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Universality of the asymptotics of the one-sided exit problem for integrated processes

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Abstract. We consider the one-sided exit problem – also called one-sided barrier problem – for (α-fractionally) integrated random walks and Lévy processes.

Our main result is that there exists a positive, non-increasing function $\alpha \mapsto \theta(\alpha)$ such that the probability that any α-fractionally integrated centered Lévy processes (or random walk) with some finite exponential moment stays below a fixed level until time T behaves as $T^{-\theta(\alpha)+o(1)}$ for large T. We also investigate when the fixed level can be replaced by a different barrier satisfying certain growth conditions (moving boundary).

This, in particular, extends Sinai's result on the survival exponent $\theta(1) = 1/4$ for the integrated simple random walk to general random walks with some finite exponential moment.

Résumé. Nous considérons le problème unilatéral de sortie – ou problème unilatéral de barrière – pour des intégrales (α-fractionnelles) de marches aléatoires et de processus de Lévy.

Notre résultat principal est l'existence d'une fonction positive, décroissante $\alpha \mapsto \theta(\alpha)$ telle que la probabilité qu'une intégrale d'un processus de Lévy α-fractionnel quelconque (ou marche aléatoire) avec certains moments exponentiels finis reste en dessous d'un niveau fixe jusqu'à un temps T se comporte comme $T^{-\theta(\alpha)+o(1)}$ pour T grand. Nous analysons aussi la possibilité de remplacer le niveau fixe par une barrière différente qui satisfait certaines conditions de croissance (marge mouvante).

Cela, en particulier, étend le résultat de Sinai sur l'exposant de survie d'une marche aléatoire simple intégrée à des marches aléatoires générales de moment exponentiel fini.

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1. Introduction

1.1. *Statement of the problem*

This article deals with the so-called one-sided exit problem – also called one-sided barrier problem. For a real-valued stochastic process $(A_t)_{t\geq 0}$, say with $A_0 = 0$, one investigates whether there is a $\theta > 0$ such that

$$\mathbb{P}\Big(\sup_{t\in[0,T]} A_t \leq 1\Big) = T^{-\theta+o(1)} \quad \text{as } T \to \infty. \tag{1}$$

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If such an exponent θ exists it is called the *survival exponent* or *persistence exponent*. The function $F \equiv 1$ acts as a barrier, which the process must not pass. We also discuss different barriers F below.

Another formulation of (1) can be obtained if the process is self-similar, i.e. (A_{ct}) and $(c^H A_t)$ have the same finite-dimensional distributions for some $H > 0$. Then the problem is equivalent to finding the lower tail probability of the supremum of the process up to time one:

$$\mathbb{P}\Big(\sup_{t\in[0,1]} A_t \le \varepsilon\Big) = \varepsilon^{\theta/H+\mathrm{o}(1)} \quad \text{as } \varepsilon \to 0. \tag{2}$$

Apart from this, we also look at the discrete version of (1):

$$\mathbb{P}\Big(\sup_{n=1,\dots,N} A_n \le 1\Big) = N^{-\theta+\mathrm{o}(1)} \quad \text{as } N \to \infty, \tag{3}$$

where $(A_n)_{n\in\mathbb{N}_0}$ is a discrete time stochastic process.

Obviously, problems (1), (2) and (3) are classical questions; they are relevant in a number of quite different applications. The most important of these is in statistical physics when studying the fractal nature of the solution of Burgers' equation, see e.g. [4,30] concerning results on this relation. Apart from this, the exponent plays a role in connection with pursuit problems (see [18] and references therein), in the study of most visited sites of a process (see e.g. [1]), in the investigation of zeros of random polynomials (see [8] and references therein and Section 4 below), and in connection with so-called sticky particles (see [35]). We refer to [18,19] for a recent overview of the applications. The question can also be encountered in the physics literature, see [21] for a summary. The discrete version (3) is studied in connection with random polymers, see [6].

It is therefore surprising that very little seems to be known about this type of problems. In fact, for (1) the exponent is known in the following cases: Brownian motion ($\theta = 1/2$, trivially obtained via the reflection principle), integrated Brownian motion ($\theta = 1/4$, [11,13,15,22,31], see also [12,16]), and fractional Brownian motion ($\theta = 1 - H$, [18, 23–25]). For Lévy processes there is a general framework for obtaining the survival exponent and even more precise information in many cases (see e.g. [2,3,5]).

It is even more surprising that for the discrete version (3) yet less seems to be known. The only case where the exponent is well understood are general random walks: the question of positivity for random walks is well studied, see e.g. [7] (e.g. $\theta = 1/2$ if the increments are centered, see for example [10]).

Further, the exponent is known for the integrated *simple* random walk ($\theta = 1/4$, [31]). Bounds for general integrated random walks are given in [6], polynomial bounds for integrated Gaussian random walks can be obtained from [18]. In several further special cases, Vysotsky [36] obtained $\theta = 1/4$. It was conjectured ([6,35]) that for any integrated random walk with finite variance the exponent is $\theta = 1/4$. This is not the case for integrated *heavy tailed* random walks and Lévy processes, cf. Simon [29]. Even though we also study integrated Lévy processes and random walks in this paper, the results and techniques are completely disjoint.

The focus of the present article is on integrated Lévy processes and integrated random walks: i.e. $A = \mathcal{I}(X)$ with $\mathcal{I}$ some integration operator and X a Lévy martingale or centered random walk. The main motivation for this work was that the exponent was known for integrated Brownian motion, but not for general integrated random walks. We will show that indeed the exponent is $\theta = 1/4$ under mild assumptions. We stress that the processes that we consider are non-Markovian.

The outline of this paper is as follows. Our main results are collected in Section 1.2. An important method in the proofs is the change of the barrier F; some tools in this connection are presented in Section 1.3 and may be of independent interest. The proof of the first main result, Theorem 1.1, is given in Section 2.2. Section 2.1 is concerned with an auxiliary result which may be of independent interest: it contains an a priori estimate for $\mathcal{I}$ being the identity. In Section 3, we prove and discuss the results concerning the change of the barrier. Using these arguments, the second main result, Theorem 1.4, in Section 3.2 follows easily. Finally, in Section 4 we give an application of our results to the question of random polynomials having no real zeros.

1.2. *Main results*

The goal of this article is to investigate the asymptotics of

$$\mathbb{P}\Big(\sup_{t\in J\cap[0,T]} A_t \le 1\Big) \quad \text{as } T\to\infty \tag{4}$$

for $J=\mathbb{N}_0$ or $J=[0,\infty)$. We show the following:

- For a fixed integration operator $\mathcal{I}$, the asymptotics of this probability for $A=\mathcal{I}(X)$ is universal over the class of Lévy processes and random walks X. The reason for this is that all of these processes can be coupled with a suitable Brownian motion. The resulting order of (4) can then be inferred from Brownian motion or any other process in this class, such as the simple random walk.
- The survial exponent exists for fractionally integrated and it is non-increasing in the order of integration. As a byproduct we show that the survival exponent of fractionally integrated Brownian motion (also called Riemann–Liouville process) is not the same as for the corresponding fractional Brownian motion (FBM).
- The problem is robust with respect to certain changes of the barrier F, which is equivalent to adding a drift to the process. In fact, adding a drift to a Gaussian process that is in its reproducing kernel Hilbert space, does not change the survival exponent of the (integrated) process.
- We exploit the connection of the one-sided exit problem to random polynomials established in [8] in order to improve the knowledge of the crucial constant appearing there.

