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# DISPERSION

## FOR NON-LINEAR RELATIVISTIC EQUATIONS. II

BY IRVING SEGAL.



### INTRODUCTION.

The present article is part of a continuing study of the theory in the large of non-linear partial differential equations satisfying the mathematically natural and physically indicated condition of relativistic invariance. Associated with any such equation is a temporally invariant function on the Cauchy data space known as the energy; many equations which have been studied have the property that this function is non-negative; this "positivity of the energy" will here be important in the derivation of precise results for specific classes of equations, although less so in the general theory.

Our main aim is to make specific and validate the general conception that solutions of such equations behave after the passage of sufficiently great intervals of time (or at sufficiently early times) like solutions of linear relativistic equations. In other words, the non-linear parts of the equation ultimately become highly attenuated to the point of negligibility. This conception underlies, in part explicitly and in part implicitly, fundamental procedures in quantum mechanics and in applied mathematics; it makes it possible to analyze the states of a complex non-linear system, incapable of description in closed mathematical form, in terms of the states of a linear system, which can often be quite explicitly described. Indeed, in the empirical situations which provide the motivation for the theoretical physical developments with which the present work is related, observations are to a large extent made in terms of the states of the so-called "free"

linear system, which is usually defined by the first-order variation of the fundamental non-linear partial differential equation in the vicinity of a distinguished solution, such as on occasion the identically vanishing solution. Thus, interstellar disturbances, presumably governed by non-linear relativistic equations, are empirically analyzed in terms of linear waves observed at much later times; and the same is effectively true of many observations on light, sound, and other wave phenomena.

As in earlier work, our theory is presented for the case of relativistic equations involving a single unknown function; such equations are also known as "scalar" equations. Similar methods are applicable also to relativistic systems, known as equations of "higher spin"; the results, however, depend on the "spin" of the equation, as well as on whether the "mass" vanishes or not, and on the number of space dimensions, to name only the most important parameters. In the present article, only the number of space dimensions will be allowed to vary. The basic methods are applicable as well to quasi-relativistic equations (i. e. those in which the highest-order derivatives define a relativistic operator); to some extent they could be applied to abstract equations of the form

$$u'(t) = Au(t) + K(u(t)),$$

where the function  $u(t)$  has values in a Hilbert space on which  $A$  is a given skew-adjoint operator, and  $K$  is a given (in general, unbounded) non-linear operator, of a quite restricted character. In practice, space-dependent coefficients are readily dealt with, if they vanish sufficiently rapidly near infinity, and lead to stronger results than in the relativistic case; on the other hand, non-linear terms involving additional space- or (lower-order) time- derivatives cause significant complications and at best weaken the results.

For the most part, the difficulties of the relativistic case majorize those of tractable generally similar equations whose non-linear terms are local and whose principal part is a constant-coefficient hyperbolic operator. In principle linear hyperbolic equations could also be treated by the present methods, but such relatively simple applications will be omitted, the stress here being entirely on the treatment of non-linearities.

In [5] it was shown that for an equation of the form

$$\square \varphi = m^2 \varphi + gF(\varphi); \quad F(0) = F'(0) = 0 \quad (g = \text{Const.}),$$

$F$  being a given function of a numerical variable, the "wave operator" exists in a strong sense under reasonably general conditions on  $F$ . The "dispersion (or S-) operator" exists in a weak sense under more stringent conditions (notably, positivity of the energy) on  $F$ . The theory was thus unsymmetrical between past and future; it left open the questions of the

univalence, continuity, differentiability (in function space) of the (non-linear) S-operator, etc.

In the present article it is shown that a solution of a positive-energy non-linear equation typically decays temporally, in  $L_\infty$  or  $L_p$  norms (for large  $p$ ) of the Cauchy datum as a function on space at approximately the same rate as the associated linear equations, provided either that the norm of the Cauchy data, or the "coupling" constant  $g$ , is sufficiently small. The use of the  $L_\infty$  or  $L_p$  norm is here essential, for the  $L_2$  norms do not decay at all, nor do the  $L_2$  norms of any of the derivatives, due to the conservation of energy; thus, pure  $L_2$  estimates are of no avail in the present connection; however, the use of  $L_p$  norms is complicated by the circumstance that even for linear hyperbolic equations, the  $L_p$  norm does not behave well under temporal propagation (in particular, temporal propagation is rarely continuous in the  $L_p$  norm; *cf.* Littman [3]). Nevertheless, it is shown that for suitable equations in  $n$ -space dimensions ( $n = 1, 2$  or  $3$ ), among which are the frequently studied equations

$$\square \varphi = m^2 \varphi + g \varphi^p \quad (m > 0, g > 0, p \text{ odd}; p > 3 \text{ if } n = 1),$$

the solution  $\varphi$  decays uniformly throughout  $n$ -space according to the estimate

$$|\varphi(x, t)| \leq \text{Const. } |t|^{-\frac{n}{2}}$$

provided the Cauchy data for  $\varphi$  has at some time a sufficient number ( $n$ -dependent but relatively small) of integrable derivatives, and  $g$  is sufficiently small.

It then follows readily from [5] that the S-operator exists in a strong sense and is univalent; it could also be deduced that it is in suitable explicit topologies continuous and differentiable, etc. (*cf.* Sec. 5); and the theory is essentially symmetrical between past and future. In other terms, if  $\varphi_0$  is any given sufficiently regular solution of the equation  $\square \varphi_0 = m^2 \varphi_0$ , there then exists a unique solution  $\varphi$  of the non-linear equation which is asymptotic to  $\varphi_0$  as  $t \rightarrow -\infty$ ; this means in particular that

$$\varphi(x) = \varphi_0(x) - \int_{-\infty}^t D_{\text{ret}}(x - x') F(\varphi(x')) dx',$$

where  $D_{\text{ret}}$  denotes the retarded elementary solution for the linear equation; and additionally, there exists a solution  $\varphi_1$  of the linear equation to which  $\varphi$  is asymptotic as  $t \rightarrow \infty$ , i. e.  $\varphi$  has also the form

$$\varphi(x) = \varphi_1(x) + \int_t^{\infty} D_{\text{adv}}(x - x') F(\varphi(x')) dx',$$

where  $D_{\text{adv}}$  is the advanced elementary solution and  $\square \varphi_1 = m^2 \varphi_1$ , etc.

It follows in particular that the concept of “adiabatic switching-on and -off of the interaction”, much treated in the theoretical physical literature, is superfluous in the present cases, at least. This concept refers to the view that, in as much as the interaction (i. e. non-linear term) is time-independent (i. e. does not depend explicitly on the time), integrals out to  $\pm \infty$  involving the interaction had no reason to exist; it was therefore necessary to introduce an explicit temporal damping into the interaction, e. g. by replacing  $F(\varphi(x))$  by  $F(\varphi(x))e^{-\varepsilon|x|}$ ,  $\varepsilon > 0$ ; then to determine the S-operator  $S(\varepsilon)$  as a function of the parameter  $\varepsilon$ , as is presumed to involve no convergence problem; and finally to move the effective times of switching-on and -off of the interaction to  $-\infty$  and  $+\infty$  respectively, by permitting  $\varepsilon$  to tend to zero, while defining the sought-for relativistic S-operator as the hoped-for limit of  $S(\varepsilon)$ . This limit does indeed exist, in the present cases, and agrees with the S-operator as indicated above.

In the case of a 3-dimensional space, the estimates of  $L_\infty$  norms serve also to establish the regularity of solutions in a number of cases in which it was previously known only that weak solutions, not necessarily globally determined by the Cauchy data at one time, existed. Thus for the special equations cited whose non-linear term is a power, it follows that if either  $g$  or the initial data are sufficiently small, then the solution of the Cauchy problem exists in the strong sense and is unique; and if, for example, additionally, the Cauchy data are infinitely differentiable of compact support, then the solution is infinitely differentiable throughout space-time, a result which appears doubtfully obtainable in any direct fashion.

Although the basic inequalities on which the foregoing results rest are non-perturbative, the specific results are either perturbative in character, as indicated, or relatively weaker than what is clearly optimal. When suitable additional *a priori* bounds are available, as in a class of zero-mass cases studied by Strauss [9], the perturbative restriction may be removed, as is exemplified in forthcoming work of Strauss.

From the standpoint of the theory of flows (i. e. one-parameter groups of automorphisms) in differentiable manifolds, the present work represents an asymptotic analysis of a flow on a certain infinite-dimensional manifold — namely, the solution manifold of the partial differential equation in question, the flow being that defined by temporal propagation in accordance with the equation — in the vicinity of an “elliptic” fixed point in the terminology used by Smale — namely the solution which vanishes identically. The induced linear flow in the tangent space at this point is that associated with a linear relativistic equation,  $\square \varphi = m^2 \varphi$  in the present case, and is unitary in a natural metric in the space, corresponding to the cited ellipticity. In these terms the result is that, for a certain class of relativistic equations in 1, 2 and 3 space dimensions, any orbit beginning

in a sufficiently small neighborhood of the fixed point is asymptotic as the parameter  $t \rightarrow \pm \infty$ , to linear orbits in the tangent plane; and conversely, given any orbit in the tangent plane which is sufficiently close to the origin, there are orbits in the non-linear manifold which are asymptotic to the given linear orbit as  $t \rightarrow \pm \infty$  (respectively). The S-operator is then the (non-linear) mapping which assigns to any tangent vector that tangent vector whose orbit is asymptotic as  $t \rightarrow + \infty$  to the same non-linear orbit that the orbit of the first tangent vector is asymptotic to as  $t \rightarrow - \infty$ .

2. TECHNICAL PRELIMINARIES. — We shall be considering in abstract form the partial differential equation

$$(2.1) \quad \square \varphi = m^2 \varphi + F(\varphi),$$

where  $\varphi(X) = \varphi(x, t)$  is a numerically-valued function on space-time,  $F$  is a given numerical function of a numerical variable, and  $\square$  denotes the d'Alembertian operator,  $\Delta - \left(\frac{\partial}{\partial t}\right)^2$ . By the abstract form of the equation is meant the equation for the vector valued-function of time,  $\Phi(t) = \varphi(\cdot, t)$ , which is namely

$$(2.2) \quad \Phi''(t) + B^2 \Phi(t) = -G(\Phi(t)),$$

where  $B$  denotes the operator  $(m^2 I - \Delta)^{\frac{1}{2}}$  in its usual formulation as a self-adjoint operator in the Hilbert space  $L_2(\mathbb{R}^n)$ , where  $n$  is the number of space dimensions; and  $G$  denotes the mapping:  $f(x) \rightarrow F(f(x))$ , defined on numerical function on  $\mathbb{R}^n$ . As indicated earlier (*cf.* e. g. [8]), it is convenient for both theory and applications to treat the integrated form of equation (2.2), which is

$$(2.3) \quad \Phi(t) = \Phi_0(t) - \int_{t_0}^t \frac{\sin[(t-s)B]}{B} G(\Phi(s)) ds,$$

whose solutions are in general solutions of equation (2.2) only in a somewhat generalized sense; the integral here, as on all later occasions in this paper, is in the absolutely convergent sense for a Banach space-valued function; the function  $\Phi_0(\cdot)$  in equation (2.3) denotes a solution of the equation, which will be called the *free* equation,

$$(2.4) \quad \Phi_0''(t) + B^2 \Phi_0(t) = 0,$$

in the correspondingly generalized sense. In the present connection, the *free energy* of a function  $\varphi(X)$  or corresponding function  $\Phi(\cdot)$ , at the time  $t$ , is defined as  $\|B\Phi(t)\|_2^2 + \|\Phi'(t)\|_2^2$ , and we shall deal throughout only with finite-energy solutions to the differential equations under consideration. This means that  $\Phi(t)$  must be in the domain of  $B$  for each  $t$ , and that  $\Phi(\cdot)$  should be differentiable in any of a variety of effectively

equivalent senses; for specificity, we may require that  $\langle \Phi(t), \Psi \rangle$  be a differentiable function of  $t$  for every fixed vector  $\Psi$  in the Hilbert space  $\mathcal{H}$  of all square-integrable numerical function (classes) on  $\mathbb{R}^n$  [which space we denote by  $L_2(\mathbb{R}^n)$ , and within which the inner product is denoted  $\langle \cdot, \cdot \rangle$ ], which is also in the domain of  $B$  and that the derivative have the form  $\langle \Omega(t), \Psi \rangle$  for some fixed vector  $\Omega(t) \in \mathcal{H}$ ; in this event,  $\Phi'(t)$  is defined as  $\Omega(t)$ .

It is readily verified that if

$$(2.5) \quad \Phi_0(t) = \frac{\cos(tB)}{B} f + \frac{\sin(tB)}{B} g,$$

where  $f$  and  $g$  are fixed vectors in the Hilbert space  $\mathcal{H}$ , then the (free) energy of  $\Phi_0(t)$  is independent of  $t$  and has the value  $\|f\|_2^2 + \|g\|_2^2$ ; it is also easily verified that equation (2.4) holds weakly relative to suitable dense classes of linear functionals; by the term *finite-energy free solution* will be meant a function  $\Phi_0(\cdot)$  of the form given by equation (2.5). Note that if  $f$  and  $g$  are in the domains of  $B^a$ , where  $a$  is a positive number, then  $B^a \Phi_0(t)$  is as a function of  $t$  also a finite-energy free solution.

The following Banach spaces will be employed here: (i) the spaces  $L_p(\mathbb{R}^n)$  consisting of all  $p$ th-power integrable functions on  $\mathbb{R}^n$ ,  $1 \leq p \leq \infty$ , in which the norm of a function  $f$  is denoted as  $\|f\|_p$ ; (ii) the spaces denoted  $\mathcal{H}_a$ , defined for  $a > 0$  as the completion of the domain of  $B^a$  relative to the new inner product  $\langle f, g \rangle' = \langle B^a f, B^a g \rangle$ , and with  $\mathcal{H}_0$  defined as  $\mathcal{H}$ . All Hilbert space norms will however be taken in  $\mathcal{H}$ , the norm of a vector  $f \in \mathcal{H}_a$  being indicated as  $\|B^a f\|$  (or  $\|B^a f\|_2$  when norms in other  $L_p$ -spaces may be under consideration).

It is convenient notationally to identify all closed translationally-invariant operators in  $\mathcal{H}$ , such as  $B$  or functions thereof, with their extensions to operators on a space containing all the  $L_p$  and  $\mathcal{H}_a$  spaces, so that the action of the operator on a given function can be denoted without reference to a particular space in which the function is considered to lie. In particular, it will be convenient to use the fact for certain operators  $l(B)$ ,  $l$  being a given function,

$$l(B) f = \frac{l(B)}{B^a} B^a f$$

for general functions  $f$  such that  $l(B) f$  and  $B^a f$  are both defined. A variety of adequate formulations and proofs of such results may be given. For specificity, and with a view to the later extension to more general differential equations, we may proceed as follows. A linear subset of  $\mathcal{H}$  may be defined to be *strongly dense* (relative to the action of the translation group on  $\mathcal{H}$ ) in case it contains an increasing sequence  $\mathcal{M}_n$  of translationally invariant closed linear subspaces of  $\mathcal{H}$ . A *generalized vector* is then defined

as a conjugate linear functional on a strongly dense domain in  $\mathcal{H}$ , which is continuous relative to any closed linear translationally invariant subspace  $\mathcal{M}$  contained in its domain. Any such vector  $f$  has a Fourier transform which is a measurable function  $\hat{f}$ , determined by condition that  $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$  for all vectors  $g$  contained in a subspace  $\mathcal{M}$ , for all  $\mathcal{M}$ ; here the notation  $\langle f, g \rangle$  denotes the number assigned to  $g$  by  $f$ , so that for  $f \in \mathcal{H}$ , the associated functional is the canonical one. It is not difficult to show that the generalized vectors form a linear vector space containing, relative to the imbedding analogous to that just indicated in the case of  $L_2$ , all the  $L_p$  spaces for  $1 \leq p \leq 2$  and all the spaces  $\mathcal{H}_a$ , for all values of  $a$  and  $m$ . If  $T$  denotes a closed translationally-invariant, densely-defined operator in  $\mathcal{H}$ , it is a multiplication operator by a measurable function in its action on Fourier transforms, and by the same action, is readily seen to carry generalized vectors into generalized vectors. The induced action of  $T$  on the  $L_p$ -spaces for  $p > 2$  may be defined and treated by duality. Equation (2.6) is readily verified for an arbitrary generalized function  $f$  or vector  $f$  in any  $L_p$ -space,  $1 \leq p \leq \infty$  <sup>(1)</sup>.

