Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2015, Vol. 51, No. 3, 1159–1189
DOI: 10.1214/14-AIHP609

ANNALES DE L'INSTITUT HENRI POINCARÉ PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

On the limiting behaviour of needlets polyspectra[1]

Valentina Cammarota and Domenico Marinucci

Department of Mathematics, University of Rome Tor Vergata, via della Ricerca Scientifica, 1, Roma, 00133, Italy.
E-mail: *marinucc@mat.uniroma2.it*

Received 18 July 2013; revised 3 February 2014; accepted 12 February 2014

Abstract. This paper provides quantitative Central Limit Theorems for nonlinear transforms of spherical random fields, in the high-frequency limit. The sequences of fields that we consider are represented as smoothed averages of spherical Gaussian eigenfunctions and can be viewed as random coefficients from continuous wavelets/needlets; as such, they are of immediate interest for spherical data analysis. In particular, we focus on so-called needlets polyspectra, which are popular tools for non-Gaussianity analysis in the astrophysical community, and on the area of excursion sets. Our results are based on Stein–Malliavin approximations for nonlinear transforms of Gaussian fields, and on an explicit derivation on the high-frequency limit of their variances, which may have some independent interest.

Résumé. Dans cet article on prouve un TCL pour des fonctionnelles nonlinéaires de champs aléatoires sur la sphère avec bornes en variation totale dans le sens de la limite en haute fréquence. Les suites de champs aléatoires que l'on considère sont des moyennes régularisées de fonctions propres gaussiennes sur la sphère qui peuvent être vues comme des coefficients aléatoires d'ondelettes/needlets continues. En particulier on se concentre sur le polyspectre en needlets lequel est un outil couramment utilisé dans l'analyse de la nongaussianité en astrophysique et dans le domaine des ensembles de niveau. Nos résultats sont basés sur des approximations de type Stein–Malliavin pour des fonctionnelles nonlinéaires de champs gaussiens ainsi que sur le calcul explicite de la limite en haute fréquence de leur variance, ce qui pourrait constituer un résultat ayant un interêt en lui même.

MSC: 60G60; 62M15; 42C15

Keywords: Spherical random fields; Stein–Malliavin approximations; Polyspectra; Excursion sets; Wavelets; Needlets

1. Introduction

1.1. *Background and notation*

Let $\{f(x), x \in S^2\}$ denote a Gaussian, zero-mean isotropic spherical random field, i.e. for some probability space $(\Omega, \Im, P)$ the application $f(x,\omega) \to \mathbb{R}$ is $\{\Im \times \mathcal{B}(S^2)\}$ measurable, $\mathcal{B}(S^2)$ denoting the Borel σ-algebra on the sphere. We shall use $\mathrm{d}\sigma(x)$ to denote the Lebesgue measure on the sphere which, in spherical coordinates, is defined as $\mathrm{d}\sigma(x) := \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi$. It is well-known that the following representation holds, in the mean square sense (see for instance [13,15–17]):

$$f(x) = \sum_{\ell=1}^{\infty} f_\ell(x), \qquad f_\ell(x) = \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x),$$

[1]Supported by ERC Grant 277742 *Pascal*.

where $\{Y_{\ell m}(\cdot)\}$ denotes the family of spherical harmonics, and $\{a_{\ell m}\}$ the array of random spherical harmonic coefficients, which satisfy $\mathbb{E}a_{\ell m}\overline{a}_{\ell' m'} = C_\ell \delta_\ell^{\ell'} \delta_m^{m'}$; here, δ_a^b is the Kronecker delta function, and the sequence $\{C_\ell\}$ represents the angular power spectrum of the field. As pointed out in [18], under isotropy the sequence C_ℓ necessarily satisfies $\sum_\ell C_\ell \frac{(2\ell+1)}{4\pi} = \mathbb{E}f^2 < \infty$ and the random field $f(x)$ is mean square continuous.

The Fourier components $\{f_\ell(x)\}$, can be viewed as random eigenfunctions of the spherical Laplacian:

$$\Delta_{S^2} f_\ell = -\ell(\ell+1) f_\ell, \quad \ell = 1, 2, \ldots;$$

the random fields $\{f_\ell(x), x \in S^2\}$ are isotropic, meaning that the probability laws of $f_\ell(\cdot)$ and $f_\ell^g(\cdot) := f_\ell(g\cdot)$ are the same for any rotation $g \in SO(3)$. Also, $\{f_\ell(\cdot)\}$ is centred Gaussian, with covariance function

$$\mathbb{E}\big[f_\ell(x) f_\ell(y)\big] = C_\ell \frac{2\ell+1}{4\pi} P_\ell\big(\cos d(x,y)\big),$$

where P_ℓ are the usual Legendre polynomials defined dy Rodrigues' formula

$$P_\ell(t) := \frac{1}{2^\ell \ell!} \frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} \big(t^2 - 1\big)^\ell,$$

and $d(x,y)$ is the spherical geodesic distance between x and y. The asymptotic behaviour of $f_\ell(x)$ and their nonlinear transforms has been studied for instance by [4,40,41], see also [21–23].

More often, however, statistical procedures to handle spherical data are based upon wavelets-like constructions, rather than standard Fourier analysis. For instance, the astrophysical/cosmological literature on these issues is vast, see [24,34] and the references therein. As is well-known, indeed, the double localization properties of wavelets (in real and harmonic domain) turn out to be of great practical value when handling real data.

In view of these motivations, we shall focus here on sequence of spherical random fields which can be viewed as averaged forms of the spherical eigenfunctions, e.g.

$$\beta_j(x) = \sum_{\ell=2^{j-1}}^{2^{j+1}} b\left(\frac{\ell}{2^j}\right) f_\ell(x), \quad j = 1, 2, 3, \ldots$$

for a smooth (e.g. C^∞) weigh function $b(\cdot)$, compactly supported in $[\frac{1}{2}, 2]$, and satisfying the partition of unity property $\sum_j b^2(\frac{\ell}{2^j}) = 1$, for all $\ell \geq 1$. The fields $\{\beta_j(x)\}$ can indeed be viewed as a representation of the coefficients from a continuous wavelet transform from $T(x)$, at scale j, see also the discussion in [20]. More precisely, consider the kernel

$$\begin{aligned}\Psi_j\big(\langle x, y\rangle\big) &:= \sum_{\ell=2^{j-1}}^{2^{j+1}} b\left(\frac{\ell}{2^j}\right) \frac{2\ell+1}{4\pi} P_\ell\big(\langle x, y\rangle\big) \\ &= \sum_{\ell=2^{j-1}}^{2^{j+1}} b\left(\frac{\ell}{2^j}\right) \sum_{m=-\ell}^{\ell} Y_{\ell m}(x) \overline{Y}_{\ell m}(y).\end{aligned}$$

Then $\Psi_j(\langle x, y\rangle)$ can be viewed as a continuous version of the needlet transform, which was introduced by Narcowich et al. in [25], and considered from the point of view of statistics and cosmological data analysis by many subsequent authors, starting from [2,19,29]. In this framework, the following localization property is now well-known: for all $M \in \mathbb{N}$, there exists a constant K_M such that

$$\big|\Psi_j\big(\langle x, y\rangle\big)\big| \leq \frac{K_M 2^{2j}}{\{1 + 2^j d(x,y)\}^M},$$

where $d(x,y) = \arccos(\langle x, y\rangle)$ is the usual geodesic distance on the sphere. Heuristically, the "needlet" field

$$\beta_j(x) = \int_{S^2} \Psi_j\big(\langle x, y\rangle\big) f(y)\, \mathrm{d}y = \sum_{\ell=2^{j-1}}^{2^{j+1}} b\left(\frac{\ell}{2^j}\right) f_\ell(x)$$

is then only locally determined, i.e., for 2^j large enough its value depends only from the behaviour of $f(y)$ in a neighbourhood of x. This is a very important property, for instance when dealing with spherical random fields which can only be partially observed, the canonical example being provided by the masking effect of the Milky Way on Cosmic Microwave Background radiation [30,31].

It is hence very natural to produce out of $\{\beta_j(x)\}$ nonlinear statistics of great practical relevance. For instance, it is readily seen that

$$\mathbb{E}\{\beta_j^2(x)\} = \sum_{\ell=2^{j-1}}^{2^{j+1}} b^2\left(\frac{\ell}{2^j}\right)\frac{2\ell+1}{4\pi}C_\ell,$$

which hence suggests a natural "local" estimator for a binned form of the angular power spectrum. More generally, we might focus on statistics of the form

$$\nu_{j;q} := \int_{S^2} H_q\bigl(\beta_j(x)\bigr)\,\mathrm{d}x,$$

where $H_q(\cdot)$ is the Hermite polynomial of qth order, which can be labelled needlets polyspectra for a straightforward analogy with the Fourier case. For $q=3$ we obtain for instance the needlets bispectrum, which was in introduced in [11] and then widely used on CMB data to study non-Gaussian behaviour, see for instance [7,32,33] for more discussion and details; for $q=4$ we obtain the needlets trispectrum, which is the natural candidate to estimate higher-order non-Gaussian behaviour such as the one introduced by cubic models through the parameter g_{NL}, see [30]. As we shall show below, the analysis of such polyspectra for arbitrary values of q provides moreover natural building blocks for other nonlinear functionals of the field $\beta_j(x)$. We shall investigate in particular quantitative Central Limit Theorems for the excursion sets, as $j\to\infty$.

Concerning this point, we stress that the limiting behaviour we consider is in the high frequency sense, e.g. assuming that a single realization of a spherical random field is observed at higher and higher resolution as more and more refined experiments are implemented. This is the setting adopted in [17], see also [1,8,14,35,39] for the related framework of fixed-domain asymptotics, and [30,31] for applications to cosmological data analysis.

1.2. *Statement of the main results*

The main technical contribution of this paper is the derivation of analytical expressions for the asymptotic variance of the needlet polyspectra $\nu_{j;q}$. To this aim, we shall impose the following mild regularity conditions on the power spectrum C_ℓ (see [17], page 257).

Condition 1. *There exists $M\in\mathbb{N}$, $\alpha>2$ and a sequence of functions $\{g_j(\cdot)\}$ such that for $2^{j-1}<\ell<2^{j+1}$*

$$C_\ell = \ell^{-\alpha} g_j\left(\frac{\ell}{2^j}\right) > 0,$$

where $0<c_0^{-1}\le g_j\le c_0$ for all $j\in\mathbb{N}$ and for some $c_1,\dots,c_M>0$ and $r=1,\dots,m$, we have

$$\sup_j \sup_{2^{-1}\le u\le 2}\left|\frac{\mathrm{d}^r}{\mathrm{d}u^r}g_j(u)\right| \le c_r.$$

Condition 1 entails a weak smoothness requirement on the behaviour of the angular power spectrum, which is satisfied by cosmologically relevant models. This condition is fulfilled for instance by models of the form

$$C_\ell = \ell^{-\alpha}G(\ell),$$

where $G(\ell)=P(\ell)/Q(\ell)$ and $P(\ell), Q(\ell)>0$ are two positive polynomials of the same order. Indeed, in the now dominant Bardeen's potential model for the angular power spectrum of the Cosmic Microwave Background radiation (which is theoretically justified by the so-called inflationary paradigm for the Big Bang Dynamics, see e.g., [6,9]) one has $C_\ell \sim (\ell(\ell+1))^{-1}$ for the observationally relevant range $\ell \leq 5 \times 10^3$ (the decay becomes faster at higher multipoles, in view of the so-called Silk damping effect, but these multipoles are far beyond observational capacity). This is clearly in good agreement with Condition 1; in what follows we denote with G the limit $G := \lim_{\ell\to\infty} G(\ell)$, which certainly exists given Condition 1.

Under Condition 1, we shall be able to show the following result (compare with [22]).

Theorem 1. *For $q \geq 2$, we have*

$$\lim_{j\to\infty} 2^{2j} \operatorname{Var}[\nu_{j;q}] = q! c_q,$$

where

$$c_2 = \frac{8\pi^2}{(\int_{1/2}^2 b^2(x)x^{1-\alpha}\,\mathrm{d}x)^2} \int_{1/2}^2 b^4(x_1)x_1^{1-2\alpha}\,\mathrm{d}x_1;$$

$$\begin{aligned}
c_3 = {} & \frac{16\pi}{(\int_{1/2}^2 b^2(x)x^{1-\alpha}\,\mathrm{d}x)^3} \int_{1/2}^2 \int_{1/2}^2 \int_{1/2}^2 \prod_{i=1}^3 b^2(x_i)x_i^{1-\alpha} \\
& \times \frac{1}{\sqrt{x_1+x_2-x_3}\sqrt{x_1-x_2+x_3}\sqrt{-x_1+x_2+x_3}\sqrt{x_1+x_2+x_3}} \\
& \times \mathbb{1}_{P_3}(x_1,x_2,x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3;
\end{aligned}$$

$$\begin{aligned}
c_4 = {} & \frac{16}{(\int_{1/2}^2 b^2(x)x^{1-\alpha}\,\mathrm{d}x)^4} \int_{1/2}^2 \int_{1/2}^2 \int_{1/2}^2 \int_{1/2}^2 \prod_{i=1}^4 b^2(x_i)x_i^{1-\alpha} \\
& \times \int_0^4 y \frac{1}{\sqrt{-x_1+x_2+y}\sqrt{x_1-x_2+y}\sqrt{x_1+x_2-y}\sqrt{x_1+x_2+y}} \\
& \times \frac{1}{\sqrt{-x_3+x_4+y}\sqrt{x_3-x_4+y}\sqrt{x_3+x_4-y}\sqrt{x_3+x_4+y}} \\
& \times \mathbb{1}_{P_3}(x_1,x_2,y)\mathbb{1}_{P_3}(y,x_3,x_4)\,\mathrm{d}y\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4;
\end{aligned}$$

and finally for $q \geq 5$

$$c_q = \frac{8\pi^2}{(\int_{1/2}^2 b^2(x)x^{1-\alpha}\,\mathrm{d}x)^q} \int_{1/2}^2 \cdots \int_{1/2}^2 \int_0^\infty \prod_{k=1}^q b^2(x_k)x_k^{1-\alpha} J_0(x_k\psi)\psi\,\mathrm{d}\psi\,\mathrm{d}x_1\cdots\mathrm{d}x_q;$$

where P_3 is the set of all $(x_1,x_2,x_3) \in \mathbb{R}^3$ that satisfy the "triangular" conditions (4.2).

Here, J_0 denotes the standard Bessel function of order zero, defined as usual by

$$J_0(x) = \sum_{k=0}^\infty \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}.$$

For each $q \geq 2$, the scaling factor for the needlets polyspectra is of order 2^{2j}. This result can be heuristically explained as follows. Needlets polyspectra can be viewed as linear combination of random polynomials of degree $q \times j$. On a compact manifold as the sphere, there exist exact cubature formulae for such polynomials, so that the integrals defining $\nu_{j;q}$ can be really expressed as finite averages sums, of cardinality 2^{2j}. In view of the uncorrelation inequality (3.1), we expect the variance of these averages to scale as the inverse of the number of summands, e.g. exactly 2^{-2j}.

Making this heuristic rigorous is indeed quite challenging, and requires a careful analysis on the behaviour of Legendre polynomials (Hilb's asymptotics, see [22,23]) and Clebsch–Gordan/Wigner's coefficients.

Once the asymptotic behaviour of the variance is established, in view of the celebrated results from Nourdin and Peccati [26] the derivation of quantitative Central Limit Theorems and Total Variation/Wasserstein distances limits requires only the analysis of fourth-order cumulants. These computations are quite standard and provided in Section 5, where it is hence shown that

Theorem 2. *For $\mathcal{N}$ a standard Gaussian random variable, as $j \to \infty$, we have that*

$$d_{\mathrm{TV}}\left(\frac{\nu_{j;q}}{\sqrt{\operatorname{Var}(\nu_{j;q})}}, \mathcal{N}\right) = \mathrm{O}\left(2^{-j}\right).$$

Here d_{TV} denotes as usual Total Variation distance between random variables, see below for details and definitions. While this result is quite straightforward given the previous computations on the asymptotic variance, it has several statistical applications for handling Gaussian random fields data, where wavelets polyspectra are widely exploited.

It is also possible to establish a more challenging result on the behaviour of excursion sets, which we expand in the L^2 sense in terms of the polyspectra. More precisely, let us define the empirical measure $\Phi_j(z)$ as follows: for all $z \in (-\infty, \infty)$ we have

$$\Phi_j(z) := \int_{S^2} \mathbb{1}_{\{\tilde{\beta}_j(x) \le z\}} \, \mathrm{d}\sigma(x),$$

where $\tilde{\beta}_j(x)$ has been normalized to have unit variance; the function $\Phi_j(z)$ provides the (random) measure of the set where $\tilde{\beta}_j$ lies below the value z. We shall hence be able to prove the following

Theorem 3. *For $\mathcal{N}$ a standard Gaussian random variable, as $j \to \infty$ we have*

$$d_W\left(\frac{\tilde{\Phi}_j(z)}{\sqrt{\operatorname{Var}[\tilde{\Phi}_j(z)]}}, \mathcal{N}\right) = \mathrm{O}\left(\frac{1}{\sqrt[4]{j}}\right).$$

Here d_W denotes Wasserstein distance between random variables. This result is close in spirit to some recent work by Viet-Hung Pham [38], who considered a Euclidean setting and traditional large-sample asymptotics; we exploit several ideas from his proof in our argument below.

1.3. *Relationship with some recent literature on random spherical eigenfunctions*

Some questions related to those considered in this paper were recently investigated in the literature for the case where the sequence of needlet fields $\{\beta_j(x)\}$ is replaced by the spherical eigenfunctions $\{f_\ell(x),\}$ e.g., focussing on

$$h_{\ell;q} := \int_{S^2} H_q\left(f_\ell(x)\right) \mathrm{d}x, \quad \ell = 1, 2, \ldots,$$

see for instance [21–23]. However, the results presented here for $\{\nu_{j;q}\}$ are qualitatively and quantitatively different from those in the literature for sequences such as $\{h_{\ell;q}\}$, and require rather independent arguments, as we shall now detail.