Let us be more precise. We let $\mathcal{X}$ denote the class of all (non-deterministic, right-continuous) martingales $(X_t)_{t\ge0}$ with independent and stationary increments, $X_0=0$, satisfying

$$\mathbb{E}\big[e^{\beta|X_1|}\big]<\infty \quad \text{for some } \beta>0.$$

If the martingale is only defined on $\mathbb{N}_0$, we set $X_t:=X_{\lfloor t\rfloor}$ for all $t\ge0$ (which does not have stationary increments, but this will not be used outside $\mathbb{N}_0$).

Let us further specify the type of functionals $\mathcal{I}$ that we consider. We let $\mathcal{I}$ be a functional of the following convolution type:

$$\mathcal{I}(X)_t=\int_0^t K(t-s)X_s\,ds, \quad t\ge0,$$

where $K:[0,\infty)\to[0,\infty)$ is a measurable function satisfying

$$K(s)\le k\big[s^{\alpha-1}+s^{\beta-1}\big], \quad s>0, \tag{5}$$

for positive constants k, α, and β with $\alpha\ge\beta$. Additionally, we need to impose a regularity assumption on the tail behavior of K. Here, we assume that for some $k_+>0$

$$K(s)\ge k_- s^{\alpha-1} \quad \text{for large } s. \tag{6}$$

Alternatively, our proof works if K is ultimately decreasing. We remark that these technical assumptions can be further relaxed.

The main example is the integration operator:

$$\mathcal{I}_1(X)_t:=\int_0^t X_s\,ds, \quad t\ge0,$$

where $K(s)\equiv1$, but our definition also includes fractional integration operators

$$\mathcal{I}_\alpha(X)_t:=\int_0^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}X_s\,ds, \quad t\ge0, \tag{7}$$

where $\alpha > 0$ and Γ denotes Euler's Gamma function. In particular, if α is an integer, $\mathcal{I}_\alpha(X)$ is the α-times integrated process. For completeness we set $\mathcal{I}_0$ to be the identity; and we recall that $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = \mathcal{I}_{\alpha+\beta}$ for $\alpha, \beta \geq 0$.

Here and below we use $f \precsim g$ (or $g \succsim f$) if $\limsup f/g < \infty$ and $f \approx g$ if $f \precsim g$ and $g \precsim f$. Further, $f \lesssim g$ (or $g \gtrsim f$) means $\limsup f/g \leq 1$, and $f \sim g$ means that $f \lesssim g$ and $g \lesssim f$.

Our first main theorem reads as follows.

Theorem 1.1. *Let $(X_t)_{t\geq 0}$ be any process from the class $\mathcal{X}$ and W be a Brownian motion. Then, for either $J = \mathbb{N}_0$ or $J = [0, \infty)$,*

$$\mathbb{P}\Big(\sup_{t\in J\cap[0,T]} \mathcal{I}(W)_t \leq 1\Big)(\log T)^{-2(1+\alpha)}$$

$$\precsim \mathbb{P}\Big(\sup_{t\in J\cap[0,T]} \mathcal{I}(X)_t \leq 1\Big) \precsim \mathbb{P}\Big(\sup_{t\in J\cap[0,T]} \mathcal{I}(W)_t \leq 1\Big)(\log T)^{2(1+\alpha)},$$

where α is as in (5).

This means the asymptotics of all processes in the class $\mathcal{X}$ are equivalent up to logarithmic factors. This is the mentioned universality result. In particular, the survival exponent (if it exists) is universal over the class $\mathcal{X}$. The proof of Theorem 1.1 is given in Section 2.2.

A particularly important case is when $\mathcal{I}$ is the usual integration operator. Then the rate of the survival probability is known for X being Brownian motion or X being the simple random walk. This entails the following corollary for general random walks, generalizing Sinai [31].

Corollary 1.2. *Let $X_1, X_2, \ldots$ be a random walk started in* 0 *with $\mathbb{E}[e^{\beta|X_1|}] < \infty$ for some $\beta > 0$ and $\mathbb{E}[X_1] = 0$. Set $A_n = \sum_{i=1}^n X_i$. Then, as $N \to \infty$,*

$$N^{-1/4}(\log N)^{-4} \precsim \mathbb{P}\Big(\sup_{n=1,\ldots,N} A_n \leq 1\Big) \precsim N^{-1/4}(\log N)^4.$$

Similarly, we obtain the result for integrated Lévy processes.

Corollary 1.3. *Let $(X_t)_{t\geq 0}$ be a real-valued Lévy process with $\mathbb{E}[e^{\beta|X_1|}] < \infty$ for some $\beta > 0$ and $\mathbb{E}[X_1] = 0$. Set $A_t := \int_0^t X_s \, \mathrm{d}s$. Then, as $T \to \infty$,*

$$T^{-1/4}(\log T)^{-4} \precsim \mathbb{P}\Big(\sup_{t\in[0,T]} A_t \leq 1\Big) \precsim T^{-1/4}(\log T)^4.$$

Theorem 1.1 implies that the survival exponent θ is the same for any process from the class $\mathcal{X}$ if it exists. Now we prove that the survival exponent does indeed exist for the particularly important case of the α-fractional integration operator (7) and that it is decreasing in α.

Theorem 1.4. *There is a non-increasing function $\theta : [0, \infty) \to (0, 1/2]$, $\theta : \alpha \mapsto \theta(\alpha)$, such that for any process X from the class $\mathcal{X}$ and any $\alpha \geq 0$*

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \mathcal{I}_\alpha(X)_t \leq 1\Big) = T^{-\theta(\alpha)+o(1)} \quad \text{as } T \to \infty.$$

We recall that $\theta(0) = 1/2$ and $\theta(1) = 1/4$. Further, $\theta(\alpha) > 1/8$ for all $\alpha \geq 0$.

The proof of Theorem 1.4 is given in Section 3.2. Theorem 1.4 does not yield new values for θ, so it remains a challenge to calculate $\theta(\alpha)$, e.g. for integers α.

Let us relate the present results to those for fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. Theorem 1.4 is concerned with the α-fractionally integrated Brownian motion (also called Riemann–Liouville process) with $\alpha := H - 1/2 > 0$ defined by

$$R_t^\alpha := \mathcal{I}_\alpha(W)_t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} W_s \, \mathrm{d}s, \quad t \geq 0, \tag{8}$$

where W is a Brownian motion. Let X^α be a fractional Brownian motion with Hurst parameter $H = \alpha + 1/2$. It is well-known that there is a close relation between X^α and R^α.

