r}\alpha(i_0,\ \ldots,\ \check{i}_r,\ \ldots,\ i_{q+1}).$$

It is easy to see that $d^2 = \delta^2 = d\delta + \delta d = 0$. We define the *hypercohomology of the covering* $\mathfrak{U}$ *with coefficients in* $(\Omega_{X/k}^{\cdot}, d)$

$$\mathrm{H}_{\mathrm{DR}}^n(\mathfrak{U}) = \mathbf{H}^n(\mathfrak{U}, \Omega_{X/k}^{\cdot})$$

to be the cohomology of the associated simple complex

$$\mathrm{C}_{\mathrm{DR}}^{\cdot}(\mathfrak{U}) = \{ \mathrm{C}^{\cdot}(\mathfrak{U}, \Omega_{X/k}^{\cdot}), d + \delta \}.$$

By the generalized Leray spectral sequence proved in EGA $\mathrm{o}_{\mathrm{III}}$, Corollary 12.4.7 we get a canonical isomorphism

$$\mathrm{H}_{\mathrm{DR}}^n(\mathfrak{U}) \cong \mathrm{H}_{\mathrm{DR}}^n(X).$$

(The assumption in EGA that the differentials of the complex are $\mathcal{O}_X$-linear is unnecessary.)

The exterior product $\Omega_{X/k}^p \otimes_k \Omega_{X/k}^{p'} \to \Omega_{X/k}^{p+p'}$ induces the cup product on $\mathrm{H}_{\mathrm{DR}}^{\cdot}(X)$. We can express it explicitly as follows (*cf.* Godement [14]) :

$$\mathrm{C}^q(\mathfrak{U}, \Omega_{X/k}^p) \otimes_k \mathrm{C}^{q'}(\mathfrak{U}, \Omega_{X/k}^{p'}) \to \mathrm{C}^{q+q'}(\mathfrak{U}, \Omega_{X/k}^{p+p'})$$

sending $\alpha \otimes \beta$ to $\alpha . \beta$ defined by

$$(\alpha.\beta)(i_0, \ldots, i_{q+q'}) = (-1)^{p'q} \alpha(i_0, \ldots, i_q) \wedge \beta(i_q, \ldots, i_{q+q'}).$$

Then it is easy to see that

$$d(\alpha.\beta) = (d\alpha).\beta + (-1)^{p+q} \alpha.(d\beta),$$
$$\delta(\alpha.\beta) = (\delta\alpha).\beta + (-1)^{p+q} \alpha.(\delta\beta).$$

Thus according to Cartan-Eilenberg [5], Exercises of Chapter XV, the complex $(\mathrm{E}_r^{p,q}, d_r)$ obtained from the double complex $\{ \mathrm{C}^q(\mathfrak{U}, \Omega_{X/k}^p), d, \delta \}$ is an associative and anti-commutative doubly-graded k-algebra with differentiation, i. e.

$$\mathrm{E}_r^{p,q} \otimes_k \mathrm{E}_r^{p',q'} \to \mathrm{E}_r^{p+p',q+q'}$$

sending $\alpha \otimes \beta$ to $\alpha . \beta$, is associative and

$$\beta.\alpha = (-1)^{(p+q)(p'+q')} \alpha.\beta,$$
$$d_r(\alpha.\beta) = (d_r\alpha).\beta + (-1)^{p+q} \alpha.(d_r\beta).$$

PROPOSITION 5.1. — *Let* X *be an abelian scheme over a field* k. *Then the spectral sequence*

$$\mathrm{H}^q(X, \Omega_{X/k}^p) \Rightarrow \mathrm{H}_{\mathrm{DR}}^n(X)$$

is degenerate, i. e. $d_r : \mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p+r,q-r+1}$ *is zero for all positive integers* r.

Proof. — The method is entirely similar to Serre ([32], VII, No. 22, Th. 11). We prove the proposition by induction on r. Using the induction hypothesis and the Künneth formula we get a canonical isomorphism

$$\mathrm{E}_r^{\cdot\cdot}(X \times X) = \mathrm{E}_r^{\cdot\cdot}(X) \otimes_k \mathrm{E}_r^{\cdot\cdot}(X).$$

The law of composition $\mu : X \times X \to X$ thus induces a comultiplication $\mu^\star : E_r^{..}(X) \to E_r^{..}(X) \otimes_k E_r^{..}(X)$. Moreover each $E_r^{p,q}(X)$ is finite dimensional, $E_r^{0,0} = k$, $E_r^{p,q} = 0$ if $p + q > 2 \dim X$, and

$$\dim_k [\, E_r^{1,0} \oplus E_r^{0,1} \,] = 2 \dim X.$$

Thus by Borel-Hopf's theorem, we have an isomorphism of Hopf algebras

$$E_r^{..} \cong \bigwedge [\, E_r^{1,0} \oplus E_r^{0,1} \,].$$

Especially $E_r^{1,0} \oplus E_r^{0,1}$ is the set of all primitive elements in $E_r^{..}$. Thus to prove d_r is zero it is enough to prove $d_r(x) = 0$, when x is primitive, i. e. $\mu^\star x = x \otimes 1 + 1 \otimes x$. Since d_r is functorial, we have $d_r \circ \mu^\star = \mu^\star \circ d_r$. Thus

$$\mu^\star \circ d_r(x) = d_r(x) \otimes 1 + 1 \otimes d_r(x),$$

i. e. $d_r(x)$ is also primitive. But x is in $E_r^{1,0} \oplus E_r^{0,1}$, hence $d_r(x)$ is in $E_r^{1+r,1-r} \oplus E_1^{r,2-r}$, whose intersection with $E_r^{1,0} \oplus E_r^{0,1}$ is zero. Thus $d_r(x) = 0$.

<div align="right">Q. E. D.</div>

Corollary 5.2. — *Let C be a proper smooth curve over a field k. Then the spectral sequence*

$$H^q(C, \Omega_{C/k}^p) \Rightarrow H_{DR}^n(C)$$

is degenerate.

Proof. — Only non-trivial part is to prove that $d_1 : E_1^{0,1} \to E_1^{1,1}$ is equal to zero. This map is equal to $d : H^1(C, \mathcal{O}_C) \to H^1(C, \Omega_{C/k}^1)$. We may assume k is algebraically closed. Let $i : C \to J$ be an imbedding into the Jacobian variety. Then by Serre ([32], VII, No. 19, Th. 9) there is a commutative diagram whose first column is an isomorphism

$$
\begin{array}{ccc}
H^1(C, \mathcal{O}_C) & \xrightarrow{\ d\ } & H^1(C, \Omega_{C/k}^1) \\
\uparrow{\scriptstyle i^\star} & & \uparrow{\scriptstyle i^\star} \\
H^1(J, \mathcal{O}_J) & \xrightarrow{\ d\ } & H^1(J, \Omega_{J/k}^1)
\end{array}
$$

By Proposition 5.1 the bottom row is zero, hence the top row is zero.