The final results will depend materially on whether  $m > 0$  or  $m = 0$ , but no assumption in this respect need be made in Section 3.

It will however be convenient to make the notational conventional convention that  $B^0 = mI$ . When  $m > 0$ , all the spaces  $\mathcal{H}_a$  for a fixed value of  $a$  are evidently isomorphic. The applications given in Section 4 will be confined to the case  $m > 0$ ; for the case  $m = 0$ , cf. [10].

Although basic definitions and results will be quoted in the present paper, free use will be made of results earlier set forth in [8], and especially [5], referred to hereafter as I; for further background concerning dispersion theory, see the latter. Frequent use will be made of the estimates developed in [1], [4], and I for the decay of free solutions. Unless otherwise indicated, all such estimates used here may be found in [4].

<sup>(1)</sup> In the extended sense indicated, many weak solutions of differential equations of the type considered here are strict solutions [e. g. the above indicated generalized free solutions satisfy equation (2.4) in a literal sense]. Of course, this could also be achieved in many cases through the use of distributions. There are non-linear equations, however, as well as singular linear equations, to which the present method applies but which are outside the scope of the theory of distributions. For example, the weak solutions of the equations  $\square \varphi = m^2 \varphi + g \varphi^p$  obtained in [7] are in general not distributions even locally, even, so far as is now known, in case the Cauchy data are infinitely differentiable of compact support; they are however strict solutions as generalized functions in the present sense.

A general theory of generalized vectors relative to a given ring of operators in a Hilbert space will be given elsewhere. One application will be the extension of a number of the considerations of the present paper to entirely abstract equations of the form  $u' = Au + K(u)$ , where  $A$  and  $K$  are given operators in a Hilbert space,  $A$  being skew-adjoint and  $K$  being a given partially defined non-linear operator.



3. ASYMPTOTIC BOUNDS FOR SOLUTIONS OF THE CAUCHY PROBLEM. — We begin by writing the basic equation

$$(3.1) \quad \Phi(t) = \Phi_0(t) - \int_{t_0}^t \frac{\sin[(t-s)B]}{B} G(\Phi(s)) ds$$

in the form

$$(3.2) \quad \Phi(t) = \Phi_0(t) - \int_0^t \frac{\sin[(t-s)B]}{B^{1+a}} B^a G(\varphi(s)) ds;$$

as noted in Section 2, this equation is valid if  $G(\Phi(s))$  is in the domain of  $B^a$  for  $t_0 < s < t$ ; the initial time  $t_0$  may be finite or infinite, but it will be convenient, and no essential loss of generality, to suppose that  $t_0 < t$ . We denote as  $E_{t,a}$  the generalized function whose Fourier transform is, as a function of  $y \in \mathbb{R}^n$ , the function  $\sin\left[t(m^2 + y^2)^{\frac{1}{2}}\right] (m^2 + y^2)^{-\frac{1+a}{2}}$ ;  $E_{t,a}$  is a function of the variable  $x \in \mathbb{R}^n$ , and its form evidently depends also on  $m$  and  $n$ , but the latter two variables will be held fixed. In case  $E_{t,a}$  is in a suitable  $L_q$ -space — and this is the only case which there is occasion to consider here — equation (3.2) may be written in the form

$$(3.3) \quad \Phi(t) = \Phi_0(t) - \int_{t_0}^t E_{t-s,a} \star B^a G(\Phi(s)) ds.$$

On taking the  $L_r$ -norm on both sides of equation (3.3) and applying the Hausdorff-Young inequality, it follows that

$$(3.4) \quad \|\Phi(t)\|_r \leq \|\Phi_0(t)\|_r + \int_{t_0}^t E_{t-s,a} \|G\| B^a G(\Phi(s)) \|_{q'} ds,$$

if:

$$1 \leq r, \quad q, q' \leq \infty; \quad 1 + r^{-1} = q^{-1} + q'^{-1}.$$

In order to obtain a bound for  $\|\Phi(t)\|_r$ , it is appropriate to relate the integrand more explicitly to  $\|\Phi(s)\|_r$ . The following assumption on the given non-linear operator  $G(\cdot)$  is adequate for this purpose and for present applications:

(3.5) *Assumption on  $G(\cdot)$ :  $h(M) < \infty$  if  $M < \infty$ , where  $h(M)$  is defined as the supremum, for  $\|B^{b+1}\Phi\|_2 \leq M$ , of  $\|B^a G(\Phi)\|_{q'} \|\Phi\|_r^{-\rho}$  ( $b$  and  $\rho$  being non-negative parameters to be specified later;  $M > 0$ ).*

Naturally, the function  $h$  depends on all the parameters  $a, b, q', p, m$  and  $n$ , in addition to  $G(\cdot)$ , but this dependence will be of no special consequence in the following. It is convenient to assume in addition that

(3.6) *Assumption on  $G(\cdot)$ :  $G(0) = 0$ ;*

the case in which  $G(0) \neq 0$  follows along lines closely similar to the present ones, and very rarely arises for a relativistic equation.

It follows from the inequality (3.4) that

$$(3.7) \quad \|\Phi(t)\|_r \leq \|\Phi(t)\|_r + \int_{t_0}^t \|E_{t-s,a}\|_q h(\|B^{b+1}\Phi(s)\|_2) \|\Phi(s)\|_r^2 ds$$

if: (3.5) and the arithmetical relations given in (3.4) hold.

In order to obtain an explicit bound on  $\|\Phi(t)\|_r$  from the inequality (3.7), it is necessary to make some assumption about the term  $h(\|B^{b+1}\Phi(s)\|_2)$ ; in practice this term will be bounded by a constant or by a given function of the  $\|\Phi(s')\|_r$  for  $s' < s$ . It will suffice for our purposes to resolve the inequality in the following fashion.

Setting  $\bar{M}(t) = \sup_{t_0 < s < t} \|B^{b+1}\Phi(s)\|_2$ , it results that

$$(3.8) \quad \|\Phi(t)\|_r \leq \|\Phi_0(t)\|_r + h(\bar{M}(t)) \int_{t_0}^t \|E_{t-s,a}\|_q \|\Phi(s)\|_r^2 ds.$$

We next introduce a function  $u(t)$  by the equation

$$(3.9) \quad u(t) = \sup_{t_0 < s < t} \|\Phi(s)\|_r (1 + |s|)^\varepsilon,$$

where  $\varepsilon$  is a parameter to be chosen later; for certain applications, it may be convenient to use  $|s|^\varepsilon$  in place of  $(1 + |s|)^\varepsilon$ . On multiplying the inequality (3.9) by  $(1 + |t|)^\varepsilon$ , then replacing  $t$  by  $t'$ , and finally taking supremums on both sides of the inequality for  $t_0 < t' < t$ , it results that

$$(3.10) \quad u(t) \leq u_0(t) + (1 + |t|)^\varepsilon h(\bar{M}(t)) u(t)^\rho \int_{t_0}^t \|E_{t-s,a}\|_q (1 + |s|)^{-\rho\varepsilon} ds.$$

The integral on the right may be estimated quite explicitly. It will suffice for present purposes to assume that

$$(3.11) \quad \|E_{t,a}\|_q \leq C(1 + |t|)^{-\delta},$$

where  $C$  and  $\delta$  are constants, and to use the

LEMMA 3.1. — *If  $a > 0$ ,  $b > 0$ , and  $\max(a, b) > 1$ , then*

$$\int_{-\infty}^t (1 + |t-s|)^{-a} (1 + |s|)^{-b} ds = o(|t|^{-c}) \quad \text{as } |t| \rightarrow \infty, \quad c = \min(a, b).$$

*Proof of lemma.* — Making the transformation  $s \rightarrow t - s$ , the integral  $I$  in question becomes

$$I = \int_0^\infty (1+s)^{-a} (1+|t-s|)^{-b} ds.$$

Assuming  $t > 0$ , as is no essential loss of generality, since evidently  $I(-t) \leq I(t)$ , we may write

$$I = \int_0^{\varepsilon t} + \int_{\varepsilon t}^t + \int_t^\infty,$$

where  $0 < \varepsilon < 1$ , and  $\varepsilon$  is held fixed. Now

$$\int_0^{\varepsilon t} \leq \int_0^{\varepsilon t} (1+s)^{-a} (t(1-\varepsilon))^{-b} ds = o(t^{-b}) \int_0^{\varepsilon t} (1+s)^{-a} ds.$$

In case  $a > 1$ , this expression is  $o(t^{-b})$ . In case  $a = 1$ , it is  $o(\log t \cdot t^{-b})$ , which is  $o(t^{-a})$  since  $b > 1$ . In case  $a < 1$ , it is

$$o(t^{-b}) \int_0^{\varepsilon t} s^{-a} ds = o(t^{-b}) o(t^{-a+1}) = o(t^{1-a-b});$$

since  $1 + \min(a, b) \leq a + b$  by virtue of the assumption that  $\max(a, b) > 1$ , this is in turn  $o(t^{-c})$ .

Similarly,

$$\int_{\varepsilon t}^t \leq o(t^{-a}) \int_{\varepsilon t}^t (1+|t-s|)^{-b} ds = o(t^{-a}) o(t^{-b+1})$$

if  $b \neq 1$ , and so is  $o(t^{-c})$ . In case  $b = 1$ , the integral is  $o(t^{-a}) o(\log t) = o(t^{-b})$  since in this case  $a > 1$ . If  $b > 1$ ,  $\int_t^\infty \leq o(t^{-a}) \int_t^\infty (1+|t-s|)^{-b} ds$ , which is  $o(t^{1-a-b})$ . Finally, if  $b \leq 1$ ,

$$\int_t^\infty \leq o(t^{-b}) \int_t^\infty (1+|t-s|)^{-a} ds = o(t^{-b}).$$

(End of proof of Lemma 3.1).

Applying Lemma 3.1 to the inequality (3.10), it follows that

$$(3.12) \quad u(t) \leq u_0(t) + Ch(\overline{M}(t)) u(t)^p$$

if :  $\max(\delta, \varphi\varepsilon) > 1$  and  $\min(\delta, \varphi\varepsilon) \geq \varepsilon$ , and the arithmetical relations given in (3.4); (3.5); and (3.11) hold;

here  $C$ , as always denotes a constant (not always the same) which is independent of  $\Phi(\cdot)$ ,  $\Phi_0(\cdot)$ , and  $G(\cdot)$ ; and  $u_0(\cdot)$  is defined in the same way as  $u(\cdot)$  but with the use of  $\Phi_0(\cdot)$  in place of  $\Phi(\cdot)$ . With a simple bound for  $h(\overline{M}(t))$  in terms of  $t$  and  $u(t)$ , this inequality may be resolved quite explicitly; towards this end, the following further assumptions, which are satisfied in the cases of interest here, are made :

$$(3.13) \quad \begin{cases} a. & u_0(t) < c_1, & -\infty < t < \infty; \\ b. & h(\overline{M}(t)) < c_2 + c_3 u(t)^\sigma, & t \in I, \end{cases}$$

where  $I$  denotes an arbitrary interval on which  $\Phi(t)$  is defined and continuous, as a mapping into  $\mathcal{H}_{b+1}$ , and satisfies equation (3.1) on this interval (the integral in question being absolutely convergent in  $\mathcal{H}_{b+1}$ ); the  $c_i$  are constants which may depend on  $\Phi_0(\cdot)$  and  $G(\cdot)$ , but are independent of

the interval I; and  $\sigma$  is a positive constant, also independent of I. (It may be helpful to remark that the following notational conventions are employed throughout : constants denoted by small Roman letters may depend on  $G(\cdot)$ ,  $\Phi_0(\cdot)$ , and  $\Phi(\cdot)$ , while constants denoted by capital Roman letters have no such dependence; all constants are assumed independent of the interval I under consideration, as long as the indicated conditions are satisfied by I.)

Combining the inequalities (3.12) and (3.13) by elementary algebra, it results that

$$(3.14) \quad u(t) \leq c + d u(t)^\omega$$

if :  $c = c_1 + Cc_2$ ,  $d = C(c_2 + c_3)$ ,  $\omega = \rho + \sigma$  and the assumptions (3.13) and those given in (3.12) are satisfied.

This inequality may be further reduced through the use of the

LEMMA 3.2. — *If  $\omega > 1$ , there exists a constant  $e(\omega)$  with the following property : if  $c$  and  $d$  are given positive constants satisfying the inequality*

$$c^{\omega-1} d < e(\omega),$$

*then there exist positive constants  $k_i = k_i(\omega, c, d)$  ( $i = 1, 2$ ),  $k_1 < k_2$ , such that if  $k > 0$  and  $k \leq c + dk^\omega$ , then either  $k \leq k_1$  or  $k \geq k_2$ .*

*Proof of lemma.* — Let  $H(k) = dk^\omega - k + c$ ; evidently,  $k \leq c + dk^\omega$  if and only if  $H(k) \geq 0$ . It follows from calculus that for the proof of the lemma, it suffices to show that if the indicated condition is satisfied, then  $H(k') < 0$ , where  $k'$  is the unique positive root of the equation  $H'(k') = 0$ . By an elementary computation,  $k' = (\omega d)^{-1/\theta}$ , where  $\theta = (\omega - 1)^{-1}$ ;  $H(k') = c - d^{-\theta} e_1(\omega)$ , where  $e_1(\omega) = (\omega - 1) \omega^{-1-\theta}$ . Evidently,  $H(k') < 0$  if and only if  $c < d^{-\theta} e_1$ ; raising both sides of the inequality to the power  $\omega - 1$ , it is equivalent to the inequality :

$$c^{\omega-1} < d^{-1} e_1^{-1} \quad \text{or} \quad dc^{\omega-1} < e, \quad \text{where} \quad e = e_1(\omega)^{\omega-1}.$$

(End of proof of Lemma 3.2)

Applying Lemma 3.2 to the inequality (3.9), it results that

(3.15) *There exists constants  $k_1$  and  $k_2$  such that  $k_1 < k_2$ , and either  $u(t) < k_1$  or  $u(t) > k_2$ , provided :  $dc^{\omega-1} < e(\omega)$ , and conditions (a) and (b) of (3.13) hold.*

This result is naturally of relatively little use unless the alternative which bounds  $u(t)$  from below can be excluded. The general idea used to achieve this is as follows, taking for specificity the case  $t_0 = -\infty$ . From the theory in I it is deduced that  $u(t) \rightarrow 0$  as  $t \rightarrow -\infty$ , for appropriate values of the disposable parameters. Now if  $u(t)$  is a continuous function

of  $t$ , it cannot jump from the interval  $[0, k_1]$  to the disjoint interval  $[k_2, \infty]$ , and so must remain in the interval  $[0, k_1]$  if originally there; thus  $u(t) \leq k_1$  for all  $t$ . The main problem is to establish the continuity; this will hold if  $b$  is chosen sufficiently large; on the other hand  $b$  must be chosen relatively small in order to obtain an effective bound of the form (3.13 b).