For α-fractionally integrated Brownian motion, the survial exponent is investigated in Theorem 1.4. Further, we recall that the survival exponent for FBM with Hurst parameter H is known to be $\theta_{\mathrm{FBM}} = 1 - H$, see [23]. In view of Theorem 1.4 (the function θ is decreasing and $\theta(1) = 1/4$), it is clear that the survival exponents of the two processes X^α and R^α cannot coincide at least for $H > 3/4$. This fact may come as a surprise since often properties of X^α are the same as those of R^α.

Corollary 1.5. *For $\alpha \in (1/4, 1/2)$, the survival exponent of α-fractionally integrated Brownian motion $R^\alpha = \mathcal{I}_\alpha(W)$ is not equal to the survival exponent of FBM with the corresponding Hurst parameter $H := \alpha + 1/2 \in (3/4, 1)$.*

1.3. *The one-sided exit problem with moving boundary for Gaussian processes*

In this section, we study the influence of the barrier on the survival exponent for a Gaussian process. As a 'barrier' we consider a function $F : [0, \infty) \to (-\infty, \infty]$ and ask when

$$\mathbb{P}\bigl(\forall t \leq T\colon\ X_t \leq F(t)\bigr) \quad \text{as } T \to \infty \tag{9}$$

has the same asymptotics as

$$\mathbb{P}(\forall t \leq T\colon\ X_t \leq 1) \quad \text{as } T \to \infty.$$

Of course, (9) is equivalent to adding a drift to the process:

$$\mathbb{P}\Bigl(\sup_{0 \leq t \leq T} \bigl(X_t - F(t) + 1\bigr) \leq 1\Bigr) \quad \text{as } T \to \infty.$$

We show that one can safely add a drift of a certain strength without changing the survival exponent. The technique can be formulated rather generally in terms of the reproducing kernel Hilbert space of the Gaussian process.

Proposition 1.6. *Let X be some centered Gaussian process attaining values in the Banach space E with reproducing kernel Hilbert space $\mathcal{H}$. Denote by $\|\cdot\|$ the norm in $\mathcal{H}$. Then, for each $f \in \mathcal{H}$ and each measurable S such that $\mathbb{P}(X \in S) \in (0, 1)$, we have*

$$\mathrm{e}^{-\sqrt{2\|f\|^2 \log(1/\mathbb{P}(X \in S))} - \|f\|^2/2} \leq \frac{\mathbb{P}(X + f \in S)}{\mathbb{P}(X \in S)} \leq \mathrm{e}^{\sqrt{2\|f\|^2 \log(1/\mathbb{P}(X \in S))} - \|f\|^2/2}.$$

This statement allows to estimate $\mathbb{P}(X + f \in S)$ by the respective probability without drift. Of course, we are interested in the set

$$S := S_T := \Bigl\{(x_t)_{0 \leq t \leq T}\colon \sup_{t \in [0,T]} \mathcal{I}(x)_t \leq 1\Bigr\},$$

where $\mathcal{I}$ is a functional as specified above. If the order of $\mathbb{P}(X \in S_T)$, when $T \to \infty$, is polynomial with exponent θ then, by Proposition 1.6, the same holds for $\mathbb{P}(X + f \in S_T)$, for $f \in \mathcal{H}$. Let us illustrate the use of Proposition 1.6 in this context.

Corollary 1.7. *Let W be a Brownian motion, $\mathcal{I}$ be a functional as specified above, $f' : [0,\infty) \to \mathbb{R}$ be a measurable function with $\int_0^\infty f'(s)^2\,\mathrm{d}s < \infty$, and set $f(t) := \int_0^t f'(s)\,\mathrm{d}s$. Let $\theta > 0$. Then*

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \mathcal{I}(W)_t \le 1\Big) = T^{-\theta+\mathrm{o}(1)} \quad \textit{if and only if} \quad \mathbb{P}\Big(\sup_{t\in[0,T]} \mathcal{I}(W+f)_t \le 1\Big) = T^{-\theta+\mathrm{o}(1)}.$$

Also, upper (lower) bounds imply upper (lower) bounds.

Example 1.8. *It is interesting to note that, for Brownian motion, one can add any continuous function f with $f(0) < 1$ and $|f(t)| \lesssim t^\gamma$, $t \to \infty$, with $\gamma < 1/2$, since for some $c > 0$*

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \big(W_t + ct^\gamma\big) \le 1\Big) \le \mathbb{P}\Big(\sup_{t\in[0,T]} \big(W_t + f(t)\big) \le 1\Big) \le \mathbb{P}\Big(\sup_{t\in[0,T]} \big(W_t - ct^\gamma\big) \le 1\Big)$$

and Corollary 1.7 yields that for any $c \in \mathbb{R}$ and $0 \le \gamma < \frac{1}{2}$ with $c < 1$ for $\gamma = 0$:

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \big(W_t + ct^\gamma\big) \le 1\Big) = T^{-1/2+\mathrm{o}(1)}. \tag{10}$$

Similarly, for any $c \in \mathbb{R}$ and $0 \le \gamma < \frac{1}{2}$ ($c < 1$ for $\gamma = 0$):

$$\mathbb{P}\left(\sup_{t\in[0,T]} \left(\int_0^t W_s\,\mathrm{d}s + ct^{1+\gamma}\right) \le 1\right) = T^{-1/4+\mathrm{o}(1)}. \tag{11}$$

We remark that (11) improves Sinai's result [31] who showed this statement for $\gamma = 0$.

Remark 1.9. *Concerning (10), we remark that [34] even shows that for any $0 \le \gamma < \frac{1}{2}$*

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \big(W_t + ct^\gamma\big) \le 1\Big) \approx T^{-1/2}.$$

However, the proof of this result is very particular for Brownian motion and it does not seem to extend on our setting. We will need the more general result in Corollary 1.7. Further, we note that similar ideas can be found in [27].

The proof of Proposition 1.6 and further examples are given in Section 3.1.

2. Proof of the universality result

2.1. *A priori estimate via Skorokhod embedding*

In the proof of Theorem 1.1, we need an a priori estimate for the supremum of X when, simultaneously, letting the barrier and the time horizon tend to zero and infinity, respectively. Here we provide a way of obtaining such an estimate. We do not require finite exponential moments in this context.

Proposition 2.1. *Let X be either a Lévy martingale or a random walk with centered increments with $\mathbb{V}(X_1) = \sigma^2 > 0$. Let $(b_t)_{t\ge 0}$ be such that $t^{-\delta} \lesssim b_t \le \mathrm{o}(t^{1/2})$, as $t \to \infty$, for some $\delta \ge 0$. Suppose that $\mathbb{E}|X_1|^{2p} < \infty$ for some $p > 2\delta + 1$ and $p \ge 2$. Then we have*

$$\mathbb{P}\Big(\sup_{s\in[0,t]} X_s \le b_t\Big) \gtrsim \sqrt{\frac{2b_t^2}{\pi\sigma^2 t}} \quad \textit{as } t \to \infty.$$

Remark 2.2. *Note that the estimate is sharp in the sense that one gets $\sim$ instead of $\gtrsim$ if X is a Brownian motion.*

Proof of Proposition 2.1. First suppose that $X=(X_t)_{t\ge 0}$ is a Lévy martingale. Fix p such that $p>2\delta+1$ and $\mathbb{E}|X_1|^{2p}<\infty$, where δ is as in the statement of the proposition.