The crucial point to realize is that the sequence of fields $\{f_\ell(x)\}$ has a very different correlation structures from the smoothed averages $\{\beta_j(x)\}$ that we consider in this paper. In particular, normalizing variances to unity we have the correlation function

$$\operatorname{Corr}\left(f_\ell(x), f_\ell(y)\right) = P_\ell\left(\langle x, y\rangle\right),$$

$P_\ell(\cdot)$ denoting as usual Legendre polynomials. At high ℓ, Legendre polynomials can be approximated by Bessel function through the Hilb's asymptotics that is widely discussed below (e.g., $P_\ell(\cos\theta) \simeq J_0(\ell\theta) + \mathrm{O}(\ell^{-2})$); Bessel functions are known to decay slowly, indeed the following upper bound is sharp for arbitrary values of θ

$$P_\ell(\cos\theta) \leq \frac{K}{\{1 + (\ell\theta)\}^{1/2}}, \quad \text{some } K > 0.$$

The correlation behaviour of the fields $\{\beta_j(x)\}$ is very different, indeed as deeply exploited throughout the present submission we have the correlation inequality

$$\operatorname{Corr}\bigl(\beta_j(x), \beta_j(y)\bigr) \leq \frac{K_M}{\{1 + 2^j d(x, y)\}^M}, \quad \text{for all } M \in \mathbb{N}, \text{ some } K_M \text{ not depending on } j, x, y.$$

To draw an analogy with the more common case of random fields on $\mathbb{R}^d$, the eigenfunctions $\{f_\ell\}$ show some sort of long range dependent behaviour (e.g., non-integrable autocorrelation functions), while the fields $\{\beta_j(x)\}$ are characterized by much quicker decay in the correlation (which is indeed integrable), and hence, in a loose sense, they exhibit some form of short range dependence. Both these statements should be taken in a very loose sense, as these fields are defined on a compact manifold (the sphere) and hence long range/short range behaviour has a rather different meaning than usual; however this heuristic argument may provide some intuition to explain the very different behaviour we observe under the two frameworks, and the different methods of proofs which are required.

To make our comparison clearer, let us pretend that $2^j = \ell$; this identification makes sense heuristically, because β_j is obtained by averaging $\{f_\ell\}$ over multipoles such that $\frac{1}{2} \leq \frac{\ell}{2^j} \leq 2$, so 2^j can be takes as a sort of representative multipole. In this setting, we have that:

(A) the variance of nonlinear transforms of eigenfunctions $\{h_{\ell;q}\}$ has a different rate for different values of $q = 2, 3, 4$, and it stabilizes for $q \geq 5$ (see [21,22]); in particular, the variance is of order ℓ^{-1} for $q = 2$, $\log\ell/\ell^2$ for $q = 4$, ℓ^{-2} for $q = 3$ or $q \geq 5$. By contrast, the rate for the variance of $\{\nu_{j;q}(x)\}$ is equal to 2^{-2j} and indeed the same for every value of q. A heuristic explanation for this difference is as follows. In both cases (e.g, for $\{h_{\ell;q}\}$ and $\{\nu_{j;q}\}$) we are actually dealing with integrals of polynomials on the sphere. As mentioned earlier, these integrals can be evaluated exactly (by means of cubature formulae, see [3,25]) as discrete sums over approximately $\ell^2 \approx 2^{2j}$ (n, say) terms, and in this setting they can be viewed as sample means over these grid points, which are at distances of order ℓ^{-1}. Of course, under short range dependence one expects the variance of a sample mean to be of order n^{-1}, and this fits indeed with the 2^{-2j} rate for the fields $\{\beta_j(x)\}$ that we shall provide below. On the other hand, by diagram formulae, one expects the correlation function of Hermite transforms of random spherical harmonics at points x, y to be of order $\{P_\ell(\langle x, y\rangle)\}^q$; after scaling, this yields expressions of the form

$$\mathrm{O}\Biggl(\frac{1}{\ell^2}\sum_{k=1}^{\ell}\sum_{y:d(x,y)\simeq k}\bigl\{P_\ell\bigl(\langle x, y\rangle\bigr)\bigr\}^q\Biggr) \simeq \mathrm{O}\Biggl(\frac{1}{\ell^2}\sum_{k=1}^{\ell} k\frac{1}{k^{q/2}}\Biggr)$$

which are summable only for $q > 4$, where indeed one finds the "short range dependence" rate ℓ^{-2}. For smaller values of q, these sums diverge; note indeed that the number of points at distances of order k/ℓ grows linearly with k, as easily verifiable by elementary arguments (compare [3]).

(B) The rates of convergence in the Total Variation bound for Hermite transforms of random spherical harmonics (as given in [23]) are different from the one presented here, and again depends on q; for $q = 4$ it is not even algebraic but logarithmic. This is strictly related to the peculiar behaviour of the correlation functions that we described in the previous point. It should be noted that while in the present submission the exact Total Variation rate is provided for Hermite transforms of arbitrary orders, the rates currently available for transforms of random spherical harmonics are slower for $q \geq 5$, and presumably not sharp; hence the results provided in this paper are stronger.

(C) The excursion sets of $\{f_\ell\}$ have again a rather different behaviour from the one established in this paper for $\{\beta_j(x)\}$. Exact Total Variation rates have not been given so far for excursion sets of $\{f_\ell\}$; a Central Limit Theorem has been given in [21] with an entirely different technique than provided here, e.g., exploting a convenient degeneracy in the Hermite expansion, which is again similar to what was found in a long range setting by [5]. The same technique cannot be used in the setting of the present paper, due to the lack of this convenient degeneracy (again because of "short range," rather than "long range," behaviour). The techniques we exploit below are then much more complex, and much closer to some recent contribution by [38] in a Euclidean setting.

We can now turn to review the background material that we will need throughout the paper.

2. Malliavin operators and quantitative Central Limit Theorems

In a number of recent papers, summarized in the monograph by Nourdin and Peccati [26], a beautiful connection has been established between Malliavin calculus and the Stein method to prove quantitative Central Limit Theorems on functional of Gaussian subordinated random processes. In this section we review briefly some notation on isonormal Gaussian processes and Malliavin operators and we state the main results on Normal approximations on Wiener chaos, which we shall exploit in the sequel of the paper; we follow closely [27].

Let $\mathfrak{H}$ be a real separable Hilbert space, with inner product $\langle\cdot,\cdot\rangle_{\mathfrak{H}}$. An *isonormal Gaussian process* over $\mathfrak{H}$ is a collection $X = \{X(h)\colon\ h \in \mathfrak{H}\}$ of jointly Gaussian random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $\mathbb{E}[X(h)X(g)] = \langle h, g\rangle_{\mathfrak{H}}$ for every $h, g \in \mathfrak{H}$. We assume that $\mathcal{F}$ is generated by X.

If A is a Polish space (e.g. complete, metric and separable), $\mathcal{A}$ the associated σ-field and μ a positive, σ-finite and non-atomic measure, then $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ is a real separable Hilbert space with inner product $\langle g, h\rangle_{\mathfrak{H}} = \int_A g(a)h(a)\mu(\mathrm{d}a)$. For every $h \in \mathfrak{H}$ it is possible to define the isonormal Gaussian process

$$X(h) = \int_A h(a)W(\mathrm{d}a) \tag{2.1}$$

to be the Wiener–Itô integral of h with respect to the Gaussian family $W = \{W(B)\colon\ B \in \mathcal{A}, \mu(B) < \infty\}$ such that for every $B, C \in \mathcal{A}$ of finite μ-measure $\mathbb{E}[W(B)W(C)] = \mu(B \cap C)$.

Throughout this paper, we shall make extensive use of Hermite polynomials $H_q(x)$. We recall the usual definition: $H_0(x) = 1$ and, for every integer $q \geq 1$,

$$H_q(x) = (-1)^q \phi^{-1}(x)\frac{\mathrm{d}^q}{\mathrm{d}x^q}\phi(x),$$

where $\phi(x)$ is the probability density function of a standard Gaussian variable.

For each $q \geq 0$ the qth *Wiener chaos* $\mathcal{H}_q$ of X is the closed linear subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ generated by the random variables of type $H_q(X(h))$, $h \in \mathfrak{H}$ such that $\|h\|_{\mathfrak{H}} = 1$.

The following property of Hermite polynomials is useful for our discussion (for a proof see [27], Proposition 2.2.1).

Proposition 2.1. *Let $Z_1, Z_2 \sim \mathcal{N}(0,1)$ be jointly Gaussian. Then, for all $n, m \geq 0$*

$$\mathbb{E}\bigl[H_n(Z_1)H_m(Z_2)\bigr] = n!\bigl\{\mathbb{E}[Z_1Z_2]\bigr\}^n \tag{2.2}$$

if $m = n$ and $\mathbb{E}[H_n(Z_1)H_m(Z_2)] = 0$ if $n \neq m$.

The next result is the well known *Wiener–Itô* decomposition of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ (see e.g. [27], Theorem 2.2.4 for a proof). Every random variable $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ admits a unique expansion of the type

$$F = \mathbb{E}[F] + \sum_{q=1}^{\infty} F_q,$$

where $F_q \in \mathcal{H}_q$ and the series converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

We denote by $\mathfrak{H}^{\otimes q}$ and $\mathfrak{H}^{\odot q}$ the qth tensor product and the qth symmetric tensor product of $\mathfrak{H}$ respectively. In particular if $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ then $\mathfrak{H}^{\otimes q}$ can be identified with $L^2(A^q, \mathcal{A}^q, \mu^q)$. For every $1 \leq p \leq q$, $f \in L^2(A^p, \mathcal{A}^p, \mu^p)$, $g \in L^2(A^q, \mathcal{A}^q, \mu^q)$ and $r = 1, \dots, p$, the *contraction* of the elements f and g is given by

$$\begin{aligned}
&f \otimes_r g(a_1, \dots, a_{p+q-2r}) \\
&\quad = \int_{A^r} f(x_1, \dots, x_r, a_1, \dots, a_{p-r}) g(x_1, \dots, x_r, a_{p-r+1}, \dots, a_{p+q-2r})\, \mathrm{d}\mu(x_1)\cdots \mathrm{d}\mu(x_r).
\end{aligned}$$

For $p=q=r$ we have $f \otimes_r g = \langle f, g\rangle_{\mathfrak{H}^{\otimes r}}$ and for $r=0$ we have $f \otimes_0 g = f \otimes g$. Denote with $f \tilde{\otimes}_r g$ the canonical symmetrization of $f \otimes_r g$.

Let $\mathcal{S}$ be the set of smooth random variables of the form

$$f\big(X(h_1), \dots, X(h_m)\big),$$

where $m \geq 1$, $f : \mathbb{R}^m \to \mathbb{R}$ is a C^∞ function such that its partial derivatives have at most polynomial growth, and $h_1, \dots, h_m \in \mathfrak{H}$.

Let $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathfrak{H}^{\odot q})$ be the $\mathfrak{H}^{\odot q}$-valued random elements Y that are $\mathcal{F}$-measurable and such that $\mathbb{E}\|Y\|^2_{\mathfrak{H}^{\odot q}} < \infty$. For $F \in \mathcal{S}$ and $q \geq 1$, the *qth Malliavin derivative* of F with respect to X is the element of $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathfrak{H}^{\odot q})$ defined by

$$D^q F = \sum_{i_1, \dots, i_q = 1}^{m} \frac{\partial^q f}{\partial x_{i_1} \cdots \partial x_{i_q}} \big(X(h_1), \dots, X(h_m)\big) h_{i_1} \otimes \cdots \otimes h_{i_q}.$$

If $q=1$, we write D instead of D^1.

Let $q \geq 1$ be an integer. We denote by $\operatorname{Dom} \delta^q$ the subset of $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathfrak{H}^{\otimes q})$ composed of those elements u such that there exists a constant $c > 0$ satisfying

$$\big|\mathbb{E}\big[\langle D^q F, u\rangle_{\mathfrak{H}^{\otimes q}}\big]\big| \leq c\sqrt{\mathbb{E}\big[F^2\big]},$$

for all $F \in \mathcal{S}$. If $u \in \operatorname{Dom} \delta^q$, then $\delta^q(u)$ is the unique element of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ characterized by the following integration by parts formula

$$\mathbb{E}\big[F \delta^q(u)\big] = \mathbb{E}\big[\langle D^q F, u\rangle_{\mathfrak{H}^{\otimes q}}\big],$$

for all $F \in \mathcal{S}$, δ^q is the *divergence operator* of order q. Let $q \geq 1$ and $f \in \mathfrak{H}^{\odot q}$. The qth *multiple integral* of f with respect to X is defined by $I_q(f) = \delta^q(f)$. If $f \in L^2(A^q, \mathcal{A}^q, \mu^q)$ is symmetric, and we regard the Gaussian space generated by the paths of W as an isonormal Gaussian process over $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$, then

$$I_q(f) = \int_{A^q} f(a_1, \dots, a_q)\, \mathrm{d}W(a_1) \cdots \mathrm{d}W(a_q)$$

(see [27], page 39).

We state now two fundamental properties of multiple integrals that we shall exploit in the sequel. For a proof see again [27], Theorem 2.7.4 and Theorem 2.7.5. Let $q \geq 1$ and $f \in \mathfrak{H}^{\odot q}$, for all $r \geq 1$, we have

$$D^r I_q(f) = \frac{q!}{(q-r)!} I_{q-r}(f) \tag{2.3}$$

if $r \leq q$ and $D^r I_p(f) = 0$ if $r > q$. For $1 \leq q \leq p$, $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$ we have

$$\mathbb{E}\big[I_p(f) I_q(g)\big] = p! \langle f, g\rangle_{\mathfrak{H}^{\otimes p}} \tag{2.4}$$

if $p = q$ and $\mathbb{E}[I_p(f) I_q(g)] = 0$ if $p \neq q$. The linear operator I_q provides an isometry from $\mathfrak{H}^{\odot q}$ onto qth Wiener chaos $\mathcal{H}_q$ of X. In fact, let $f \in \mathfrak{H}$ be such that $\|f\|_{\mathfrak{H}} = 1$, then for any integer $q \geq 1$, we have

$$H_q\big(X(f)\big) = I_q\big(f^{\otimes q}\big), \tag{2.5}$$

see once more [27], Theorem 2.7.7. In particular, if $f \in L^2(A, \mathcal{A}, \mu)$, for $(a_1, \dots, a_q) \in A^q$, we have

$$f^{\otimes q}(a_1, \dots, a_q) = f(a_1) \cdots f(a_q).$$

The following well-known *product formula* implies, in particular, that the product of two multiple integrals is indeed a finite sum of multiple integrals. In fact for $p, q \geq 1$, $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$ we have

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{q}{r}\binom{p}{r} I_{p+q-2r}(f \tilde{\otimes}_r g). \tag{2.6}$$

For a proof see [27], Theorem 2.7.10.

We say that $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ belongs to $\operatorname{Dom} L$ if $\sum_{q=1}^{\infty} q^2 \mathbb{E}[F_q^2] < \infty$. For such an F we define the *Ornstein–Uhlenbeck operator* $LF = -\sum_{q=1}^{\infty} qF_q$. The *pseudo-inverse* of L is defined as $L^{-1}F = -\sum_{q=1}^{\infty} \frac{1}{q} F_q$.

Let $\mathcal{N}$ be a standard Gaussian random variable and define as usual the Kolmogorov, Total Variation and Wasserstein distance, between $\mathcal{N}$ and a random variable F, as

$$d_W(F,\mathcal{N}) = \sup_{h\in \mathrm{Lip}(1)} \bigl|\mathbb{E}\bigl[h(F)\bigr] - \mathbb{E}\bigl[h(\mathcal{N})\bigr]\bigr|,$$

$$d_{\mathrm{TV}}(F,\mathcal{N}) = \sup_{B\in\mathcal{B}(\mathbb{R})} \bigl|\mathbb{P}(F\in B) - \mathbb{P}(\mathcal{N}\in B)\bigr|,$$

$$d_{\mathrm{Kol}}(F,\mathcal{N}) = \sup_{z\in\mathbb{R}} \bigl|\mathbb{P}(F\leq z) - \mathbb{P}(\mathcal{N}\leq z)\bigr|.$$

The connection between stochastic calculus and probability metrics is summarized in the following proposition ([27], Theorem 5.1.3). Let $\mathbb{D}^{1,2}$ be the space of Gaussian subordinated random variables whose Malliavin derivative has finite second moment; we have that:

Proposition 2.2. *Let $F \in \mathbb{D}^{1,2}$, such that $\mathbb{E}[F] = 0$ and $\mathbb{E}[F^2] = \sigma^2 > 0$. Then*

$$d_W(F,\mathcal{N}) \leq \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \mathbb{E}\bigl[\bigl|\sigma^2 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}\bigr|\bigr].$$

Assuming that F has a density we have

$$d_{\mathrm{TV}}(F,\mathcal{N}) \leq \frac{2}{\sigma^2} \mathbb{E}\bigl[\bigl|\sigma^2 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}\bigr|\bigr],$$

$$d_{\mathrm{Kol}}(F,\mathcal{N}) \leq \frac{1}{\sigma^2} \mathbb{E}\bigl[\bigl|\sigma^2 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}\bigr|\bigr].$$

3. Needlets random fields and Wiener chaoses

As motivated earlier, in this paper we shall focus on sequences of needlet random fields, defined by a sequence of spherical random fields which can be viewed as averaged forms of the spherical eigenfunctions, e.g. they take the form

$$\beta_j(x) = \sum_{\ell=2^{j-1}}^{2^{j+1}} b\left(\frac{\ell}{2^j}\right) f_\ell(x), \qquad \tilde{\beta}_j(x) := \frac{\beta_j(x)}{\sqrt{\mathbb{E}[\beta_j^2(x)]}}, \qquad j = 1, 2, 3, \dots$$

for a weight function $b(\cdot)$ such that $b(\cdot)$ is smooth ($b(\cdot) \in C^\infty$) compactly supported in $[\frac{1}{2}, 2]$, and satisfies the partition of unity property $\sum_j b^2(\frac{\ell}{2^j}) = 1$, for all $\ell \geq 1$, see also [20].

The following property is well-known and gives an upper bound on the correlation coefficient of $\{\beta_j(\cdot)\}$, for a proof see [17], Lemma 10.8.

Proposition 3.1. *Under Condition 1, for all $M \in \mathbb{N}$, there exists a constant $K_M > 0$, not depending on j, x, and y, such that the following inequality holds*

$$\left|\mathrm{Corr}\left[\beta_j(x), \beta_j(y)\right]\right| \le \frac{K_M}{(1+2^j d(x,y))^M}, \tag{3.1}$$

where $d(x,y) = \arccos(\langle x, y\rangle)$ is the geodesic distance on the sphere.