This problem is dealt with as follows.

LEMMA 3.3. — Let  $W(t)$  ( $t \geq 0$ ) be a bounded continuous one-parameter semi-group of operators on the Banach space  $\mathcal{L}$ . Let  $K$  denote a semi-Lipschitzian operator from  $\mathcal{L}$  into  $\mathcal{L}$  (i. e.  $\|K(x) - K(y)\|/\|x - y\|$  is bounded on bounded subsets of  $\mathcal{L} \times \mathcal{L}$  which are disjoint from the diagonal). Let  $T$  denote the supremum of the values  $t' > t_0$  such that the equation

$$(\star) \quad x(t) = W(t - t_0) x_0 + \int_{t_0}^t W(t - s) K(x(s)) ds \quad (x_0 \text{ given in } \mathcal{L}; t_0 \text{ given in } \mathbb{R}^1)$$

has a continuous solution in the interval  $t_0 \leq t < t'$ . Then either  $T = \infty$ , or  $\|K(x(s))\|/\|x(s)\|$  is unbounded for  $t_0 < s < T$ ,  $x(s) \neq 0$ .

*Proof of lemma.* — By general theory (cf. [8]) we know that equation  $(\star)$  has a unique continuous solution in some interval  $t_0 < t < t_1$ , and that the solution exists and is continuous in the largest interval in which  $\|x(t)\|$  remains bounded. It suffices therefore to show that if  $\|K(x(s))\|/\|x(s)\|$  remains bounded, then so also does  $\|x(s)\|$ . Now within the interval of existence, i. e. for  $t_0 < t < T$ , a direct estimate shows that

$$\|x(t)\| \leq C \|x_0\| + CC' \int_{t_0}^t \|x(s)\| ds,$$

provided that  $\|K(x(s))\|/\|x(s)\| < C'$  for  $t_0 \leq s \leq t$ ; from this it follows by the Gronwall inequality that

$$\|x(t)\| \leq C \|x_0\| \exp[CC'(t - t_0)].$$

Thus,  $\|x(s)\|$  is bounded in any interval  $t_0 < t < t'$  in which  $\|K(x(s))\|/\|x(s)\|$  is bounded, showing that  $t' < T$ . (End of proof of Lemma 2.3).

LEMMA 3.4. — With the same hypotheses and notation as Lemma 3.3, let  $N$  denote a pseudo-norm on  $L$  such that :

$$(i) \quad N(x) \leq \text{const. } \|x\|;$$

$$(ii) \quad \|K(x)\| \leq p(N(x)) \|x\|, \text{ where the function } p \text{ is bounded on bounded sets.}$$

Suppose also that if  $x(s)$  is a continuous solution of equation  $(\star)$  throughout an interval  $I$ , then either  $N(x(s))h(s) < k$  or  $N(x(s))h(s) > k'$ , for all  $s \in I$ ,

where  $k$  and  $k'$  are constants independent of  $I$ , such that  $k < k'$ , and  $h$  is a given continuous function such that  $h(s) > 0$ . Finally, suppose that  $N(x_0)h(t_0) < k$ . Then equation  $(\star)$  admits a continuous solution in the infinite interval  $t > t_0$ , and  $N(x(t))h(t) < k$  in this interval.

*Proof of lemma.* — If equation  $(\star)$  admits a continuous solution  $x(t)$  for all  $t > t_0$ , then  $N(x(t))h(t)$  is a continuous function of  $t$ , for the pseudo-norm  $N$  is dominated by the norm  $\|\cdot\|$ . The range of  $N(x(t))h(t)$  for  $t > t_0$  is therefore connected, and so must be contained either in the interval  $[0, k]$  or  $[k', \infty]$ ; since  $N(x_0)h(t_0) < k$ , it is the former alternative which obtains.

Now suppose, as the basis of an argument by contradiction, that the supremum  $T$  of the values  $t'$  such that equation  $(\star)$  admits a continuous solution in the interval  $t_0 \leq t \leq t'$ , is finite. Then  $\|u(t)\| \rightarrow \infty$  as  $t \rightarrow T$ ,  $t < T$ . On the other hand,  $N(x(t))h(t)$  is a continuous function of  $t$  for  $t < T$ ; by the same argument as that given for the case  $T = \infty$ ,  $N(x(t)) \leq k$  throughout the interval  $t_0 \leq t < T$ . Since

$$\|K(x)\| \leq p(N(x))\|x\|$$

for arbitrary  $x$ , where  $p(\cdot)$  is bounded on bounded sets, it follows that  $\|K(x(t))\|/\|x(t)\|$  is bounded throughout the interval  $t_0 \leq t < T$  (where defined). According to Lemma 3.3, this is impossible. (*End of proof of Lemma 3.4.*)

LEMMA 3.5. — Let  $G$  denote a semi-Lipschitzian mapping from  $\mathcal{H}_{b+1}$  into  $\mathcal{H}_b$  ( $b$  being real) such that  $\|B^b G(\Phi)\|_2 \leq p(N(\Phi))\|B^{b+1}\Phi\|_2$ , where  $N$  is a pseudo-norm on  $\mathcal{H}_{b+1}$  such that  $N(\Phi) \leq \text{const.}\|B^{b+1}\Phi\|_2$ , for arbitrary  $\Phi$ , and  $p(\cdot)$  is bounded on bounded sets; suppose also that  $G(0) = 0$ . Let  $\Phi_0(\cdot)$  be a given continuous mapping from  $\mathbb{R}^1$  into  $\mathcal{H}_{b+1}$  and assume further that if  $\Phi(t)$  is any continuous solution of the equation (2.3) in an interval  $I$ , then either  $N(\Phi(s))h(s) < k$  or  $N(\Phi(s))h(s) > k'$  for all  $s \in I$  where  $k, k'$ , and  $h(\cdot)$  are as in Lemma 3.4; suppose also that  $N(\Phi_0(t_0))h(t_0) < k$ . Then equation  $(\star)$  admits a continuous solution in the infinite interval  $t > t_0$ , and  $N(\Phi(t))h(t) < k$  in this interval.

*Proof of lemma.* — This result is a virtually immediate specialization of Lemma 3.4 to the spaces and one-parameter groups considered in I. Specifically,  $\mathcal{L}$  is the Hilbert space direct sum  $\mathcal{H}_{b+1} \oplus \mathcal{H}_b$ ;  $W(t)$  is the isometry whose matrix relative to the indicated decomposition of  $\mathcal{L}$  has the form

$$W(t) = \begin{pmatrix} \cos(tB) & \frac{\sin(tB)}{B} \\ -B \sin(tB) & \cos(tB) \end{pmatrix}.$$

and  $K$  is the mapping

$$(\Phi, \Psi) \rightarrow (0, -G(\Phi)) \quad (\Phi \in \mathcal{X}_b)$$

from  $\mathcal{L}$  into  $\mathcal{L}$ . The details of the specialization are readily supplied (*cf.* I). (*End of proof of Lemma 3.5.*)

In order to apply the foregoing to the Cauchy problem with data given at time  $-\infty$ , a local existence theorem at  $t = -\infty$ , involving greater regularity than was explicitly developed in I, is needed in certain cases. The following result is essentially a corollary to Theorem 1 of I.

LEMMA 3.6. — *Let  $a, b, \varepsilon, \delta, \rho$ , and  $\tau$  be given positive numbers; let  $r, q, q'$  be extended real numbers in the interval  $[1, \infty]$  such that  $1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{q'}$ ; let  $G$  denote a semi-Lipschitzian mapping from  $\mathcal{X}_{b+1}$  into  $\mathcal{X}_b$ ; let  $\Phi_0(\cdot)$  be a given finite-energy free solution such that  $B^b \Phi_0(\cdot)$  is also such, and which satisfies the relation:  $\|\Phi_0(t)\|_r = o(|t|^{-\varepsilon})$ ,  $t \rightarrow -\infty$ . Then under the assumptions (a) — (d) below, there exist unique continuous functions  $\Phi(\cdot)$  and  $\dot{\Phi}(\cdot)$  from  $\mathbb{R}^1$  into  $\mathcal{X}_{b+1}$  and  $\mathcal{X}_b$  respectively, such that  $\dot{\Phi}(\cdot)$  is the derivative of  $\Phi(\cdot)$  in the sense that  $\frac{d}{dt} \langle \Phi(t) f, g \rangle = \langle \dot{\Phi}(t) f, g \rangle$  whenever  $f, g \in \mathcal{X}_b \wedge \mathcal{X}_{b+1}$ , and satisfying the relations*

$$\begin{aligned} \Phi(t) &= \Phi_0(t) + \int_{-\infty}^t \frac{\sin[(t-s)B]}{B} G(\Phi(s)) ds, \\ \dot{\Phi}(t) &= \dot{\Phi}_0(t) + \int_{-\infty}^t \cos[(t-s)B] G(\Phi(s)) ds, \\ \|\Phi(t)\|_r &= o(|t|^{-\varepsilon}), \end{aligned}$$

where the integrals involved are absolutely convergent in  $\mathcal{X}_{b+1}$  and  $\mathcal{X}_b$  respectively, for all  $t < t_0$  for some finite  $t_0$ .

ASSUMPTIONS :

(a) The mapping  $t \rightarrow E_{t,a}$  is continuous and bounded into  $L_q$ .

(b)  $\|B^a G(\Phi)\|_{q'} \leq h(\|B^{b+1} \Phi\|_2) \|\Phi\|_r^{\rho}$ ,

where  $h(\cdot)$  is bounded on bounded sets.

(c)  $\|B^b(G(\Phi) - G(\Psi))\|_2 \leq \|B^{b+1}(\Phi - \Psi)\|_2 [\max(\|\Phi\|_r, \|\Psi\|_r)]^{\tau} C(\Phi, \Psi)$ , where  $C(\Phi, \Psi)$  is bounded on any set of functions  $\Phi$  and  $\Psi$  on which  $\|B^{b+1} \Phi\|_2$  and  $\|B^{b+1} \Psi\|_2$  are bounded.

(d)  $\rho c > 1 + \varepsilon$ ,  $\tau \varepsilon > 1$ .

*Proof of lemma.* — We use Theorem 1 of I; employing the notation of the proof of Lemma 3.5, we set  $W(t, s) = W(t - s)$ ,  $u_0(s) = (\Phi_0(s), \dot{\Phi}(s))$ ,

$N(u) = \|\Phi\|_r$  if  $u = (\Phi, \Psi)$ ; and  $f(t) = c|t|^{-\varepsilon}$ , with the constant  $c$  chosen so that  $N(u_0(t)) \leq f(t)$  (it is no essential loss of generality to assume  $t < 0$ , since only a local result at  $t = -\infty$  is in question). We check conditions (i)-(iv) of the cited theorem in order.

*Ad (i) :*

$$\begin{aligned} N(W(t, s) K(u, s)) &= \left\| \frac{\sin(t-s) B}{B} G(\Phi) \right\|_r \quad \text{if } u = (\Phi, \Psi), \\ &= \left\| \frac{\sin(t-s) B}{B^{1+a}} B^a G(\Phi) \right\|_r \leq \|E_{t-s, a, m}\|_q \|B^a G(\Phi)\|_{q'}. \end{aligned}$$

Using the boundedness of  $\|E_{t-s, a, m}\|_q$  as a function of  $t$ , and assumption (b), the expression in question is bounded by

$$\text{const. } \|\Phi\|_r^\rho h(\|B^{b+1}\Phi\|_2).$$

The question is then whether, replacing the  $\varepsilon$  in the cited theorem by  $e$  in order to avoid confusion with the present  $\varepsilon$ ,

$$(f(t))^{-1} \int_{-\infty}^t ((1+e)f(s))^\rho ds \rightarrow 0$$

as  $t \rightarrow -\infty$ , and this is found to be the case if  $(\rho-1)\varepsilon > 1$ , as assumed.

*Ad (ii) :*

$$\begin{aligned} &\|W(t, s) (K(u, s) - K(v, s))\| \\ &= \left\| \left( \frac{\sin(t-s) B}{B} (G(\Phi) - G(\Psi)), \cos(t-s) B (G(\Phi) - G(\Psi)) \right) \right\|_e \\ &\leq \|B^b \sin(t-s) B (G(\Phi) - G(\Psi))\|_2 + \|B^b \cos(t-s) B (G(\Phi) - G(\Psi))\|_2 \\ &\leq 2 \|B^b (G(\Phi) - G(\Psi))\|_2. \end{aligned}$$

Applying assumption (c), it suffices if  $\int_{-\infty}^t s^{-\varepsilon\tau} ds < \infty$ , which is satisfied if  $\tau\varepsilon > 1$  as assumed.

*Ad (iii) :* this has been explicitly assumed.

*Ad (iv) :* the question is that of the continuity, as functions of  $s, t$ , and  $\Phi$ , of  $\frac{\sin(t-s) B}{B} G(\Phi)$  and of  $\cos(t-s) B G(\Phi)$ , as mappings into  $\mathcal{X}_{b+1}$  and  $\mathcal{X}_b$  respectively. Both of these follow from the continuity of  $\sin(t-s) B$  and  $\cos(t-s) B$  as functions of  $s$  and  $t$ , in the strong operator topology, together with the continuity of  $G(\cdot)$  which follows from its assumed semi-Lipschitzian character.

It remains only to show the uniqueness within the indicated class, which is slightly broader than the class for which it is indicated in I. In view of condition (ii), if  $\Psi(\cdot)$  satisfies near  $t = -\infty$  the same conditions as  $\Phi(\cdot)$ ,



then on subtracting the defining integral equations for the two functions, taking norms in  $\mathcal{H}_{b+1}$ , and estimating directly, it results that

$$\|B^{b+1}(\Phi(t) - \Psi(t))\|_2 \leq \int_{-\infty}^t o(s^{-\varepsilon}) \|B^{b+1}(\Phi(s) - \Psi(s))\|_2 ds;$$

by virtue of integrability near  $-\infty$  of  $s^{-\varepsilon}$ , it follows that the bounded function  $\|B^{b+1}(\Phi(t) - \Psi(t))\|_2$  vanishes identically on any interval  $(-\infty, t_0)$  on which both are defined and satisfy the stated relations.

The foregoing developments may be combined in a variety of ways in accordance with the particularities of the equation under consideration. The following partial summaries largely cover in principle the applications to be made later in this paper.

