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On the coupling property of Lévy processes

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Abstract. We give necessary and sufficient conditions guaranteeing that the coupling for Lévy processes (with non-degenerate jump part) is successful. Our method relies on explicit formulae for the transition semigroup of a compound Poisson process and earlier results by Mineka and Lindvall–Rogers on couplings of random walks. In particular, we obtain that a Lévy process admits a successful coupling, if it is a strong Feller process or if the Lévy (jump) measure has an absolutely continuous component.

Résumé. Nous donnons les conditions nécessaires et suffisantes pour le succès du couplage entre des processus de Lévy (avec partie de sauts non-dégénérée). Notre méthode est basée sur les formules explicites pour le semigroupe de transition d'un processus de Poisson composé, et les résultats de Mineka et Lindvall–Rogers sur le couplage d'une marche aléatoire. En particulier, nous montrons qu'un processus de Lévy admet un couplage, s'il est un processus fortement fellerien ou si la mesure de Lévy (mesure de sauts) possède une composante absolument continue.

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1. Introduction and main results

The coupling method is a powerful tool in the study of Markov processes and interacting particle systems. There are some comprehensive books on this topic now, see e.g. [2,7,13,15]. Let $(X_t)_{t\ge 0}$ be a Markov process on $\mathbb{R}^d$ with transition probability function $\{P_t(x,\cdot)\}_{t\ge 0, x\in\mathbb{R}^d}$. An $\mathbb{R}^{2d}$-valued process $(X'_t, X''_t)_{t\ge 0}$ is called *a coupling of the Markov process* $(X_t)_{t\ge 0}$, if both $(X'_t)_{t\ge 0}$ and $(X''_t)_{t\ge 0}$ are Markov processes which have the same transition functions $P_t(x,\cdot)$ but possibly different initial distributions. In this case, $(X'_t)_{t\ge 0}$ and $(X''_t)_{t\ge 0}$ are called the *marginal processes* of the coupling process; the coupling time is defined by $T := \inf\{t \ge 0\colon\ X'_t = X''_t\}$. The coupling $(X'_t, X''_t)_{t\ge 0}$ is said to be *successful* if T is a.s. finite. A Markov process $(X_t)_{t\ge 0}$ admits a *successful coupling* (also: enjoys the *coupling property*) if for any two initial distributions μ_1 and μ_2, there exists a successful coupling with marginal processes possessing the same transition probability functions $P_t(x,\cdot)$ and starting from μ_1 and μ_2, respectively. It is known, see [7,12], that the coupling property is equivalent to the statement that

$$\lim_{t\to\infty} \|\mu_1 P_t - \mu_2 P_t\|_{\mathrm{Var}} = 0 \quad \text{for } \mu_1 \text{ and } \mu_2 \in \mathscr{P}\big(\mathbb{R}^d\big). \tag{1.1}$$

As usual, $\mu P(A) = \int P(x,A)\mu(\mathrm{d}x)$ is the left action of the semigroup and $\|\cdot\|_{\mathrm{Var}}$ stands for the total variation norm. If a Markov process admits a successful coupling, then it also has the Liouville property, i.e. every bounded harmonic function is constant; in this context a function f is harmonic, if $Lf = 0$ where L is the generator of the Markov process. See [3,4] and references therein for this result and more details on the coupling property.

The aim of this paper is to study the coupling property of Lévy processes by using explicit conditions on Lévy measures. Our work is mainly motivated by the recent paper [14], which contains some interesting results on the coupling property of Ornstein–Uhlenbeck processes; the paper [14] uses mainly the conditional Girsanov theorem on Poisson space and assumes that the corresponding Lévy measure has a non-trivial absolutely continuous part. Our technique here is completely different from F.-Y. Wang's paper [14]. We use an explicit expression of the compound Poisson semigroup and combine this with the Mineka and Lindvall–Rogers couplings for random walks, see [8].