Since $\{f_\ell(x)\}$ is Gaussian for each $x \in S^2$, then $\tilde{\beta}_j(x)$ is a standard Gaussian random variable and $\beta_j(x)$ is centred with variance

$$\mathbb{E}\left[\beta_j^2(x)\right] = \sum_{\ell=2^{j-1}}^{2^{j+1}} b^2\left(\frac{\ell}{2^j}\right) C_\ell \frac{2\ell+1}{4\pi} P_\ell\left(\langle x, x\rangle\right) = \sum_{\ell=2^{j-1}}^{2^{j+1}} b^2\left(\frac{\ell}{2^j}\right) C_\ell \frac{2\ell+1}{4\pi},$$

with $c_1 2^{j(2-\alpha)} \le \mathbb{E}[\beta_j^2(x)] \le c_2 2^{j(2-\alpha)}$. From Proposition 3.1, for the covariance function we have

$$\mathbb{E}\left[\beta_j(x)\beta_j(y)\right] = \sum_{\ell=2^{j-1}}^{2^{j+1}} b^2\left(\frac{\ell}{2^j}\right) C_\ell \frac{2\ell+1}{4\pi} P_\ell\left(\langle x, y\rangle\right) \le \frac{K_M}{(1+2^j d(x,y))^M} \sum_{\ell=2^{j-1}}^{2^{j+1}} b^2\left(\frac{\ell}{2^j}\right) C_\ell \frac{2\ell+1}{4\pi}. \tag{3.2}$$

As in [20], we exploit here the fact that the field $\{\tilde{\beta}_j(\cdot)\}$ can be expressed as an isonormal Gaussian process. Let

$$B_j = \sum_{\ell=2^{j-1}}^{2^{j+1}} b^2\left(\frac{\ell}{2^j}\right) C_\ell \frac{2\ell+1}{4\pi}$$

and for all $x \in S^2$ let us define

$$\tilde{\Theta}_j\left(\langle x, \cdot\rangle\right) := \frac{1}{\sqrt{B_j}} \sum_{\ell=2^{j-1}}^{2^{j+1}} b\left(\frac{\ell}{2^j}\right) \sqrt{C_\ell} \frac{2\ell+1}{4\pi} P_\ell\left(\langle x, \cdot\rangle\right) =: \frac{1}{\sqrt{B_j}} \Theta_j\left(\langle x, \cdot\rangle\right). \tag{3.3}$$

We have that $\tilde{\Theta}_j(\langle x, \cdot\rangle)$ is in the Hilbert space $\mathfrak{H} = L^2(S^2, \mathrm{d}\sigma(y))$ and we can represent $\{\tilde{\beta}_j(\cdot)\}$ as

$$\tilde{\beta}_j(x) = \int_{S^2} \tilde{\Theta}_j\left(\langle x, y\rangle\right) W\left(\mathrm{d}\sigma(y)\right), \quad x \in S^2,$$

where W is Gaussian white noise on the sphere as in formula (2.1). In fact the covariance function is given by

$$\begin{aligned}
\tilde{\rho}_j\left(\langle x, y\rangle\right) &:= \mathbb{E}\left[\tilde{\beta}_j(x)\tilde{\beta}_j(y)\right] = \left\langle \tilde{\Theta}_j\left(\langle x, z\rangle\right)\tilde{\Theta}_j\left(\langle z, y\rangle\right)\right\rangle_{\mathfrak{H}} = \int_{S^2} \tilde{\Theta}_j\left(\langle x, z\rangle\right)\tilde{\Theta}_j\left(\langle z, y\rangle\right) \mathrm{d}\sigma(z) \\
&= \frac{1}{B_j} \sum_{\ell=2^{j-1}}^{2^{j+1}} b^2\left(\frac{\ell}{2^j}\right) C_\ell \frac{2\ell+1}{4\pi} P_\ell\left(\langle x, y\rangle\right) =: \frac{1}{B_j} \rho_j\left(\langle x, y\rangle\right).
\end{aligned} \tag{3.4}$$

It follows immediately that the transformed process $\{H_q(\tilde{\beta}_j(\cdot))\}$ belongs to the qth order Wiener chaos generated by the Gaussian measure governing f_ℓ and so does any linear transform including

$$\nu_{j;q} = \int_{S^2} H_q\left(\tilde{\beta}_j(x)\right) \mathrm{d}\sigma(x).$$

Let $\mathbb{1}_{\{\cdot\}}$ be the usual the indicator function, clearly $\mathbb{1}_{\{\tilde{\beta}_j(x)\leq z\}}$ belongs for each x and $z\in S^2$ to the L^2 space of square integrable functions of Gaussian random variables and we can write

$$\mathbb{1}_{\{\tilde{\beta}_j(x)\leq z\}} = \sum_{q=0}^{\infty} \frac{\mathcal{J}_q(z)}{q!} H_q\big(\tilde{\beta}_j(x)\big),$$

where the right hand side converges in the L^2 sense i.e.

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{q=N}^{\infty} \frac{\mathcal{J}_q(z)}{q!} H_q\big(\tilde{\beta}_j(x)\big)\right]^2 = 0$$

uniformly w.r.t. x, z. It is possible to provide analytic expressions of the coefficients $\{\mathcal{J}_q(\cdot)\}$, indeed for $q\geq 1$

$$\mathcal{J}_q(z) = \int_{\mathbb{R}} \mathbb{1}_{(u\leq z)} H_q(u)\phi(u)\,\mathrm{d}u = -H_{q-1}(z)\phi(z)$$

and $\mathcal{J}_0(z)=\Phi(z)$ where ϕ, Φ denote, respectively, the density and the cumulative distribution function of the standard Gaussian (see [21,23]). Let us define the empirical measure $\varPhi_j(z)$ as follows: for all $z\in(-\infty,\infty)$ we have

$$\varPhi_j(z) := \int_{S^2} \mathbb{1}_{\{\tilde{\beta}_j(x)\leq z\}}\,\mathrm{d}\sigma(x).$$

The function $\varPhi_j(z)$ provides the (random) measure of the set where $\tilde{\beta}_j$ lies below the value z. The value $\varPhi_j(z)$ at $z=0$ is related to the so-colled defect (or 'signed area') of the function $\tilde{\beta}_j : S^2\to\mathbb{R}$, which is defined by

$$\mathcal{D}_j := \operatorname{meas}\big(\tilde{\beta}_j^{-1}(0,\infty)\big) - \operatorname{meas}\big(\tilde{\beta}_j^{-1}(-\infty,0)\big)$$

and is hence the difference between the areas of positive and negative inverse image of $\tilde{\beta}_j$. By a straightforward transformation we have $\mathcal{D}_j = 4\pi - 2\varPhi_j(0)$. Instead $4\pi - \varPhi_j(z)$ provides the area of the excursion set $\{x\colon\ \tilde{\beta}_j(x) > z\}$.

4. On the variance of $\nu_{j;q}$

In this section we obtain, for all fixed $q\geq 2$, the explicit value for the limit of $2^{2j}\operatorname{Var}[\nu_{j;q}]$ as $j\to\infty$.

Theorem 4. *For $q>4$, we have*

$$\lim_{j\to\infty} 2^{2j}\operatorname{Var}[\nu_{j;q}] = q!c_q,$$

where

$$c_q = \frac{8\pi^2}{(\int_{1/2}^2 b^2(x)x^{1-\alpha}\,\mathrm{d}x)^q} \int_{1/2}^2 \cdots \int_{1/2}^2 \int_0^\infty \prod_{k=1}^q b^2(x_k)x_k^{1-\alpha} J_0(x_k\psi)\psi\,\mathrm{d}\psi\,\mathrm{d}x_1\cdots\mathrm{d}x_q.$$

Remark 4.1. *It is obvious that $c_q\geq 0$ for all $q>0$. In the sequel, se shall assume that the inequality is strict when needed, e.g., in Theorem 8.*

Our proof is close to the argument by [22]; in particular let us start by recalling the following fact on the asymptotic behaviour of Legendre polynomials (see for instance [36,40,41]).

Lemma 4.1 (Hilb's asymptotics). *For any $\varepsilon > 0$, $C > 0$ we have*

$$P_{\ell_k}(\cos\theta) = \left(\frac{\theta}{\sin\theta}\right)^{1/2} J_0\big((\ell_k + 1/2)\theta\big) + \delta_k(\theta),$$

where

$$\delta_k(\theta) \ll \begin{cases} \theta^2, & 0 < \theta < 1/\ell_k, \\ \theta^{1/2}\ell_k^{-3/2}, & \theta > 1/\ell_k \end{cases}$$

uniformly w.r.t. $\ell_k \geq 1$ and $\theta \in [0, \pi - \varepsilon]$.

Lemma 4.2. *Let $q > 4$. For $\ell = 2^j$, $\ell_k \in [2^{j-1}, 2^{j+1}]$ where $k = 1, \dots, q$, as $j \to \infty$, we have*

$$\ell^2 \int_0^{\pi/2} P_{\ell_1}(\cos\theta) \cdots P_{\ell_q}(\cos\theta) \sin\theta \, \mathrm{d}\theta = \int_0^{\ell\pi/2} \prod_{k=1}^q J_0\left(\frac{\ell_k + 1/2}{\ell}\psi\right)\psi \, \mathrm{d}\psi + \mathrm{O}\left(\frac{1}{\sqrt{\ell}}\right). \tag{4.1}$$

Proof. From Lemma 4.1 we have

$$\begin{aligned}
&\int_0^{\pi/2} P_{\ell_1}(\cos\theta) \cdots P_{\ell_q}(\cos\theta) \sin\theta \, \mathrm{d}\theta \\
&\quad = \int_0^{\pi/2} \prod_{k=1}^q \left[\left(\frac{\theta}{\sin\theta}\right)^{1/2} J_0\big((\ell_k + 1/2)\theta\big) + \delta_k(\theta)\right] \sin\theta \, \mathrm{d}\theta \\
&\quad = \int_0^{\pi/2} \Bigg[\prod_{k=1}^q \delta_k(\theta) + \sum_{k=1}^q \left(\frac{\theta}{\sin\theta}\right)^{1/2} J_0\big((\ell_k + 1/2)\theta\big) \prod_{k' \neq k} \delta_{k'}(\theta) + \cdots \\
&\qquad + \left(\frac{\theta}{\sin\theta}\right)^{q/2} \prod_{k=1}^q J_0\big((\ell_k + 1/2)\theta\big)\Bigg] \sin\theta \, \mathrm{d}\theta.
\end{aligned}$$

Let, for $k = 1, \dots, q$,

$$\begin{aligned}
A_{q-k,k} &:= \int_0^{\pi/2} \left(\frac{\theta}{\sin\theta}\right)^{k/2} \prod_{m=1}^k J_0\big((\ell_m + 1/2)\theta\big) \prod_{m'=k+1}^q \delta_{m'}(\theta) \sin\theta \, \mathrm{d}\theta, \\
A_{q,0} &:= \int_0^{\pi/2} \prod_{m'=1}^q \delta_{m'}(\theta) \sin\theta \, \mathrm{d}\theta.
\end{aligned}$$

• For $k = q$, with the change of variable $\psi = \ell\theta$, we have

$$\begin{aligned}
A_{0,q} &= \frac{1}{\ell} \int_0^{\ell\pi/2} \left(\frac{\psi/\ell}{\sin(\psi/\ell)}\right)^{q/2} \prod_{m=1}^q J_0\left(\frac{\ell_m + 1/2}{\ell}\psi\right) \sin(\psi/\ell) \, \mathrm{d}\psi \\
&= \frac{1}{\ell^2} \int_0^{\ell\pi/2} \left(\frac{\psi/\ell}{\sin(\psi/\ell)}\right)^{q/2-1} \prod_{m=1}^q J_0\left(\frac{\ell_m + 1/2}{l}\psi\right) \psi \, \mathrm{d}\psi.
\end{aligned}$$

For $\psi \in [0, \ell\pi/2]$, we write $(\frac{\psi/\ell}{\sin(\psi/\ell)})^{q/2-1} = 1 + \mathrm{O}(\frac{\psi^2}{\ell^2})$, that is

$$A_{0,q} = \frac{1}{\ell^2} \int_0^{\ell\pi/2} \prod_{m=1}^q J_0\left(\frac{\ell_m + 1/2}{\ell}\psi\right)\psi \, \mathrm{d}\psi + \mathrm{O}\left(\frac{1}{\ell^4} \int_0^{\ell\pi/2} \prod_{m=1}^q J_0\left(\frac{\ell_m + 1/2}{\ell}\psi\right)\psi^3 \, \mathrm{d}\psi\right).$$

We consider now the error term. Since for $x \in [0,2]$ we have $J_0(x) \in (0,1]$, if $\varepsilon = \frac{2}{2+1/2}$, for $\psi \in (0,\varepsilon]$ we have $J_0(\frac{\ell_m+1/2}{\ell}\psi) \in (0,1]$. Recalling that $|J_0(x)| \le x^{-1/2}$, we have

$$\begin{aligned}
&\int_0^{\ell\pi/2} \prod_{m=1}^q \left| J_0\left(\frac{\ell_m+1/2}{\ell}\psi\right)\right| \psi^3 \,\mathrm{d}\psi \\
&\quad = \int_0^{\varepsilon} \prod_{m=1}^q \left| J_0\left(\frac{\ell_m+1/2}{\ell}\psi\right)\right| \psi^3 \,\mathrm{d}\psi + \prod_{m=1}^q \left(\frac{\ell_m+1/2}{\ell}\right)^{-1/2} \int_\varepsilon^{\ell\pi/2} \psi^{3-q/2}\,\mathrm{d}\psi \\
&\quad \le \varepsilon^4 + \prod_{m=1}^q \left(\frac{\ell_m+1/2}{\ell}\right)^{-1/2} \times \begin{cases} \frac{1}{8(q-8)}(16\varepsilon^{4-q/2} - 2^{q/2}(\ell\pi)^{4-q/2}) & \text{if } q \ne 8, \\ \log(\frac{\ell\pi}{2}) - \log(\varepsilon) & \text{if } q = 8, \end{cases}
\end{aligned}$$

so that

$$A_{0,q} = \frac{1}{\ell^2}\int_0^{\ell\pi/2} \prod_{m=1}^q J_0\left(\frac{\ell_m+1/2}{\ell}\psi\right)\psi\,\mathrm{d}\psi + \begin{cases} \mathrm{O}(\ell^{-4}+\ell^{-q/2}) & \text{if } q\ne 8, \\ \mathrm{O}(\ell^{-4}+\ell^{-4}\log(\frac{\ell\pi}{2})) & \text{if } q = 8. \end{cases}$$

• For $A_{q,0}$, since, in view of Lemma 4.1, $\delta_m(\theta) \ll \theta^{1/2}\ell_m^{-3/2}$, we obtain

$$A_{q,0} = \int_0^{\pi/2} \prod_{m'=1}^q \delta_{m'}(\theta)\sin\theta\,\mathrm{d}\theta \ll \left(\frac{1}{2^{j-1}}\right)^{(3/2)q} \int_0^{\pi/2} \theta^{q/2}\sin\theta\,\mathrm{d}\theta = \mathrm{O}\big(\ell^{-(3/2)q}\big).$$

• For $k = 1,\dots,q-1$,

$$\begin{aligned}
A_{q-k,k} &\ll \left(\frac{1}{2^{j-1}}\right)^{(3/2)(q-k)} \int_0^{\pi/2} \theta^{(1/2)(q-k)} \left(\frac{\theta}{\sin\theta}\right)^{k/2} \prod_{m=1}^k J_0\big((\ell_m+1/2)\theta\big)\sin\theta\,\mathrm{d}\theta \\
&= \left(\frac{1}{2^{j-1}}\right)^{(3/2)(q-k)} \left(\frac{\pi}{2}\right)^{(1/2)(q-k)} A_{0,k}.
\end{aligned}$$

□

Remark 4.2. *Note that formula (4.1) is meaningful only if $\ell_1,\dots,\ell_q$ satisfy the following "polygonal" conditions, i.e., for $q \ge 4$ and for all $k = 1,\dots,q$,*

$$\ell_k \le \ell_1 + \cdots + \ell_{k-1} + \ell_{k+1} + \cdots + \ell_q, \tag{4.2}$$

while otherwise we have

$$\int_0^{\pi/2} P_{\ell_1}(\cos\theta)\cdots P_{\ell_q}(\cos\theta)\sin\theta\,\mathrm{d}\theta = 0.$$

We exploit Lemma 4.2 to prove the following:

Lemma 4.3. *For $\ell = 2^j$, $q > 4$ and $\tilde\gamma(\lfloor \ell x_k\rfloor,\ell) = b^2(\frac{\lfloor \ell x_k\rfloor}{\ell})\frac{2\lfloor \ell x_k\rfloor+1}{4\pi\ell}(\frac{\lfloor \ell x_k\rfloor}{\ell})^{-\alpha}G(\lfloor \ell x_k\rfloor)$, we have*

$$\begin{aligned}
&\lim_{\ell\to\infty} \ell^2 \int_{1/2}^2 \cdots \int_{1/2}^2 \int_0^{\pi/2} \prod_{k=1}^q \tilde\gamma\big(\lfloor \ell x_k\rfloor,\ell\big) P_{\lfloor \ell x_k\rfloor}(\cos\theta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}x_1\cdots\mathrm{d}x_q \\
&\quad = \left(\frac{G}{2\pi}\right)^q \int_{1/2}^2\cdots\int_{1/2}^2\int_0^\infty \prod_{k=1}^q b^2(x_k)x_k^{1-\alpha} J_0(x_k\psi)\psi\,\mathrm{d}\psi\,\mathrm{d}x_1\cdots\mathrm{d}x_q.
\end{aligned}$$

Proof. From Lemma 4.2, we have

$$\int_{1/2}^{2}\cdots\int_{1/2}^{2}\lim_{\ell\to\infty}\ell^2\prod_{k=1}^{q}\tilde{\gamma}\big(\lfloor \ell x_k\rfloor,\ell\big)\int_0^{\pi/2}\prod_{k=1}^{q}P_{\lfloor \ell x_k\rfloor}(\cos\theta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}x_1\cdots\mathrm{d}x_q$$
$$=\int_{1/2}^{2}\cdots\int_{1/2}^{2}\lim_{\ell\to\infty}\prod_{k=1}^{q}\tilde{\gamma}\big(\lfloor \ell x_k\rfloor,\ell\big)\int_0^{\ell\pi/2}\prod_{k=1}^{q}J_0\biggl(\frac{\lfloor \ell x_k\rfloor+1/2}{\ell}\psi\biggr)\psi\,\mathrm{d}\psi\,\mathrm{d}x_1\cdots\mathrm{d}x_q.$$