Embedding. We apply a Monroe [26] embedding. On an appropriate filtered probability space (possibly one needs to enlarge the underlying probability space), one can define a (right-continuous) family of finite *minimal* stopping times $(\tau(t))_{t\ge 0}$ and a Brownian motion (W_t) such that almost surely

$$X_t = W_{\tau(t)}$$

for all times $t\ge 0$. By [26], the minimality of the stoping times $\tau(t)$ is equivalent to the fact that $(W_{s\wedge\tau(t)})_{s\ge 0}$ is uniformly integrable for all $t\ge 0$. Consequently, $\mathbb{E}[\tau(t)]=\mathbb{E}[W^2_{\tau(t)}]=\mathbb{E}[X_t^2]=\sigma^2 t<\infty$. Moreover, by the Burkholder–Davis–Gundy (BDG) inequality (see the version in [28], Proposition 2.1), one has

$$\mathbb{E}\bigl[\tau(1)^p\bigr]\le c_1\mathbb{E}\bigl[|W_{\tau(1)}|^{2p}\bigr]=c_1\mathbb{E}\bigl[|X_1|^{2p}\bigr]<\infty, \tag{12}$$

where $c_1=c_1(p)$ is a constant that depends only on p.

Since (X_t) has stationary and independent increments, the embedding can be established such that $(\tau(t))_{t\ge 0}$ itself has stationary and independent increments, see [26]. Hence, $(\tau(t)-\sigma^2 t)$ is a martingale and we conclude with the BDG inequality that

$$\mathbb{E}\bigl[\bigl|\tau(t)-\sigma^2 t\bigr|^p\bigr]\le \mathbb{E}\Bigl[\sup_{0\le s\le\lceil t\rceil}\bigl|\tau(s)-\sigma^2 s\bigr|^p\Bigr]\le c_2\mathbb{E}\bigl[\bigl[\tau(\cdot)-\sigma^2\cdot\bigr]_{\lceil t\rceil}^{p/2}\bigr],$$

where $c_2=c_2(p)$ is an appropriate constant and $[\cdot]$ denotes the classical bracket process. Next, we apply the triangle inequality ($p/2\ge 1$) together with the stationarity of the increments of $(\tau(t)-\sigma^2 t)$ to conclude that

$$\begin{aligned}
\mathbb{E}\bigl[\bigl[\tau(\cdot)-\sigma^2\cdot\bigr]_{\lceil t\rceil}^{p/2}\bigr]^{2/p} &= \mathbb{E}\Biggl[\Biggl(\sum_{n=1}^{\lceil t\rceil}\bigl[\tau(\cdot)-\sigma^2\cdot\bigr]_n-\bigl[\tau(\cdot)-\sigma^2\cdot\bigr]_{n-1}\Biggr)^{p/2}\Biggr]^{2/p}\\
&\le \sum_{n=1}^{\lceil t\rceil}\mathbb{E}\bigl[\bigl(\bigl[\tau(\cdot)-\sigma^2\cdot\bigr]_n-\bigl[\tau(\cdot)-\sigma^2\cdot\bigr]_{n-1}\bigr)^{p/2}\bigr]^{2/p}\\
&\le \lceil t\rceil\mathbb{E}\bigl[\bigl[\tau(\cdot)-\sigma^2\cdot\bigr]_1^{p/2}\bigr]^{2/p}.
\end{aligned}$$

Hence,

$$\mathbb{E}\bigl[\bigl|\tau(t)-\sigma^2 t\bigr|^p\bigr]\le c_2\lceil t\rceil^{p/2}\mathbb{E}\bigl[\bigl[\tau(\cdot)-\sigma^2\cdot\bigr]_1^{p/2}\bigr].$$

It remains to verify the finiteness of the latter expectation. First observe that by the BDG inequality

$$\mathbb{E}\bigl[\bigl[\tau(\cdot)-\sigma^2\cdot\bigr]_1^{p/2}\bigr]\le c_3\mathbb{E}\Bigl[\sup_{s\in[0,1]}\bigl|\tau(s)-\sigma^2 s\bigr|^p\Bigr]\le 2^p c_3\bigl(\mathbb{E}\bigl[\tau(1)^p\bigr]+\sigma^{2p}\bigr),$$

where $c_3=c_3(p)$ is an appropriate constant. By (12), $\mathbb{E}[\tau(1)^p]$ is finite, and there exists a constant c_4 depending on p and the $2p$th moment of X_1 such that for all $t>0$

$$\mathbb{E}\bigl[\bigl|\tau(t)-\sigma^2 t\bigr|^p\bigr]\le c_4\lceil t\rceil^{p/2}. \tag{13}$$

Estimate of the probability. Fix $\varepsilon>0$ and observe that

$$\mathbb{P}\Bigl(\sup_{s\in[0,t]}X_s\le b_t\Bigr)\ge \mathbb{P}\Bigl(\sup_{s\in[0,(1+\varepsilon)t\sigma^2]}W_s\le b_t\Bigr)-\mathbb{P}\bigl(\tau(t)\ge(1+\varepsilon)\sigma^2 t\bigr). \tag{14}$$

Note that the first term on the right hand side of the latter equation can be computed explicitly:

$$\mathbb{P}\Bigl(\sup_{s\in[0,(1+\varepsilon)t\sigma^2]}W_s\le b_t\Bigr)=\sqrt{\frac{2}{\pi}}\int_0^{b_t/\sqrt{(1+\varepsilon)t\sigma^2}}\mathrm{e}^{-y^2/2}\,\mathrm{d}y\sim\sqrt{\frac{2}{\pi}}\frac{b_t}{\sqrt{(1+\varepsilon)t\sigma^2}}.$$

In the last step, we used that $b_t^2/t \to 0$. Conversely, the second term in (14) can be controlled via the Chebyshev inequality and (13):

$$\mathbb{P}\big(\tau(t) - \sigma^2 t \geq \varepsilon\sigma^2 t\big) \leq \frac{\mathbb{E}[|\tau(t) - \sigma^2 t|^p]}{(\varepsilon\sigma^2 t)^p} \leq c_4 \frac{\lceil t\rceil^{p/2}}{(\varepsilon\sigma^2 t)^p} \approx t^{-p/2}.$$

By the choice of p, the second term on the right hand side of (14) is of lower order than the first term. We obtain the lower bound in the proposition by letting ε tend to zero.

If X is a centered random walk, the same argument goes through with the bracket denoting the sum of the squared increments. □

The last result yields the following corollary for the integrated process $\mathcal{I}(X)$. It assures that the decrease of the survival probability is always at most polynomial.