**THEOREM 3.1.** — *Let  $a, b$  and  $\varepsilon$  be given non-negative numbers; let  $\Phi_0(t)$  be a given solution of the free equation such that  $B^b\Phi_0(t)$  is of finite energy; suppose that*

$$\|\Phi_0(t)\|_r = o(|t|^{-\varepsilon}), \quad |t| \rightarrow \infty \quad (r \geq 1).$$

Let  $G$  be a given semi-Lipschitzian operator from  $\mathcal{H}_{b+1}$  into  $\mathcal{H}_b$ . Suppose that for some real  $t_0$ , there exist unique continuous functions  $\Phi(t)$  and  $\dot{\Phi}(t)$  from  $(-\infty, t_0]$  into  $\mathcal{H}_{b+1}$  and  $\mathcal{H}_b$  respectively, such that

$$\frac{d}{dt} \langle \Phi(t), f, g \rangle = \langle \dot{\Phi}(t), f, g \rangle$$

for  $f, g \in \mathcal{H}_b \wedge \mathcal{H}_{b+1}$ , and satisfying the relations :

$$\left. \begin{aligned} \text{(i)} \quad \Phi(t) &= \Phi_0(t) + \int_{-\infty}^t \frac{\sin[(t-s)B]}{B} G(\Phi(s)) ds \\ \text{(ii)} \quad \dot{\Phi}(t) &= \dot{\Phi}_0(t) + \int_{-\infty}^t \cos[(t-s)B] G(\Phi(s)) ds \\ \text{(iii)} \quad \|\Phi(t)\|_r &= o(|t|^{-\varepsilon}), \quad t \rightarrow -\infty \end{aligned} \right\} \quad -\infty < t \leq t_0,$$

where the integrals involved are absolutely convergent in  $\mathcal{H}_{b+1}$  and  $\mathcal{H}_b$ , respectively.

Then there exist unique continuous functions  $\Phi(t)$  and  $\dot{\Phi}(t)$  from  $\mathbb{R}^1$  to  $\mathcal{H}_{b+1}$  and  $\mathcal{H}_b$ , satisfying the equations (i) and (ii) for all  $t$ , and satisfying the relation (iii) for  $t \rightarrow +\infty$  as well as for  $t \rightarrow -\infty$ , under the following further assumptions :

- (a)  $1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{q'}$ ;
- (b)  $\|G_{i,a}\|_q = o(|t|^{-\delta}), \quad t \rightarrow \infty$ ;
- (c)  $\max(\delta, \rho\varepsilon) > 1, \quad \min(\delta, \rho\varepsilon) \geq \varepsilon$ ;

(d) There exist constants  $c_1$  and  $c_2$  with the property that for any  $t_0$  such that the relations (i) and (ii) hold and for  $s < t_0$ ,

$$\|B^a G(\Phi(s))\|_{q'} \leq \|\Phi(s)\|_r^{\rho} (c_2 + c_3 (\sup_{t < s} (1 + |s|)^{\varepsilon} \|\Phi(s)\|_r)^{\sigma})$$

( $c_2$  and  $c_3$  being independent of  $t_0$ );

(e)  $dc^{\omega-1} < e(\omega)$ , where  $\omega = \rho + \sigma$  and  $c = c_1 + Cc_2$ ,  $d = C(c_2 + c_3)$ ,  $C$  being a fixed constant independent of  $G$  and  $\Phi_0(\cdot)$ , and

$$c_1 = \sup_s (1 + |s|)^{\varepsilon} \|\Phi(s)\|_r;$$

(f)  $\|B^b G(\Phi)\|_2 \leq p(\|\Phi\|_r) B^{b+1} \Phi\|_2$ , where  $p(\cdot)$ , is bounded on bounded sets;

(g)  $\|\Phi\|_r \leq \text{const.} \|B^{b+1} \Phi\|_2$ .

*Proof.* — The assumptions involved in (3.12) above are implied by (a), (b) and (c), while assumption (d) is tantamount to assumption (3.13), leading to the validity of (3.14); (e) then gives (3.15), and it remains only to deal with the continuity aspects treated in Lemma 3.5. Taking  $h(s) = (1 + |s|)^{\varepsilon}$  and  $N(\Phi) = \|\Phi\|_r$ , the hypotheses of Lemma 3.5 are supplied by assumptions (f) and (g) in combination with (3.15), and the global existence and indicated decay rate for  $\Phi(\cdot)$  follow. (*End of proof of Theorem 3.1.*)

**COROLLARY 3.1.** — *With the combined hypotheses of Lemma 3.6 and Theorem 3.1, and the assumption that  $p(l) < \text{const.} l^{\tau}$ , where  $\tau' \varepsilon > 1$ , there exists a unique finite-energy free solution  $\Phi_1(\cdot)$  satisfying the same conditions as  $\Phi_0(\cdot)$ , such that*

$$\Phi(t) = \Phi_1(t) + \int_t^{\infty} \frac{\sin[t-s]B}{B} G(\Phi(s)) ds,$$

i. e.  $\Phi_1$  stands in the same relation to  $\Phi$  near  $t = \infty$  as  $\Phi_0$  does near  $t = -\infty$ .

*Proof.* — Let  $\Phi_1(\cdot)$  be defined by the equation

$$\Phi_1(t) = \Phi_0(t) - \int_{-\infty}^{\infty} \frac{\sin[(t-s)B]}{B} G(\Phi(s)) ds;$$

the integral in question here is convergent in  $L_r$ ; indeed,

$$|t|^{\varepsilon} \int_{-\infty}^{\infty} \left\| \frac{\sin[(t-s)B]}{B} G(\Phi(s)) \right\|_r ds \leq |t|^{\varepsilon} \int_{-\infty}^{\infty} \|G_{t-s,a}\|_{q'} \|B^a G(\Phi(s))\|_{q'} ds;$$

now  $\|G_{t-s,a}\|_{q'} \leq C(1 + |t-s|)^{-\delta}$ , while by condition (d) of Theorem 3.1,

$$\|B^a G(\Phi(s))\|_{q'} \leq \|\Phi(s)\|_r^{\rho} (c_1 + c_2 u(t)^{\sigma}) \leq c'(1 + |s|)^{-\rho\varepsilon};$$

thus

$$|t|^{\varepsilon} \|\Phi_1(t)\|_r \leq |t|^{\varepsilon} \int_{-\infty}^{\infty} (1 + |t-s|)^{-\delta} (1 + |s|)^{-\rho\varepsilon} ds = o(1),$$

showing that  $\Phi_1(t) \in L_r$ , and that  $\|\Phi_1(t)\|_r = o(|t|^{-\varepsilon})$ .

It remains to show that  $\Phi_1(\cdot)$  is a solution of the free equation such that  $B^b\Phi_1(\cdot)$  is of finite energy. Expanding  $\sin[(t-s)B]$  by the addition rule, it results that

$$\Phi_1(t) = \Phi_0(t) - \cos(tB) \int_{-\infty}^{\infty} \frac{\sin(sB)}{B} G(\Phi(s)) ds + \frac{\sin tB}{B} \int_{-\infty}^{\infty} \cos(sB) G(\Phi(s)) ds,$$

provided the integrals in question are convergent, in  $\mathcal{H}_b$  and  $\mathcal{H}_{b+1}$  respectively; and it then results that  $\|B^{b+1}\Phi_1(t)\|_2$  is finite.

To establish the convergence of these integrals, note that since  $\|\cos(tB)\|$  and  $\|\sin(tB)\|$  are bounded uniformly, for  $t \in \mathbb{R}^1$ , it suffices to show that

$$\int_{-\infty}^{\infty} \|B^b G(\Phi(s))\|_2 ds < \infty.$$

By condition (f), the latter integral is bounded by

$$c \int_{-\infty}^{\infty} \|\Phi(s)\|_r' \|B^{b+1}\Phi(s)\|_2 ds,$$

which is finite provided that  $\|B^{b+1}\Phi(s)\|_2$  remains bounded. Now

$$B^{b+1}\Phi(t) = B^{b+1}\Phi_0(t) + \int_{-\infty}^t \sin[t-s]B B^b G(\Phi(s)) ds,$$

provided the integral in question is convergent in  $\mathcal{H}$ ; thus

$$\|B^{b+1}\Phi(t)\|_2 \leq \|B^{b+1}\Phi_0(t)\|_2 + \int_{-\infty}^t \|B^b G(\Phi(s))\|_2 ds,$$

which, by (f), is bounded by

$$\|B^{b+1}\Phi_0(t)\|_2 + \int_{-\infty}^t \|\Phi(s)\|_r' \|B^{b+1}\Phi(s)\|_2 ds.$$

It follows that  $\|B^{b+1}\Phi(s)\|_2$  remains bounded provided that  $\int_{-\infty}^{\infty} \|\Phi(s)\|_r' ds$  is finite, as is the case. (*End of proof of Corollary 3.1.*)

A comprehensive statement for the case in which  $m > 0$  is as follows.

**THEOREM 3.2.** — *Let  $b, g$ , and  $\varepsilon$  be given non-negative numbers; let  $\Phi_0(\cdot)$  be a given solution of the free equation such that  $B^b\Phi_0(\cdot)$  is (also) of finite energy, and such that  $\|\Phi_0(t)\|_r = o(|t|^{-\varepsilon})$  for  $|t| \rightarrow \infty$  ( $r \geq 1$ ). Let  $G$  be a given semi-Lipschitzian operator from  $\mathcal{H}_{b+1}$  into  $\mathcal{H}_b$ . Then, provided the assumptions below are satisfied, there exist unique continuous functions  $\Phi(\cdot)$  and  $\dot{\Phi}(\cdot)$  from  $\mathbb{R}^1$  into  $\mathcal{H}_{b+1}$  and  $\mathcal{H}_b$ , respectively, and a (finite-energy) free solution  $\Phi_1(\cdot)$  such that :*

$$\Phi(t) = \Phi_0(t) - g \int_{-\infty}^t \frac{\sin[(t-s)B]}{B} G(\Phi(s)) ds$$

(integral convergent in  $\mathcal{H}_{b+1}$ );

$$\dot{\Phi}(t) = \dot{\Phi}_0(t) - g \int_{-\infty}^t \cos[(t-s)B] G(\Phi(s)) ds$$

(integral convergent in  $\mathcal{H}_b$ );

$$\langle \dot{\Phi}(t) x, y \rangle = \frac{d}{dt} \langle \Phi(t) x, y \rangle,$$

for arbitrary  $x$  and  $y$  in  $\mathcal{H}_{b+1}$ ;

$$\begin{aligned} \|\Phi(t)\|_r &= o(|t|^{-\varepsilon}), & |t| \rightarrow \infty; \\ \|\Phi_1(t)\|_r &= o(|t|^{-\varepsilon}), & |t| \rightarrow \infty \end{aligned}$$

and  $B^b \Phi_1(\cdot)$  is of finite energy;

$$\Phi(t) = \Phi_1(t) + g \int_t^\infty \frac{\sin[(t-s)B]}{B} G(\Phi(s)) ds$$

(integral convergent in  $L_r \wedge \mathcal{H}_{b+1}$ ).

ASSUMPTIONS. — There exist non-negative numbers  $q, q', \delta, \tau$  and  $\tau'$  such that  $1 \leq q, q' \leq \infty, 1 + r^{-1} = q^{-1} + q'^{-1}$ , and

(i) the mapping  $t \rightarrow E_{t,a}$  is continuous from  $\mathbb{R}^1$  into  $L_q$ , and  $\|E_{t,a}\|_q = o(|t|^{-\delta})$ , where  $\delta \geq \varepsilon$ ;

(ii)  $b + 1 > n(2^{-1} - r^{-1})$  (equality permitted if  $r < \infty$ );

(iii)  $\|B^a G(\Phi)\|_{q'} \leq c \|B^{b+1} \Phi\|_2 \|\Phi\|_r^\rho$ , where  $\rho \varepsilon > 1 + \varepsilon$ ;

(iv)  $\|B^b G(\Phi)\|_2 \leq c \|\Phi\|_r^{\tau'} \|B^{b+1} \Phi\|_2$ , where  $\tau' \varepsilon > 1$ ;

(v)  $\|B^b(G(\Phi) - G(\Psi))\|_2 \leq \|B^{b+1}(\Phi - \Psi)\|_2 (\|\Phi\|_r + \|\Psi\|_r)^\tau$   
 $\times C(\|B^{b+1} \Phi\|_2, \|B^{b+1} \Psi\|_2)$ ,

where  $C(\cdot, \cdot)$  is bounded on bounded sets, and  $\tau \varepsilon > 1$ ;

(vi) either  $g$  is sufficiently small, or the initial datum  $\Phi_0(\cdot)$  is sufficiently small in the energy norm  $e(B^b \Phi_0(\cdot))$  of  $B^b \Phi_0(\cdot)$  and the norm

$$M(\Phi_0(\cdot)) = \sup_t (1 + |t|)^\varepsilon \|\Phi_0(t)\|_r;$$

(vii) there exists a function  $d'(\cdot, \cdot)$ , bounded on bounded sets, such that  $d'(\alpha, \beta) \rightarrow 0$  as  $\alpha, \beta \rightarrow 0$ , and such that if  $\Phi(\cdot)$  satisfies the first three equations indicated above for all values of  $t < t_0$ , then

$$\|B^b G(\Phi(s))\|_2 \leq d'(e(B^b \Phi_0(\cdot)), M(\Phi_0(\cdot))) \|\Phi(s)\|_r^\sigma,$$

where  $\sigma \varepsilon > 1$  [ $d'(\cdot, \cdot)$  being independent of  $t_0$ ].

Remark. — The norm  $\sup_t (1 + |t|)^\varepsilon \|\Phi_0(t)\|_r$  can be replaced by a norm which depends only on the Cauchy data at one time of the form  $\|B^{a+1} \Phi_0(t)\|_1 + \|B^d \dot{\Phi}(t)\|_1$ , where  $d$  is a constant dependent on  $m$  and  $n$ , and the time  $t$  is arbitrary.

Note that Assumption (i) is implied by a simple arithmetical condition by virtue of [4].

*Proof.* — The verification of the hypotheses of Lemma 3.6 proceeds as follows : (a) is implied by (i); (b) is implied by (iii); (c) is implied by (v); (d) is implied by (iii) and (v). In the case of Theorem 3.1, hypotheses (a) and (b) are explicitly assumed, and (c) is immediate from (i) and (iii). Hypothesis (d) may be derived as follows : by (vii),

$$\|B^b G(\Phi(s))\|_2 \leq d' \|\Phi(s)\|_r^\sigma$$

[where  $d' = d'(e(B^b \Phi_0(\cdot)), M(\Phi_0(\cdot)))$ ]; on the other hand,

$$\|B^{b+1} \Phi(t)\|_2 \leq \|B^{b+1} \Phi_0(s)\|_2 + \int_{-\infty}^t \|B^b G(\Phi(s))\|_2 ds;$$

by an estimation used earlier; thus

$$\|B^{b+1} \Phi(t)\|_2 \leq c' + d' \int_{-\infty}^t \|\Phi(s)\|_r^\sigma ds;$$

it follows that  $\|B^{b+1} \Phi(t)\|_2 \leq c' + d' u(t)^\sigma$ , and substituting in (iii) there results the *a priori* bound postulated in (d). Condition (e) is readily deduced from (vi) and (vii), noting that  $c' < e(B^b \Phi_0(\cdot))$ . Hypothesis (f) and the supplementary condition of Corollary 3.1 are implied by (iv). Condition (g) follows from (ii) by a Soboleff-type inequality (*cf.* [2]).

4. SCALAR RELATIVISTIC EQUATIONS. — While specific equations are readily subsumed under the foregoing theorems, the methods involved may be made more concrete and the particularities of the situation better taken advantage of by a more direct approach using the methods of the previous section. This is illustrated by the treatment in the present section of scalar relativistic equations in (space) dimensions  $n = 1, 2$  and  $3$ , classified according to the dimensions.