Set $v(\ell,x_1,\dots,x_q)=\int_0^{\ell\pi/2}\prod_{k=1}^{q}J_0(\frac{\lfloor \ell x_k\rfloor+1/2}{\ell}\psi)\psi\,\mathrm{d}\psi$, by dominated convergence we obtain that

$$\lim_{\ell\to\infty}v(\ell,x_1,\dots,x_q)$$
$$=\lim_{\ell\to\infty}\int_0^{\infty}\prod_{k=1}^{q}J_0\biggl(\frac{\lfloor \ell x_k\rfloor+1/2}{\ell}\psi\biggr)\psi\,\mathrm{d}\psi-\lim_{\ell\to\infty}\int_{2\pi}^{\infty}\prod_{k=1}^{q}J_0\biggl(\frac{\lfloor \ell x_k\rfloor+1/2}{\ell}\psi\biggr)\psi\mathbb{1}_{\{\psi\in[\ell\pi/2,\infty)\}}\,\mathrm{d}\psi$$
$$=\int_0^{\infty}\prod_{k=1}^{q}J_0(x_k\psi)\psi\,\mathrm{d}\psi$$

in fact, there exists a finite real number M such that

$$\biggl|\prod_{k=1}^{q}J_0\biggl(\frac{\lfloor \ell x_k\rfloor+1/2}{\ell}\psi\biggr)\psi\mathbb{1}_{\{\psi\in[\ell\pi/2,\infty)\}}\biggr|\le\biggl|\prod_{k=1}^{q}J_0\biggl(\frac{\lfloor \ell x_k\rfloor+1/2}{\ell}\psi\biggr)\psi\biggr|$$
$$\le\begin{cases}\varepsilon & \text{if } \psi\in[0,\varepsilon],\\ \prod_{k=1}^{q}(\frac{\ell}{\lfloor \ell x_k\rfloor+1/2})^{1/2}\psi^{1-q/2}<M & \text{if } \psi\in[\varepsilon,\infty].\end{cases}$$

This leads to

$$\lim_{\ell\to\infty}\prod_{k=1}^{q}\tilde{\gamma}\big(\lfloor \ell x_k\rfloor,\ell\big)v(\ell,x_1,\dots,x_q)=\biggl(\frac{G}{2\pi}\biggr)^q\int_0^{\infty}\prod_{k=1}^{q}b^2(x_k)x_k^{1-\alpha}J_0(x_k\psi)\psi\,\mathrm{d}\psi.$$

On the other hand, we apply again dominated convergence to the sequence of measurable functions

$$u_\ell(x_1,\dots,x_q)=\prod_{k=1}^{q}\tilde{\gamma}\big(\lfloor \ell x_k\rfloor,\ell\big)\int_0^{\pi/2}\prod_{k=1}^{q}P_{\lfloor \ell x_k\rfloor}(\cos\theta)\sin\theta\,\mathrm{d}\theta$$

on the set $[\frac{1}{2},2]^q$. Since, from Lemma 4.2, for all ℓ and all $(x_1,\dots,x_q)\in[\frac{1}{2},2]^q$, we have

$$\big|u_\ell(x_1,\dots,x_q)\big|\le\prod_{k=1}^{q}\big|\tilde{\gamma}\big(\lfloor \ell x_k\rfloor,\ell\big)\big|\biggl|\int_0^{\pi/2}\prod_{k=1}^{q}P_{\lfloor \ell x_k\rfloor}(\cos\theta)\sin\theta\,\mathrm{d}\theta\biggr|$$
$$\le\prod_{k=1}^{q}\big|\tilde{\gamma}\big(\lfloor \ell x_k\rfloor,\ell\big)\big|\biggl|\int_0^{\ell\pi/2}\prod_{k=1}^{q}J_0\biggl(\frac{\lfloor \ell x_k\rfloor+1/2}{\ell}\psi\biggr)\psi\,\mathrm{d}\psi+1\biggr|$$
$$\le\prod_{k=1}^{q}\big|\tilde{\gamma}\big(\lfloor \ell x_k\rfloor,\ell\big)\big|\biggl[\int_0^{\varepsilon}\psi\,\mathrm{d}\psi+\int_{\varepsilon}^{\ell\pi/2}\prod_{k=1}^{q}\biggl(\frac{\ell}{\lfloor \ell x_k\rfloor+1/2}\biggr)^{1/2}\psi^{1-q/2}\,\mathrm{d}\psi+1\biggr],$$

where $\varepsilon=\frac{2}{2+1/2}$, there exists a finite real number M' such that for all ℓ and for all $(x_1,\dots,x_q)\in[\frac{1}{2},2]^q$

$$\prod_{k=1}^{q}\big|\tilde{\gamma}\big(\lfloor \ell x_k\rfloor,\ell\big)\big|\biggl[\varepsilon^2+\prod_{k=1}^{q}\biggl(\frac{\ell}{\lfloor \ell x_k\rfloor+1/2}\biggr)^{1/2}\frac{4\varepsilon^{2-q/2}-2^{q/2}(\ell\pi)^{2-q/2}}{2(q-4)}+1\biggr]\le M'$$

and this leads to

$$\int_{1/2}^{2} \cdots \int_{1/2}^{2} \lim_{\ell\to\infty} u_\ell(x_1,\dots,x_q)\,\mathrm{d}x_1\cdots \mathrm{d}x_q = \lim_{\ell\to\infty} \int_{1/2}^{2} \cdots \int_{1/2}^{2} u_\ell(x_1,\dots,x_q)\,\mathrm{d}x_1\cdots \mathrm{d}x_q. \qquad \square$$

Remark 4.3. *The previous discussion yields the following corollary: for $q > 4$, $\ell = 2^j$, $x_k \in [\frac{1}{2}, 2]$ with $k = 1,\dots,q$, we have*

$$\lim_{\ell\to\infty} \ell^2 \int_0^{\pi/2} P_{\lfloor \ell x_1\rfloor}(\cos\theta)\cdots P_{\lfloor \ell x_q\rfloor}(\cos\theta)\sin\theta\,\mathrm{d}\theta = \int_0^\infty J_0(x_1\psi)\cdots J_0(x_q\psi)\psi\,\mathrm{d}\psi.$$

For $q = 3$ it is well-known that, if $x_1, x_2, x_3 > 0$, we have

$$\int_0^\infty J_0(x_1\psi)J_0(x_2\psi)J_0(x_3\psi)\psi\,\mathrm{d}\psi = \begin{cases} \frac{1}{2\pi\Delta}, & \text{if } |x_1-x_2| < x_3 < x_1+x_2, \\ 0, & \text{if } 0 < x_3 \le |x_1-x_2| \text{ or } x_3 \ge x_1+x_2, \end{cases}$$

where $\Delta = \frac{1}{4}\sqrt{[x_3^2-(x_1-x_2)^2][(x_1+x_2)^2-x_3^2]}$ is equal to the area of a triangle whose sides are x_1, x_2 and x_3, see [10], *formula* 6.578.9.

Before proving Theorem 4, we introduce some further notation i.e.

$$B_\ell = \sum_{\ell_1=2^{j-1}}^{2^{j+1}} b^2\left(\frac{\ell_1}{\ell}\right)\frac{2\ell_1+1}{4\pi}\ell_1^{-\alpha}G(\ell_1),$$

and we prove the last lemma:

Lemma 4.4. *For $\ell = 2^j$, we have that*

$$\lim_{\ell\to\infty} \ell^{\alpha-2}B_\ell = \frac{G}{2\pi}\int_{1/2}^{2} b^2(x)x^{1-\alpha}\,\mathrm{d}x.$$

Proof. We first note that

$$\begin{aligned}\lim_{\ell\to\infty} \ell^{\alpha-2}B_\ell &= \lim_{\ell\to\infty}\frac{\ell}{\ell^{2-\alpha}}\sum_{\ell_1=\ell/2}^{2\ell}\int_{\ell_1/\ell}^{(\ell_1+1)/\ell} b^2\left(\frac{\lfloor \ell x\rfloor}{\ell}\right)\lfloor \ell x\rfloor^{-\alpha}\frac{2\lfloor \ell x\rfloor+1}{4\pi}G\bigl(\lfloor \ell x\rfloor\bigr)\,\mathrm{d}x \\ &= \lim_{\ell\to\infty}\int_{1/2}^{2} b^2\left(\frac{\lfloor \ell x\rfloor}{\ell}\right)\left(\frac{\lfloor \ell x\rfloor}{\ell}\right)^{-\alpha}\frac{2\lfloor \ell x\rfloor+1}{2\ell}\frac{G(\lfloor \ell x\rfloor)}{2\pi}\,\mathrm{d}x,\end{aligned}$$

and using dominated convergence, we have the statement. $\square$

Proof of Theorem 4.

$$\operatorname{Var}[\nu_{j;q}] = \mathbb{E}\left[\left(\int_{S^2} H_q\bigl(\tilde\beta_j(x)\bigr)\,\mathrm{d}x\right)^2\right] = \int_{S^2\times S^2}\mathbb{E}\bigl[H_q\bigl(\tilde\beta_j(x_1)\bigr)H_q\bigl(\tilde\beta_j(x_2)\bigr)\bigr]\,\mathrm{d}\sigma(x_1)\,\mathrm{d}\sigma(x_2)$$

by Proposition 2.1, for $\ell = 2^j$, we have

$$\begin{aligned}\operatorname{Var}[\nu_{j;q}] &= q!\int_{S^2\times S^2}\bigl\{\mathbb{E}\bigl[\tilde\beta_j(x_1)\tilde\beta_j(x_2)\bigr]\bigr\}^q\,\mathrm{d}\sigma(x_1)\,\mathrm{d}\sigma(x_2) = q!B_\ell^{-q}\int_{S^2\times S^2}\bigl\{\mathbb{E}\bigl[\beta_j(x_1)\beta_j(x_2)\bigr]\bigr\}^q\,\mathrm{d}\sigma(x_1)\,\mathrm{d}\sigma(x_2) \\ &= q!B_\ell^{-q}\int_{S^2\times S^2}\left\{\sum_{\ell_1=2^{j-1}}^{2^{j+1}} b^2\left(\frac{\ell_1}{\ell}\right)\frac{2\ell_1+1}{4\pi}\ell_1^{-\alpha}G(\ell_1)P_{\ell_1}\bigl(\langle x_1,x_2\rangle\bigr)\right\}^q\,\mathrm{d}\sigma(x_1)\,\mathrm{d}\sigma(x_2).\end{aligned}$$

Let $\tilde{\gamma}(\ell_k,\ell) := b^2(\frac{\ell_k}{\ell})\frac{2\ell_k+1}{4\pi\ell}(\frac{\ell_k}{\ell})^{-\alpha}G(\ell_k)$ where, for all $k=1,\dots,q$, $\ell_k \in [2^{j-1},2^{j+1}]$; we have

$$\operatorname{Var}[\nu_{j;q}] = q!\ell^{-\alpha q+q}B_\ell^{-q}\sum_{\ell_1\cdots\ell_q}\tilde{\gamma}(\ell_1,\ell)\cdots\tilde{\gamma}(\ell_q,\ell)$$
$$\times\int_{S^2\times S^2}P_{\ell_1}\bigl(\langle x_1,x_2\rangle\bigr)\cdots P_{\ell_q}\bigl(\langle x_1,x_2\rangle\bigr)\,\mathrm{d}\sigma(x_1)\,\mathrm{d}\sigma(x_2),$$

where

$$\int_{S^2\times S^2}P_{\ell_1}\bigl(\langle x_1,x_2\rangle\bigr)\cdots P_{\ell_q}\bigl(\langle x_1,x_2\rangle\bigr)\,\mathrm{d}\sigma(x_1)\,\mathrm{d}\sigma(x_2) = 8\pi^2\int_0^\pi P_{\ell_1}(\cos\theta)\cdots P_{\ell_q}(\cos\theta)\sin\theta\,\mathrm{d}\theta.$$

Then

$$\begin{aligned}\operatorname{Var}[\nu_{j;q}] &= q!8\pi^2\ell^{-\alpha q+q}B_\ell^{-q}\sum_{\ell_1\cdots\ell_q}\tilde{\gamma}(\ell_1,\ell)\cdots\tilde{\gamma}(\ell_q,\ell)\int_0^\pi P_{\ell_1}(\cos\theta)\cdots P_{\ell_q}(\cos\theta)\sin\theta\,\mathrm{d}\theta\\ &= q!8\pi^2\ell^{-\alpha q+q}B_\ell^{-q}\sum_{\substack{\ell_1\cdots\ell_q\\ \sum l_k\ \text{even}}}\tilde{\gamma}(\ell_1,\ell)\cdots\tilde{\gamma}(\ell_q,\ell)2\int_0^{\pi/2} P_{\ell_1}(\cos\theta)\cdots P_{\ell_q}(\cos\theta)\sin\theta\,\mathrm{d}\theta\end{aligned}$$

since

$$\begin{aligned}&\int_0^\pi P_{\ell_1}(\cos\theta)\cdots P_{\ell_q}(\cos\theta)(\cos\theta)\sin\theta\,\mathrm{d}\theta\\ &\quad=\begin{cases}2\int_0^{\pi/2}P_{\ell_1}(\cos\theta)\cdots P_{\ell_q}(\cos\theta)\sin\theta\,\mathrm{d}\theta, & \text{for } \sum_{k=1}^q l_k \text{ even},\\ 0, & \text{for } \sum_{k=1}^q l_k \text{ odd}.\end{cases}\end{aligned}$$

Also

$$\begin{aligned}\operatorname{Var}[\nu_{j;q}] &= q!8\pi^2\ell^{-\alpha q+q}B_\ell^{-q}\sum_{\ell_1\cdots\ell_q}\tilde{\gamma}(\ell_1,\ell)\cdots\tilde{\gamma}(\ell_q,\ell)\int_0^{\pi/2} P_{\ell_1}(\cos\theta)\cdots P_{\ell_q}(\cos\theta)\sin\theta\,\mathrm{d}\theta\\ &= q!8\pi^2\ell^{-\alpha q+2q}B_\ell^{-q}\sum_{\ell_1=\ell/2}^{2\ell}\cdots\sum_{\ell_q=\ell/2}^{2\ell}\int_{\ell_1/\ell}^{(\ell_1+1)/\ell}\cdots\int_{\ell_q/\ell}^{(\ell_q+1)/\ell}\tilde{\gamma}\bigl(\lfloor\ell x_1\rfloor,\ell\bigr)\cdots\tilde{\gamma}\bigl(\lfloor\ell x_q\rfloor,\ell\bigr)\\ &\quad\times\int_0^\pi P_{\lfloor\ell x_1\rfloor}(\cos\theta)\cdots P_{\lfloor\ell x_q\rfloor}(\cos\theta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}x_1\cdots\mathrm{d}x_q\\ &= q!8\pi^2\ell^{-\alpha q+2q}B_\ell^{-q}\int_{1/2}^{(2\ell+1)/\ell}\cdots\int_{1/2}^{(2\ell+1)/\ell}\tilde{\gamma}\bigl(\lfloor\ell x_1\rfloor,\ell\bigr)\cdots\tilde{\gamma}\bigl(\lfloor\ell x_q\rfloor,\ell\bigr)\\ &\quad\times\int_0^\pi P_{\lfloor\ell x_1\rfloor}(\cos\theta)\cdots P_{\lfloor\ell x_q\rfloor}(\cos\theta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}x_1\cdots\mathrm{d}x_q\end{aligned}$$

and then

$$\begin{aligned}\lim_{\ell\to\infty}\ell^2\operatorname{Var}[\nu_{j;q}] &= \lim_{\ell\to\infty}q!8\pi^2\ell^{-\alpha q+2q}B_\ell^{-q}\ell^2\int_{1/2}^2\cdots\int_{1/2}^2\tilde{\gamma}\bigl(\lfloor\ell x_1\rfloor,\ell\bigr)\cdots\tilde{\gamma}\bigl(\lfloor\ell x_q\rfloor,\ell\bigr)\\ &\quad\times\int_0^\pi P_{\lfloor\ell x_1\rfloor}(\cos\theta)\cdots P_{\lfloor\ell x_q\rfloor}(\cos\theta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}x_1\cdots\mathrm{d}x_q.\end{aligned}$$

The statement follows by applying Lemma 4.3 and Lemma 4.4. □

For the cases $q=2,3,4$ we write a different proof based on the representation of the integral of the product of spherical harmonics in terms of Wigner's $3j$ coefficients.

Theorem 5. *For $q=2$, we have*

$$\lim_{j\to\infty} 2^{2j}\operatorname{Var}[\nu_{j;2}] = 2!c_2,$$

where

$$c_2 = \frac{8\pi^2}{(\int_{1/2}^2 b^2(x)x^{1-\alpha}\,\mathrm{d}x)^2}\int_{1/2}^2 b^4(x_1)x_1^{1-2\alpha}\,\mathrm{d}x_1.$$

Proof. For $\ell=2^j$ and $\ell_1,\ell_2\in[2^{j-1},2^{j+1}]$ we have as before

$$\begin{aligned}\operatorname{Var}[\nu_{j;2}] &= 2!B_\ell^{-2}\int_{S^2\times S^2}\bigl\{\mathbb{E}\bigl[\beta_j(x_1)\beta_j(x_2)\bigr]\bigr\}^2\,\mathrm{d}\sigma(x_1)\,\mathrm{d}\sigma(x_2)\\ &= 2!8\pi^2\ell^{-2\alpha+2}B_\ell^{-2}\sum_{\ell_1\ell_2} b^2\biggl(\frac{\ell_1}{\ell}\biggr)b^2\biggl(\frac{\ell_2}{\ell}\biggr)\frac{2\ell_1+1}{4\pi\ell}\frac{2\ell_2+1}{4\pi\ell}\biggl(\frac{\ell_1\ell_2}{\ell^2}\biggr)^{-\alpha}G(\ell_1)G(\ell_2)\\ &\quad\times\int_0^\pi P_{\ell_1}(\cos\theta)P_{\ell_2}(\cos\theta)\sin\theta\,\mathrm{d}\theta,\end{aligned}$$

from the orthogonality property of Legendre polynomials, we have

$$\begin{aligned}\operatorname{Var}[\nu_{j;2}] &= 2!8\pi^2\ell^{-2\alpha+2}B_\ell^{-2}\sum_{\ell_1=\ell/2}^{2\ell} b^4\biggl(\frac{\ell_1}{\ell}\biggr)\biggl(\frac{2\ell_1+1}{4\pi\ell}\biggr)^2\biggl(\frac{\ell_1}{\ell}\biggr)^{-2\alpha}G^2(\ell_1)\frac{2}{2\ell_1+1}\\ &= 2!8\pi^2\ell^{-2\alpha+2}B_\ell^{-2}\int_{1/2}^{(2\ell+1)/\ell} b^4\biggl(\frac{\lfloor\ell x_1\rfloor}{\ell}\biggr)\frac{2\lfloor\ell x_1\rfloor+1}{2\ell}\biggl(\frac{\lfloor\ell x_1\rfloor}{\ell}\biggr)^{-2\alpha}\biggl(\frac{G(\lfloor\ell x_1\rfloor)}{2\pi}\biggr)^2\,\mathrm{d}x_1.\end{aligned}$$

So we see that

$$\begin{aligned}&\lim_{\ell\to\infty}\ell^2\operatorname{Var}[\nu_{j;2}]\\ &\quad= \lim_{\ell\to\infty} 2!8\pi^2\ell^{-2\alpha+2}B_\ell^{-2}\ell^2\int_{1/2}^2 b^4\biggl(\frac{\lfloor\ell x_1\rfloor}{\ell}\biggr)\frac{2\lfloor\ell x_1\rfloor+1}{2\ell}\biggl(\frac{\lfloor\ell x_1\rfloor}{\ell}\biggr)^{-2\alpha}\biggl(\frac{G(\lfloor\ell x_1\rfloor)}{2\pi}\biggr)^2\,\mathrm{d}x_1\end{aligned}$$

and by applying Lemma 4.4 and dominated convergence we arrive at the statement. □

We introduce now the Wigner's $3j$ coefficients

$$\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix},\quad -(2\ell_i+1)\le m_i\le 2\ell_i+1, i=1,2,3.$$

The Wigner's $3j$ coefficients are zero unless the triangle conditions $|\ell_i-\ell_r|\le\ell_k\le\ell_i+\ell_r$ for $i,r,k=1,2,3$ are satisfied and $m_1+m_2+m_3=0$, see [17], Section 3.5.3 for further details. When $m_1=m_2=m_3=0$, the analytic expression reduces to

$$\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0 & 0\end{pmatrix} = \begin{cases}\frac{(-1)^{(\ell_1+\ell_2-\ell_3)/2}((\ell_1+\ell_2+\ell_3)/2)!}{((\ell_1+\ell_2-\ell_3)/2)!((\ell_1-\ell_2+\ell_3)/2)!((-\ell_1+\ell_2+\ell_3)/2)!} & \\ \quad\times\bigl[\frac{(\ell_1+\ell_2-\ell_3)!(\ell_1-\ell_2+\ell_3)!(-\ell_1+\ell_2+\ell_3)!}{(\ell_1+\ell_2+\ell_3+1)!}\bigr]^{1/2}, & \ell_1+\ell_2+\ell_3 \text{ even},\\ 0, & \ell_1+\ell_2+\ell_3 \text{ odd},\end{cases} \tag{4.3}$$

see [37], equations 8.1.2.12 and 8.5.2.32.