Corollary 2.3. *Let X be either a Lévy martingale or a random walk with centered increments, $\mathcal{I}$ as in the introduction, and α as in (5). Assume that $\mathbb{E}|X_1|^{2p} < \infty$ for some $p > 2\alpha + 1$ and $p \geq 2$. Then*

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \mathcal{I}(X)_t \leq 1\Big) \gtrsim T^{-(\alpha+1/2)}.$$

Proof. Let α be as in (5) and note that there is a constant $c \in (0,\infty)$ with $\int_0^t K(s)\,\mathrm{d}s \leq ct^\alpha$ for all $t \geq 1$. We conclude that for $T \geq 1$

$$\sup_{t\in[0,T]} \mathcal{I}(X)_t \leq cT^\alpha \sup_{t\in[0,T]} X_t;$$

so that the assertion of the corollary readily follows from Proposition 2.1. □

The estimate from the previous corollary is in general far from optimal. We shall use it as an a priori estimate.

Finally, we recall the following result for Brownian motion with drift. It can be obtained from the distribution of the first hitting time of Brownian motion with a line, which is explicitly known, see e.g. [33], p. 217.

Lemma 2.4. *Let $\sigma > 0$ and W be a Brownian motion. Then*

$$\mathbb{P}\bigg(\sigma W_t \leq 1 - \frac{t}{\sqrt{T}}, \forall t \leq T\bigg) \gtrsim T^{-1/2}.$$

2.2. *Proof of Theorem* 1.1

Here we give the proof of Theorem 1.1.

Proof of Theorem 1.1. For an arbitrary fixed process X from the class $\mathcal{X}$ and a Wiener process W, we shall show that

$$\mathbb{P}\Big(\sup_{t\in J\cap[0,T]} \mathcal{I}(X)_t \leq 1\Big) \geq c(\log T)^{-2(\alpha+1)}\mathbb{P}\Big(\sup_{t\in J\cap[0,T]} \mathcal{I}(W)_t \leq 1\Big) \tag{15}$$

for T large enough and some constant $c > 0$. The opposite bound follows by the same method when exchanging the roles of W and X.

Step 1: In the first step, we derive one of the key techniques used in the proof (an appropriate coupling of X and σW with $\sigma > 0$ and $\sigma^2 = \mathbb{V}(X_1)$) from the Komlós–Major–Tusnády (KMT) coupling. Since X_1 has some finite exponential moments one can apply the KMT theorem [14]: there exist positive constants β_1, β_2 such that for every $T \in \mathbb{N}_0$ there exists a coupling of X and σW with

$$\mathbb{E}\Big[\exp\Big(\beta_1 \sup_{t\in\{0,\dots,T\}} |X_t - \sigma W_t|\Big)\Big] \leq \exp\big(\beta_2 \log(T \vee \mathrm{e})\big). \tag{16}$$

As we indicate next, we can take the supremum in the last equation equally well over the interval $[0,T]$ with $T\in(0,\infty)$, possibly with different constants β_1,β_2. Indeed, let $\beta_3>0$. If X is a Lévy martingale then $\exp(\frac{\beta_3}{2}|X_t|)$ is a non-negative submartingale, and we get with Doob's inequality that

$$\mathbb{E}\Big[\exp\Big(\beta_3\sup_{t\in[0,1]}|X_t|\Big)\Big]=\mathbb{E}\Big[\Big(\sup_{t\in[0,1]}\mathrm{e}^{\beta_3/2|X_t|}\Big)^2\Big]\le 4\mathbb{E}\big[\mathrm{e}^{\beta_3|X_1|}\big].$$

Consequently,

$$\mathbb{E}\Big[\sup_{t\in\{1,\dots,T\}}\exp\Big(\beta_3\sup_{s\in[t-1,t]}|X_s-X_{t-1}|\Big)\Big]\le T\,\mathbb{E}\Big[\exp\Big(\beta_3\sup_{t\in[0,1]}|X_t|\Big)\Big]\le 4T\,\mathbb{E}\big[\mathrm{e}^{\beta_3|X_1|}\big];$$

and the right hand side is finite as long as β_3 is sufficiently small. One gets an analogous estimate when replacing the Lévy process by the Wiener process. Now, an application of the triangle inequality and Hölder inequality, together with straightforward calculations yield the mentioned stronger version of (16).

Let α be as in (5). We fix $\rho>\alpha+1/2$. By the exponential Chebyshev inequality, we get for $T\ge\mathrm{e}$ and arbitrary $a>0$

$$\mathbb{P}\Big(\sup_{t\in[0,T]}|X_t-\sigma W_t|\ge a\Big)\le \mathrm{e}^{-\beta_1 a}T^{\beta_2},$$

which implies for $a_T:=\frac{\beta_2+\rho}{\beta_1}\log T$ that

$$\mathbb{P}\Big(\sup_{t\in[0,T]}|X_t-\sigma W_t|\ge a_T\Big)\le T^{-\rho}. \tag{17}$$

Step 2: In order to prove (15), we consider a particular scenario for which $\sup_{t\in[0,T]}\mathcal{I}(X)_t\le 1$ is satisfied. We couple X and σW on the time interval $[0,T_0]$ (where T_0 is chosen presently) as described above. Moreover, we apply the same coupling for the two processes $(X_t-X_{T_0})_{t\in[T_0,T]}$ and $(\sigma W_t-\sigma W_{T_0})_{t\in[T_0,T]}$. Certainly, both couplings can be established on a common probability space in such a way that the random variables involved in the first coupling are independent from the ones involved in the second coupling.

We fix $\delta_1,\delta_2>0$ with $\delta_2\int_0^{\delta_1}K(s)\,\mathrm{d}s\ge 1$ and consider the barriers

$$\bar g_T(t):=1-\frac{t}{\sqrt{T_0}}+a_T\quad\text{and}\quad g_T(t):=1-\frac{t}{\sqrt{T_0}},$$

where $T_0=T_0(T)=\lceil(2a_T+\delta_2\sigma+1)^2\rceil$. Then $\bar g_T(T_0)\le -a_T-\delta_2\sigma$.

As we will show next, for any sufficiently large T, the event $\{\sup_{t\in J\cap[0,T]}\mathcal{I}(X)_t\le 1\}$ occurs at least if all of the following events occur:

$$E_1=\big\{X\le \bar g_T \text{ on } [0,T_0]\big\},\qquad E_2=\Big\{\sup_{t\in[0,T_0]}X_t\le c_1T_0^{-\alpha}\Big\},$$

$$E_3=\Big\{\sup_{t\in J\cap[0,T-T_0]}\mathcal{I}(W_{\cdot+T_0}-W_{T_0})\le 1\Big\}\quad\text{and}\quad E_4=\Big\{\sup_{t\in[T_0,T]}\big|X_t-X_{T_0}-\sigma(W_t-W_{T_0})\big|\le a_T\Big\},$$

where $c_1>0$ is a finite constant with $\int_0^t K(s)\,\mathrm{d}s\le c_1^{-1}t^{\alpha}$ for all $t\ge 1$. Indeed, E_1 and E_2 imply that

$$\int_0^{T_0-\delta_1}K(t-s)X_s\,\mathrm{d}s\le 0\quad\text{for } t\ge T_0\quad\text{and}\quad X\le-\delta_2\sigma\quad\text{on } [T_0-\delta_1,T_0], \tag{18}$$

as long as T (or equivalently T_0) is sufficiently large. The second statement in (18) is immediately clear from E_1; to see that the first statement in (18) holds on $E_1\cap E_2$ note that there are $0<d_1<1$ such that $X_t\le c(-a_T)$ for all