<div align="right">Q. E. D.</div>

For later purposes we now describe $H_{DR}^1(X)$ in terms of Čech cocyles explicitly. Let $\mathfrak{U} = \{U(i)\}$ be an affine open covering. Then as we remarked above we have a canonical isomorphism

$$H_{DR}^1(X) = Z_{DR}^1(\mathfrak{U})/B_{DR}^1(\mathfrak{U}),$$

where $Z_{DR}^1(\mathfrak{U})$ is the set of pairs (f, ω) for f in $C^1(\mathfrak{U}, \mathcal{O}_X)$ and ω in $C^0(\mathfrak{U}, \Omega_{X/k}^1)$ such that

$$\delta f = 0, \quad df - \delta\omega = 0 \quad \text{and} \quad d\omega = 0,$$

i. e.

$$f(j, k) - f(i, k) + f(i, j) = 0,$$
$$df(i, j) = \omega(i) - \omega(j),$$
$$d\omega(i) = 0$$

for all i, j and k, while $B_{DR}^1(\mathfrak{U})$ is the set of pairs of the form $(\delta g, dg)$ for g in $C^0(\mathfrak{U}, \mathcal{O}_X)$, where

$$(\delta g)(i, j) = g(i) - g(j),$$
$$(dg)(i) = d(g(i)).$$

Suppose k is a ring of characteristic p. Let X be a scheme over k. Then it is easy to see that

$$H_{DR}^{\cdot}(X^{(p)}) = (k, \sigma) \otimes_k H_{DR}^{\cdot}(X).$$

Hence the k-morphism $F : X \to X^{(p)}$ induces a homomorphism of k-modules

$$F^* : (k, \sigma) \otimes_k H_{DR}^{\cdot}(X) \to H_{DR}^{\cdot}(X)$$

or a σ-homomorphism F from $H_{DR}^{\cdot}(X)$ into itself sending α in $H_{DR}^{\cdot}(X)$ to $F\alpha = F^*(1 \otimes \alpha)$. In particular, if k is a perfect field of characteristic p, we can give $H_{DR}^{\cdot}(X)$ a structure of left $k[F]$-module, where $k[F]$ is the non-commutative ring defined by $F\lambda = \lambda^\sigma F$ for all λ in k.

The homomorphism $F^*\Omega_{X^{(p)}/k}^{\cdot} \to \Omega_{X/k}^{\cdot}$ induced by F coincides with taking p-th power in degree 0 and is equal to zero in higher degrees. Using this fact it is easy to express F in terms of Čech cocycles. For example, if (f, ω) is an element of $Z_{DR}^1(\mathfrak{U})$, then the class of $(f, \omega) \mod B_{DR}^1(\mathfrak{U})$ is sent to the class of $(f^p, 0) \mod B_{DR}^1(\mathfrak{U})$, where $(f^p)(i, j) = f(i, j)^p$.

DEFINITION 5.3. — *For a scheme X over a field k of characteristic p, we define a homomorphism of k-vector spaces*

$$F : (k, \sigma) \otimes_k H^1(X, \mathcal{O}_X) \to H_{DR}^1(X)$$

or a σ-homomorphism of k-vector spaces

$$F : H^1(X, \mathcal{O}_X) \to H_{DR}^1(X)$$

as follows : if f is in $Z^1(\mathfrak{U}, \mathcal{O}_X)$ for an affine open covering $\mathfrak{U}$, then F sends the cohomology class of f to the class of $(f^p, 0)$ in $Z_{DR}^1(\mathfrak{U})$.

It is well defined. In fact if $f = \delta g$ for some g in $C^0(\mathfrak{U}, \mathcal{O}_X)$, then $(\delta g^p, 0) = (\delta g^p, dg^p)$. If X is normal, then F is injective. In fact if $(f^p, 0) = (\delta g, dg)$ for some g in $C^0(\mathfrak{U}, \mathcal{O}_X)$, then passing to a finer covering

if necessary, we may assume that $g = h^p$ for some h in $C^0(\mathfrak{U}, \mathcal{O}_X)$ since X is normal. Hence $f = \delta h$.

From now on we are only interested in $H^1_{DR}(X)$. Note that $E_1^{1,0} = H^0(X, \Omega^1_{X/k})$ and $E_1^{0,1} = H^1(X, \mathcal{O}_X)$. Hence $E_2^{1,0}$ is equal to the kernel of $d : H^0(X, \Omega^1_{X/k}) \to H^0(X, \Omega^2_{X/k})$ and $E_2^{0,1}$ is the kernel of $d : H^1(X, \mathcal{O}_X) \to H^1(X, \Omega^1_{X/k})$. The exact sequence of terms of low degree of this spectral sequence is

(1) $$0 \to H^0(X, \Omega^1_{X/k})_{d=0} \to H^1_{DR}(X) \to H^1(X, \mathcal{O}_X).$$

It can be easily seen that the map F in Definition 5.3 coincides on the image of the third arrow in (1) with the map induced by the map F defined before Definition 5.3, since this latter F kills $E_2^{1,0}$.

If X is an abelian scheme over k or a proper smooth curve over k, then by Proposition 5.1 and Corollary 5.2 we have an exact sequence

(1') $$0 \to H^0(X, \Omega^1_{X/k}) \to H^1_{DR}(X) \to H^1(X, \mathcal{O}_X) \to 0.$$

PROPOSITION 5.4. — *If X is a normal k-scheme, then the sequence*

(2) $$0 \to H^0(X, \Omega^1_{X/k})_{d=0} \to H^1_{DR}(X) \xrightarrow{F} (k, \sigma^{-1}) \otimes_k H^1_{DR}(X)$$

is exact.

Proof. — We prove it in terms of Čech cocyles. Suppose $\mathfrak{U}$ is an affine open covering of X. Suppose (f, ω) is in $Z^1_{DR}(\mathfrak{U})$ such that $F(f, \omega) = (\delta g, dg)$ for some g in $C^0(\mathfrak{U}, \mathcal{O}_X)$, i. e. $f^p = \delta g$ and $dg = 0$. Then by passing to a finer affine open covering if necessary we may assume that $g = h^p$ for some h in $C^0(\mathfrak{U}, \mathcal{O}_X)$, since X is normal. Hence $f = \delta h$. On the other hand since $df + \delta \omega = 0$ and $d\omega = 0$ by definition, there is an ω_0 in $H^0(X, \Omega^1_{X/k})_{d=0}$ such that $\omega = dh + \omega_0$. Thus $(f, \omega) = (0, \omega_0) + (\delta h, dh)$.