Lemma 4.5. *For every fixed $(x_1, x_2, x_3) \in P_3$, define*

$$g_\ell(x_1, x_2, x_3) = \begin{pmatrix} \lfloor \ell x_1 \rfloor & \lfloor \ell x_2 \rfloor & \lfloor \ell x_3 \rfloor \\ 0 & 0 & 0 \end{pmatrix}^2,$$

we have that

$$\lim_{\ell\to\infty} \ell^2 g_\ell(x_1, x_2, x_3) = \frac{2}{\pi} \frac{1}{\sqrt{x_1 + x_2 - x_3}\sqrt{x_1 - x_2 + x_3}\sqrt{-x_1 + x_2 + x_3}\sqrt{x_1 + x_2 + x_3}},$$

where the limit is defined for all ℓ such that $\lfloor \ell x_1 \rfloor + \lfloor \ell x_2 \rfloor + \lfloor \ell x_3 \rfloor$ is even.

Proof. Let $\lambda_0 = \lfloor \ell x_1 \rfloor + \lfloor \ell x_2 \rfloor + \lfloor \ell x_3 \rfloor$, $\lambda_1 = -\lfloor \ell x_1 \rfloor + \lfloor \ell x_2 \rfloor + \lfloor \ell x_3 \rfloor$, $\lambda_2 = \lfloor \ell x_1 \rfloor - \lfloor \ell x_2 \rfloor + \lfloor \ell x_3 \rfloor$ and $\lambda_3 = \lfloor \ell x_1 \rfloor + \lfloor \ell x_2 \rfloor - \lfloor \ell x_3 \rfloor$, from (4.3), by applying Stirling's formula

$$\ell! = \sqrt{2\pi}\ell^{\ell+1/2}\mathrm{e}^{-\ell} + O(\ell^{-1})$$

we see that

$$\begin{aligned}
&\lim_{\ell\to\infty} \ell^2 g_\ell(x_1, x_2, x_3) \\
&\quad= \lim_{\ell\to\infty} \ell^2 \left[\frac{\sqrt{2\pi}(\lambda_0/2)^{\lambda_0/2+1/2}\mathrm{e}^{-\lambda_0/2}}{\prod_{i=1}^3 \sqrt{2\pi}(\lambda_i/2)^{\lambda_i/2+1/2}\mathrm{e}^{-\lambda_i/2}} \right]^2 \frac{\prod_{i=1}^3 \sqrt{2\pi}\lambda_i^{\lambda_i+1/2}\mathrm{e}^{-\lambda_i}}{\sqrt{2\pi}(\lambda_0+1)^{\lambda_0+3/2}\mathrm{e}^{-\lambda_0-1}} \\
&\quad= \lim_{\ell\to\infty} \ell^2 \mathrm{e} \frac{2\pi(2\pi)^{3/2}}{(2\pi)^3\sqrt{2\pi}} 2^{-\lambda_0+2+\sum_{i=1}^3 \lambda_i} \frac{\lambda_0^{\lambda_0+1}\prod_{i=1}^3 \lambda_i^{\lambda_i+1/2}}{(\lambda_0+1)^{\lambda_0+3/2}\prod_{i=1}^3 \lambda_i^{\lambda_i+1}} \\
&\quad= \lim_{\ell\to\infty} \frac{2\mathrm{e}}{\pi} \frac{\ell^2}{\sqrt{\lambda_1\lambda_2\lambda_3}} \frac{\lambda_0}{(\lambda_0+1)^{3/2}} \left(1 + \frac{1}{\lambda_0}\right)^{-\lambda_0} \\
&\quad= \lim_{\ell\to\infty} \frac{2\mathrm{e}}{\pi} \frac{\ell^2}{\sqrt{(-\lfloor \ell x_1 \rfloor + \lfloor \ell x_2 \rfloor + \lfloor \ell x_3 \rfloor)(\lfloor \ell x_1 \rfloor - \lfloor \ell x_2 \rfloor + \lfloor \ell x_3 \rfloor)(\lfloor \ell x_1 \rfloor + \lfloor \ell x_2 \rfloor - \lfloor \ell x_3 \rfloor)}} \\
&\qquad \times \frac{\lfloor \ell x_1 \rfloor + \lfloor \ell x_2 \rfloor + \lfloor \ell x_3 \rfloor}{(\lfloor \ell x_1 \rfloor + \lfloor \ell x_2 \rfloor + \lfloor \ell x_3 \rfloor + 1)^{3/2}} \left(1 + \frac{1}{\lfloor \ell x_1 \rfloor + \lfloor \ell x_2 \rfloor + \lfloor \ell x_3 \rfloor}\right)^{-(\lfloor \ell x_1 \rfloor + \lfloor \ell x_2 \rfloor + \lfloor \ell x_3 \rfloor)} \\
&\quad= \frac{2}{\pi} \frac{1}{\sqrt{x_1 + x_2 - x_3}\sqrt{x_1 - x_2 + x_3}\sqrt{-x_1 + x_2 + x_3}\sqrt{x_1 + x_2 + x_3}}.
\end{aligned}$$

□

Remark 4.4. *Note that for $\lfloor \ell x_1 \rfloor = \lfloor \ell x_2 \rfloor = \lfloor \ell x_3 \rfloor = \ell$ we have the same result as in* [21], *Lemma* A.1, *in fact*

$$\lim_{\ell\to\infty} \ell^2 \begin{pmatrix} \ell & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix}^2 = \lim_{\ell\to\infty} \frac{2\mathrm{e}}{\pi} \frac{\ell^2}{\sqrt{\ell^3}} \frac{3\ell}{(3\ell+1)^{3/2}} \left(1 + \frac{1}{3\ell}\right)^{-3\ell} = \frac{2}{\pi\sqrt{3}}.$$

Theorem 6. *For $q = 3$, we have*

$$\lim_{j\to\infty} 2^{2j} \operatorname{Var}[\nu_{j;3}] = 3! c_3,$$

where

$$\begin{aligned}
c_3 &= \frac{16\pi}{(\int_{1/2}^2 b^2(x) x^{1-\alpha}\,\mathrm{d}x)^3} \int_{1/2}^2 \cdots \int_{1/2}^2 \prod_{i=1}^3 b^2(x_i) x_i^{1-\alpha} \\
&\quad \times \frac{1}{\sqrt{x_1 + x_2 - x_3}\sqrt{x_1 - x_2 + x_3}\sqrt{-x_1 + x_2 + x_3}\sqrt{x_1 + x_2 + x_3}} \mathbb{1}_{P_3}(x_1, x_2, x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3.
\end{aligned}$$

Proof. For $\ell = 2^j$ and $\ell_1, \ell_2, \ell_3 \in [2^{j-1}, 2^{j+1}]$ we have

$$\begin{aligned}\operatorname{Var}[\nu_{j;3}] &= 3! B_\ell^{-3} \int_{S^2\times S^2} \{\mathbb{E}[\beta_j(x_1)\beta_j(x_2)]\}^3 \,\mathrm{d}\sigma(x_1)\,\mathrm{d}\sigma(x_2) \\ &= 3! 8\pi^2 B_\ell^{-3} \sum_{\ell_1\ell_2\ell_3} \prod_{i=1}^{3} b^2\left(\frac{\ell_i}{\ell}\right) \frac{2\ell_i+1}{4\pi} \ell_i^{-\alpha} G(\ell_i) \int_0^\pi P_{\ell_1}(\cos\theta) P_{\ell_2}(\cos\theta) P_{\ell_3}(\cos\theta) \sin\theta \,\mathrm{d}\theta.\end{aligned}$$

By expressing Legendre polynomials in terms of spherical harmonics and by applying the well-known formula for the integral of the product of three spherical harmonics over the sphere (see [17], Section 3.5.3 for a proof), we obtain

$$\int_0^\pi P_{\ell_1}(\cos\theta) P_{\ell_2}(\cos\theta) P_{\ell_3}(\cos\theta) \sin\theta \,\mathrm{d}\theta = 2 \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^2,$$

and then, from (4.3),

$$\begin{aligned}\operatorname{Var}[\nu_{j;3}] &= 3! 8\pi^2 B_\ell^{-3} \sum_{\substack{\ell_1\ell_2\ell_3 \\ \sum l_k \text{ even}}} \prod_{i=1}^{3} b^2\left(\frac{\ell_i}{\ell}\right) \frac{2\ell_i+1}{4\pi} \ell_i^{-\alpha} G(\ell_i) 2 \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^2 \\ &= 3! 8\pi^2 \ell^{-3\alpha+6} B_\ell^{-3} \sum_{\substack{\ell_1\ell_2\ell_3 \\ \sum l_k \text{ even}}} \int_{l_1/l}^{(l_1+1)/l} \cdots \int_{l_3/l}^{(l_3+1)/l} \prod_{i=1}^{3} b^2\left(\frac{\lfloor \ell x_i\rfloor}{\ell}\right) \frac{2\lfloor \ell x_i\rfloor+1}{2\ell} \\ &\quad\times \left(\frac{\lfloor \ell x_i\rfloor}{\ell}\right)^{-\alpha} \frac{G(\lfloor \ell x_i\rfloor)}{2\pi} 2 \begin{pmatrix} \lfloor \ell x_1\rfloor & \lfloor \ell x_2\rfloor & \lfloor \ell x_3\rfloor \\ 0 & 0 & 0 \end{pmatrix}^2 \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3.\end{aligned}$$

Applying dominated convergence again and Lemma 4.5,

$$\begin{aligned}&\lim_{\ell\to\infty} \ell^2 \operatorname{Var}[\nu_{j;3}] \\ &\quad= \lim_{\ell\to\infty} 3! 8\pi^2 \ell^{-3\alpha+6} B_\ell^{-3} \sum_{\substack{\ell_1\ell_2\ell_3 \\ \sum l_k \text{ even}}} \int_{l_1/l}^{(l_1+1)/l} \cdots \int_{l_3/l}^{(l_3+1)/l} \prod_{i=1}^{3} b^2\left(\frac{\lfloor \ell x_i\rfloor}{\ell}\right) \frac{2\lfloor \ell x_i\rfloor+1}{2\ell} \left(\frac{\lfloor \ell x_i\rfloor}{\ell}\right)^{-\alpha} \\ &\qquad\times \frac{G(\lfloor \ell x_i\rfloor)}{2\pi} \frac{2\mathrm{e}}{\pi} 2 \frac{\ell^2}{\sqrt{(-\lfloor \ell x_1\rfloor + \lfloor \ell x_2\rfloor + \lfloor \ell x_3\rfloor)(\lfloor \ell x_1\rfloor - \lfloor \ell x_2\rfloor + \lfloor \ell x_3\rfloor)(\lfloor \ell x_1\rfloor + \lfloor \ell x_2\rfloor - \lfloor \ell x_3\rfloor)}} \\ &\qquad\times \frac{\lfloor \ell x_1\rfloor + \lfloor \ell x_2\rfloor + \lfloor \ell x_3\rfloor}{(\lfloor \ell x_1\rfloor + \lfloor \ell x_2\rfloor + \lfloor \ell x_3\rfloor + 1)^{3/2}} \left(1 + \frac{1}{\lfloor \ell x_1\rfloor + \lfloor \ell x_2\rfloor + \lfloor \ell x_3\rfloor}\right)^{-(\lfloor \ell x_1\rfloor + \lfloor \ell x_2\rfloor + \lfloor \ell x_3\rfloor)} \\ &\qquad\times \mathbb{1}_{P_3}(x_1,x_2,x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \\ &\quad= \lim_{\ell\to\infty} 3! 8\pi^2 \ell^{-3\alpha+6} B_\ell^{-3} \int_{1/2}^{(2l+1)/l} \cdots \int_{1/2}^{(2l+1)/l} \prod_{i=1}^{3} b^2\left(\frac{\lfloor \ell x_i\rfloor}{\ell}\right) \frac{2\lfloor \ell x_i\rfloor+1}{2\ell} \left(\frac{\lfloor \ell x_i\rfloor}{\ell}\right)^{-\alpha} \\ &\qquad\times \frac{G(\lfloor \ell x_i\rfloor)}{2\pi} \frac{2\mathrm{e}}{\pi} \frac{\ell^2}{\sqrt{(-\lfloor \ell x_1\rfloor + \lfloor \ell x_2\rfloor + \lfloor \ell x_3\rfloor)(\lfloor \ell x_1\rfloor - \lfloor \ell x_2\rfloor + \lfloor \ell x_3\rfloor)(\lfloor \ell x_1\rfloor + \lfloor \ell x_2\rfloor - \lfloor \ell x_3\rfloor)}} \\ &\qquad\times \frac{\lfloor \ell x_1\rfloor + \lfloor \ell x_2\rfloor + \lfloor \ell x_3\rfloor}{(\lfloor \ell x_1\rfloor + \lfloor \ell x_2\rfloor + \lfloor \ell x_3\rfloor + 1)^{3/2}} \left(1 + \frac{1}{\lfloor \ell x_1\rfloor + \lfloor \ell x_2\rfloor + \lfloor \ell x_3\rfloor}\right)^{-(\lfloor \ell x_1\rfloor + \lfloor \ell x_2\rfloor + \lfloor \ell x_3\rfloor)} \\ &\qquad\times \mathbb{1}_{P_3}(x_1,x_2,x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3.\end{aligned}$$

Then, by dominated convergence again and Lemma 4.4, we arrive at the statement. $\square$

Theorem 7. *For* $q=4$,

$$\lim_{j\to\infty} 2^{2j}\operatorname{Var}[\nu_{j;4}] = 4!c_4,$$

where

$$\begin{aligned}
c_4 = {}& \frac{16}{(\int_{1/2}^2 b^2(x)x^{1-\alpha}\,\mathrm{d}x)^4}\int_{1/2}^2\cdots\int_{1/2}^2\prod_{i=1}^4 b^2(x_i)x_i^{1-\alpha}\\
&\times\int_0^4 y\frac{1}{\sqrt{-x_1+x_2+y}\sqrt{x_1-x_2+y}\sqrt{x_1+x_2-y}\sqrt{x_1+x_2+y}}\\
&\times\frac{1}{\sqrt{-x_3+x_4+y}\sqrt{x_3-x_4+y}\sqrt{x_3+x_4-y}\sqrt{x_3+x_4+y}}\\
&\times \mathbb{1}_{P_3}(x_1,x_2,y)\mathbb{1}_{P_3}(y,x_3,x_4)\,\mathrm{d}y\,\mathrm{d}x_1\cdots\mathrm{d}x_4.
\end{aligned}$$

Proof. For $\ell=2^j$ and $\ell_1,\ell_2,\ell_3,\ell_4\in[2^{j-1},2^{j+1}]$ we have

$$\begin{aligned}
\operatorname{Var}[\nu_{j;4}] &= 4!B_\ell^{-4}\int_{S^2\times S^2}\{\mathbb{E}[\beta_j(x_1)\beta_j(x_2)]\}^4\,\mathrm{d}\sigma(x_1)\,\mathrm{d}\sigma(x_2)\\
&= 4!8\pi^2 B_\ell^{-4}\sum_{\ell_1\ell_2\ell_3\ell_4}\prod_{i=1}^4 b^2\left(\frac{\ell_i}{\ell}\right)\frac{2\ell_i+1}{4\pi}\ell_i^{-\alpha}G(\ell_i)\int_0^\pi\prod_{i=1}^4 P_{\ell_i}(\cos\theta)\sin\theta\,\mathrm{d}\theta.
\end{aligned}$$

From the product formula

$$Y_{\ell_1 0}(\theta,\phi)Y_{\ell_2 0}(\theta,\phi)=\sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi}}\sum_{L=|\ell_1-\ell_2|}^{\ell_1+\ell_2}\sqrt{2L+1}\begin{pmatrix}\ell_1&\ell_2&L\\0&0&0\end{pmatrix}^2 Y_{L0}(\theta,\phi),$$

and the orthogonality property of spherical harmonics, we obtain the following formula for the integral of the product of four spherical harmonics over the sphere

$$\begin{aligned}
&\int_0^{2\pi}\int_0^\pi\prod_{i=1}^4 Y_{\ell_i 0}(\theta,\phi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\\
&\quad=\sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi}}\sqrt{\frac{(2\ell_3+1)(2\ell_4+1)}{4\pi}}\sum_{L_1=|\ell_1-\ell_2|}^{\ell_1+\ell_2}\sqrt{2L_1+1}\begin{pmatrix}\ell_1&\ell_2&L_1\\0&0&0\end{pmatrix}^2\\
&\qquad\times\sum_{L_2=|\ell_3-\ell_4|}^{\ell_3+\ell_4}\sqrt{2L_2+1}\begin{pmatrix}\ell_3&\ell_4&L_2\\0&0&0\end{pmatrix}^2\delta_{L_1}^{L_2}\\
&\quad=4\pi\prod_{i=1}^4\sqrt{\frac{2\ell_i+1}{4\pi}}\sum_L(2L+1)\begin{pmatrix}\ell_1&\ell_2&L\\0&0&0\end{pmatrix}^2\begin{pmatrix}\ell_3&\ell_4&L\\0&0&0\end{pmatrix}^2,
\end{aligned}$$

that is

$$\int_0^\pi\prod_{i=1}^4 P_{\ell_i}(\cos\theta)\sin\theta\,\mathrm{d}\theta = 2\sum_L(2L+1)\begin{pmatrix}\ell_1&\ell_2&L\\0&0&0\end{pmatrix}^2\begin{pmatrix}\ell_3&\ell_4&L\\0&0&0\end{pmatrix}^2.$$