$t \in [d_1 T_0, T_0]$. Fix any $d_1 < d_2 < 1$. Then

$$\int_0^{T_0-\delta_1} K(t-s)X_s\,\mathrm{d}s \le \int_0^{d_1 T_0} K(t-s)\,\mathrm{d}s + \int_{d_1 T_0}^{d_2 T_0} K(t-s)c(-a_T)\,\mathrm{d}s + 0$$
$$\le \frac{c_1}{T_0^\alpha}\int_0^{d_1 T_0} k(t-s)^{\alpha-1}\,\mathrm{d}s + c(-a_T)\int_{d_1 T_0}^{d_2 T_0} k_-(t-s)^{\alpha-1}\,\mathrm{d}s,$$

where we used (5) and (6). Now, due to the additional a_T in the second term, it can be seen easily that the last expression is negative for sufficiently large T (and thus T_0 and a_T). This shows (18).

On the other hand, given that also E_4 occurs, one has for $t \in [T_0, T]$,

$$X_t \le X_{T_0} + \sigma(W_t - W_{T_0}) + a_T \le -\delta_2\sigma + \sigma(W_t - W_{T_0}),$$

so that

$$\int_{T_0}^t K(t-s)X_s\,\mathrm{d}s \le \sigma\int_{T_0}^t K(t-s)[W_s - W_{T_0} - \delta_2]\,\mathrm{d}s.$$

Assuming additionally E_3, we conclude with (18) that, for all $t \in J \cap [T_0, T]$,

$$\begin{aligned}
\mathcal{I}(X)_t &= \int_0^{T_0-\delta_1} K(t-s)X_s\,\mathrm{d}s + \int_{T_0-\delta_1}^{T_0} K(t-s)X_s\,\mathrm{d}s + \int_{T_0}^t K(t-s)X_s\,\mathrm{d}s \\
&\le 0 + \int_{T_0-\delta_1}^{T_0} K(t-s)(-\delta_2\sigma)\,\mathrm{d}s + \sigma\int_{T_0}^t K(t-s)[W_s - W_{T_0} - \delta_2]\,\mathrm{d}s \\
&= -\sigma\delta_2\int_{T_0-\delta_1}^t K(t-s)\,\mathrm{d}s + \sigma\int_0^{t-T_0} K(t-T_0-s)[W_{s+T_0} - W_{T_0}]\,\mathrm{d}s \\
&\le -\sigma\delta_2\int_0^{\delta_1} K(s)\,\mathrm{d}s + \sigma\cdot 1 \\
&\le -\sigma + \sigma \le 1
\end{aligned}$$

as long as T is sufficiently large. Note that $\mathcal{I}(X)_t \le 1$ also holds on $J \cap [0, T_0]$ due to E_2, see Corollary 2.3.

Step 3: It remains to estimate the probability of $E_1 \cap \cdots \cap E_4$. First we estimate $\mathbb{P}(E_1 \cap E_2)$.

Note that, for any choice of n and $0 \le t_1 < \cdots < t_n$, the random variables $(X_{t_i})_{i=1}^n$ are associated (cf. [9]), as they are sums of independent random variables. Thus, the events $\mathbb{1}_{E_1}$ and $\mathbb{1}_{E_2}$ can both be written as limits of decreasing functions of associated random variables and are thus also associated. Hence, we have $\mathbb{P}(E_1 \cap E_2) \ge \mathbb{P}(E_1)\cdot\mathbb{P}(E_2)$. By Corollary 2.3, we have $\mathbb{P}(E_2) \gtrsim T_0^{-\alpha-1/2}$. Moreover, the event E_1 occurs whenever the events

$$E_1' = \bigl\{\forall t \in [0, T_0]\colon \sigma W_t \le g_T(t)\bigr\} \quad\text{and}\quad E_1'' = \Bigl\{\sup_{t\in[0,T_0]} |X_t - \sigma W_t| \le a_T\Bigr\}$$

occur; and we thus have

$$\mathbb{P}(E_1) \ge \mathbb{P}\bigl(E_1' \cap E_1''\bigr) \ge \mathbb{P}\bigl(E_1'\bigr) - \mathbb{P}\bigl(E_1''^c\bigr).$$

By Lemma 2.4 and by inequality (17), one has $\mathbb{P}(E_1') \gtrsim T_0^{-1/2}$ and $\mathbb{P}(E_1''^c) \lesssim T_0^{-\rho}$, respectively, so that $\mathbb{P}(E_1) \gtrsim T_0^{-1/2}$. Altogether we thus obtain

$$\mathbb{P}(E_1 \cap E_2) \gtrsim T_0^{-(\alpha+1)} \approx (\log T)^{-2(\alpha+1)}. \tag{19}$$

Moreover, $E_3 \cap E_4$ is independent of $E_1 \cap E_2$ and

$$\mathbb{P}(E_3 \cap E_4) \geq \mathbb{P}(E_3) - \mathbb{P}(E_4^c) \gtrsim \mathbb{P}\Big(\sup_{t \in J \cap [0,T]} \mathcal{I}(W)_t \leq 1\Big),$$

since $\mathbb{P}(E_4^c) \leq T^{-\rho}$ is of lower order than $\mathbb{P}(E_3) \gtrsim T^{-(\alpha+1/2)}$, see Corollary 2.3. Combining this with (19) finishes the proof. □

3. Drift and barriers

3.1. *The influence of a drift on Gaussian processes*

In this section, we study the influence of a drift on the survival exponent. The first aim is to prove Proposition 1.6.

Proof of Proposition 1.6. In the notation of [20], the Cameron–Martin formula says that

$$\mathbb{P}(X + f \in S) = \mathbb{E}\big[\mathbb{1}_{\{X \in S\}} \mathrm{e}^{\langle z, X\rangle - \|f\|^2/2}\big], \tag{20}$$

where $z = z(f)$ is an element of the L_2-completion of the dual of E.

Upper bound. Let $p > 1$ and $1/p + 1/q = 1$. We use the Hölder inequality in (20) to get

$$\mathbb{P}(X + f \in S) \leq \big(\mathbb{E}\big[\mathbb{1}_{\{X \in S\}}^p\big]\big)^{1/p} \big(\mathbb{E}\big[\mathrm{e}^{q\langle z, X\rangle}\big]\big)^{1/q} \mathrm{e}^{-\|f\|^2/2}.$$

Recall that $\langle z, X\rangle$ is a centered Gaussian random variable with variance $\|f\|^2$. Therefore, we get

$$\mathbb{P}(X + f \in S) \leq \mathbb{P}(X \in S)^{1/p} \mathrm{e}^{q\|f\|^2/2 - \|f\|^2/2}. \tag{21}$$

Optimizing in p shows that the best choice is

$$1/p := 1 - \sqrt{\frac{\|f\|^2}{2\log(1/\mathbb{P}(X \in S))}} < 1.$$

Plugging this into (21) shows the upper bound in the proposition.