A.  $n = 1$  : Only the case  $m > 0$  can reasonably be treated, since it is evident that even for the free equation, there is no decay when  $m = 0$ , in any  $L_r$  norm. (However, the boundedness of solutions of the non-linear equations under consideration, throughout space-time, could be explored by similar methods.) Supposing now that  $m > 0$ , differing choices for the parameter  $a$  will lead to somewhat different, overlapping but not entirely comparable results.

Consider to begin with the results which can be obtained with the assumption that  $a = 0$ . In this case,  $\|E_{t,a}\|_q = o(|t|^{-\varepsilon(q)})$ , where  $\varepsilon(q) = q^{-1}$  if  $q > 4$ , and  $m > 0$  (it may be deduced that if  $2 \leq q < 4$ , then  $\|E_{t,0}\|_q = o(|t|^{-\varepsilon(q)})$  provided  $\varepsilon(q) < \frac{q-2}{2q}$ ). The existence of the

wave operator may be established from either Lemma 6, or with adequate generality for present purposes, Theorem 2 of I. Assuming then that  $F(l)$  is a  $C^1$  function of the real variable  $l$  such that  $|F(l)| \leq g|l|^p$  and  $|F'(l)| \leq c|l|^{p-1}$ , then there exists a unique solution of the (integrated form of the) differential equation

$$\square \varphi = m^2 \varphi + F(\varphi),$$

for all times  $t < \text{some time } t_0 > -\infty$ , and throughout space, which is asymptotic to a given free solution  $\varphi_0$  of the equation

$$\square \varphi_0 = m^2 \varphi_0,$$

in the sense of the conditions given in Lemma 6, provided  $\varphi_0$  decays uniformly throughout space at the rate  $t^{-\frac{1}{2}}$ . In order to be assured that the solution  $\varphi$  may be extended to a unique solution throughout space-time it suffices to assume that  $F(\cdot)$  has the form  $F(l) = H'(l)$  for some function  $H(\cdot)$  which is semi-bounded [8]. Assuming this, the decay of the solution as  $t \rightarrow +\infty$  may be treated by specializing the inequality (3.4) to the inequality

$$(4.1) \quad \|\Phi(t)\|_r \leq \|\Phi_0(t)\|_r + \int_{-\infty}^t (1+|t-s|)^{-\varepsilon(q)} \|F(\Phi(s))\|_{q'} ds,$$

where  $1+r^{-1} = q^{-1} + q'^{-1}$ . Assuming that  $pq' = r$ , it then follows that

$$(4.2) \quad \|\Phi(t)\|_r \leq \|\Phi_0(t)\|_r + g \int_{-\infty}^t (1+|t-s|)^{-\varepsilon(q)} (1+|s|)^{-p\varepsilon(q)} ds.$$

Introducing  $u(t)$  as earlier, it results that

$$(4.3) \quad u(t) \leq u_0(t) + gu(t)^p (1+|t|)^{\varepsilon(q)} \int_{-\infty}^t (1+|t-s|)^{-\varepsilon(q)} (1+|s|)^{-p\varepsilon(q)} ds;$$

assuming further that  $p\varepsilon(q) > 1$ , it results that

$$(4.4) \quad u(t) \leq u_0(t) + gu(t)^p.$$

To see that  $u(\cdot)$  is continuous, note that if  $t > t'$ ,

$$\|\Phi(t) - \Phi(t')\|_r \leq \int_{t'}^t \|E_{t-s,0}\|_q \|F(\Phi(s))\|_{q'} ds \leq c' \int_{t'}^t \|F(\Phi(s))\|_{q'} ds.$$

As shown in [8], the assumption that  $F = H'$ , where  $H'$  is bounded from below, implies the global boundedness of the free energy, in particular the boundedness of  $\|B\Phi(t)\|_2$  for  $t \in \mathbb{R}^1$ . In one space-dimension, the  $L_r$  norm for  $2 \leq r \leq \infty$  is bounded by the norm in  $\mathcal{BC}_1$ ; this means that

$$\|F(\Phi(s))\|_{q'} \leq c'' \left( \int |\Phi(s)|^r \right)^{\frac{1}{q'}} \leq c''' \|B\Phi(s)\|_2^2,$$

from which it follows that  $\Phi(t) \rightarrow \Phi(t')$  in  $L_r$  as  $t \rightarrow t'$ .

The numerical conditions involved here may be summarized as follows :

$$pq' = r, \quad p\varepsilon(q) > 1, \quad 2 \leq q < \infty, \quad 1 + r^{-1} = q^{-1} + q'^{-1}, \quad r, q, q' \geq 1.$$

Since  $p > 4$  by assumption, there exists  $q$  such that  $p > q > 4$ , and choosing any such  $q$ , it follows that  $p\varepsilon(q) > 1$ . Now set

$$r = \frac{q(p-1)}{q-1}, \quad q' = \frac{q(p-1)}{p(q-1)};$$

then  $pq' = r$ , and it is easily checked that  $1 + r^{-1} = q^{-1} + q'^{-1}$ . Finally,  $q' = \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{q}\right)^{-1} > 1$ , since  $p > q$ ; and  $r = pq' > 1$ , since  $p > 4$  and  $q' > 1$ .

It follows that  $\|\Phi(t)\|_r = o(|t|^{-\gamma})$ , for any  $\gamma < \frac{1}{4}$ , provided that either  $g$  or  $\sup_{t \in \mathbb{R}^1} \|\Phi_0(t)\|_\infty (1 + |t|)^{\frac{1}{2}}$  is sufficiently small. The result may be stated formally as follows.

LEMMA 4.1. — Suppose  $n = 1$ ,  $m > 0$ , and that  $F(l)$  is a given  $C^1$  function of the real variable  $l$  such that  $|F(l)| < g|l|^p$  and  $|F'(l)| < c|l|^{p-1}$ , for some  $p > 4$ ; and suppose that  $F(l) = H'(l)$  for some function  $H(\cdot)$  which is bounded from below. Let  $\varphi_0$  be a given finite-energy solution of the equation  $\square \varphi_0 = m^2 \varphi_0$  which decays uniformly in space at a rate no slower than  $|t|^{-\frac{1}{2}}$  (e. g. it suffices if the Cauchy data for  $\varphi_0$  at some time have two integrable derivatives).

Then there exists a unique global solution of the integrated form of the equation

$$\square \varphi = m^2 \varphi + F(\varphi)$$

which is asymptotic to  $\varphi_0$  near  $t = -\infty$ , and whose  $L_r$  norm (over space) decays temporally at a rate no slower than  $|t|^{-\frac{1}{4+\delta}}$  for arbitrary  $\delta > 0$ , for any value of  $r$  in the range :  $p < r < \frac{4}{3}(p-1)$ , provided that either  $g$  is sufficiently small, or  $\sup_{t \in \mathbb{R}^1} \|\Phi_0(t)\|_\infty (1 + |t|)^{\frac{1}{2}}$  is sufficiently small.

This has been fully demonstrated except for the parenthetical sufficient condition for the required temporal decay of  $\varphi_0$ . To establish this, note that for any time  $s$ ,

$$\Phi_0(t) = \frac{\cos[(t-s)B]}{B^2} B^2 \Phi_0(s) + \frac{\sin[(t-s)B]}{B^2} B \dot{\Phi}_0(s).$$

Estimating in  $L_\infty$  as earlier, it results that

$$\|\Phi_0(t)\|_\infty \leq \|E_{t-s,1}\|_\infty \|B \dot{\Phi}_0(s)\|_1 + \|K_{t-s,1}\|_\infty \|B^2 \Phi_0(s)\|_1,$$

where  $K_{t-s,1}$  is the kernel for  $\cos [(t-s)B]B^{-2}$ ; and both  $L_\infty$  norms here are  $o(|t|^{-\frac{1}{2}})$ .

It should be recalled that the assertion “ $\varphi$  is asymptotic to  $\varphi_0$  near  $t_0$ ” is defined to mean that the three relations and associated conditions given in Lemma 6, with  $-\infty$  replaced by  $t_0$ , are satisfied. In the present case, this means that the mapping  $t \rightarrow \Phi(\cdot, t), \dot{\Phi}(\cdot, t)$  is continuous in the energy norm; that the following equations held (with convergence of the indicated integrals in the energy norm):

$$\begin{aligned}\Phi(\cdot, t) &= \Phi_0(\cdot, t) - \int_{-\infty}^t \frac{\sin[(t-s)B]}{B} F(\Phi(\cdot, s)) ds, \\ \dot{\Phi}(\cdot, t) &= \dot{\Phi}_0(\cdot, t) - \int_{-\infty}^t \cos[(t-s)B] F(\Phi(\cdot, s)) ds\end{aligned}$$

and that  $\Phi(\cdot, t)$  decays in the indicated fashion, in the specified auxiliary  $L_r$  norm.

The foregoing result is readily adapted to the case in which the data are prescribed not at time  $-\infty$ , but a finite time; and this adaptation is more significant, since Theorem 1 itself — but not its proof or the adaptation to the Cauchy problem with data at a finite time — will be superseded by the results to be obtained shortly through the choice  $a=1$ . The advantage of the following corollary over a similar one to be derived from this choice is that less regularity is required of the Cauchy data.

**THEOREM 4.1.** — *Suppose  $n=1$ ,  $m>0$ , and that  $F(\cdot)$  is a given  $C^1$  function of the real variable  $l$  such that  $|F(l)| \leq g|l|^p$  with  $p>4$ , and  $F'(l) = o(|l|^{p'})$  as  $|l| \rightarrow \infty$ , for some finite  $p'$ ; and suppose that  $F(l) = H'(l)$  for some function  $H(\cdot)$  which is bounded from below. Let  $f$  and  $g$  be given functions on space in  $\mathcal{X}$ , and  $\mathcal{X}_0$  respectively such that the solution  $\varphi_0$  of the equation  $\square \varphi_0 = m^2 \varphi_0$  such that  $\varphi_0(\cdot, t) = f$  and  $\dot{\varphi}_0(\cdot, t) = g$  satisfies the inequality  $\|\varphi(\cdot, t)\|_r = o(|t|^{-\frac{1}{4+\delta}})$  for every  $\delta>0$ . (It suffices if  $Bf \in L_1$  and  $g \in L_1$ .)*

*Then there exists a unique global solution of the equation*

$$\Phi(t) = \cos tBf + \frac{\sin tB}{B}g + \int_0^t \frac{\sin[(t-s)B]}{B} F(\Phi(s)) ds,$$

*for any  $r$  in the range  $p < r < \frac{4}{3}(p-1)$  and  $\|\Phi(t)\|_r = o(|t|^{-\frac{1}{4+\delta}})$  for all  $\delta>0$ , provided that either  $g$  or  $\sup_{t \in \mathbb{R}^1} \|\Phi(t)\|_\infty (1+|t|)^{\frac{1}{4}}$  is sufficiently small.*

Consider first the existence of a unique global solution; it suffices to show that the mapping  $\Phi \rightarrow F(\Phi)$  is semi-Lipschitzian from  $\mathcal{X}_1$  to  $\mathcal{X}_0$ , i. e. that

$$\|F(\Phi) - F(\Psi)\|_2 \leq K(\|B\Phi\|_2, \|B\Psi\|_2) \|B(\Phi - \Psi)\|_2,$$



where  $K(\cdot, \cdot)$  is bounded on bounded sets. Now by the mean-value theorem  $F(\Phi) - F(\Psi) = F'(\Omega)(\Phi - \Psi)$ , where

$$\Omega = a\Phi + b\Psi \quad (a, b \geq 0, a + b = 1);$$

since  $F'(\cdot)$  is continuous,  $F'(\cdot) = F_1(\cdot) + F_2(\cdot)$ , where  $|F_1(\lambda)| \leq C_1$  and  $|f_2(\lambda)| \leq C_2 |\lambda|^{p'}$ ; and

$$\|F_1(\Phi) - F_1(\Psi)\|_2 \leq C_1 \|\Phi - \Psi\|_2 \leq C_1 \|B(\Phi - \Psi)\|_2.$$

Now writing  $T$  for the operation of multiplication by  $F_2(\Omega)$ , it results that

$$\begin{aligned} F(\Phi) - F(\Psi) &= G_1(\Omega)(\Phi - \Psi) + T(\Phi - \Psi); & T(\Phi - \Psi) &= TB^{-1}(B(\Phi - \Psi)), \\ \|F(\Phi) - F(\Psi)\|_2 &\leq \|TB^{-1}\| \cdot \|B(\Phi - \Psi)\|_2 + C_1 \|B(\Phi - \Psi)\|_2. \end{aligned}$$

Now the Hilbert-Schmidt norm of an operator bounds its conventional norm (or bound as an operator in Hilbert space), and the Hilbert-Schmidt norm of the product of a convolution and a multiplication operator, where the convolution is by a function in  $L_2$  and the multiplication is by a function in  $L_2$ , is easily seen to be Hilbert-Schmidt, and to have Hilbert-Schmidt norm equal to the product of the  $L_2$ -norms of the functions in question. Since  $B^{-1}$  is the operation of convolution with a function in  $L_2$ , by the Plancherel theorem, it follows that

$$\|TB^{-1}\| \leq C \|F_2(\Omega)\|_2 \leq C' \|\Omega\|^{p'} \leq C' (\|\Phi\| + \|\Psi\|)^{p'}.$$

Now using the fact that the  $L_r$  norm in one-dimensional space is dominated by the  $\mathcal{H}_1$ -norm, if  $2 \leq r < \infty$ , it follows that

$$\|TB^{-1}\| \leq C' (\|B\Phi\|_2 + \|B\Psi\|_2)^{p'}.$$

Observe next the sufficiency of the indicated condition for the decay of  $\Phi_0$ . Since  $\Phi_0(t) = \cos(tB)f + \frac{\sin tB}{B}g$ ,

$$\|\Phi_0(t)\|_r \leq K_{t,0} \|g\|_{q'} + \|E_{t,0}\|_q \|g\|_{q'},$$

where  $r, q$  and  $q'$  are as in the proof of Theorem 4.1. As shown there,  $q' > 1$ ; also,  $q' = (1 - p^{-1})(1 - q^{-1})^{-1} \leq \frac{4}{3}$ , since  $q > 4$ ; thus,  $1 < q' < 2$ , and since  $\|Bf\|_1$  and  $\|Bf\|_2$  are finite, so also is  $\|Bf\|_{q'}$ . Similarly,  $\|g\|_{q'}$  is finite, and employing the bounds cited earlier from [ ], it follows that

$$\|\Phi_0(t)\|_r = o\left(|t|^{-\frac{1}{4+\delta}}\right) \quad \text{for all } \delta > 0.$$

The remainder of the argument is virtually identical with that given in the proof of Theorem 4.1, apart from the replacement of the time  $t = -\infty$  by the time  $t = 0$ .