We can write the variance as

$$\operatorname{Var}[\nu_{j;4}] = 4!8\pi^2 B_\ell^{-4} \sum_{l_1 l_2 l_3 l_4} \prod_{i=1}^{4} b^2\left(\frac{\ell_i}{\ell}\right)\frac{2\ell_i+1}{4\pi}\ell_i^{-\alpha}G(\ell_i)$$

$$\times 2 \sum_{\substack{L \\ l_1+l_2+L \text{ even} \\ l_3+l_4+L \text{ even}}} (2L+1)\begin{pmatrix} \ell_1 & \ell_2 & L \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} \ell_3 & \ell_4 & L \\ 0 & 0 & 0 \end{pmatrix}^2.$$

Since $\max\{|\ell_1-\ell_2|, |\ell_3-\ell_4|\} \le L \le \min\{\ell_1+\ell_2, \ell_3+\ell_4\}$ where $|\ell_i-\ell_k| \ge 0$ and $\ell_i+\ell_k \le 4\ell$, we can write

$$\operatorname{Var}[\nu_{j;4}] = 4!8\pi^2 \ell^{-4\alpha+10} B_\ell^{-4} \sum_{\ell_1\ell_2\ell_3\ell_4} \int_{\ell_1/\ell}^{(\ell_1+1)/\ell} \cdots \int_{\ell_4/\ell}^{(\ell_4+1)/\ell} \prod_{i=1}^{4} b^2\left(\frac{\lfloor \ell x_i\rfloor}{\ell}\right)\frac{2\lfloor \ell x_i\rfloor+1}{2\ell}$$

$$\times \left(\frac{\lfloor \ell x_i\rfloor}{\ell}\right)^{-\alpha}\frac{G(\lfloor \ell x_i\rfloor)}{2\pi} 2 \sum_{\substack{L=0 \\ l_1+l_2+L \text{ even} \\ l_3+l_4+L \text{ even}}}^{4\ell} \int_{L/\ell}^{(L+1)/\ell} \frac{2\lfloor \ell y\rfloor+1}{2\ell}$$

$$\times \begin{pmatrix} \lfloor \ell x_1\rfloor & \lfloor \ell x_2\rfloor & \lfloor \ell y\rfloor \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} \lfloor \ell x_3\rfloor & \lfloor \ell x_4\rfloor & \lfloor \ell y\rfloor \\ 0 & 0 & 0 \end{pmatrix}^2 \mathrm{d}y\,\mathrm{d}x_1\cdots\mathrm{d}x_4.$$

Then, by dominated convergence and Lemma 4.5, we have

$$\lim_{\ell\to\infty} \ell^2 \operatorname{Var}[\nu_{j;4}]$$

$$= \lim_{\ell\to\infty} 4!8\pi^2 \ell^{-4\alpha+8} B_\ell^{-4} \sum_{\ell_1\ell_2\ell_3\ell_4} \int_{\ell_1/\ell}^{(\ell_1+1)/\ell} \cdots \int_{\ell_4/\ell}^{(\ell_4+1)/\ell} \prod_{i=1}^{4} b^2\left(\frac{\lfloor \ell x_i\rfloor}{\ell}\right)\frac{2\lfloor \ell x_i\rfloor+1}{2\ell}$$

$$\times \left(\frac{\lfloor \ell x_i\rfloor}{\ell}\right)^{-\alpha}\frac{G(\lfloor \ell x_i\rfloor)}{2\pi} 2 \sum_{\substack{L=0 \\ l_1+l_2+L \text{ even} \\ l_3+l_4+L \text{ even}}}^{4\ell} \int_{L/\ell}^{(L+1)/\ell} \frac{2\lfloor \ell y\rfloor+1}{2\ell}\frac{4\mathrm{e}^2}{\pi^2}$$

$$\times \frac{\ell^2}{\sqrt{(-\lfloor \ell x_1\rfloor+\lfloor \ell x_2\rfloor+\lfloor \ell y\rfloor)(\lfloor \ell x_1\rfloor-\lfloor \ell x_2\rfloor+\lfloor \ell y\rfloor)(\lfloor \ell x_1\rfloor+\lfloor \ell x_2\rfloor-\lfloor \ell y\rfloor)}}$$

$$\times \frac{\lfloor \ell x_1\rfloor+\lfloor \ell x_2\rfloor+\lfloor \ell y\rfloor}{(\lfloor \ell x_1\rfloor+\lfloor \ell x_2\rfloor+\lfloor \ell y\rfloor+1)^{3/2}}\left(1+\frac{1}{\lfloor \ell x_1\rfloor+\lfloor \ell x_2\rfloor+\lfloor \ell y\rfloor}\right)^{-(\lfloor \ell x_1\rfloor+\lfloor \ell x_2\rfloor+\lfloor \ell y\rfloor)}$$

$$\times \frac{\ell^2}{\sqrt{(-\lfloor \ell x_3\rfloor+\lfloor \ell x_4\rfloor+\lfloor \ell y\rfloor)(\lfloor \ell x_3\rfloor-\lfloor \ell x_4\rfloor+\lfloor \ell y\rfloor)(\lfloor \ell x_3\rfloor+\lfloor \ell x_4\rfloor-\lfloor \ell y\rfloor)}}$$

$$\times \frac{\lfloor \ell x_3\rfloor+\lfloor \ell x_4\rfloor+\lfloor \ell y\rfloor}{(\lfloor \ell x_3\rfloor+\lfloor \ell x_4\rfloor+\lfloor \ell y\rfloor+1)^{3/2}}\left(1+\frac{1}{\lfloor \ell x_3\rfloor+\lfloor \ell x_4\rfloor+\lfloor \ell y\rfloor}\right)^{-(\lfloor \ell x_3\rfloor+\lfloor \ell x_4\rfloor+\lfloor \ell y\rfloor)}$$

$$\times \mathbb{1}_{P_3}(x_1,x_2,y)\mathbb{1}_{P_3}(x_3,x_4,y)\,\mathrm{d}y\,\mathrm{d}x_1\cdots\mathrm{d}x_4$$

$$= \lim_{\ell\to\infty} 4!8\pi^2 \ell^{-4\alpha+8} B_\ell^{-4} \frac{1}{2}\int_{1/2}^{(2\ell+1)/\ell} \cdots \int_{1/2}^{(2\ell+1)/\ell} \prod_{i=1}^{4} b^2\left(\frac{\lfloor \ell x_i\rfloor}{\ell}\right)\frac{2\lfloor \ell x_i\rfloor+1}{2\ell}$$

$$\times \left(\frac{\lfloor \ell x_i\rfloor}{\ell}\right)^{-\alpha}\frac{G(\lfloor \ell x_i\rfloor)}{2\pi}\int_0^{(4\ell+1)/\ell} \frac{2\lfloor \ell y\rfloor+1}{2\ell}\frac{4\mathrm{e}^2}{\pi^2}$$

$$\times \frac{\ell^2}{\sqrt{(-\lfloor \ell x_1 \rfloor + \lfloor \ell x_2 \rfloor + \lfloor \ell y \rfloor)(\lfloor \ell x_1 \rfloor - \lfloor \ell x_2 \rfloor + \lfloor \ell y \rfloor)(\lfloor \ell x_1 \rfloor + \lfloor \ell x_2 \rfloor - \lfloor \ell y \rfloor)}}$$
$$\times \frac{\lfloor \ell x_1 \rfloor + \lfloor \ell x_2 \rfloor + \lfloor \ell y \rfloor}{(\lfloor \ell x_1 \rfloor + \lfloor \ell x_2 \rfloor + \lfloor \ell y \rfloor + 1)^{3/2}} \left(1 + \frac{1}{\lfloor \ell x_1 \rfloor + \lfloor \ell x_2 \rfloor + \lfloor \ell y \rfloor}\right)^{-(\lfloor \ell x_1 \rfloor + \lfloor \ell x_2 \rfloor + \lfloor \ell y \rfloor)}$$
$$\times \frac{\ell^2}{\sqrt{(-\lfloor \ell x_3 \rfloor + \lfloor \ell x_4 \rfloor + \lfloor \ell y \rfloor)(\lfloor \ell x_3 \rfloor - \lfloor \ell x_4 \rfloor + \lfloor \ell y \rfloor)(\lfloor \ell x_3 \rfloor + \lfloor \ell x_4 \rfloor - \lfloor \ell y \rfloor)}}$$
$$\times \frac{\lfloor \ell x_3 \rfloor + \lfloor \ell x_4 \rfloor + \lfloor \ell y \rfloor}{(\lfloor \ell x_3 \rfloor + \lfloor \ell x_4 \rfloor + \lfloor \ell y \rfloor + 1)^{3/2}} \left(1 + \frac{1}{\lfloor \ell x_3 \rfloor + \lfloor \ell x_4 \rfloor + \lfloor \ell y \rfloor}\right)^{-(\lfloor \ell x_3 \rfloor + \lfloor \ell x_4 \rfloor + \lfloor \ell y \rfloor)}$$
$$\times \mathbb{1}_{P_3}(x_1, x_2, y)\mathbb{1}_{P_3}(x_3, x_4, y)\, \mathrm{d}y\, \mathrm{d}x_1 \cdots \mathrm{d}x_4.$$

Once again, applying dominated convergence and Lemma 4.4, we have the statement. □

5. Quantitative Central Limit Theorems for $\nu_{j;q}$

We start by recalling that $H_q(\tilde{\beta}_j(x))$ belongs to the qth order Wiener chaos and so does the linear transform $\nu_{j;q}$. Inside a fixed Wiener chaos it is possible to get explicit estimates on the speed of convergence to the Gaussian law for the Kolmogorov, Total Variation and Wasserstein distance by applying Proposition 2.2 and by explicitly relate norms of Malliavin operators with moments and cumulants. In fact, for $\mathcal{N}$ standard Gaussian, we have

$$d\left(\frac{\nu_{j;q}}{\sqrt{\mathrm{Var}(\nu_{j;q})}}, \mathcal{N}\right) \le 2\sqrt{\frac{q-1}{3q}\frac{\mathrm{cum}_4(\nu_{j;q})}{\mathrm{Var}^2(\nu_{j;q})}},$$

where d is the Kolmogorov, Total Variation or Wasserstein distance and cum_4 is the fourth-order cumulant of $\nu_{j;q}$. See [27], Theorem 5.2.6 for more discussion and a full proof.

Quantitative Central Limit Theorems for $\nu_{j;q}$ then follow easily from the results of Section 4 and by computing the fourth-order cumulant as in [20], Section 5.1. The arguments are indeed quite standard but nevertheless for completeness we report them below.

We start by expressing the 4th order cumulant as an integral over $(S^2)^4$, using the well-known Diagram formula, see [17], Proposition 4.15 for further details.

Fix a set of integers $\alpha_1, \dots, \alpha_p$, a diagram is a graph with α_1 vertexes labelled by 1, α_2 vertexes labelled by $2, \dots, \alpha_p$ vertexes labelled by p, such that each vertex has degree 1. We can view the vertexes as belonging to p different rows and the edges may connect only vertexes with different labels, i.e. there are no flat edges on the same row. The set of such graphs that are connected (i.e. such that it is not possible to partition the vertexes into two subsets A and B such that no edge connect a vertex in A with a vertex in B) is denoted by $\Gamma_c(\alpha_1, \dots, \alpha_p)$. Given a diagram $\gamma \in \Gamma_c$, $\eta_{ik}(\gamma)$ is the number of edges between the vertexes labelled by i and the vertexes labelled by k in γ. The following proposition holds:

Proposition 5.1 (Diagram formula for Hermite polynomials). *Let $(Z_1, \dots, Z_p)$ be a centered Gaussian vector, and let $H_{l_1}, \dots, H_{l_p}$ be Hermite polynomials of degrees $l_1, \dots, l_p$ (≥ 1) respectively. Then*

$$\mathrm{cum}\big(H_{l_1}(Z_1), \dots, H_{l_p}(Z_p)\big) = \sum_{\gamma \in \Gamma_c(l_1, \dots, l_p)} \prod_{1 \le i \le j \le p} \big\{\mathbb{E}[Z_i Z_j]\big\}^{\eta_{ij}(\gamma)}.$$

For a proof see [28], Section 7.3.

Theorem 8. *For $\mathcal{N}$ standard Gaussian variable and for all q such that $c_q > 0$, as $j \to \infty$, we have that*

$$d_{\mathrm{TV}}\left(\frac{\nu_{j;q}}{\sqrt{\mathrm{Var}(\nu_{j,q})}}, \mathcal{N}\right), d_W\left(\frac{\nu_{j;q}}{\sqrt{\mathrm{Var}(\nu_{j,q})}}, \mathcal{N}\right) = \mathrm{O}\big(2^{-j}\big).$$

Proof. In view of Proposition 5.1, for $p=4$ and $l_1=\cdots=l_4=q$, we obtain

$$\begin{aligned}
\operatorname{cum}_4[\nu_{j;q}] &= \operatorname{cum}_4\biggl[\int_{S^2} H_q\bigl(\tilde\beta_j(x_1)\bigr)\,\mathrm{d}\sigma(x_1)\cdots\int_{S^2} H_q\bigl(\tilde\beta_j(x_4)\bigr)\,\mathrm{d}\sigma(x_4)\biggr]\\
&= \int_{(S^2)^4} \operatorname{cum}_4\bigl[H_q\bigl(\tilde\beta_j(x_1)\bigr)\cdots H_q\bigl(\tilde\beta_j(x_4)\bigr)\bigr]\,\mathrm{d}\sigma(x_1)\cdots\mathrm{d}\sigma(x_4)\\
&= \int_{(S^2)^4} \sum_{\gamma\in\Gamma_c(q,q,q,q)}\prod_{(i,k)\in\gamma}\bigl\{\mathbb{E}\bigl[\tilde\beta_j(x_i)\tilde\beta_j(x_k)\bigr]\bigr\}^{\eta_{ik}(\gamma)}\,\mathrm{d}\sigma(x_1)\cdots\mathrm{d}\sigma(x_4)\\
&= \frac{1}{B_j^{2q}}\int_{(S^2)^4} \sum_{\gamma\in\Gamma_c(q,q,q,q)}\prod_{(i,k)\in\gamma}\bigl\{\mathbb{E}\bigl[\beta_j(x_i)\beta_j(x_k)\bigr]\bigr\}^{\eta_{ik}(\gamma)}\,\mathrm{d}\sigma(x_1)\cdots\mathrm{d}\sigma(x_4),
\end{aligned}$$

since $\sum_{(i,k)\in\gamma}\eta_{ik}(\gamma)=2q$. Now we apply formula (3.2) and we obtain

$$\begin{aligned}
&\operatorname{cum}_4[\nu_{j;q}]\\
&\quad\le \frac{1}{B_j^{2q}}\sum_{\gamma\in\Gamma_c(q,q,q,q)}\int_{(S^2)^4}\prod_{(i,k)\in\gamma}\biggl\{\frac{K_M}{(1+2^j d(x_i,x_k))^M}\sum_l b^2\biggl(\frac{l}{2^j}\biggr)C_l\frac{2l+1}{4\pi}\biggr\}^{\eta_{ik}(\gamma)}\,\mathrm{d}\sigma(x_1)\cdots\mathrm{d}\sigma(x_4)\\
&\quad= \tilde C_M^{2q}\sum_{\gamma\in\Gamma_c(q,q,q,q)}\int_{(S^2)^4}\prod_{(i,k)\in\gamma}\frac{1}{(1+2^j d(x_i,x_k))^{M\eta_{ik}(\gamma)}}\,\mathrm{d}\sigma(x_1)\cdots\mathrm{d}\sigma(x_4).
\end{aligned}$$

To compute the integral we note that for spherical symmetry we can assume without loss of generality that e.g. x_3 is the North Pole denoted by p_N, and we get

$$\begin{aligned}
&\int_{(S^2)^4}\prod_{(i,k)\in\gamma}\frac{1}{(1+2^j d(x_i,x_k))^{M\eta_{ik}(\gamma)}}\,\mathrm{d}\sigma(x_1)\cdots\mathrm{d}\sigma(x_4)\\
&\quad\le \int_{(S^2)^4}\frac{1}{(1+2^j d(x_1,x_2))^M}\frac{1}{(1+2^j d(x_2,x_3))^M}\frac{1}{(1+2^j d(x_3,x_4))^M}\frac{1}{(1+2^j d(x_1,x_4))^M}\,\mathrm{d}\sigma(x_1)\cdots\mathrm{d}\sigma(x_4)\\
&\quad\le 4\pi\int_{(S^2)^3}\frac{1}{(1+2^j d(x_1,x_2))^M}\frac{1}{(1+2^j d(x_2,p_N))^M}\frac{1}{(1+2^j d(p_N,x_4))^M}\,\mathrm{d}\sigma(x_1)\,\mathrm{d}\sigma(x_2)\,\mathrm{d}\sigma(x_4)\\
&\quad\le 4\pi C2^{-2j}\int_{(S^2)^2}\frac{1}{(1+2^j d(x_1,x_2))^M}\frac{1}{(1+2^j d(x_2,p_N))^M}\,\mathrm{d}\sigma(x_1)\,\mathrm{d}\sigma(x_2)\\
&\quad\le \mathrm{const}\,2^{-6j}
\end{aligned}$$

since, for example, for $M>2$

$$\begin{aligned}
\int_{S^2}\frac{1}{(1+2^j d(p_N,x_4))^M}\,\mathrm{d}\sigma(x_4) &= \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\frac{\theta\sin\theta}{(1+2^j\theta)^M}\,\mathrm{d}\theta \le 2\pi\int_0^{\infty}\frac{\theta}{(1+2^j\theta)^M}\,\mathrm{d}\theta\\
&= 2\pi\biggl[\int_0^{2^{-j}}\frac{\theta}{(1+2^j\theta)^M}\,\mathrm{d}\theta+\int_{2^{-j}}^{\infty}\frac{\theta}{(1+2^j\theta)^M}\,\mathrm{d}\theta\biggr]\\
&\le 2\pi\biggl[\int_0^{2^{-j}}\theta\,\mathrm{d}\theta+2^{-jM}\int_{2^{-j}}^{\infty}\theta^{1-M}\,\mathrm{d}\theta\biggr]\\
&= 2\pi\biggl[2^{-1-2j}+\frac{2^{-2j}}{M-2}\biggr]\le \mathrm{const}\,2^{-2j}.
\end{aligned}$$

□

6. A quantitative Central Limit Theorem for the empirical measure

In the next theorem we obtain a bound on the Wasserstein distance for the speed of convergence of $\Phi_j(z)$ to the Gaussian law.