Q. E. D.

The Cartier operator is defined in Cartier [6] and Seshadri [36] as follows [6]:

DEFINITION 5.5. — *Let k be a perfect field of characteristic p and let X be a k-prescheme such that $\mathcal{O}_{X,x}$ has a p-base for all points x in X (e. g. X is smooth over k). Then there is a surjective homomorphism of sheaves of rings*

$$V : \quad \Omega^{\bullet}_{X/k, d=0} \to \Omega^{\bullet}_{X/k}$$

characterized by :

(i) *the kernel of V is equal to the subsheaf of exact differentials;*

(ii) *V coincides in degree 0 to the extraction of p-th root (noting that $\Omega^0_{X/k, d=0} = \mathcal{O}^p_X$);*

(iii) *if ω is in $\Omega^1_{X/k,x,d=0}$ for a point x of X, then $V(\omega) = \omega$ if and only if there exists an element f in $\mathcal{O}^\star_{X,x}$ such that $\omega = \dfrac{df}{f}$.*

The point here is that if $\{y_1, y_2, \ldots, y_n\}$ is a p-base of the local ring $\mathcal{O}_{X,x}$ at a point x of X such that y_i are units in $\mathcal{O}_{X,x}$, then

$$\Omega^1_{X/k,x,d=0} = d\mathcal{O}_{X,x} \oplus \sum_{1 \leqslant i \leqslant n} \mathcal{O}^p_{X,x}\left(\frac{dy_i}{y_i}\right).$$

In particular if X is a smooth scheme over a perfect field k of characteristic p, then from Definition 5.5, we have an exact sequence

(3) $$0 \to d\mathcal{O}_X \to \Omega^1_{X/k,d=0} \xrightarrow{V} \Omega^1_{X/k} \to 0$$

of abelian sheaves, where V is a σ^{-1}-homomorphism from the $\mathcal{O}^p_X$-module to the $\mathcal{O}_X$-module.

DEFINITION 5.6. — *Let X be a smooth scheme over a perfect field k of characteristic p (or more generally X has a p-base at every point). We define a homomorphism of k-vector spaces*

$$V : \ H^1_{DR}(X) \to (k, \sigma) \otimes_k H^0(X, \Omega^1_{X/k})$$

or a σ^{-1}-homomorphism of k-vector spaces

$$V : \ H^1_{DR}(X) \to H^0(X, \Omega^1_{X/k})$$

as follows : if (f, ω) is in $Z^1_{DR}(\mathfrak{U})$ for an affine open covering $\mathfrak{U}$, i. e.

$$f(j, k) - f(i, k) + f(i, j) = 0,$$
$$df(i, j) = \omega(i) - \omega(j),$$
$$d\omega(i) = 0$$

for all i, j and k, then V sends the cohomology class of (f, ω) to $V\omega$ in $H^0(X, \Omega^1_{X/k})$.

In fact $d\omega(i) = 0$ hence $V\omega(i)$ is well defined. The second equality implies that $V\omega(i) = V\omega(j)$ for all i and j. Moreover if (f, ω) is in $B^1_{DR}(\mathfrak{U})$, i. e. if there exists g in $C^0(\mathfrak{U}, \mathcal{O}_X)$ such that $f = \delta g$ and $\omega = dg$, then $V\omega = V(dg) = 0$.

PROPOSITION 5.7. — *If X is a smooth scheme over a perfect field k of characteristic p, then the sequence*

(4) $$0 \to (k, \sigma) \otimes_k H^1(X, \mathcal{O}_X) \xrightarrow{F} H^1_{DR}(X) \xrightarrow{V} (k, \sigma) \otimes_k H^0(X, \Omega^1_{X/k})$$

is exact.

Proof. — The injectivity of F is shown in Definition 5.3. Suppose (f, ω) is an element of $Z^1_{DR}(\mathfrak{U})$ for an affine open covering $\mathfrak{U}$, such that $V(f, \omega) = 0$.

Then passing to a finer covering if necessary, we may assume that $\omega(i) = dg(i)$ for all i for some g in $C^0(\mathfrak{U}, \mathcal{O}_X)$ by Definition 5.5 (i). Since $df + \delta\omega = 0$, we get $d(f - \delta g) = 0$. Hence passing to a finer covering if necessary we have $f - \delta g = h^p$ for some h in $Z^1(\mathfrak{U}, \mathcal{O}_X)$. Thus $(f, \omega) = (h^p, 0) + (\delta g, dg)$ and we are done.

<div align="right">Q. E. D.</div>

There is an example of a smooth scheme X over a perfect field k of characteristic p, for which

$$H^0(X, \Omega^1_{X/k})_{d=0} \neq H^0(X, \Omega^1_{X/k})$$

(*cf.* Mumford [28]).

By definition 5.6 we have a σ^{-1}-homomorphism

$$H^1_{DR}(X) \xrightarrow{V} H^0(X, \Omega^1_{X/k}) \xrightarrow{d} H^0(X, \Omega^2_{X/k}).$$

We can define V^2 from the kernel of $d \circ V$ in $H^1_{DR}(X)$ to $H^0(X, \Omega^1_{X/k})$. Hence we have a σ^{-2}-homomorphism

$$\ker(d \circ V) \xrightarrow{V^2} H^0(X, \Omega^1_{X/k}) \xrightarrow{d} H^0(X, \Omega^2_{X/k}).$$

Similarly we can define a $\sigma^{-(n+1)}$-homomorphism V^{n+1} from the kernel of $d \circ V^n$ in $H^1_{DR}(X)$ to $H^0(X, \Omega^1_{X/k})$. $V(\ker[d \circ V^n])$ is contained in $\ker(d \circ V^{n-1})$. Hence

$$\bigcap_n \ker(d \circ V^n) \quad \text{in } H^1_{DR}(X)$$

is stable under V, where $H^0(X, \Omega^1_{X/k})_{d=0}$ is considered as a subspace of $H^1_{DR}(X)$ by (1).

There is an example of a smooth scheme X over a perfect field k of characteristic p, for which

$$\ker(d \circ V^{n-1}) \neq \ker(d \circ V^n).$$

The following is due to Mumford. Start from the projective plane $\mathbf{P}_2$ and consider the rational differential

$$\omega = x^{p^n}\left(\frac{dy}{y}\right),$$

where (x, y) is the affine coordinate of $\mathbf{P}_2$. Then

$$d\omega = 0, \qquad V\omega = x^{p^{n-1}}\left(\frac{dy}{y}\right), \qquad dV\omega = 0, \qquad \ldots, \qquad V^{n-1}\omega = x^p\left(\frac{dy}{y}\right),$$

$$dV^{n-1}\omega = 0, \qquad V^n\omega = x\left(\frac{dy}{y}\right), \qquad dV^n\omega = dx \wedge \left(\frac{dy}{y}\right) \neq 0.$$

We now apply the method in Mumford [28] to this ω.