Consider now the results of taking  $a = 1$  in the preceding section. Since  $\|E_{t,1}\|_\infty = o(|t|^{-\frac{1}{2}})$ , it follows on taking  $r = q = \infty$  and  $q' = 1$  in equation (3.4) that

$$(4.5) \quad \|\Phi(t)\|_\infty \leq \|\Phi_0(t)\|_\infty + C \int_{-\infty}^t (1 + |t-s|)^{-\frac{1}{2}} \|BF(\Phi(s))\|_1 ds.$$

Now

$$\begin{aligned} \|BF(\Phi)\|_1 &\leq C(\|F(\Phi)\|_1 + \|F'(\Phi)\|_2 \|\text{grad } \Phi\|_1) \\ &\leq C(\|F_1(\Phi)\|_2 \|\Phi\|_2 + \|F'(\Phi)\|_2 \|\text{grad } \Phi\|_2), \end{aligned}$$

where  $F_1(l) = \frac{F(l)}{l}$ ,  $l \neq 0$  and  $F_1(0) = 0$ . In the conserved and positive definite energy case, i. e.  $F(\cdot)$  has the form  $F(l) = H'(l)$ , where  $H(\cdot)$  is bounded from below, both  $\|\Phi(s)\|_2$  and  $\|\text{grad } \Phi(s)\|_2$  are bounded, and it follows that

$$(4.6) \quad \|\Phi(t)\|_\infty \leq \|\Phi_0(t)\|_\infty + geC \int_{-\infty}^t (1 + |t-s|)^{-\frac{1}{2}} \|\Phi(s)\|_\infty^{p-2} ds,$$

making the same assumptions on the function  $F(\cdot)$  as in Theorem 4.1, and denoting by  $e$  the (conserved) energy, i. e.

$$\|\text{grad } \Phi(s)\|_2^2 + m^2 \|\Phi(s)\|_2^2 + \|\dot{\Phi}(s)\|_2^2 + \int H(\Phi(s))$$

(the latter integral extending over space).

Taking  $\varepsilon = \frac{1}{2}$  in the procedure of Section 3, it follows that if  $\frac{p-2}{2} > 1$ , i. e. if  $p > 4$ , then

$$u(t) \leq u_0(t) + geCu(t)^{p-2}.$$

The same argument as in the case  $a = 0$  now establishes the

**THEOREM 4.2.** — *Let  $n, m, F(\cdot)$ , and  $\varphi_0$  be as in Lemma 4.1. Then there exists a unique global solution of the integrated form of the equation*

$$\square \varphi = m^2 \varphi + F(\varphi)$$

*which is asymptotic to  $\varphi_0$  near  $t = -\infty$ , and such that  $\|\Phi(t)\|_\infty = o(|t|^{-\frac{1}{2}})$ , provided that one of the following is sufficiently small :  $g, e$ , and*

$$\sup_{t \in \mathbb{R}^1} \|\Phi_0(t)\|_\infty (1 + |t|)^{\frac{1}{2}}.$$

*Remark.* — More specifically, the statement : “  $(\star)$  holds if one of  $a_1, a_2, \dots, a_j$  is sufficiently small ” is defined as an abbreviation for the statement : “ Given all of the  $a_i$  except one, say  $a_{i_0}$ , there exists a positive number  $\delta$  such that if  $|a_{i_0}| < \delta$ , then  $(\star)$  holds ”. In the present situation, the  $\delta$  may be chosen to be uniform for any compact set of the  $a_i$  excluding  $a_{i_0}$ , but not necessarily uniform throughout the range of values of the  $a_i$ .

The decay obtained by Theorem 4.2 is sufficiently rapid to insure the existence of the dispersion operator. It may clarify the earlier argument towards this to illustrate it in the present concrete context.

Consider the integral,  $\int_{-\infty}^{\infty} \frac{\sin[(t-s)B]}{B} F(\Phi(s)) ds$ , with regard to its absolute convergence in the space  $\mathcal{X}$ . In this space, the norm of the integrand is

$$\begin{aligned} \|\sin[(t-s)B] F(\Phi(s))\|_2 &\leq \|F(\Phi(s))\|_2 \leq g \left( \int |\Phi(s)|^{2p} \right)^{\frac{1}{2}} \\ &\leq g e^{\frac{1}{2}} \|\Phi(s)\|^{p-1} = o(|s|^{-\frac{p-1}{2}}) \quad \text{as } |s| \rightarrow \infty. \end{aligned}$$

It is easily seen that the integrand is a continuous function, so it follows that the integral exists. The function  $\Phi_1(\cdot)$ ,

$$\Phi_1(t) = \Phi_0(t) - \int_{-\infty}^{\infty} \frac{\sin[(t-s)B]}{B} F(\Phi(s)) ds$$

is therefore well-defined, and is easily seen to be differentiable in the sense indicated in Lemma 3.6 with derivative

$$\dot{\Phi}_1(t) = \dot{\Phi}_0(t) - \int_{-\infty}^{\infty} \cos[(t-s)B] F(\Phi(s)) ds,$$

the integral here being convergent absolutely in  $\mathcal{X}_0$ .

The appropriate decay for  $\|\Phi_1(t)\|_{\infty}$  follows from the estimate

$$\begin{aligned} \|\Phi_1(t)\|_{\infty} &\leq o(|t|^{-\frac{1}{2}}) + \int_{-\infty}^{\infty} \|E_{t-s,1}\|_{\infty} \|BF(\Phi(s))\|_1 ds \\ &\leq o(|t|^{-\frac{1}{2}}) + C \int_{-\infty}^{\infty} (1+|t-s|)^{-\frac{1}{2}} (1+|s|)^{-\frac{p-2}{2}} ds = o(|t|^{-\frac{1}{2}}), \end{aligned}$$

which serves first to show that  $\|\Phi_1(t)\|_{\infty}$  is finite, and then to establish the decay. It follows from the definition of  $\Phi_1(\cdot)$  that  $\Phi(\cdot)$  is asymptotic to  $\bar{\Phi}_1(\cdot)$  as  $t \rightarrow \infty$ :

$$\begin{aligned} \Phi(t) &= \Phi_1(t) + \int_t^{\infty} \frac{\sin[(t-s)B]}{B} F(\Phi(s)) ds, \\ \dot{\Phi}(t) &= \dot{\Phi}_1(t) + \int_t^{\infty} \cos[(t-s)B] F(\Phi(s)) ds, \end{aligned}$$

the integrals being absolutely convergent in  $\mathcal{X}_1$  and  $\mathcal{X}_0$  respectively. This result may be summarized as

**COROLLARY 4.2 A.** — *With the hypotheses and notation of Theorem 4.2, there exists a (unique) function  $\varphi_1$  with the same properties as  $\varphi_0$ , to which  $\varphi$  is asymptotic as  $t \rightarrow +\infty$ .*

As earlier, it is possible to adapt the results obtained in the case in which data are prescribed at time  $t = -\infty$  to that in which they are prescribed a finite time.

COROLLARY 4.2 B. — *With the hypotheses and notation of Theorem 4.2, there exists for any given time  $t_0$  a unique global solution of the integrated form of the equation*

$$\square \varphi = m^2 \varphi + F(\varphi)$$

whose Cauchy data at time  $t_0$  are the same as those of  $\varphi_0$ , and this solution decays at the rate :

$$\|\Phi(t)\|_\infty = o(|t|^{-\frac{1}{2}}).$$

As  $t \rightarrow +\infty$ ,  $\varphi$  is asymptotic to (unique) solutions  $\varphi_{\pm\infty}$  of the free equation, satisfying the same conditions as  $\varphi_0$ .

*Remark.* — The class of Cauchy data in question here, which have finite energy and are such that the corresponding free solution  $\varphi_0$  (having the given data at a specified time) satisfies the inequality  $\|\Phi_0(t)\|_\infty = o(|t|^{-\frac{1}{2}})$ , has the convenient property of being invariant under both finite and infinite temporal propagation, provided either the data or the “coupling constant”  $g$  are sufficiently small. More specifically, the free-to-interacting, and interacting-to-free wave operators, and corresponding dispersion operators, have this invariance feature. This is of course not the case for the data of compact support of any designated regularity, or those whose Fourier transforms have compact support; the former is invariant only under finite temporal propagation, for suitably restricted types of regularity and the latter is in general invariant only for the free equation.

B.  $n = 2$  : In this case the free solutions typically decay at the rate  $|t|^{-\frac{1}{2}}$  when  $m = 0$ , and at the rate  $|t|^{-1}$  when  $m > 0$ ; only the latter case will be treated here. Taking  $a = 1$ , and noting that  $\|E_{t,1}\|_\infty = o(|t|^{-1})$ , it results as earlier that

$$\|\Phi(t)\|_\infty \leq \|\Phi_0(t)\|_\infty + C \int_{-\infty}^t (1 + |t-s|)^{-1} \|BF(\Phi(s))\|_1 ds.$$

Now assuming that the energy remains bounded for all times, it follows as in the case  $n = 1$ , but with the changed value  $\varepsilon = 1$ , that

$$u(t) \leq u_0(t) + cu(t)^{p-2} \quad (\text{if } p > 3).$$

Again the energy norm dominates all  $L_r$  norms for  $2 \leq r < \infty$ ; it does not however dominate the  $L_\infty$  norm, so that  $u(\cdot)$  need not be continuous. It is therefore necessary to modify the procedure so as to employ in place of the  $L_\infty$  norm an  $L_r$  norm for a sufficiently large finite value of  $r$ .

Noting that  $\|E_{t,1}\|_r \leq \text{const.} \|E_{t,1}\|_\infty^{1-\frac{2}{r}}$  since  $\|E_{t,1}\|_2$  is bounded as a function of  $t$  by the Plancherel theorem, and setting  $\varepsilon = 1 - \frac{2}{r}$ , it results that, with the new definition of  $u(t)$  :

$$u(t) = \sup_{s < t} (1 + |s|)^\varepsilon \|\Phi(s)\|_r,$$

$$u(t) \leq u_0(t) + cu(t)^{p-2} \quad \text{if } (p-2)\left(1 - \frac{2}{r}\right) \geq 1.$$

From this estimate and the continuity applicable for  $r < \infty$ , it follows as in the case  $n = 1$  that

**THEOREM 4.3.** — *Suppose  $n = 2$ ,  $m > 0$ , and that  $F(l)$  is a  $C^1$  function of the real variable  $l$  such that  $|F(l)| \leq g|l|^p$ ,  $|F'(l)| \leq g|l|^{p-1}$ , for some  $p > 3$ . Suppose also that  $F(\lambda) = H'(\lambda)$  where  $H(\cdot)$  is bounded from below.*

Let  $\varphi_0$  be a given finite-energy solution of the equation

$$\square \varphi_0 = m^2 \varphi_0$$

such that

$$|t|^{1-\frac{2}{r}} \|\varphi(\cdot, t)\|_r = c < \infty \quad (t \in \mathbb{R}^1).$$

Then if either  $g, c$ , or the energy of  $\varphi_0$  is sufficiently small, there exists a unique global solution  $\varphi$  of the (integrated form of the) equation

$$\square \varphi = m^2 \varphi + F(\varphi)$$

which is asymptotic to  $\varphi_0$  near  $t = -\infty$ , whose  $L_r$  norm decays like  $t^{-(1-\frac{2}{r})}$ , as  $|t| \rightarrow \infty$ , and which is asymptotic near  $t = \infty$  to a finite-energy free solution, satisfying the same decay law.

As earlier this result may be adapted to the case in which data are prescribed at a finite time instead of the time  $-\infty$ , in full analogy with the relation of Corollary 4.2 to Theorem 4.2.

C.  $n = 3$ ,  $m > 0$  : Taking  $a = 1$ , and noting that

$$\|E_{t,1}\|_q = o\left(|t|^{-1+\frac{1}{q}}\right) \quad \text{for } q > 4,$$

it follows as earlier that

$$\|\Phi(t)\|_\infty \leq \|\Phi(t)\|_0 + c \int_{-\infty}^t (1 + |t-s|)^{-1+\frac{1}{q}} \|\text{BF}(\Phi(s))\|_{q'} ds;$$

assuming boundedness of the energy, it follows that

$$\|\text{BF}(\Phi(s))\|_{q'} \leq c(\|F'(\Phi(s))\|_{q''} + \|F_1(\Phi(s))\|_{q''}), \quad \frac{1}{q'} = \frac{1}{2} + \frac{1}{q''}$$

$[F_1(l) = F(l)l^{-1}$  for  $l \neq 0$ ;  $F_1(0) = 0]$ ; setting  $\varepsilon = 1 - \frac{1}{q}$ , it results that

$$u(t) \leq u_0(t) + cu(t)^\rho, \quad \rho = p - 1 - \frac{2}{q'}$$

provided  $\rho\left(1 - \frac{1}{q}\right) \geq 1$ . The constant  $c$  in the last inequality is proportional to the constant  $g$  such that  $|F(l)| \leq g|l|^\rho$ ,  $|F'(l)| \leq g|l|^{\rho-1}$ , and to a positive power of the free energy  $e$  of  $\varphi_0$  (= total energy of  $\varphi$ , since  $\varphi$  is asymptotic to  $\varphi_0$  near  $t = -\infty$ ).

It now needs only the continuity of  $u(t)$ , together with the assumption that a certain monomial in  $g$ ,  $e$  and  $\sup_t |u_0(t)|$  is sufficiently small, to conclude the global existence of  $\varphi(\cdot)$  and its satisfaction of the estimate  $\|\varphi(\cdot, t)\|_\infty = o\left(|t|^{-1+\frac{1}{q'}}\right)$ ,  $|t| \rightarrow \infty$ . However, in general this continuity will fail, nor will the use of an  $L_r$  norm with  $r < \infty$  alter the situation, since when  $n = 3$  these norms are not bounded by the energy norm. It is necessary to modify the basic Hilbert space  $H_{b+1}$  in which the solution is sought; this necessitates a further restriction on  $\varphi_0$ , and depends on the development of suitable *a priori* estimates for higher-order derivatives of  $\varphi$ . It suffices for the present to take  $b = 1$ , and to assume further that  $B\Phi_0(\cdot)$  is again of finite energy.

In order to use these assumptions in showing the continuity of  $u(t)$ , it is necessary to show that the solution  $\Phi(\cdot)$  which is asymptotic to  $\Phi_0(\cdot)$  near  $t = -\infty$ , likewise has its values in the space  $\mathcal{H}_{b+1}$ . To this end, Lemma 3.6 may be employed. Suppose  $r = \infty$ ,  $q > 4$ , and  $q'$  is such that  $\frac{1}{q} + \frac{1}{q'} = 1$ ; let  $G$  denote the map,  $\varphi(x) \rightarrow F(\varphi(x))$ . To show that  $G$  is semi-Lipschitzian from  $\mathcal{H}_2 \rightarrow \mathcal{H}_1$  it suffices to show that (c) holds, and that  $\|\Phi\|_\infty \leq \text{const.} \|\mathcal{B}^2\Phi\|_2$ . The latter inequality follows by Fourier analysis combined with the Schwarz inequality. To treat (c), note that  $\|\mathcal{B}f\|_2 \leq \|\text{grad } f\|_2 + \|f\|_2$ . Now

$$\begin{aligned} \|\text{grad}(F(\Phi) - F(\Psi))\|_2 &= \|F'(\Phi)\text{grad } \varphi - F'(\Psi)\text{grad } \Psi\|_2 \\ &\leq \|(\text{grad } \Phi - \text{grad } \Psi)F'(\Phi)\|_2 + \|\text{grad } \Psi(F'(\Phi) - F'(\Psi))\|_2; \\ \|(\text{grad } \Phi - \text{grad } \Psi)F'(\Phi)\|_2 &\leq \|\text{grad } \Phi - \text{grad } \Psi\|_2 \|F'(\Phi)\|_\infty \\ &\leq c \|\mathcal{B}(\Phi - \Psi)\|_2 \|\Phi\|_\infty^{p-1} \leq c \|\mathcal{B}^2(\Phi - \Psi)\|_2 \|\Phi\|_\infty^{p-1}, \end{aligned}$$

since  $m > 0$ . For the remaining term.

$$\begin{aligned} \|\text{grad } \Psi(F'(\Phi) - F'(\Psi))\|_2 &\leq \|\text{grad } \Psi\|_2 \|F'(\Phi) - F'(\Psi)\|_\infty; \\ \|F'(\Phi) - F'(\Psi)\|_\infty &\leq \|\Phi - \Psi\|_\infty \|F''(\Omega)\|_\infty \leq c \|\Phi - \Psi\|_\infty (\|\Phi\|_\infty + \|\Psi\|_\infty)^{p-2}, \end{aligned}$$

where  $\Omega$  is as earlier, and the assumption is made that  $F$  is  $C^2$  and that  $\|F''(l)\| \leq c|l|^{p-2}$ ; finally,  $\|\Phi - \Psi\|_\infty \leq C \|\mathcal{B}^2(\Phi - \Psi)\|_2$ . Similar estimates apply to  $\|F(\Phi) - F(\Psi)\|_2$ , and it follows that (c) holds with  $\tau = p - 2$ .