Theorem 9. *For $\mathcal{N}$ standard Gaussian, as $j \to \infty$ we have*

$$d_W\left(\frac{\Phi_j(z)}{\sqrt{\mathrm{Var}[\Phi_j(z)]}}, \mathcal{N}\right) = \mathrm{O}\left(\frac{1}{\sqrt[4]{j}}\right).$$

We start by proving the following lemma.

Lemma 6.1. *For integers $q, q' \geq 2$ we have that*

$$\mathbb{E}\big[\big(\langle D\nu_{j;q}, -DL^{-1}\nu_{j;q'}\rangle_{\mathfrak{H}}\big)^2\big] \leq \mathrm{const}\, 2^{-6j} q^2 \sum_{r=1}^{q\wedge q'} (r-1)!^2 \binom{q-1}{r-1}^2 \binom{q'-1}{r-1}^2 (q+q'-2r)!,$$

$$\mathrm{Var}\big[\langle D\nu_{j;q}, -DL^{-1}\nu_{j;q}\rangle_{\mathfrak{H}}\big] \leq \mathrm{const}\, 2^{-6j} q^2 \sum_{r=1}^{q-1} (r-1)!^2 \binom{q-1}{r-1}^4 (2q-2r)!.$$

Proof. Since $H_q(\tilde{\beta}_j(x))$ is in the qth order Wiener chaos, from (2.5), we obtain

$$\begin{aligned}\nu_{j;q} &= \int_{S^2} \mathrm{d}x \int_{(S^2)^q} \prod_{i=1}^q \tilde{\Theta}_j(\langle x, y_i\rangle) W(\mathrm{d}\sigma(y_i)) = \int_{(S^2)^q} g_{q,j}(y_1, \ldots, y_q) W(\mathrm{d}\sigma(y_1)) \cdots W(\mathrm{d}\sigma(y_q)) \\ &= I_q(g_{q,j}(y_1, \ldots, y_q)),\end{aligned}$$

where

$$g_{q,j}(y_1, \ldots, y_q) = \int_{S^2} \prod_{i=1}^q \tilde{\Theta}_j(\langle x, y_i\rangle)\, \mathrm{d}\sigma(x)$$

and, from formula (2.3),

$$D\nu_{j;q} = \frac{q!}{(q-1)!} I_{q-1}(g_{q,j}(y_1, \ldots, y_{q-1}, z)) = q \int_{(S^2)^{q-1}} g_{q,j}(y_1, \ldots, y_{q-1}, z) W(\mathrm{d}\sigma(y_1)) \cdots W(\mathrm{d}\sigma(y_{q-1})).$$

Applying the definition of the pseudo-inverse of L, we obtain

$$\begin{aligned}\langle D\nu_{j;q}, -DL^{-1}\nu_{j;q'})\rangle_{\mathfrak{H}} &= \frac{1}{q'} \langle D\nu_{j;q}, D\nu_{j;q'}\rangle_{\mathfrak{H}} \\ &= q\langle I_{q-1}(g_{q,j}(y_1, \ldots, y_{q-1}, z)), I_{q'-1}(g_{q',j}(y_1, \ldots, y_{q'-1}, z))\rangle_{\mathfrak{H}} \\ &= q \int_{S^2} I_{q-1}(g_{q,j}(y_1, \ldots, y_{q-1}, z)) I_{q'-1}(g_{q',j}(y_1, \ldots, y_{q'-1}, z))\, \mathrm{d}\sigma(z)\end{aligned}$$

and by the multiplication formula (2.6)

$$\begin{aligned}&\langle D\nu_{j;q}, -DL^{-1}\nu_{j;q'}\rangle_{\mathfrak{H}} \\ &\quad= q \sum_{r=0}^{q\wedge q'-1} r! \binom{q-1}{r} \binom{q'-1}{r} \int_{S^2} I_{q+q'-2-2r}(g_{q,j}(y_1, \ldots, y_{q-1}, z) \tilde{\otimes}_r g_{q',j}(y_1, \ldots, y_{q'-1}, z))\, \mathrm{d}\sigma(z)\end{aligned}$$

$$= q \sum_{r=0}^{q\wedge q'-1} r! \binom{q-1}{r} \binom{q'-1}{r} I_{q+q'-2-2r}\big(g_{q,j}(y_1,\ldots,y_q) \tilde{\otimes}_{r+1} g_{q',j}(y_1,\ldots,y_{q'})\big)$$

$$= q \sum_{r=1}^{q\wedge q'} (r-1)! \binom{q-1}{r-1} \binom{q'-1}{r-1} I_{q+q'-2r}\big(g_{q,j}(y_1,\ldots,y_q) \tilde{\otimes}_{r} g_{q',j}(y_1,\ldots,y_{q'})\big).$$

From the isometry property (2.4) we have

$$\begin{aligned}
&\mathbb{E}\big[\big(\langle Dv_{j;q}, -DL^{-1}v_{j;q'}\rangle_{\mathfrak{H}}\big)^2\big] \\
&= q^2 \sum_{r=1}^{q\wedge q'} (r-1)!^2 \binom{q-1}{r-1}^2 \binom{q'-1}{r-1}^2 \big(q+q'-2r\big)! \big\| g_{q,j}(y_1,\ldots,y_q) \tilde{\otimes}_r g_{q',j}(y_1,\ldots,y_{q'})\big\|^2_{\mathfrak{H}^{\otimes q+q'-2r}} \\
&\le q^2 \sum_{r=1}^{q\wedge q'} (r-1)!^2 \binom{q-1}{r-1}^2 \binom{q'-1}{r-1}^2 \big(q+q'-2r\big)! \big\| g_{q,j}(y_1,\ldots,y_q) \otimes_r g_{q',j}(y_1,\ldots,y_{q'})\big\|^2_{\mathfrak{H}^{\otimes q+q'-2r}}.
\end{aligned}$$

Applying Lemma 6.2.1 in [27], we write

$$\begin{aligned}
&\operatorname{Var}\big[\langle Dv_{j;q}, -DL^{-1}v_{j,q}\rangle_{\mathfrak{H}}\big] \\
&\le q^2 \sum_{r=1}^{q-1} (r-1)!^2 \binom{q-1}{r-1}^4 (2q-2r)! \big\| g_{q,j}(y_1,\ldots,y_q) \otimes_{q-r} g_{q,j}(y_1,\ldots,y_q)\big\|^2_{\mathfrak{H}^{\otimes 2r}}.
\end{aligned}$$

We determine now the explicit form for the contractions:

$$\begin{aligned}
&g_{q,j}(y_1,\ldots,y_q) \otimes_r g_{q',j}(y_1,\ldots,y_{q'}) \\
&= \int_{(S^2)^r} g_{q,j}(y_1,\ldots,y_{q-r},t_1,\ldots,t_r) g_{q',j}(y_{q-r+1},\ldots,y_{q+q'-2r},t_1,\ldots,t_r)\,\mathrm{d}\sigma(t_1)\cdots\mathrm{d}\sigma(t_r) \\
&= \int_{(S^2)^r} \left[\int_{S^2} \prod_{n=1}^{q-r} \tilde{\Theta}_j\big(\langle x_1, y_n\rangle\big) \prod_{i=1}^{r} \tilde{\Theta}_j\big(\langle x_1, t_i\rangle\big)\,\mathrm{d}\sigma(x_1)\right] \\
&\quad \times \left[\int_{S^2} \prod_{m=q-r+1}^{q+q'-2r} \tilde{\Theta}_j\big(\langle x_2, y_m\rangle\big) \prod_{i=1}^{r} \tilde{\Theta}_j\big(\langle x_2, t_i\rangle\big)\,\mathrm{d}\sigma(x_2)\right] \mathrm{d}\sigma(t_1)\cdots\mathrm{d}\sigma(t_r) \\
&= B_j^{-(q+q')/2} \int_{(S^2)^2} \mathrm{d}\sigma(x_1)\,\mathrm{d}\sigma(x_2) \prod_{n=1}^{q-r} \Theta_j\big(\langle x_1, y_n\rangle\big) \prod_{m=q-r+1}^{q+q'-2r} \Theta_j\big(\langle x_2, y_m\rangle\big) \\
&\quad \times \int_{(S^2)^r} \prod_{i=1}^{r} \Theta_j\big(\langle x_1, t_i\rangle\big) \Theta_j\big(\langle x_2, t_i\rangle\big)\,\mathrm{d}\sigma(t_1)\cdots\mathrm{d}\sigma(t_r) \\
&= B_j^{-(q+q')/2} \int_{(S^2)^2} \mathrm{d}\sigma(x_1)\,\mathrm{d}\sigma(x_2) \prod_{n=1}^{q-r} \Theta_j\big(\langle x_1, y_n\rangle\big) \prod_{m=q-r+1}^{q+q'-2r} \Theta_j\big(\langle x_2, y_m\rangle\big) \rho_j^r\big(\langle x_1, x_2\rangle\big),
\end{aligned}$$

for Θ_j and ρ_j as in (3.3) and (3.4). It follows that

$$\begin{aligned}
&\big\| g_{q,j}(y_1,\ldots,y_q) \otimes_r g_{q',j}(y_1,\ldots,y_{q'})\big\|^2_{\mathfrak{H}^{\otimes q+q'-2r}} \\
&= B_j^{-(q+q')} \int_{(S^2)^{q+q'-2r}} \mathrm{d}\sigma(y_1)\cdots\mathrm{d}\sigma(y_{q+q'-2r}) \Bigg[\int_{(S^2)^4} \prod_{n=1}^{q-r} \Theta_j\big(\langle x_1, y_n\rangle\big) \Theta_j\big(\langle x_3, y_n\rangle\big)
\end{aligned}$$

$$\times \prod_{m=q-r+1}^{q+q'-2r} \Theta_j\big(\langle x_2, y_m\rangle\big)\Theta_j\big(\langle x_4, y_m\rangle\big)\rho_j^r\big(\langle x_1, x_2\rangle\big)\rho_j^r\big(\langle x_3, x_4\rangle\big)\,\mathrm{d}\sigma(x_1)\cdots \mathrm{d}\sigma(x_4)\Bigg]$$

$$= B_j^{-(q+q')}\int_{(S^2)^4} \rho_j^{q-r}\big(\langle x_1, x_3\rangle\big)\rho_j^{q'-r}\big(\langle x_2, x_4\rangle\big)\rho_j^r\big(\langle x_1, x_2\rangle\big)\rho_j^r\big(\langle x_3, x_4\rangle\big)\,\mathrm{d}\sigma(x_1)\cdots \mathrm{d}\sigma(x_4).$$

Since $\rho_j(\langle x, y\rangle) \le B_j$ and from (3.2)

$$\int_{S^2} \rho_j^p\big(\langle x, y\rangle\big)\,\mathrm{d}\sigma(x) \le \int_{S^2}\left(\frac{K_M}{(1+2^j d(x,y))^M} B_j\right)^p \mathrm{d}\sigma(x) \le B_j^p K_M^p 2^{-2j},$$

we have

$$\begin{aligned}
&\big\| g_{q,j}(y_1,\dots,y_q) \otimes_r g_{q',j}(y_1,\dots,y_{q'})\big\|^2_{\mathfrak{H}^{\otimes q+q'-2r}} \\
&\quad = B_j^{-(q+q')} B_j^r \int_{(S^2)^4} \rho_j^{q-r}\big(\langle x_1, x_3\rangle\big)\rho_j^{q'-r}\big(\langle x_2, x_4\rangle\big)\rho_j^r\big(\langle x_1, x_2\rangle\big)\,\mathrm{d}\sigma(x_1)\cdots \mathrm{d}\sigma(x_4) \\
&\quad \le \mathrm{const}\, 2^{-6j},
\end{aligned}$$

and analogously

$$\begin{aligned}
&\big\| g_{q,j}(y_1,\dots,y_q) \otimes_{q-r} g_{q,j}(y_1,\dots,y_q)\big\|^2_{\mathfrak{H}^{\otimes 2r}} \\
&\quad = B_j^{-2q} \int_{(S^2)^4} \rho_j^{q-r}\big(\langle x_1, x_3\rangle\big)\rho_j^{q-r}\big(\langle x_2, x_4\rangle\big)\rho_j^r\big(\langle x_1, x_2\rangle\big)\rho_j^r\big(\langle x_3, x_4\rangle\big)\,\mathrm{d}\sigma(x_1)\cdots \mathrm{d}\sigma(x_4) \\
&\quad \le \mathrm{const}\, 2^{-6j}.
\end{aligned}$$

□

Proof of Theorem 9. Let us introduce the following notation:

$$\tilde{\Phi}_{j,N}(z) = 2^j \sum_{q=2}^{N} \frac{\mathcal{J}_q(z)}{q!}\nu_{j;q}, \qquad \sigma_N^2 = \frac{\mathrm{Var}[\tilde{\Phi}_{j,N}(z)]}{\mathrm{Var}[2^j \Phi_j(z)]}, \qquad \mathcal{N}_N \sim \mathcal{N}\big(0, \sigma_N^2\big).$$

We have that

$$\begin{aligned}
d_W\left(\frac{\Phi_j(z)}{\sqrt{\mathrm{Var}[\Phi_j(z)]}}, \mathcal{N}\right) &\le d_W\left(\frac{\Phi_j(z)}{\sqrt{\mathrm{Var}[\Phi_j(z)]}}, \frac{\tilde{\Phi}_{j,N}(z)}{\sqrt{\mathrm{Var}[2^j \Phi_j(z)]}}\right) \\
&\quad + d_W\left(\frac{\tilde{\Phi}_{j,N}(z)}{\sqrt{\mathrm{Var}[2^j \Phi_j(z)]}}, \mathcal{N}_N\right) + d_W(\mathcal{N}_N, \mathcal{N}).
\end{aligned}$$

• For the first term we apply the properties of the Wasserstein distance to get:

$$\begin{aligned}
&d_W\left(\frac{\Phi_j(z)}{\sqrt{\mathrm{Var}[\Phi_j(z)]}}, \frac{\tilde{\Phi}_{j,N}(z)}{\sqrt{\mathrm{Var}[2^j \Phi_j(z)]}}\right) \\
&\quad \le \left\{\mathbb{E}\left[\frac{\Phi_j(z)}{\sqrt{\mathrm{Var}[\Phi_j(z)]}} - \frac{\tilde{\Phi}_{j,N}(z)}{\sqrt{\mathrm{Var}[2^j \Phi_j(z)]}}\right]^2\right\}^{1/2} \\
&\quad = \frac{1}{\sqrt{\mathrm{Var}[2^j \Phi_j(z)]}}\left\{\mathbb{E}\left[2^j \int_{S^2} \sum_{q=N+1}^{\infty} \frac{\mathcal{J}_q(z)}{q!} H_q\big(\tilde{\beta}_j(x)\big)\,\mathrm{d}\sigma(x)\right]^2\right\}^{1/2},
\end{aligned}$$

and since $2^j \Phi_j(z) - \tilde{\Phi}_{j,N}(z)$ belongs to the Hilbert space of Gaussian subordinated random variables, with continuous inner product $\langle X, Y\rangle := \mathbb{E}[XY]$, we have

$$d_W\left(\frac{\Phi_j(z)}{\sqrt{\operatorname{Var}[\Phi_j(z)]}}, \frac{\tilde{\Phi}_{j,N}(z)}{\sqrt{\operatorname{Var}[2^j \Phi_j(z)]}}\right) \le \frac{1}{\sqrt{\operatorname{Var}[2^j \Phi_j(z)]}} \left\{ \sum_{q=N+1}^{\infty} \frac{\mathcal{J}_q^2(z)}{(q!)^2} 2^{2j} \mathbb{E}\left[\nu_{j;q}^2\right] \right\}^{1/2}.$$

Since for any finite z, as $q \to \infty$, the asymptotic formula $\mathrm{e}^{-z^2/4} H_q(z) \le \operatorname{const} q^{q/2} \mathrm{e}^{-q/2}$ holds (see e.g. [12], formula (4.14.9)), by applying the Stirling's approximation to the factorial $(q-1)!$ we have (see [38]),

$$\frac{\mathcal{J}_q^2(z)}{q!} = \frac{\phi(z)}{q!}\left[\mathrm{e}^{-z^2/4} H_{q-1}(z)\right]^2 \le \operatorname{const} \frac{\phi(z)}{q\sqrt{q-1}}.$$

From this we obtain the first bound, in fact form Theorem 1, we have

$$\begin{aligned} d_W\left(\frac{\Phi_j(z)}{\sqrt{\operatorname{Var}[\Phi_j(z)]}}, \frac{\tilde{\Phi}_{j,N}(z)}{\sqrt{\operatorname{Var}[2^j \Phi_j(z)]}}\right) &\le \operatorname{const} \left\{ \sum_{q=N+1}^{\infty} \frac{1}{q\sqrt{q-1}} \frac{2^{2j}}{q!} \mathbb{E}\left[\nu_{j;q}^2\right] \right\}^{1/2} \\ &\le \operatorname{const} N^{-1/4}. \end{aligned}$$

• To bound the second term, we apply now Proposition 2.2 and we get

$$d_W\left(\frac{\tilde{\Phi}_{j,N}(z)}{\sqrt{\operatorname{Var}[2^j \Phi_j(z)]}}, \mathcal{N}_N\right) \le \frac{\sqrt{2}}{\sigma_N \sqrt{\pi}} \frac{1}{\operatorname{Var}[2^j \Phi_j(z)]} \mathbb{E}\left[\left|\operatorname{Var}\left[\tilde{\Phi}_{j,N}(z)\right] - \left\langle D\tilde{\Phi}_{j,N}(z), -DL^{-1}\tilde{\Phi}_{j,N}(z)\right\rangle_{\mathfrak{H}}\right|\right].$$