DEFINITION 5.8. — *Let* X *be a smooth scheme over a perfect field* k *of characteristic* p. *We define a left* A-*module structure on* $\bigcap_n \ker(d \circ V^n)$ *in* $H^1_{DR}(X)$ *by operating with* $W(k)$ *through reduction mod* p *and the canonical* k-*vector space structure, by operating with* F *in the manner described before Definition* 5.3, *and by operating with* V *via the Cartier operator.*

This is well defined by Proposition 5.4 and Proposition 5.7. If X is an abelian scheme over k or a proper and smooth curve over k, then by Proposition 5.1 and Corollary 5.2 we get a left A-module structure on $H^1_{DR}(X)$ itself. Furthermore Proposition 5.4 and Proposition 5.7 imply that

$$H^0(X, \Omega^1_{X/k}) = VH^1_{DR}(X) = {}_FH^1_{DR}(X)$$

and

$$FH^1_{DR}(X) = {}_VH^1_{DR}(X).$$

Let k be a perfect field of characteristic p. As before we denote by C the contravariant functor of Witt covectors from the category of k-preschemes to that of left A-modules.

DEFINITION 5.9. — *We define a morphism of* set *functors* (*not of group functors*)

$$S : C \to C$$

by

$$(Sx)_m = \begin{cases} 0 & \text{for} \quad m = -1, \\ x_{m+1} & \text{for} \quad m \leq -2 \end{cases}$$

for all k-*preschemes* T *and points* x *in* C(T).

It is not difficult to see that

(5) $$V \circ S = id_C;$$

(6) $$S \circ F = F \circ S;$$

(7) $$S \circ \{a\} = \{a^p\} \circ S \quad \text{for all } a \text{ in } k,$$

where $\{a\}$ is the element in $W(k)$ whose first component is a and the rest of whose components are zero;

(8) $$(x - SVx)_m = \begin{cases} x_{-1} & \text{for} \quad m = -1, \\ 0 & \text{for} \quad m \leq -2 \end{cases}$$

for all k-prescheme T and x in C(T);

(9) $$[S(x+y) - Sx - Sy]_m = 0 \quad \text{for} \quad m \leq -2$$

for all k-preschemes T and x and y in C(T).

Let X be a k-prescheme. The functor C defines a sheaf of left A-modules C_X on X in the Zariski topology. We define a homomorphism of abelian sheaves

$$\Delta : \quad C_X \to \Omega^1_{X/k}$$

by

$$\Delta u = du^{(-1)} = \sum_{i \geqq 0} u^{p^i - 1}_{-1-i}\, du_{-1-i}$$

for an open set U of X and a section u in C(U).

It is easy to see that

(10) $d \circ \Delta = 0;$

(11) $\Delta \circ \{a\} = a^{p-1} \Delta \quad \text{for } a \text{ in } k,$

where the right hand side is the scalar multiplication by an element of k on $\Omega^1_{X/k}$.

We remark here that the Cartier operator is well defined on the closed rational differential forms even if X is not smooth over k, e. g. X is normal, i. e.

$$V : \quad \Omega^1_{k(X)/k,\, d=0} \to \Omega^1_{k(X)/k}.$$

However it is not guaranteed that $V(\Omega^1_{X/k,x,\,d=0})$ is contained in $\Omega^1_{X/k,x}$.

From (10) the image of Δ is contained in $\Omega^1_{X/k,\,d=0}$. From the form of Δ it is easily seen that $V \circ \Delta$ sends C_X into $\Omega^1_{X/k}$, even if X is not smooth,

(12) $V \circ \Delta = \Delta \circ V.$

In fact for an open set U on X and a section u in C(U) we get

$$\Delta(Vu) = \sum_{i \geqq 0} (Vu)^{p^i - 1}_{-1-i}\, d(Vu)_{-1-i} = \sum_{i \geqq 0} u^{p^i - 1}_{-2-i}\, du_{-2-i}$$

$$= V\left[\sum_{i \geqq 0} u^{p^i - 1}_{-1-i}\, du_{-1-i}\right] = V(\Delta u);$$

(13) $\Delta \circ F = 0.$

Moreover if X is normal, then the kernel of Δ is equal to FC_X.

In fact suppose $\Delta u = 0$ for an element u in $C_{X,x}$. If $u_m = 0$ for $m \leqq -2$, then $\Delta u = du_{-1}$, hence $u_{-1} = v^p_{-1}$ for some v_{-1} in $k(X)$. But v_{-1} is actually in $\mathcal{O}_{X,x}$ since it is normal. We prove by induction on the length of u. Suppose we already proved the statement for u of length no more than n. Then $\Delta Vu = V\Delta u = 0$. Hence by induction hypothesis we get $Vu = Fw$ for some w in $C_{X,x}$. Then $u - FSw = u - SFw = u - SVu$ has length 1, since $V(u - SVu) = 0$. Moreover $\Delta(u - FSw) = \Delta u = 0$. Hence $u - FSw = Fw'$ for some w'. Thus we are done.

Similarly we can define a homomorphism of abelian sheaves

$$(d \log) : \quad G_{m\,X} \to \Omega^1_{X/k}$$

by

$$(d \log)\,(u) = \frac{du}{u}$$

for an open set U of X and a section u in $G_m(U)$.

It is easy to see that

(14) $d \circ (d \log) = 0$;

(15)   V leaves image elements of $(d \log)$ fixed;

(16) $(d \log) \circ F = 0$.

Moreover if X is normal, then the kernel of $(d \log)$ is $FG_{m\,X}$.

Let $I(X)$ be the left A-module given in Definition 4.1.

THEOREM 5.10. — *Let k be a perfect field of characteristic p.*

(i) *Let X be a k-scheme such that $H^0(X, \mathcal{O}_X) = k$. Then there is a canonical homomorphism of left $k[F]$-modules*

$$\varphi : \quad {}_p I(X) \to H^1_{DR}(X)$$

such that $\varphi({}_F I(X))$ is contained in $H^0(X, \Omega^1_{X/k})_{d=0}$.