Lower bound. Here we let $p > 1$ and use the reverse Hölder inequality in (20) to get

$$\mathbb{P}(X + f \in S) \geq \big(\mathbb{E}\big[\mathbb{1}_{\{X \in S\}}^{1/p}\big]\big)^{p} \big(\mathbb{E}\big[\mathrm{e}^{-1/(p-1)\langle z, X\rangle}\big]\big)^{-(p-1)} \mathrm{e}^{-\|f\|^2/2}.$$

As above, we calculate the second expectation and optimize in p to find that the best choice is

$$p := 1 + \sqrt{\frac{\|f\|^2}{2\log(1/\mathbb{P}(X \in S))}} > 1.$$

Using this shows the lower bound. □

As a further example for a Gaussian process, let us consider the α-fractionally integrated Brownian motion defined in (8). Here, one can add drift functions up to $|f(t)| \preceq t^\gamma$, $\gamma < H = \alpha + 1/2$.

Corollary 3.1. *Let $R^\alpha = \mathcal{I}_\alpha(W)$ be an α-fractionally integrated Brownian motion, and let $f' : [0, \infty) \to \mathbb{R}$ be a function with $\int_0^\infty f'(s)^2 \, \mathrm{d}s < \infty$. Let $\theta > 0$ and define*

$$g(t) := \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t - s)^\alpha f'(s) \, \mathrm{d}s, \quad t \geq 0.$$

Then

$$\mathbb{P}\Big(\sup_{t\in[0,T]} R_t^\alpha \le 1\Big) = T^{-\theta+o(1)} \quad \textit{if and only if} \quad \mathbb{P}\Big(\sup_{t\in[0,T]} \big(R_t^\alpha + g(t)\big) \le 1\Big) = T^{-\theta+o(1)}.$$

Also, upper (lower) bounds imply upper (lower) bounds.

We demonstrate the method of changing the barrier with the following important example.

Example 3.2. *Let us consider the barrier*

$$F(t) := \begin{cases} \infty, & 0 \le t < 1, \\ 0, & 1 \le t \le T \end{cases}$$

for the process $R^\alpha = \mathcal{I}_\alpha(W)$ *defined in* (8).

Corollary 3.3. *Let* R^α *be the* α*-fractionally integrated Brownian motion. Then, for a* $\theta > 0$,

$$\mathbb{P}\Big(\sup_{t\in[1,T]} R_t^\alpha \le 0\Big) = T^{-\theta+o(1)}$$

if and only if

$$\mathbb{P}\Big(\sup_{t\in[0,T]} R_t^\alpha \le 1\Big) = T^{-\theta+o(1)}.$$

Also, upper (lower) bounds imply upper (lower) bounds.

This corollary will be an important part of the proof of Theorem 1.4.

Proof of Corollary 3.3. We can e.g. use the function $f' := \mathbb{1}_{[0,1]}\Gamma(\alpha+1)(\alpha+1)$, for which $\int_0^\infty f'(s)^2\,\mathrm{d}s < \infty$. Then $g(t) = \Gamma(\alpha+1)(\alpha+1)\mathcal{I}_{\alpha+1}(\mathbb{1}_{[0,1]})_t = t^{\alpha+1} - (t-1)^{\alpha+1} \ge 1$ for all $t \ge 1$ and all $\alpha > 0$, i.e.

$$F(t) \ge 1 - g(t) \quad \text{for all } t \in [0,T];$$

and thus

$$\mathbb{P}\big(\forall t\in[0,T]\colon\ R_t^\alpha + g(t) \le 1\big) \le \mathbb{P}\big(\forall t\in[0,T]\colon\ R_t^\alpha \le F(t)\big) = \mathbb{P}\Big(\sup_{t\in[1,T]} R_t^\alpha \le 0\Big).$$

Corollary 3.1 therefore implies one bound in the assertion.

The opposite estimate can be obtained via Slepian's lemma (cf. Theorem 3, Section 14 in [20]): Since R^α is a centered Gaussian process with positive covariances, one obtains with the argument in Section 2.4 of [32] that

$$\begin{aligned}\mathbb{P}\Big(\sup_{t\in[0,T]} R_t^\alpha \le 1\Big) &\ge \mathbb{P}\Big(\sup_{t\in[0,1]} R_t^\alpha \le 1\Big)\mathbb{P}\Big(\sup_{t\in[1,T]} R_t^\alpha \le 1\Big) \\ &\ge \mathbb{P}\Big(\sup_{t\in[0,1]} R_t^\alpha \le 1\Big)\mathbb{P}\Big(\sup_{t\in[1,T]} R_t^\alpha \le 0\Big).\end{aligned}$$

□

3.2. *Proof of Theorem* 1.4

Here we give the proof of Theorem 1.4. Due to Theorem 1.1 it is sufficient to consider the question of the survival exponent in the case when X is a Brownian motion. Therefore, we consider $R^\alpha = \mathcal{I}_\alpha(X)$, where X is a Brownian motion.

Proof of the existence: We use an approach from [18] involving the Lamperti transform. However, we employ the new drift argument developed in Corollary 3.3 which could also be used in [18] to bypass the argumentation on pages 235–238. Note that the Lamperti transform of R^α,

$$Y_t := \mathrm{e}^{-t(\alpha+1/2)} R^\alpha_{\mathrm{e}^t}, \quad t \geq 0,$$

is a continuous, zero mean, stationary Gaussian process with positive correlations $\mathbb{E}[Y_t Y_0] \geq 0$. Therefore, Slepian's lemma and the standard subadditivity argument (see Proposition 3.1 in [18]) show that the following limit exists and equals the supremum:

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\Big(\sup_{t\in[0,T]} Y_t \leq 0\Big) = \sup_{T>0} \frac{1}{T} \log \mathbb{P}\Big(\sup_{t\in[0,T]} Y_t \leq 0\Big).$$

We shall prove that this limit is actually a representation for $\theta(\alpha)$. To see this, observe that

$$\begin{aligned}\mathbb{P}\Big(\sup_{t\in[0,\log T]} Y_t \leq 0\Big) &= \mathbb{P}\Big(\sup_{t\in[0,\log T]} \mathrm{e}^{-t(\alpha+1/2)} R^\alpha_{\mathrm{e}^t} \leq 0\Big) \\ &= \mathbb{P}\Big(\sup_{t\in[0,\log T]} R^\alpha_{\mathrm{e}^t} \leq 0\Big) = \mathbb{P}\Big(\sup_{t\in[1,T]} R^\alpha_t \leq 0\Big).\end{aligned}$$

Thus the existence follows from Corollary 3.3.

Proof of the monotonicity: Let $\gamma \geq 0$ and $0 < \alpha < 1$. We will show that $\theta(\gamma) \geq \theta(\gamma+\alpha)$.