Consider now condition (a). The mappings  $t \rightarrow \frac{\sin [t(m^2 + y^2)^{\frac{1}{2}}]}{m^2 + y^2}$  is easily seen to be continuous into  $L_q(\mathbb{R}^3)$ ; applying the  $L_q$ -Fourier transform, as is possible since the assumption that  $q > 4$  implies that  $q' < \frac{4}{3}$ , it follows that the mapping  $t \rightarrow E_{t,1}$  is continuous into  $L_q(\mathbb{R}^3)$ .

For condition (b), note that

$$\begin{aligned} \|\mathbf{B}\mathbf{F}(\Phi)\|_{q'} &\leq C(\|\text{grad}\mathbf{F}(\Phi)\|_{q'} + \|\mathbf{F}(\Phi)\|_{q'}); \\ \|\text{grad}\mathbf{F}(\Phi)\|_{q'} &= \|\mathbf{F}'(\Phi)\text{grad}\Phi\|_{q'} \leq \|\mathbf{F}'(\Phi)\|_{q'} \|\text{grad}\Phi\|_2, \end{aligned}$$

with  $\frac{1}{q'} = \frac{1}{2} + \frac{1}{q'}$ , by Hölder's inequality;

$$\|\mathbf{F}'(\Phi)\|_{q'} \leq c \left( \int |\Phi|^{(p-1)} \right)^{\frac{1}{q'}} \leq c' \|\mathbf{B}\Phi\|_2 \|\Phi\|_{\infty}^{p-\frac{2}{q'}},$$

since  $\int |\Phi|^2 \leq \text{const.} \|\mathbf{B}\Phi\|_2^2$ . The term  $\|\mathbf{F}(\Phi)\|_{q'}$  may be estimated in a similar fashion, and it follows that (b) holds with  $\rho = p - \frac{2}{q'}$ .

Now setting  $\varepsilon = 1 - \frac{1}{q}$ , conditions (d) become

$$\left(p - \frac{2}{q}\right) \left(1 - \frac{1}{q}\right) > 2 - \frac{1}{q}, \quad (p-2)\varepsilon > 1.$$

In case  $p > 3$ , the second condition is readily satisfied by choosing  $q$  sufficiently large, in which case it is easily seen that the first condition is also satisfied. In order to treat the interesting case  $p = 3$  (as well as some lower values), a variant of Lemma 3.6 will be established.

**LEMMA 4.2.** — *Let  $\Phi_0(\cdot)$  be a given finite-energy abstract solution of the equation  $\Phi_0'' + \mathbf{B}^2\Phi_0 = 0$  ( $n = 3$ ), such that  $\|\Phi_0(t)\|_{\infty} \leq c|1 + |t|^{-\frac{3}{2}}$ ,  $t \in \mathbb{R}^1$  and such that (each component of)  $\text{grad}\Phi_0(t)$  is also of finite energy and has its  $L_{\infty}$  norm similarly bounded. Let  $\mathbf{F}(l)$  be a given realvalued  $C^2$  function of the real variable  $l$  such that  $|\mathbf{F}^{(j)}(l)| \leq c|l|^{p-j}$  ( $j = 0, 1, 2$ ) for some  $p > \frac{8}{3}$ . Then the conclusion of Lemma 3.6 is satisfied with  $b = 1$  (and  $r = \infty$ ,  $\varepsilon = \frac{3}{2}$ ).*

*Proof.* — The problem is to show that not only does  $\Phi(\cdot)$  exist as a continuous mapping into  $\mathcal{H}_1$ , etc., as given by [1], but that under the additional hypotheses concerning  $\text{grad}\Phi_0(t)$ ,  $\Phi(\cdot)$  is continuous into  $\mathcal{H}_2$ ,  $\dot{\Phi}(\cdot)$  is continuous into  $\mathcal{H}_1$ , etc. To this end it suffices, by an argument given in [8], Section III, Lemma 3.1, to show that the equation

$$\Psi(t) = \text{grad}\Phi_0(t) + \int_{-\infty}^t \frac{\sin[(t-s)\mathbf{B}]}{\mathbf{B}} \mathbf{F}'(\Phi(s)) \Psi(s) ds,$$

where  $\Phi(\cdot)$  is the solution obtained for  $b = 0$  [which equation is the abstract integrated form of the equation for  $\text{grad } \Phi(t)$ ] is soluble, along with the corresponding equation for  $\dot{\Psi}(t)$ , in the energy norm. According to ([8], § 3 of Part I), these equations are appropriately soluble provided

$$\int_{-\infty}^{\infty} \|W(-s) K_s W(s)\| ds < \infty,$$

where  $W(s)$  is as earlier, and  $K_s$  denotes the mapping

$$(\Psi_1, \Psi_2) \rightarrow (0, F'(\Phi(s)) \Psi_1)$$

from  $\mathcal{H}_1 \oplus \mathcal{H}_0$  into itself.

Since  $W(s)$  is unitary, the integral in question reduces to  $\int_{-\infty}^{\infty} \|K_s\| ds$ .

Now

$$\|F(\Phi) \Psi\|_2 \leq \|F'(\Phi)\|_{\infty} \|\Psi\|_2 \leq c \|F'(\Phi)\|_{\infty} \|B\Psi\|_2 \quad (\text{since } m > 0),$$

showing that  $\|K_t\| \leq c \|F'(\Phi(t))\|_{\infty}$ . Thus it suffices to show that  $\int_{-\infty}^{\infty} \|\Phi(s)\|^{p-1} ds < \infty$ . Now the general theory of (I) gives the inequality:  $\|\Phi(s)\|_{\infty} = o(|s|^{-\frac{3}{2}})$  near  $s = -\infty$ , so that the last condition is satisfied if  $(p-1)\left(\frac{3}{2}\right) > 1$ , which is the case. (*End of proof of Lemma 4.2.*)

*Resumption of proof of Theorem 4.3.* — It has now been established that, under the assumptions of Theorem 4.3, there exist  $\Phi(t)$  and  $\dot{\Phi}(t)$  defined for  $t < t_0$  and satisfying the conditions given in Lemma 3.6. The remaining hypotheses of Theorem 3.1 may be verified as follows.

Conditions (a), (b) and (c) are immediate, with  $\delta = \varepsilon$ . Condition (d) follows, with  $\sigma = 0$ , from the estimations involved in the verification of condition (b) of Lemma 3.6, on noting that  $\|B\Phi(s)\|_2$  remain bounded by the assumptions made (of the boundedness of the energy,  $\geq \|B\Phi(s)\|_2^2$ ). Condition (e) is satisfied if a certain monomial with positive coefficients in the energy of  $\Phi_0$ , the supremum of  $|1 + |t||^{\varepsilon} \|\Phi_0(t)\|_{\infty}$ , and  $g$ , where  $|F^{(j)}(l)| \leq g |l|^{p-j}$  ( $j = 0, 1, 2$ ), is sufficiently small, and hence if one of the three indicated quantities is sufficiently small, the other two being held fixed. Condition (f) has already been checked.

The foregoing may be summarized as follows.

**THEOREM 4.4.** — *Suppose  $n = 3$ ,  $m > 0$ , and  $F(l)$  is a  $C^2$  function of the real variable  $l$  such that  $|F^{(j)}(l)| \leq g |l|^{p-j}$  ( $j = 0, 1, 2$ ) for some  $p > \frac{8}{3}$ . Suppose also that  $F(\lambda) = H'(\lambda)$ , where  $H(\cdot)$  is bounded from below. Suppose  $4 < q < \infty$ , and let  $\varepsilon = 1 - \frac{1}{q}$ .*



Let  $\varphi_0$  be a given finite-energy solution of the equation  $\square \varphi_0 = m^2 \varphi_0$ ,  $m > 0$ , such that  $\text{grad } \varphi_0$  is also of finite energy, and suppose that the  $L_\infty$  norms of  $\Phi_0(t)$  and of the components of  $\text{grad } \Phi_0(t)$  are bounded by  $\text{const.} (1 + |t|)^{-\epsilon}$ . Then if either  $g$ , or  $\varphi_0$  in a certain norm is sufficiently small, there exists a unique global solution  $\varphi$  of the abstract integrated form of the equation

$$\square \varphi = m^2 \varphi + F(\varphi),$$

which is asymptotic to  $\varphi_0$  near  $t = -\infty$ ; whose  $L_\infty$  norm decays like  $|t|^{-\epsilon}$ ; and is asymptotic near  $t = \infty$  to a finite-energy free solution satisfying the same decay law.

The decay law  $\|\Phi(t)\|_\infty = o(|t|)^{-\epsilon}$  is materially weaker than the generic result  $\|\Phi_0(t)\|_\infty = o(|t|^{-\frac{3}{2}})$  for sufficiently regular free solutions; there is however no apparent reason why the solutions of the non-linear equation should decay significantly more slowly than the corresponding free equation. Indeed, an improved decay estimate can be obtained through the consideration of the case  $a = 2$  in Theorem 3.1, with a positive value for  $\sigma$ , or by a related argument from Theorem 4.4 as in

**COROLLARY 4.4 A.** — *With the notation and hypotheses of Theorem 4.4, suppose that  $p > \frac{7}{2}$ . Then*

$$\|\Phi(t)\|_\infty \leq c(1 + |t|)^{-\frac{3}{2}} \quad \text{and} \quad \|\Phi_1(t)\|_\infty \leq c(1 + |t|)^{-\frac{3}{2}}.$$

*Proof.* — Writing now

$$\Phi(t) = \Phi_0(t) + \int_{-\infty}^t \frac{\sin[(t-s)B]}{B^3} B^2 F(\Phi(s)) ds,$$

and taking  $L_\infty$  norms, it follows that

$$\|\Phi(t)\|_\infty \leq \|\Phi_0(t)\|_\infty + c \int_{-\infty}^t (1 + |t-s|)^{-\frac{3}{2}} \|B^2 F(\Phi(s))\|_1$$

(noting that  $\|E_{t,2}\|_\infty \leq c(1 + |t|)^{-\frac{3}{2}}$ ). To conclude the argument, it suffices to show that  $\|B^2 F\Phi(s)\|_1 = o(s^{-\frac{3}{2}})$ , for Lemma 3.1 then applies. Now

$$\begin{aligned} B^2 F(\Phi(s)) &= (m^2 - \Delta) F(\Phi(s)) \\ &= m^2 F(\Phi(s)) - F'(\Phi(s)) \Delta \Phi(s) - F''(\Phi(s)) (\text{grad } \Phi(s))^2; \end{aligned}$$

$$m^2 \|F(\Phi(s))\|_1 \leq c \int |\Phi(s)|^p \leq c' \|\Phi(s)\|_\infty^{p-2};$$

$$\begin{aligned} \|F''(\Phi(s)) (\text{grad } \Phi(s))^2\|_1 &\leq \|F''(\Phi(s))\|_\infty \|\text{grad } \Phi(s)\|_2^2 \leq c \|\Phi(s)\|_\infty^{p-2}; \\ \|F'(\Phi(s)) \Delta \Phi(s)\|_1 &\leq \|F'(\Phi(s))\|_2 \|\Delta \Phi(s)\|_2; \end{aligned}$$

$$\|F'(\Phi(s))\|_2 \leq c \left( \int |\Phi(s)|^{2(p-1)} \right)^{\frac{1}{2}} \leq c' \|\Phi(s)\|_\infty^{p-2} \|\Phi\|_2.$$

To conclude the proof it therefore suffices to show that  $\|\Delta\Phi(s)\|_2$  is bounded as a function of  $s$ , for  $\|\Phi(s)\|_\infty^{p-2} = o\left(s^{-\left(1-\frac{1}{q}\right)(p-2)}\right)$ , and if  $q$  is sufficiently large and  $p > \frac{7}{2}$  as assumed, the exponent in question exceeds  $\frac{3}{2}$ .

To this end, apply  $B^2$  to both sides of the defining equation for  $\Phi(\cdot)$ , and move  $B^2$  through the integral sign, as is valid in case the resulting integral is absolutely convergent. It follows that

$$\|B^2\Phi(t)\|_2 \leq \|B^2\Phi_0(t)\|_2 + \int_{-\infty}^{\infty} \|BF(\Phi(s))\|_2 ds.$$

Now

$$\begin{aligned} \|BF(\Phi(s))\|_2 &\leq \|\text{grad } F(\Phi(s))\|_2 + \|F(\Phi(s))\|_2; \\ \|\text{grad } F(\Phi(s))\|_2 &= \|(\text{grad } \Phi(s)) F'(\Phi(s))\|_2 \\ &\leq c \|F'(\Phi(s))\|_\infty \leq c' \|\Phi(s)\|_\infty^{p-1} \leq c(1+|s|)^{-(p-1)}, \end{aligned}$$

which is integrable as a function of  $s$ . A similar estimate shows that  $\|F(\Phi(s))\|_2$  is an integrable function of  $s$ , and completes the proof that  $\|B^2\Phi(s)\|_2$ , and hence  $\|\Delta\Phi(s)\|_2$ , is bounded as a function of  $s$ .

The proof that  $\Phi_1(t)$  satisfies the same decay law as does  $\Phi(t)$  (in  $L_\infty$ ) is similar to that given earlier for the same conclusion with other values for the parameters.

The interesting case in which  $p = 3$  is not covered by Corollary 4.4 A but may be treated as a corollary to Theorem 3.1 with the value  $a = 2$ .

**COROLLARY 4.4 B.** — Suppose  $n = 3$ ,  $m > 0$ ,  $F$ , and  $\varphi_0$  are as in Theorem 4.4, with  $\varepsilon = \frac{3}{2}$ , and that  $p \geq 3$ . Then the same conclusion holds with  $\varepsilon$  replaced by  $\frac{3}{2}$ .

*Proof.* — Consider first Lemma 3.6, with the values  $a = 1$ ,  $b = 1$ ,  $\varepsilon = \frac{3}{2}$  and observe that the hypotheses are satisfied *a fortiori* by the same argument as in the case  $\varepsilon = 1 - \frac{1}{q}$ . In applying Theorem 3.1, the only hypothesis which is not immediate or previously established is (d). Employing the same estimates as those for the preceding corollary, it follows that it suffices to show the existence of an *a priori* inequality of the form

$$\|\Delta\Phi(s)\|_2 \leq c_1 + c_2 u(s)^\sigma,$$

with constants  $c_1$  and  $c_2$  independent of the interval on which the solution  $\varphi$  is considered. The argument of the preceding corollary shows that if  $v(t) = \sup_{s < t} \|B^2\Phi(s)\|_2$ , then

$$v(t) \leq v_0 + c \int_{-\infty}^t \|\Phi(s)\|_\infty^{p-1} ds \leq c' + cv(t)^{p-1},$$

since  $\int_{-\infty}^{\infty} (1 + |s|)^{-(p-1)(\frac{3}{2})} ds < \infty$ . The decisive limitation on  $p$  is implied by condition (c), where the condition  $\rho\varepsilon \geq \varepsilon$  implies that  $\rho \geq 1$ , which is satisfied for  $p \geq 3$ .