(ii) *If X is a normal k-scheme such that $H^0(X, \mathcal{O}_X) = k$, then φ is injective. Moreover φ is a homomorphism of left A-modules in the sense that in the image of φ the Cartier operator V is well defined, and that if we give the image of φ the left A-module structure by operating with V through the Cartier operator, then φ is a homomorphism of left A-modules. φ maps ${}_F I(X)$ injectively into $H^0(X, \Omega^1_{X/k})_{d=0}$.*

(iii) *If X is a k-scheme proper and smooth over k, then φ is injective and the image is equal to $\bigcap_n \ker(d \circ V^n)$. φ maps ${}_F I(X)$ isomorphically onto $\bigcap_n \ker(d \circ V^n) \cap H^0(X, \Omega^1_{X/k})$. Moreover φ is a homomorphism of left A-modules.*

Proof. — We recall that

$$_p I(X) = {}_p H^1(X, C_X) + \left\{ \bar{k} \otimes_{\mathbf{Z}} {}_p H^1(\overline{X}, G_{m\overline{X}}) \right\}^{Gal(\bar{k}/k)}.$$

We first define the map

$$\overline{\varphi}_2 : \quad {}_p H^1(\overline{X}, G_{m\overline{X}}) \to H^0(\overline{X}, \Omega^1_{\overline{X}/\overline{k}})_{d=0}.$$

Let $\alpha = \{\alpha(i, j)\}$ be a 1-cocycle representing an element of ${}_p H^1(\overline{X}, G_{m\overline{X}})$ with respect to an affine open covering $\mathfrak{U} = \{U(i)\}$ of X. Then by

definition there is a 0-cochain $\beta = \{\beta(i)\}$ such that $\alpha(i,j)^p = \beta(i)/\beta(j)$. Then $d\beta(i)/\beta(i) = d\beta(j)/\beta(j)$ determine a global section of $\Omega^1_{\overline{X}/\overline{k}}$. $\overline{\varphi}_2$ sends the cohomology class of α to this global section. This map is well defined. In fact if for a cohomologous 1-cocycle $\alpha \cdot \partial\lambda = \{\alpha(i,j)\,\lambda(i)/\lambda(j)\}$ we have

$$[\alpha(i,j)\,\lambda(i)/\lambda(j)]^p = \beta'(i)/\beta'(j)$$

for some 0-cochain β', then we get $\lambda(i)^p\beta(i)/\beta'(i) = \lambda(j)^p\beta(j)/\beta'(j)$ which determine a global section of $G_{m\overline{X}}$, thus by assumption an element of $\overline{k}^\star$. Thus we can write $\beta'(i) = \pi(i)^p\beta(i)$ for a 0-cochain π. Thus

$$d\beta'(i)/\beta'(i) = d\beta(i)/\beta(i).$$

This map is obviously additive. We extend $\overline{\varphi}_2$ by linearity to

$$\overline{\varphi}_2 : \quad \overline{k} \otimes_{\mathbf{Z}} {}_p H^1(\overline{X}, G_{m\overline{X}}) \to H^0(\overline{X}, \Omega^1_{\overline{X}/\overline{k}})_{d=0}.$$

Since $\overline{\varphi}_2 \circ F = F \circ \overline{\varphi}_2$ and $V[d\beta(i)/\beta(i)] = d\beta(i)/\beta(i)$, we conclude that $\overline{\varphi}_2$ is $A^{\overline{k}}$-linear. If X is normal, then $\overline{\varphi}_2(1 \otimes \overline{\alpha}) = 0$ implies that $d\beta(i) = 0$. Then passing to a finer covering if necessary, we may assume that $\beta(i)$ is in $FG_{m\overline{X}}(U(i)) = \mathcal{O}^\star_{\overline{X}}(U(i))^p$. Hence $\alpha(i,j) = \beta(i)^{p-1}/\beta(j)^{p-1}$, thus α is cohomologous to 0. Finally since $H^1_{DR}(\overline{X}) = \overline{k} \otimes_k H^1_{DR}(X)$ and since $\overline{\varphi}_2$ is canonical, it descends to a canonical homomorphism of A-modules φ_2 we are looking for.

We next define the map $\varphi_1 : {}_p H^1(X, C_X) \to H^1_{DR}(X)$. Let $\alpha = \{\alpha(i,j)\}$ be a 1-cocycle representing an element of ${}_p H^1(X, C_X)$ with respect to an affine open covering $\mathfrak{U} = \{U(i)\}$ of X. By definition there is a 0-cochain $\beta = \{\beta(i)\}$ such that $p\alpha(i,j) + \beta(i) - \beta(j) = 0$. Since $p = VF$, there exists a 1-cocycle $\mu = \{\mu(i,j)\}$ such that $\mu(i,j)_m = 0$ for $m \leq -2$, and

$$F\alpha(i,j) + S\beta(i) - S\beta(j) = \mu(i,j).$$

Applying Δ on both sides we get

$$\Delta S\beta(i) - \Delta S\beta(j) = \Delta\mu(i,j).$$

But the right hand side is equal to $d\mu(i,j)_{-1}$, hence $(\mu_{-1}, \Delta S\beta)$ determines an element of $Z^1_{DR}(\mathfrak{U})$. We define

$$\varphi_1(\overline{\alpha}) = \text{the class of } (\mu_{-1}, \Delta S\beta) \qquad \mathrm{mod}\, B^1_{DR}(\mathfrak{U}),$$

where the bar denotes the cohomology class. This map is well defined. In fact if for a cohomologous 1-cocycle

$$\alpha + \delta a = \{\alpha(i,j) + a(i) - a(j)\},$$

we have

$$p[\alpha(i,j) + a(i) - a(j)] + \beta'(i) - \beta'(j) = 0$$

for a o-cochain β', then

$$\beta'(i) - \beta(i) + pa(i) = \beta'(j) - \beta(j) + pa(j)$$

determine a global section of C_X, i. e. there is an element ρ in $C(k)$ such that $\beta'(i) - \beta(i) - \rho + pa(i) = 0$ for all i. Hence there exists a o-cochain $b = \{b(i)\}$ such that $b(i)_m = 0$ for $m \leq -2$ and

$$S\beta'(i) - S\beta(i) - S\rho + Fa(i) = b(i).$$

If we define $\mu' = \{\mu'(i,j)\}$ by

$$\mu'(i,j) = F[a(i,j) + a(i) - a(j)] + S\beta'(i) - S\beta'(j)$$

we get

$$\mu'(i,j) = \mu(i,j) + b(i) - b(j)$$

and

$$S\beta'(i) = S\beta(i) + S\rho - Fa(i) + b(i).$$

Thus we get $\Delta S\beta'(i) = \Delta S\beta(i) + \Delta b(i)$. Noting that $\Delta b(i) = db(i)_{-1}$, we get $(\mu'_{-1}, \Delta S\beta') = (\mu_{-1}, \Delta S\beta) + (\delta b_{-1}, db_{-1})$. Thus they determine the same element in $H^1_{DR}(X)$.