If $R^\gamma_s \leq \mathbb{1}_{[0,1]}(s)$, for all $s \in [0, T]$, then

$$R^{\alpha+\gamma}_t = \mathcal{I}_\alpha\big(R^\gamma\big)_t \leq \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \alpha(t-s)^{\alpha-1}\,\mathrm{d}s = \frac{t^\alpha}{\alpha\Gamma(\alpha)} \leq \frac{1}{\alpha\Gamma(\alpha)}, & t \leq 1, \\ \frac{1}{\Gamma(\alpha)} \int_0^1 \alpha(t-s)^{\alpha-1}\,\mathrm{d}s = \frac{t^\alpha-(t-1)^\alpha}{\alpha\Gamma(\alpha)} \leq \frac{1}{\alpha\Gamma(\alpha)}, & t \geq 1, \end{cases}$$

since $\alpha < 1$. Therefore, using also the self-similarity (set $\lambda := (\alpha\Gamma(\alpha))^{1/(\alpha+\gamma+1/2)}$),

$$\begin{aligned}&\mathbb{P}\big(\forall s \leq T\colon\ R^\gamma_s \leq \mathbb{1}_{[0,1]}(s)\big) \\ &\quad\leq \mathbb{P}\left(\forall s \leq T\colon\ R^{\alpha+\gamma}_s \leq \frac{1}{\alpha\Gamma(\alpha)}\right) = \mathbb{P}\big(\forall s \leq \lambda T\colon\ R^{\alpha+\gamma}_s \leq 1\big) = T^{-\theta(\alpha+\gamma)+\mathrm{o}(1)}.\end{aligned}$$

The left-hand side is treated with Corollary 3.1 (using $f' = (\gamma+1)\Gamma(\gamma+1)\mathbb{1}_{[0,1]}$) showing that it behaves as $T^{-\theta(\gamma)+\mathrm{o}(1)}$. This shows the monotonicity of the survival exponent.

Finally, due to the monotonicity and the relation to random polynomials (cf. Section 4 below), one gets $\theta(\alpha) \geq b/4 > 1/8$, by Theorem 3.2 in [17].

4. Application to the question of random polynomials having no real zeros

Let us finally give an application of our results to the study of zeros of random polynomials. The connection to the one-sided exit problem was established in [8]: for i.i.d. normal random variables $\xi_1, \xi_2, \ldots$ one has

$$\mathbb{P}\left(\sum_{i=0}^{2n} \xi_i x^i \leq 0\ \forall x \in \mathbb{R}\right) = n^{-b+\mathrm{o}(1)}, \quad n \to \infty,$$

where

$$b := -4 \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\Big(\sup_{t\in[0,T]} Y^\infty_t \leq 0\Big)$$

and Y^∞ is the stationary Gaussian process with correlation function

$$\operatorname{corr}_\infty(\tau) := \mathbb{E}\big[Y_0^\infty Y_\tau^\infty\big] = \frac{2\mathrm{e}^{-\tau/2}}{1+\mathrm{e}^{-\tau}}.$$

It was shown that $0.5 < b < 1.29$ (see [8,17]). Here we show the following connection to our problem and an improvement for the numerical value of b.

Corollary 4.1. *For the decreasing function θ defined in Theorem 1.4 we have*

$$\theta(\alpha) \geq b/4 \quad \textit{for all } \alpha \geq 0.$$

In particular, $b \leq 4 \cdot \theta(1) = 1$.

This fact gives a further motivation to find values for $\theta(\alpha)$, $\alpha \notin \{0, 1\}$.

Proof of Corollary 4.1. Note that it is sufficient to show the corollary for integer α, since θ is decreasing. Consider the Lamperti transforms of the processes $R^n := \mathcal{I}_n(W)$, where W is a Brownian motion, normalized by the square root of its variance:

$$Y_t^n := n!\sqrt{2n+1}\mathrm{e}^{-(n+1/2)t} R_{\mathrm{e}^t}^n.$$

This is a stationary, zero mean Gaussian process. One can calculate its correlation function ($\tau \geq 0$):

$$\begin{aligned}\operatorname{corr}_n(\tau) &:= \mathbb{E}\big[Y_0^n Y_\tau^n\big] = n!^2(2n+1)\mathrm{e}^{-(n+1/2)\tau}\mathbb{E}\big[R_1^n R_{\mathrm{e}^\tau}^n\big] \\ &= (2n+1)\mathrm{e}^{-(n+1/2)\tau}\int_0^1 \big(\mathrm{e}^\tau - u\big)^n (1-u)^n \,\mathrm{d}u,\end{aligned}$$

where the last equal sign is obtained from the following stochastic integral representation of R^n, which is immediate from (8):

$$R_t^\alpha = \int_0^t \frac{1}{\Gamma(\alpha+1)}(t-u)^\alpha \,\mathrm{d}W_u, \quad t \geq 0.$$

It is elementary to see that

$$(2n+1)\mathrm{e}^{-(n+1/2)\tau}\int_0^1 \big(\mathrm{e}^\tau - u\big)^n (1-u)^n \,\mathrm{d}u \leq \frac{2\mathrm{e}^{-\tau/2}}{1+\mathrm{e}^{-\tau}}, \quad \tau \geq 0, n \geq 1,$$

with equality at $\tau = 0$. Indeed, note that

$$\mathrm{e}^{-n\tau}\int_0^1 \big(\mathrm{e}^\tau - u\big)^n (1-u)^n \,\mathrm{d}u = \int_0^1 \big[\sqrt{(1-\mathrm{e}^{-\tau}u)(1-u)}\big]^{2n} \,\mathrm{d}u \leq \int_0^1 \left(\frac{1-\mathrm{e}^{-\tau}u+1-u}{2}\right)^{2n} \mathrm{d}u.$$

Integrating the latter expression gives

$$\frac{1}{2n+1}\frac{2}{\mathrm{e}^{-\tau}+1}\left(1-\left(1-\frac{\mathrm{e}^{-\tau}+1}{2}\right)\right)^{2n+1} \leq \frac{1}{2n+1}\frac{2}{\mathrm{e}^{-\tau}+1}.$$

This implies that, for all $n \geq 1$,

$$\operatorname{corr}_n(0) = \operatorname{corr}_\infty(0) \quad \text{and} \quad \operatorname{corr}_n(\tau) \leq \operatorname{corr}_\infty(\tau), \quad \tau \geq 0.$$

Therefore, by Slepian's lemma,

$$
\begin{aligned}
b &= -4 \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\Big(\sup_{t\in[0,T]} Y_t^\infty \le 0\Big) \\
&\le -4 \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\Big(\sup_{t\in[0,T]} Y_t^n \le 0\Big) \\
&= -4 \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\Big(\sup_{t\in[0,T]} R_{\mathrm{e}^t}^n \le 0\Big) \\
&= -4 \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\Big(\sup_{t\in[1,\mathrm{e}^T]} R_t^n \le 0\Big) \\
&= -4 \lim_{T\to\infty} \frac{1}{\log T} \log \mathbb{P}\Big(\sup_{t\in[1,T]} R_t^n \le 0\Big) \\
&= 4 \cdot \theta(n),
\end{aligned}
$$

where the last step follows from Corollary 3.3. □

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