Since, in view of (2.2), we have

$$\begin{aligned} \operatorname{Var}\left[\tilde{\Phi}_{j,N}(z)\right] &= \sum_{q=2}^{N} \sum_{q'=2}^{N} \frac{\mathcal{J}_q(z)}{q!} \frac{\mathcal{J}_{q'}}{q'!} 2^{2j} \operatorname{Cov}[\nu_{j;q}, \nu_{j;q'}] \\ &= \sum_{q=2}^{N} \sum_{q'=2}^{N} \frac{\mathcal{J}_q(z)}{q!} \frac{\mathcal{J}_{q'}}{q'!} 2^{2j} \delta_q^{q'} q! \int_{S^2\times S^2} \left\{\mathbb{E}\left[\tilde{\beta}_j(x)\tilde{\beta}_j(y)\right]\right\}^q \mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y) \\ &= \sum_{q=2}^{N} \frac{\mathcal{J}_q^2(z)}{(q!)^2} 2^{2j} \operatorname{Var}[\nu_{j;q}], \end{aligned}$$

we write

$$\begin{aligned} & d_W\left(\frac{\tilde{\Phi}_{j,N}(z)}{\sqrt{\operatorname{Var}[2^j \Phi_j(z)]}}, \mathcal{N}_N\right) \\ &\quad \le \frac{\sqrt{2}}{\sigma_N\sqrt{\pi}} \frac{2^{2j}}{\operatorname{Var}[2^j\Phi_j(z)]} \sum_{q=2}^{N} \frac{\mathcal{J}_q(z)}{q!} \mathbb{E}\left[\left| \frac{\mathcal{J}_q(z)}{q!} \operatorname{Var}[\nu_{j;q}] - \sum_{q'=2}^{N} \frac{\mathcal{J}_{q'}(z)}{q'!} \left\langle D\nu_{j;q}, -DL^{-1}\nu_{j;q'}\right\rangle_{\mathfrak{H}} \right|\right] \\ &\quad \le \frac{\sqrt{2}}{\sigma_N\sqrt{\pi}} \frac{2^{2j}}{\operatorname{Var}[2^j\Phi_j(z)]} \sum_{q=2}^{N} \frac{\mathcal{J}_q^2(z)}{(q!)^2} \mathbb{E}\left[\left|\operatorname{Var}[\nu_{j;q}] - \left\langle D\nu_{j;q}, -DL^{-1}\nu_{j;q}\right\rangle_{\mathfrak{H}}\right|\right] \\ &\qquad + \frac{\sqrt{2}}{\sigma_N\sqrt{\pi}} \frac{2^{2j}}{\operatorname{Var}[2^j\Phi_j(z)]} \sum_{q=2}^{N} \frac{\mathcal{J}_q(z)}{q!} \sum_{q\ne q'} \frac{\mathcal{J}_{q'}(z)}{q'!} \mathbb{E}\left[\left|\left\langle D\nu_{j;q}, -DL^{-1}\nu_{j;q'}\right\rangle_{\mathfrak{H}}\right|\right]. \end{aligned}$$

By Theorem 2.9.1 in [27] and by Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
& d_W\left(\frac{\tilde{\Phi}_{j,N}(z)}{\sqrt{\operatorname{Var}[2^j \Phi_j(z)]}}, \mathcal{N}_N\right) \\
& \quad \le \frac{\sqrt{2}}{\sigma_N \sqrt{\pi}} \frac{2^{2j}}{\operatorname{Var}[2^j \Phi_j(z)]} \sum_{q=2}^{N} \frac{\mathcal{J}_q^2(z)}{(q!)^2} \left\{\operatorname{Var}\left[\langle D\nu_{j;q}, -DL^{-1}\nu_{j;q}\rangle_{\mathfrak{H}}\right]\right\}^{1/2} \\
& \qquad + \frac{\sqrt{2}}{\sigma_N \sqrt{\pi}} \frac{2^{2j}}{\operatorname{Var}[2^j \Phi_j(z)]} \sum_{q=2}^{N} \frac{\mathcal{J}_q(z)}{q!} \sum_{q \neq q'} \frac{\mathcal{J}_{q'}(z)}{q'!} \left\{\mathbb{E}\left[\left(\langle D\nu_{j;q}, -DL^{-1}\nu_{j;q'}\rangle_{\mathfrak{H}}\right)^2\right]\right\}^{1/2}.
\end{aligned}
$$

Finally, in view of Lemma 6.1, we write

$$
\begin{aligned}
& d_W\left(\frac{\tilde{\Phi}_{j,N}(z)}{\sqrt{\operatorname{Var}[2^j \Phi_j(z)]}}, \mathcal{N}_N\right) \\
& \quad \le \text{const} \frac{\sqrt{2}}{\sigma_N \sqrt{\pi}} \frac{2^{2j}}{\operatorname{Var}[2^j \Phi_j(z)]} 2^{-3j} \left\{ \sum_{q=2}^{N} \frac{\mathcal{J}_q^2(z)}{(q!)^2} q \sqrt{\sum_{r=1}^{q-1} (r-1)!^2 \binom{q-1}{r-1}^4 (2q-2r)!} \right. \\
& \qquad \left. + \sum_{q=2}^{N} \frac{\mathcal{J}_q(z)}{q!} \sum_{q \neq q'} \frac{\mathcal{J}_{q'}(z)}{q'!} q \sqrt{\sum_{r=1}^{q \wedge q'} (r-1)!^2 \binom{q-1}{r-1}^2 \binom{q'-1}{r-1}^2 (q+q'-2r)!} \right\}.
\end{aligned}
$$

We now bound the two sums by reproducing in our case calculations analog to those performed in [38]:

$$
\begin{aligned}
& \sum_{r=1}^{q \wedge q'} (r-1)!^2 \binom{q-1}{r-1}^2 \binom{q'-1}{r-1}^2 (q+q'-2r)! \\
& \quad = (q-1)!(q'-1)! \sum_{r=1}^{q \wedge q'} \binom{q-1}{r-1} \binom{q'-1}{r-1} \binom{q+q'-2r}{q-r} \\
& \quad \le (q-1)!(q'-1)! \sum_{r=1}^{q \wedge q'} \binom{q-1}{r-1} \binom{q'-1}{r-1} 2^{q+q'-2r} \\
& \quad = (q-1)!(q'-1)! 2^{q+q'-2} \sum_{r=0}^{q \wedge q'-1} \binom{q-1}{r} \binom{q'-1}{r} 2^{-2r} \\
& \quad \le (q-1)!(q'-1)! 2^{q+q'-2} \left[\sum_{r=0}^{q \wedge q'-1} \binom{q-1}{r} 2^{-r}\right] \left[\sum_{r=0}^{q \wedge q'-1} \binom{q'-1}{r} 2^{-r}\right] \\
& \quad \le (q-1)!(q'-1)! 2^{q+q'-2} \left[\sum_{r=0}^{q-1} \binom{q-1}{r} 2^{-r}\right] \left[\sum_{r=0}^{q'-1} \binom{q'-1}{r} 2^{-r}\right] \\
& \quad = (q-1)!(q'-1)! 2^{q+q'-2} (1+1/2)^{q-1} (1+1/2)^{q'-1} = (q-1)!(q'-1)! 3^{q+q'-2},
\end{aligned} \tag{6.1}
$$

and likewise

$$\sum_{r=1}^{q-1}(r-1)!^2\binom{q-1}{r-1}^4(2q-2r)! \le \big[(q-1)!\big]^2 3^{2q-2}. \tag{6.2}$$

Since for any finite z, as $q\to\infty$, we have

$$\frac{\mathcal{J}_q(z)}{q!}=\frac{\phi(z)H_{q-1}(z)}{q!}\le \operatorname{const}\frac{\sqrt{\phi(z)}}{\sqrt{q!}\sqrt{q}(q-1)^{1/4}},$$

from (6.1) and (6.2), we obtain

$$\begin{aligned}
&\sum_{q=2}^{N}\frac{\mathcal{J}_q(z)}{q!}\sum_{q\neq q'}\frac{\mathcal{J}_{q'}(z)}{q'!}q\sqrt{\sum_{r=1}^{q\wedge q'}(r-1)!^2\binom{q-1}{r-1}^2\binom{q'-1}{r-1}^2\big(q+q'-2r\big)!}\\
&\quad\le\sum_{q=2}^{N}\frac{\mathcal{J}_q(z)}{q!}\sum_{q\neq q'}\frac{\mathcal{J}_{q'}(z)}{q'!}q\sqrt{(q-1)!\big(q'-1\big)!3^{q+q'-2}}\\
&\quad\le\operatorname{const}\sum_{q=2}^{N}\frac{3^{(q-1)/2}}{\sqrt{q}}\sum_{q'=2}^{N}\frac{3^{(q'-1)/2}}{\sqrt{q'}}\le\operatorname{const}3^N,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{q=2}^{N}\frac{\mathcal{J}_q^2(z)}{q!^2}q\sqrt{\sum_{r=1}^{q-1}(r-1)!^2\binom{q-1}{r-1}^4(2q-2r)!}\\
&\quad\le\sum_{q=2}^{N}\frac{\mathcal{J}_q^2(z)}{q!^2}q(q-1)!3^{q-1}\le\operatorname{const}\sum_{q=2}^{N}\frac{3^{q-1}}{q}\le\operatorname{const}3^N.
\end{aligned}$$

Since $\sigma_N^2\ge 1-\operatorname{const}N^{-1/2}$, it follows that the second term, as $N\to\infty$ is at most equal to

$$d_W\left(\frac{\tilde{\Phi}_{j,N}(z)}{\sqrt{\operatorname{Var}[2^j\Phi_j(z)]}},\mathcal{N}_N\right)\le\frac{\operatorname{const}}{\sqrt{1-N^{-1/2}}}\frac{3^N}{2^j}.$$

• For the third term, by Proposition 3.6.1 in [27],

$$\begin{aligned}
d_W(\mathcal{N}_N,\mathcal{N})&\le\sqrt{\frac{2}{\pi}}\frac{1}{1\vee\sqrt{\operatorname{Var}[\tilde{\Phi}_{j,N}(z)]/\operatorname{Var}[2^j\Phi_j(z)]}}\left|1-\frac{\operatorname{Var}[\tilde{\Phi}_{j,N}(z)]}{\operatorname{Var}[2^j\Phi_j(z)]}\right|\\
&=\sqrt{\frac{2}{\pi}}\left|\frac{\sum_{q=N+1}^{\infty}(\mathcal{J}_q^2(z)/q!)2^{2j}\mathbb{E}[\nu_{j,q}^2]/q!}{\sum_{q=2}^{\infty}(\mathcal{J}_q^2(z)/q!)2^{2j}\mathbb{E}[\nu_{j,q}^2]/q!}\right|\\
&\le\operatorname{const}N^{-1/2}.
\end{aligned}$$

Summing up the three bounds, choosing the speed $N=\log(2^j)/2$ and observing that the dominant term is $N^{-1/4}$, we arrive at the statement. □

Remark 6.1. *To obtain a bound on the Kolmogorov distance, it is enough to recall the standard inequality*

$$d_{\mathrm{Kol}}(F,\mathcal{N})\le 2\sqrt{d_W(F,\mathcal{N})}.$$

Acknowledgements

We thank Giovanni Peccati and Igor Wigman for many useful discussions on these issues; usual disclaimers apply.

References

[1] E. Anderes. On the consistent separation of scale and variance for Gaussian random fields. *Ann. Statist.* **38** (2) (2010) 870–893. MR2604700
[2] P. Baldi, G. Kerkyacharian, D. Marinucci and D. Picard. Asymptotics for spherical needlets. *Ann. Statist.* **37** (3) (2009) 1150–1171. MR2509070
[3] P. Baldi, G. Kerkyacharian, D. Marinucci and D. Picard. Subsampling needlet coefficients on the sphere. *Bernoulli* **15** (2009) 438–463. MR2543869
[4] G. Blum, S. Gnutzmann and U. Smilansky. Nodal domains statistics: A criterion for quantum chaos. *Phys. Rev. Lett.* **88** (2002) 114101.
[5] H. Dehling and M. S. Taqqu. The empirical process of some long-range dependent sequences with applications to U-statistics. *Ann. Statist.* **17** (4) (1989) 1767–1783. MR1026312
[6] S. Dodelson. *Modern Cosmology*. Academic Press, San Diego, 2003.
[7] S. Donzelli, F. K. Hansen, M. Liguori, D. Marinucci and S. Matarrese. On the linear term correction for needlets/wavelets non-Gaussianity estimators. *Astrophysical Journal* **755** (2012) 19.
[8] C. Durastanti, X. Lan and D. Marinucci. Needlet–Whittle estimates on the unit sphere. *Electron. J. Stat.* **7** (2013) 597–646. MR3035267
[9] R. Durrer. *The Cosmic Microwave Background*. Cambridge Univ. Press, Cambridge, 2008.
[10] I. S. Gradshteyn and I. M. Ryzhik. *Tables of Integrals, Series and Products*, 7th edition. Academic Press, Elsevier, 2007. MR2360010
[11] X. Lan and D. Marinucci. The needlets bispectrum. *Electron. J. Stat.* **2** (2008) 332–367. MR2411439
[12] N. N. Lebedev. *Special Functions and Their Applications*. Prentice-Hall Inc., Englewood Cliffs, NJ, 1965. MR0174795
[13] N. Leonenko. *Limit Theorems for Random Fields with Singular Spectrum. Mathematics and Its Applications* **465**. Kluwer Academic, Dordrecht, 1999. MR1687092
[14] W.-L. Loh. Fixed-domain asymptotics for a subclass of matérn-type Gaussian random fields. *Ann. Statist.* **33** (5) (2005) 2344–2394. MR2211089
[15] A. Malyarenko. *Invariant Random Fields on Spaces with a Group Action. Probability and Its Applications*. Springer, Heidelberg, 2012. MR2977490
[16] A. Malyarenko. Invariant random fields in vector bundles and application to cosmology. *Ann. Inst. Henri Poincaré Probab. Stat.* **47** (4) (2011) 1068–1095. MR2884225
[17] D. Marinucci and G. Peccati. *Random Fields on the Sphere. Representation, Limit Theorem and Cosmological Applications*. Cambridge Univ. Press, Cambridge, 2011. MR2840154
[18] D. Marinucci and G. Peccati. Mean square continuity on homogeneous spaces of compact groups. *Electron. Commun. Probab.* **18** (2013) no. 37, 10. MR3064996
[19] D. Marinucci, D. Pietrobon, A. Balbi, P. Baldi, P. Cabella, G. Kerkyacharian, P. Natoli, D. Picard and N. Vittorio. Spherical needlets for CMB data analysis. *Monthly Notices of the Royal Astronomical Society* **383** (2) (2008) 539–545.
[20] D. Marinucci and S. Vadlamani. High-frequency asymptotics for Lipschitz–Killing curvatures of excursion sets on the sphere, 2013. Available at arXiv:1303.2456.
[21] D. Marinucci and I. Wigman. On the area of excursion sets of spherical Gaussian eigenfunctions. *J. Math. Phys.* **52** (2011) 093301. MR2867816
[22] D. Marinucci and I. Wigman. The defect variance of random spherical harmonics. *J. Phys. A* **44** (2011) 355206.
[23] D. Marinucci and I. Wigman. On nonlinear functionals of random spherical eigenfunctions. *Comm. Math. Phys.* **327** (3) (2014) 849–872. MR3192051
[24] J. D. McEwen, P. Vielva, Y. Wiaux, R. B. Barreiro, I. Cayón, M. P. Hobson, A. N. Lasenby, E. Martínez-González and J. L. Sanz. Cosmological applications of a wavelet analysis on the sphere. *J. Fourier Anal. Appl.* **13** (4) (2007) 495–510. MR2329015
[25] F. J. Narcowich, P. Petrushev and J. D. Ward. Localized tight frames on spheres. *SIAM J. Math. Anal.* **38** (2006) 574–594. MR2237162
[26] I. Nourdin and G. Peccati. Stein's method on Wiener chaos. *Probab. Theory Related Fields* **145** (1–2) (2009) 75–118. MR2520122
[27] I. Nourdin and G. Peccati. *Normal Approximations Using Malliavin Calculus: From Stein's Method to Universality*. Cambridge Univ. Press, Cambridge, 2012. MR2962301
[28] G. Peccati and M. S. Taqqu. *Wiener Chaos: Moments, Cumulants and Diagrams. A Survey with Computer Implementations*. Springer, Milan, 2008. MR2791919
[29] D. Pietrobon, A. Balbi and D. Marinucci. Integrated Sachs–Wolfe effect from the cross correlation of WMAP3 year and the NRAO VLA sky survey data: New results and constraints on dark energy. *Phys. Rev. D* **74** (2006) 043524.
[30] Planck Collaboration. Planck 2013 results. XXIV. Constraints on primordial non-Gaussianity, 2013. Available at arXiv:1303.5084.
[31] Planck Collaboration. Planck 2013 results. XXII. Isotropy and statistics of the CMB, 2013. Available at arXiv:1303.5083.
[32] O. Rudjord, F. K. Hansen, X. Lan, M. Liguori, D. Marinucci and S. Matarrese. An estimate of the primordial non-Gaussianity parameter f_{NL} using the needlet bispectrum from WMAP. *Astrophysical Journal* **701** (1) (2009) 369–376.
[33] O. Rudjord, F. K. Hansen, X. Lan, M. Liguori, D. Marinucci and S. Matarrese. Directional variations of the non-Gaussianity parameter f_{NL}. *Astrophysical Journal* **708** (2) (2010) 1321–1325.
[34] J.-L. Starck, F. Murtagh and J. Fadili. *Sparse Image and Signal Processing: Wavelets, Curvelets, Morphological Diversity*. Cambridge Univ. Press, Cambridge, 2010. MR2643260

[35] M. L. Stein. *Interpolation of Spatial Data. Some Theory for Kriging. Springer Series in Statistics*. Springer, New York, 1999. MR1697409
[36] G. Szego. *Orthogonal Polynomials*, 4th edition. *Colloquium Publications*. American Mathematical Society, Providence, RI, 1975. MR0372517
[37] D. A. Varshalovich, A. N. Moskalev and V. K. Khersonskii. *Quantum Theory of Angular Momentum*. World Scientific, Teaneck, NJ, 1988. MR1022665
[38] V.-H. Pham. On the rate of convergence for central limit theorems of sojourn times of Gaussian fields. *Stochastic Process. Appl.* **123** (6) (2013) 2158–2174. MR3038501
[39] D. Wang and W.-L. Loh. On fixed-domain asymptotics and covariance tapering in Gaussian random field models. *Electron. J. Stat.* **5** (2011) 238–269. MR2792553
[40] I. Wigman. On the distribution of the nodal sets of random spherical harmonics. *J. Math. Phys.* **50** (1) (2009) 013521. MR2492631
[41] I. Wigman. Fluctuation of the nodal length of random spherical harmonics. *Comm. Math. Phys.* **298** (3) (2010) 787–831. MR2670928