We now claim that for a in k, $\varphi_1(\{a\}\bar{\alpha}) = a\varphi_1(\bar{\alpha})$. In fact we have $p\{a\}\alpha(i,j) + \{a\}\beta(i) - \{a\}\beta(j) = 0$. Then

$$F\{a\}\alpha(i,j) + S\{a\}\beta(i) - S\{a\}\beta(j) = \{a^p\}\mu(i,j).$$

Since

$$[\{a^p\}\mu(i,j)]_{-1} = a\mu(i,j)_{-1} \quad \text{and} \quad \Delta S\{a\}\beta(i) = \Delta\{a^p\}S\beta(i) = a\Delta S\beta(i),$$

we get

$$\varphi_1(\{a\}\bar{\alpha}) = \text{the class of } (a\mu_{-1}, a\Delta S\beta) = a\varphi_1(\bar{\alpha}).$$

φ_1 is obviously additive. Next we claim that $\varphi_1(F\bar{\alpha}) = F\varphi_1(\bar{\alpha})$. In fact we have $pF\alpha(i,j) + F\beta(i) - F\beta(j) = 0$. Thus

$$F(F\alpha(i,j)) + SF\beta(i) - SF\beta(j) = F\mu(i,j).$$

Hence $\varphi_1(F\bar{\alpha}) = $ the class of $((F\mu)_{-1}, \Delta SF\beta) = $ the class of $((\mu_{-1})^p, 0) = $ the class of $F(\mu_{-1}, \Delta S\beta) = F\varphi_1(\bar{\alpha})$.

Finally we claim that $\varphi_1(V\bar{\alpha}) = V\varphi_1(\bar{\alpha})$. In fact we have

$$p(V\alpha(i,j)) + V\beta(i) - V\beta(j) = 0.$$

Thus

$$F(V\alpha(i,j)) + SV\beta(i) - SV\beta(j) = [SV\beta(i) - \beta(i)] - [SV\beta(j) - \beta(j)].$$

If we denote $b(i) = SV\beta(i) - \beta(i)$, then $b(i)_m = 0$ for $m \leq -2$. Hence

$$\varphi_1(V\bar{\alpha}) = \text{the class of } (\delta b_{-1}, \Delta SV\beta)$$
$$= \text{the class of } (0, \Delta\beta) + (\delta b_{-1}, \Delta b) = \text{the class of } (0, \Delta\beta).$$

On the other hand,

$$V \varphi_1(\overline{\alpha}) = \text{the class of } V(\mu_{-1}, \Delta S \beta)$$
$$= \text{the class of } (0, V \Delta S \beta) = \text{the class of } (0, \Delta V S \beta) = \text{the class of } (0, \Delta \beta).$$

Hence we are done.

Suppose X is normal and $\varphi_1(\overline{\alpha}) = 0$. Then $(\mu_{-1}, \Delta S \beta) = (\delta c_{-1}, dc_{-1})$ for a 0-cochain $c = \{c(i)\}$ such that $c(i)_m = 0$ for $m \leq -2$. We get

$$\mu(i, j) = c(i) - c(j) \qquad \text{and} \qquad \Delta S \beta(i) = \Delta c(i).$$

Hence $F \alpha(i, j) + S \beta(i) - S \beta(j) = c(i) - c(j)$ and there exists a 0-cochain $a = \{a(i)\}$ such that $S \beta(i) = c(i) + F a(i)$, if we pass to a finer covering if necessary, since X is normal. Thus $F[\alpha(i, j) + a(i) - a(j)] = 0$. Since X is reduced we conclude that α is cohomologous to 0. Thus φ_1 is injective when X is normal.

We now show that $\varphi_1({}_F H^1(X, C_X))$ is contained in $H^0(X, \Omega^1_{X/k})_{d=0}$. In fact if $\alpha = \{\alpha(i, j)\}$ is a representing 1-cocycle of an element of ${}_F H^1(X, C_X)$ with respect to an affine open covering, then there exists a 0-cochain $\lambda = \{\lambda(i)\}$ such that $F \alpha(i, j) + \lambda(i) - \lambda(j) = 0$. Hence $p \alpha(i, j) + V \lambda(i) - V \lambda(j) = 0$. Therefore we get

$$F \alpha(i, j) + SV \lambda(i) - SV \lambda(j) = [SV \lambda(i) - \lambda(i)] - [SV \lambda(j) - \lambda(j)].$$

If we denote $b(i) = SV \lambda(i) - \lambda(i)$, then we have $b(i)_m = 0$ for $m \leq -2$, $\beta(i) = V \lambda(i)$ and $\mu(i, j) = b(i) - b(j)$. Hence

$$\varphi_1(\overline{\alpha}) = \text{the class of } (\mu_{-1}, \Delta S \beta) = \text{the class of } (\delta b_{-1}, \Delta SV \lambda)$$
$$= \text{the class of } [(0, \Delta \lambda) + (\delta b_{-1}, db_{-1})] = \text{the class of } (0, \Delta \lambda).$$

Taking the sum of φ_1 and φ_2 we have a canonical homomorphism of left $k[F]$-modules (left A-modules, if X is normal)

$$\varphi : {}_p I(X) \to H^1_{DR}(X)$$

such that $\varphi({}_F I(X)) \subseteq H^0(X, \Omega^1_{X/k})_{d=0}$. Moreover we have shown that if X is normal, then

(★) $$(\ker \varphi) \cap {}_p H^1(X, C_X) = 0;$$

(★★) $$(\ker \overline{\varphi}) \cap \{ 1 \otimes_Z {}_p H^1(\overline{X}, G_{m\overline{X}}) \}^{\text{Gal}(\overline{k}/k)} = 0.$$

φ is injective if and only if the base extension $\overline{\varphi}$ is injective. Hence we may assume k is algebraically closed. Since $\ker \varphi$ is an A-submodule of ${}_p I(X)$ and since V acts bijectively on $k \otimes_Z {}_p H^1(X, G_{mX})$ we see that

(★★★) $$(\ker \varphi) \cap \{ k \otimes_Z {}_p H^1(X, G_{mX}) \}$$

is generated over k by elements left fixed by V, i. e. elements in (★★), which is zero. Hence (★★★) is zero. On the other hand V acts nilpo-

tently on $_pH^1(X, C_X)$. Hence for all u in $\ker\varphi$ there exists an integer N such that $V^N u$ is in $k \otimes_Z {}_pH^1(X, G_{mX})$. But $V^N u$ is also in $\ker\varphi$. Hence $V^N u$ is in the intersection ($\bigstar\bigstar\bigstar$) which is zero. Thus u is in ($\bigstar$) which is also zero.