As earlier, similar arguments may be applied to solutions of the Cauchy problem with data given at finite, rather than at infinite, times, as in

**COROLLARY 4.4 C.** — Suppose  $n$ ,  $m$ , and  $F$  are as in Theorem 4.4, with  $\varepsilon = \frac{3}{2}$ , and let there be given at time  $t$  finite-energy Cauchy data such that the corresponding solution of the (abstract, integrated form of the) equation  $\square \varphi_0 = m^2 \varphi_0$  has the same properties as are hypothesized for  $\varphi$  in Theorem 4.4. (This is the case, e. g., if a certain number of derivatives of the data exist and are in  $L_1 \cap L_2$  over space.) Then if  $p \geq 3$  and if either the data are sufficiently small (in norm) or the coupling is sufficiently weak (i. e. the constant  $g$  such that  $|F^{(j)}(l)| \leq |g| |l|^{p-j}$  is sufficiently small), there exists a unique solution  $\varphi$  of the (abstract, integrated form of the) equation

$$\square \varphi = m^2 \varphi + F(\varphi)$$

having the prescribed data, globally in space and time; this solution decays at the rate  $\|\Phi(t)\|_{\infty} = o(|t|^{-\frac{3}{2}})$ ; and as  $t \rightarrow \pm\infty$ , is asymptotic to solutions of the free equation satisfying the same regularity and decay conditions as  $\varphi_0$ .

The argument is essentially the same as earlier.

It would of course be highly desirable to extend these results to ones which are independent of the weakness of the coupling, or norm of the initial datum, i. e. non-perturbative in nature. *A priori* information concerning the temporal decay of low-order norms could be used to make such an extension, but such *a priori* results are presently known only in special cases (cf. the next section, and the references to the work of Strauss therein). Some non-perturbative results can be established by using inequalities of the form

$$u(t) \leq u_0(t) + cu(t)^{\sigma},$$

where  $\sigma < 1$  rather than  $> 1$  as earlier; they are quite insufficient for establishing the existence of dispersion, but more precise than the exponential bounds typically derivable from general theory. An example of this genre is

**THEOREM 4.5.** — With the hypotheses and notation of Theorem 4.4, and the assumption that  $p < 5$ , the solution  $\varphi$  of the (integrated form of the) equation

$$\square \varphi = m^2 \varphi + F(\varphi)$$

having the same Cauchy data at a prescribed time  $t_0$  as  $\varphi_0$ , has the property that the map,  $t \rightarrow (\Phi(t), \dot{\Phi}(t))$  is continuous from  $\mathbb{R}^1$  into  $\mathcal{X}_2 \oplus \mathcal{X}_1$ , and  $\|\Phi(t)\|_\infty = o(|t|^\delta)$ ,  $\delta$  being arbitrarily small.

*Proof.* — By the earlier theory, it suffices, for the demonstration that  $\Phi(t) \in \mathcal{X}_2$  and  $\dot{\Phi}(t) \in \mathcal{X}_1$ , and that these functions of  $t$  are continuous, to note that this is initially the case at  $t = t_0$ , that the mapping  $\Phi \rightarrow F(\Phi)$  is semi-Lipschitzian from  $\mathcal{X}_2$  into  $\mathcal{X}_1$ , and that conditions (i) and (ii) of Lemma 3.4 hold, with  $N((f, g)) = \|f\|_\infty$ ; and to show that  $\|\Phi(t)\|_\infty$  is bounded on every bounded interval. The only non-trivial point in question here is the local boundedness of  $\|\Phi(t)\|_\infty$ ; this follows from the estimate:  $\|\Phi(t)\|_\infty = o(|t|^\delta)$ ; it will suffice therefore to establish this estimate.

Writing as earlier.

$$\|\Phi(t)\|_\infty \leq \|\Phi_0(t)\|_\infty + \left| \int_{t_0}^t \|E_{t-s,1}\|_q \|BF(\Phi(s))\|_{q'} ds \right|,$$

where  $\frac{1}{q} + \frac{1}{q'} = 1$  and  $q > 4$ ; setting  $\varepsilon = 1 - \frac{1}{q}$ , and

$$u(t) = \sup_{t_0 < s < t} \|\Phi(s)\|_\infty (1 + |t|)^\varepsilon$$

(assuming, as is evidently no loss of generality, that  $t > t_0$ ), it results that

$$u(t) \leq u_0(t) + c(1 + |t|)^\varepsilon \int_{t_0}^t (1 + |t-s|)^{-\varepsilon} \|B\Phi(s)\|_2 \|F'(\Phi(s))\|_{q''} ds,$$

where

$$\frac{1}{q'} = \frac{1}{2} + \frac{1}{q''}; \quad \|F'(\Phi(s))\|_{q''} \leq c \left( \int |\Phi(s)|^{(p-1)q''} \right)^{\frac{1}{q''}};$$

writing  $(p-1)q'' = d + [(p-1)q'' - d]$ , where  $2 \leq d \leq 6$ , and noting that  $\int |\Phi(s)|^d ds$  is bounded by Soboleff's inequality and energy conservation, as is  $\|B\Phi(s)\|_2$ , it results that

$$u(t) \leq u_0(t) + c(1 + |t|)^\varepsilon \int_{t_0}^t (1 + |t-s|)^{-\varepsilon} (1 + |s|)^{-\alpha\varepsilon} ds u(t)^\alpha,$$

where for an appropriate choice of  $d$  in the indicated range  $\alpha = p-1 - \frac{d}{q''}$  satisfies the inequality  $0 \leq \alpha < 1$ , provided  $p$  is in the hypothesized range. Noting that if  $a$  and  $b$  are arbitrary numbers in the interval  $(0, 1)$ , then

$$\int_{t_0}^t (1 + |t-s|)^{-a} (1 + |s|)^{-b} ds \leq \int_{t_0}^t |t-s|^{-a} |s|^{-b} ds = Ct^{1-a-b};$$

if  $t_0 = 0$ , as it is evidently no essential loss of generality to assume, it follows that

$$u(t) \leq u_0(t) + c(1 + |t|)^\gamma u(t)^\alpha, \quad \gamma = 1 - \alpha\varepsilon.$$

To reduce this inequality, define  $v(t) = u(t)(1 + |t|)$ ; on choosing  $\beta = -\frac{1 - \varepsilon\alpha}{1 - \alpha}$ , the inequality for  $u(t)$  is equivalent to the inequality

$$v(t) \leq v_0(t) + cv(t)^\alpha,$$

which implies that  $v(t)$  is bounded, inasmuch as  $v_0(t)$  is such, and  $\alpha < 1$ . This result gives for  $\|\Phi(t)\|_\infty$  the inequality

$$\|\Phi(t)\|_\infty \leq c(1 + |t|)^{-\varepsilon} (1 + |t|)^{\frac{1 - \alpha\varepsilon}{1 - \alpha}} = c(1 + |t|)^{\frac{1 + \varepsilon}{1 - \alpha}},$$

and the exponent  $\frac{1 - \varepsilon}{1 - \alpha}$  may be made arbitrarily small by choosing  $q$  sufficiently large.

5. DISCUSSION OF FURTHER DEVELOPMENTS. — A. *The case of zero mass.* — The foregoing methods remain applicable when  $m = 0$ , but the results are weaker and less general, owing to the less rapid decay of the free solutions, to the elimination of the boundedness of the  $L_2$ -norm over space of the solution at a fixed time (which in the case  $m = 0$  does not follow from energy boundedness), etc. For example, an application of Theorem 1 of I to the derivation of an analogue to Lemma 4.1 may be made with the choices:  $N(u) = \|\Phi\|_r$  if  $u = (\Phi, \Psi)$ ,  $r = 6 + \delta$ , where  $\delta$  must be chosen sufficiently small, and  $\varepsilon = 1 - \frac{2}{r}$ ; but this results in a slower decay rate, for a less convenient norm, and in addition these choices are effective only for  $p > 3$ .

On the other hand, the *a priori* estimates obtained by Strauss [(9), (10)] appear to depend on the vanishing of the mass, and provide major simplifications in the treatment, leading in fact to non-perturbative results for a certain class of relativistic equations.

B. *The question of optimal linear decay rates.* — As shown by Brodsky [1] in the case  $m = 0$  and later authors for  $m > 0$ , sufficiently regular solutions of the equation  $\square\varphi = m^2\varphi$  decay at the rate  $|\varphi(x, t)| = o(|t|^{-\varepsilon})$ ,  $\varepsilon = n - \frac{n - \text{sgn } m}{2}$ , uniformly throughout space. The study of the asymptotics of the non-linear equations would be facilitated by a more rapid rate of decay, but it is probable that these estimates are optimal in the rather strong sense that any (finite-energy) solution which decays at a more

rapid rate must be identically zero. Added in proof: This conjecture has now been confirmed by recent work of W. Littman.

C. *Unresolved special equations of interest.* — In the case  $m > 0$ , the foregoing results are fairly complete in the perturbative realm as regards power interactions, i. e. the case  $F(l) = gl^p$ , where  $p$  is an odd integer, in the dimensions  $n = 1, 2, 3$ , except for the case  $n = 1$  and  $p = 3$ . It is probable that a combination of the present results with the methods employed in I for dealing with higher values of  $n$  will permit the extension of these results to the case  $n > 3$ , which should provide some mathematical illumination.

In the case  $m = 0$ , the situation as regards decay and dispersion remains unclear for the interesting case  $n = p = 3$ , despite knowledge of the existence of regular global solutions and the applicability of Strauss' *a priori* estimates; conceivably, just as in the case of the Schrödinger equation with a Coulomb potential, dispersion does not take place in a strict sense; indeed, this relatively singular potential is connected with a vanishing mass. An extremely interesting equation, which combines the complication  $m = 0$  with that of singularity of the non-linear term relative to the energy norm (but in a different way from earlier, i. e. through the intervention of first-order derivatives in the non-linear term) is that of Yang and Mills [11], which is in addition notably symmetrical, a feature which should facilitate the development of *a priori* estimates other than that deriving from the conservation of energy.

D. *Regularity of the S-operator and the inverse problem for dispersion.* — As shown in [6], the tangent space to the solution manifold of any of a general class of evolutionary partial differential equations, in the theory of infinite-dimensional manifolds, is naturally isomorphic to the solution space of the first-order variational equation in the vicinity of the original solution in question. It can be deduced that when the S-operator exists and is sufficiently regular, its Frechet-Gateaux differential  $\partial_\varphi S$  at a point  $\varphi_0$  (= solution of the free equation) is the S-operator  $S_{\varphi_0}$  associated with the linear equation

$$\square \psi = m^2 \psi + F'(\varphi) \psi,$$

where

$$\varphi(X) = \varphi_0(X) + \int_{-\infty}^t D_{\text{ret}}(X - X') F(\varphi(X')) dx'$$

defines  $\varphi(X)$ . At this point it could be shown that S is a differentiable map relative to appropriately chosen spaces, and that  $S_{\varphi_0}$  is itself differentiable as a function of  $\varphi_0$ , etc.

This aspect will not be developed here, but it is relevant to the natural question of whether in principle the function  $F$  is determined by the operator  $S$ , i. e. the so-called inverse problem of dispersion theory. A basically affirmative response is physically indicated and is confirmed in the simplest analogous linear cases, notably that of potential scattering in one dimension. The continuity of the solution of a non-linear equation as a function of the non-linear operator involved has been treated in [6]; there is little doubt that this treatment could be extended to differentiability considerations, including that of  $\partial_F S$ , in the vicinity of  $\varphi_0 = 0$ , and with  $F$  restricted to lie in an appropriate space such that  $S$  exists and is adequately regular, as may be set up on the basis of the present results. A first step towards the inverse dispersion problem is the treatment of the univalence of the derivative  $\partial_F S$  at the point  $F = 0$ . This derivative may be computed as the mapping

$$G \rightarrow T_G, \quad \text{where } T_G: \Phi_0 \rightarrow \Psi_0, \quad \Psi_0(t) = \int_{-\infty}^{\infty} \frac{\sin[(t-s)B]}{B} G(\Phi_0(s)) ds;$$

the univalence is then the question of whether  $G = 0$  provided

$$\int_{-\infty}^{\infty} \frac{\sin[(t-s)B]}{B} G(\varphi_0(s)) ds = 0.$$

It is plausible that this should be the case for an appropriate class of non-linear functions  $G$ , such as perhaps those of the form

$$G \in C^\infty; G'(l) \geq 0, \quad G = H' \quad \text{with } H(l) \geq 0; \quad |G(l)| \leq g|l|^p,$$

inasmuch as an analogous linearized question in a non-relativistic setting (i. e. a question of time-dependent potential scattering) is that of whether the vanishing of  $\int_{-\infty}^{\infty} e^{itH} V(t) e^{-itH} dt$  implies the vanishing of  $V(\cdot)$ . This is indeed the case if  $V(t) \geq 0$ , which condition is analogous to the condition  $G'(l) \geq 0$ .

#### REFERENCES.

- [1] A. R. BRODSKY, *Asymptotic decay of solutions to the relativistic wave equation...*, Doctoral dissertation, Department of Mathematics, M. I. T., Cambridge, Mass., 1964.
- [2] A. P. CALDERON, *Lebesgue spaces of differentiable functions and distributions (Proc. Symp. Pure Math., vol. IV, 1961, p. 33-49; Amer. Math. Soc., Providence)*.
- [3] W. LITTMAN, *The wave operator and  $L_p$  norms (J. Math. Mech., vol. 12, 1963, p. 55-63)*.
- [4] S. NELSON, *Asymptotic behavior of certain fundamental solutions to the Klein-Gordon equation*, Doctoral dissertation, Department of Mathematics, M. I. T., Cambridge, Mass., 1966.

- [5] I. SEGAL, *Quantization and dispersion for non-linear relativistic equations*, p. 79-108; Proc. Conf. on Math. Theory of El. Particles, publ. M. I. T. Press, Cambridge, Mass., 1966.
- [6] I. SEGAL, *Differential operators in the manifold of solutions of a non-linear differential equation* (*J. Math. pures et appl.*, t. 44, 1965, p. 71-132).
- [7] I. SEGAL, *The global Cauchy problem for a relativistic scalar field with power interaction* (*Bull. Soc. Math. Fr.*, t. 91, 1963, p. 129-135).
- [8] I. SEGAL, *Non-linear semi-groups* (*Ann. Math.*, vol. 78, 1963, p. 339-364).
- [9] W. A. STRAUSS, *La décroissance asymptotique des solutions des équations d'onde non linéaires* (*C. R. Acad. Sc.*, t. 256, 1963, p. 2749-2750); *Les opérateurs d'onde pour les équations d'onde non linéaires indépendantes du temps* (*Ibid.*, t. 256, 1963, p. 5045-5046).
- [10] W. A. STRAUSS, To appear in *J. Functional Analysis*.
- [11] C. N. YANG and R. C. MILLS, *Phys. Rev.*, vol. 96, 1954, p. 191.

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