We now identify the image of φ, when X is smooth and proper over k. We may obviously assume k is algebraically closed. For simplicity we write

$$L = \bigcap_n \ker(d \circ V^n).$$

Then since X is proper, L is finite dimensional. Thus there is a canonical decomposition $L = L_1 \oplus L_2$, where V acts nilpotently on L_1, while it acts bijectively on L_2. L_2 is moreover generated by V-invariants.

Suppose the cohomology class of (f, ω) in $Z^1_{DR}(\mathfrak{U})$ is V-invariant. Then there is g in $C^0(\mathfrak{U}, \mathcal{O}_X)$ such that $(o, V\omega) = (f, \omega) + (\delta g, dg)$. Replacing (f, ω) by $(f, \omega) + (\delta g, dg)$, we may assume $(o, V\omega) = (f, \omega)$. Hence $f = o$ and $V\omega = \omega$. Passing to a finer open covering if necessary, we see that $\omega(i) = d\beta(i)/\beta(i)$ for some β in $C^0(\mathfrak{U}, G_{mX})$. $\omega(i) = \omega(j)$ implies that $d(\beta(i)/(\beta j)) = o$. Hence $\beta(i)/\beta(j) = \alpha(i, j)^p$ for some α in $Z^1(\mathfrak{U}, G_{mX})$.

On the other hand suppose the cohomology class of (f, ω) in $Z^1_{DR}(\mathfrak{U})$ is killed by V^n, i. e. $(o, V^n\omega) = (\delta g, dg)$ for some g in $C^0(\mathfrak{U}, \mathcal{O}_X)$. Applying V again we get $(o, V^{n+1}\omega) = o$, i. e. $V^{n+1}\omega = o$. We now show that passing to a finer covering if necessary, there exists some u in $C^0(\mathfrak{U}, C_X)$ such that $\omega(i) = \Delta u(i)$. Since $V(V^n\omega(i)) = o$, we get

$$V^n\omega(i) = du(i)_{-n-1} = V^n(u(i)^{p^n-1}_{-n-1} du(i)_{-n-1}).$$

Hence

$$V^n[\omega(i) - u(i)^{p^n-1}_{-n-1} du(i)_{-n-1}] = o.$$

We can proceed in this way and get the required result.

By definition we have $df(i, j) = \Delta u(i) - \Delta u(j)$. If we denote by y the element of $C^1(\mathfrak{U}, C_X)$ such that $y_m = o$ for $m \leq -2$ and $y(i, j)_{-1} = f(i, j)$, then we have $\Delta y(i, j) = \Delta u(i) - \Delta u(j)$. Hence passing to a finer open covering if necessary, there exists some α in $Z^1_{DR}(\mathfrak{U})$ such that

$$y(i, j) - u(i) + u(j) = F\alpha(i, j).$$

If we write $(1 - SV)u = b$ and $\beta = Vu$, then $b_m = o$ for $m \leq -2$

$$f(i, j) = [F\alpha(i, j) + S\beta(i) - S\beta(j) + b(i) - b(j)]_{-1},$$
$$\omega(i) = \Delta S\beta(i) + \Delta b(i)$$

and

$$p\alpha(i, j) + \beta(i) - \beta(j) = o.$$

Hence

$$\text{the class of } (f, \omega) = \varphi(\overline{\alpha}) + \text{the class of } (\delta b_{-1}, db_{-1}) = \varphi(\overline{\alpha}).$$

<div align="right">Q. E. D.</div>

COROLLARY 5.11. — *Let* X *be an abelian scheme over a perfect field* k *of characteristic* p. *Then there is a canonical isomorphism of left* A-*modules*

$$\psi : \quad M(_pX) \xrightarrow{\sim} H^1_{DR}(X).$$

Moreover under ψ *the submodule* $_FM(_pX) = VM(_pX) \simeq (k, \sigma^{-1}) \otimes_k M(_FX)$ *is mapped onto the subspace* $H^0(X, \Omega^1_{X/k})$, *i. e. under* ψ *the exact sequences*

$$0 \to (k, \sigma^{-1}) \otimes_k M(_FX) \xrightarrow{''} M(_pX) \to M(_VX) \to 0$$

and

$$0 \to H^0(X, \Omega^1_{X/k}) \to H^1_{DR}(X) \to H^1(X, \mathcal{O}_X) \to 0.$$

correspond.

Remark. — This was conjectured in Grothendieck [15].

Proof. — As was proved in Proposition 4.1, there is a canonical isomorphism of left A-modules $M(_pX) \simeq {}_pI(X)$. By Theorem 5.10 the right hand side is canonically isomorphic by φ to $H^1_{DR}(X)$ as left A-modules. Hence by composition we get ψ.

<div align="right">Q. E. D.</div>

COROLLARY 5.12. — *Let* k *be a perfect field of characteristic* p. *Let* X *be a smooth and proper* k-*scheme such that* $X(k)$ *is non-empty. Then there is a canonical injection of left* A-*modules*

$$\xi : \quad DM(_pPic_{X/k}) \hookrightarrow H^1_{DR}(X)$$

such that the image of ξ *is equal to* $\bigcap_n \ker(d \circ V^n)$. *Moreover*

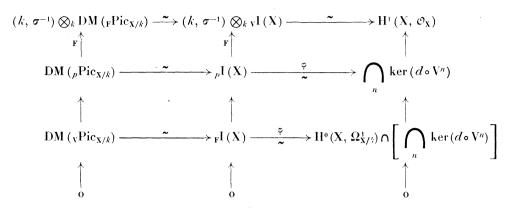$$\xi[DM(_VPic_{X/k})] = H^0(X, \Omega^1_{X/k}) \cap \left[\bigcap_n \ker(d \circ V^n) \right].$$

If X *is an abelian scheme, then* ξ *is an isomorphism and*

$$\xi[DM(_VPic_{X/k})] = H^0(X, \Omega^1_{X/k}).$$

Proof. — By Theorem 4.4 the left hand side is isomorphic as a left A-module to $_pI(X)$ and $DM(_VPic_{X/k})$ is mapped onto $_FI(X)$. The rest is obvious from Theorem 5.10.

Remark. — Let X be as in Corollary 5.12.   Then we have a commutative diagram

$$(k, \sigma^{-1}) \otimes_k \mathrm{DM}\,({}_{\mathrm{F}}\mathrm{Pic}_{X/k}) \xrightarrow{\sim} (k, \sigma^{-1}) \otimes_k {}_{\mathrm{V}}\mathrm{I}\,(X) \xrightarrow{\quad\sim\quad} \mathrm{H}^1\,(X, \mathcal{O}_X)$$

with vertical maps F, F, and an unlabelled map, and

$$\mathrm{DM}\,({}_p\mathrm{Pic}_{X/k}) \xrightarrow{\quad\sim\quad} {}_p\mathrm{I}\,(X) \xrightarrow[\sim]{\varphi} \bigcap_n \ker\,(d \circ V^n)$$

with vertical maps, and

$$\mathrm{DM}\,({}_{\mathrm{V}}\mathrm{Pic}_{X/k}) \xrightarrow{\quad\sim\quad} {}_{\mathrm{F}}\mathrm{I}\,(X) \xrightarrow[\sim]{\varphi} \mathrm{H}^0\,(X, \Omega^1_{X/k}) \cap \left[\bigcap_n \ker\,(d \circ V^n)\right]$$

with vertical maps from 0, 0, 0.

of left A-modules whose columns are exact and whose rows are isomorphisms

Remark. — d_1 is a homomorphism from $\mathrm{H}^1\,(\mathcal{O}_X)$ to $\mathrm{H}^1\,(\Omega^1_{X/k})$.   From the $\ker\,(d_1)$ there is a homomorphism d_2 to $\mathrm{H}^0\,(\Omega^2_{X/k})$. $\ker\,(d_2)$ is the image of $\mathrm{H}^1_{\mathrm{DR}}(X)$ under the canonical projection to $\mathrm{H}^1\,(\mathcal{O}_X)$.   Let X be a proper smooth scheme with a k-valued point.   Then from $\ker\,(d_2)$ we have a homomorphism $\overline{d \circ V}$ to
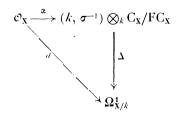
$$\mathrm{H}_0\,(\Omega^2_{X/k})/d \circ V\,[\,\ker\,(d)\ \text{in}\ \mathrm{H}^0\,(\Omega^1_{X/k})\,].$$

From $\ker\left(\overline{d \circ V}\right)$ we have a homomorphism $\overline{d \circ V^2}$ to $\mathrm{H}^0\,(\Omega^2_{X/k})/d \circ V^2[\ker\,(dV)$ in $\mathrm{H}^0\,(\Omega^1_{X/k})]$.   Proceeding in this fashion we get a homomorphism $\overline{d \circ V^{n+1}}$ from $\ker\left(\overline{d \circ V^n}\right)$ to $\mathrm{H}^0\,(\Omega^2_{X/k})/d \circ V^{n+1}[\ker\,(d \circ V^n)$ in $\mathrm{H}^0\,(\Omega^1_{X/k})]$. $\bigcap_n \ker\left(\overline{d \circ V^n}\right)$ is the image of $\bigcap_n \ker\,(d \circ V^n)$ in $\mathrm{H}^1_{\mathrm{DR}}(X)$ under the canonical projection to $\mathrm{H}^1\,(\mathcal{O}_X)$.

On the other hand if we denote by α the composite homomorphism

$$\mathcal{O}_X = (k, \sigma^{-1}) \otimes_k C_{-1,X} \xrightarrow{i} (W\,(k), \sigma^{-1}) \otimes_{W(k)} C_X \xrightarrow{\rho} (k, \sigma^{-1}) \otimes_k C_X/FC_X,$$

where i is the canonical injection and ρ is the canonical projection, we get a commutative diagram of homomorphisms of sheaves of k-vector spaces

$$\mathcal{O}_X \xrightarrow{\alpha} (k, \sigma^{-1}) \otimes_k C_X/FC_X$$

with maps d and λ to $\Omega^1_{X/k}$.

Hence we get a homomorphism of k-vector spaces

$$H^1(\mathcal{O}_X) \xrightarrow{\alpha} (k, \sigma^{-1}) \otimes_k H^1(C_X/FC_X)$$

with maps d_1 and Δ to $H^1(\Omega^1_{X/k})$.

It is not difficult to see that an element f in $H^1(\mathcal{O}_X)$ is killed by α if and only if f is in $\bigcap_n \ker(\overline{d \circ V^n})$, i. e. f is in the image of $\bigcap_n \ker(d \circ V^n)$ in $H^1_{DR}(X)$ under the canonical projection.
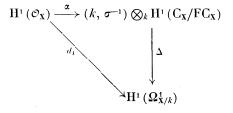
On the other hand we have a commutative diagram

$$0 \longrightarrow H^0(\Omega^1_{Alb(X)/k}) \longrightarrow H^1_{DR}(Alb(X)) \longrightarrow H^1(\mathcal{O}_{Alb(X)}) \longrightarrow 0$$
$$0 \longrightarrow H^0(\Omega^1_{X/k})_{d=0} \longrightarrow H^1_{DR}(X) \longrightarrow H^1(\mathcal{O}_X)$$

whose rows are exact and whose columns are injective. The image of $H^1(\mathcal{O}_{Alb(X)}) \hookrightarrow H^1(\mathcal{O}_X)$ is characterized as $\bigcap_n \ker(\beta_n) = H^1(W_X)/VH^1(W_X)$ where $\{\beta_n\}$ are the Bockstein operators (cf. Serre [33], and Mumford [29], lecture 27). Since by Proposition 5.1 all the cohomology operations d_1, d_2, $\{\overline{d \circ V^n}\}$ are zero for $Alb(X)$, we see that the cohomology operations d_1, d_2, $\{\overline{d \circ V^n}\}$ are dependent on $\{\beta_n\}$. We can prove this fact directly as follows. From the remark above, it is enough to prove that the elements of $H^1(\mathcal{O}_X)$ killed by β_n for all n are killed by α. From the proof of Theorem 4.4 (3) it is enough to prove that

$$H^1(C_{-1,X}) \cap \left[\bigcap_n V^n H^1(C_X) \right] \xrightarrow{\rho} H^1(C_X/FC_X)$$

is a zero map. More generally we can prove that the homomorphism

$$\bigcap_n V^n H^1(C_X) \xrightarrow{\rho} H^1(C_X/FC_X)$$

is zero. Since we have the exact sequence

$$(W(k), \sigma) \otimes_{W(k)} H^1(C_X) \xrightarrow{F} H^1(C_X) \xrightarrow{\rho} H^1(C_X/FC_X)$$

it is enough to prove that $\bigcap_n V^n H^1(C_X) \subseteq FH^1(C_X)$.

But since $H^1(C_x)$ is $W(k)$-cofinite type and V-torsion by Theorem 4.4 (1), the argument dual to Serre ([33], p. 13, Prop. 2) shows that V-divisible part of $H^1(C_x)$ is equal to the p-divisible part of $H^1(C_x)$. But the latter is obviously contained in $FH^1(C_x)$.

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