A Lévy process $(X_t)_{t\ge 0}$ on $\mathbb{R}^d$ is a stochastic process with stationary and independent increments and càdlàg (right continuous with finite left limits) paths. It is well known that X_t is a (strong) Markov process whose infinitesimal generator is, for $f \in C_b(\mathbb{R}^d)$, of the form

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^d q_{i,j}\frac{\partial^2 f(x)}{\partial x_i\,\partial x_j} + \int\left[f(x+z) - f(x) - \frac{z\cdot\nabla f(x)}{1+|z|^2}\right]\nu(\mathrm{d}z) + b\cdot\nabla f(x),$$

where $Q = (q_{i,j})_{i,j=1}^d$ is a positive semi-definite matrix, $b \in \mathbb{R}^d$ is the drift vector and ν is the Lévy or jump measure; the Lévy measure ν is a σ-finite measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ such that $\int(1\wedge|z|^2)\nu(\mathrm{d}z) < \infty$. Note that the Lévy triplet (b, Q, ν) characterizes, up to indistinguishability, the process $(X_t)_{t\ge 0}$ uniquely. Our standard reference for Lévy processes is the monograph [11]. We write $P_t(x, A) = P_t(A - x)$, $A \in \mathscr{B}(\mathbb{R}^d)$, for the transition probability of X_t.

Let μ and ν be two bounded measures on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. We define $\mu\wedge\nu := \mu - (\mu-\nu)^+$, where $(\mu-\nu)^\pm$ is the Jordan–Hahn decomposition of the signed measure $\mu - \nu$. In particular, $\mu\wedge\nu = \nu\wedge\mu$ and it is easy to see that $\mu\wedge\nu(\mathbb{R}^d) = \frac{1}{2}[\mu(\mathbb{R}^d) + \nu(\mathbb{R}^d) - \|\mu-\nu\|_{\mathrm{Var}}]$, cf. [2]. We can now state our main result.

Theorem 1.1. *Let $(X_t)_{t\ge 0}$ be a d-dimensional Lévy process with Lévy triplet (b, Q, ν). For every $\varepsilon > 0$, define ν_ε by*

$$\nu_\varepsilon(B) = \begin{cases} \nu(B), & \nu(\mathbb{R}^d) < \infty; \\ \nu\{z \in B\colon |z| \ge \varepsilon\}, & \nu(\mathbb{R}^d) = \infty. \end{cases} \tag{1.2}$$

Assume that there exist $\varepsilon, \delta > 0$ such that

$$\inf_{x\in\mathbb{R}^d, |x|\le\delta} \nu_\varepsilon \wedge (\delta_x * \nu_\varepsilon)(\mathbb{R}^d) > 0. \tag{1.3}$$

Then, there exists a constant $C = C(\varepsilon, \delta, \nu) > 0$ such that for all $x, y \in \mathbb{R}^d$ and $t > 0$,

$$\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{Var}} \le \frac{C(1+|x-y|)}{\sqrt{t}} \wedge 2.$$

In particular, the Lévy process X_t admits a successful coupling.

Remark 1.2. *Condition (1.3) guarantees that*

$$\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{Var}} = \mathrm{O}(t^{-1/2}) \quad \text{as } t\to\infty$$

holds locally uniformly for all $x, y \in \mathbb{R}^d$. This order of convergence is known to be optimal for compound Poisson processes, see [14], Remark 3.1. In [14] it is pointed out that a pure jump Lévy process admits a successful coupling only if the Lévy measure has a non-discrete support, in order to make the process more active. Condition (1.3) is one possibility to guarantees sufficient jump activity; intuitively it will hold if for sufficiently small values of $\varepsilon, \delta > 0$ and all $x \in \mathbb{R}^d$ with $|x| \le \delta$ we have $x + \mathrm{supp}(\nu_\varepsilon) \cap \mathrm{supp}(\nu_\varepsilon) \ne \varnothing$; here $\mathrm{supp}(\nu_\varepsilon)$ is the support of the measure ν_ε.

In order to see that (1.3) is sharp, we consider an one-dimensional compound Poisson process with Lévy measure ν supported on $\mathbb{Z}$. Then, for any $\delta \in (0,1)$ and $x \in \mathbb{R}^d$ with $|x| \le \delta$, $\nu\wedge(\delta_x * \nu)(\mathbb{Z}) = 0$. On the other hand, all functions satisfying $f(x+n) = f(x)$ for $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}$ are harmonic. By [3,4], this process cannot have the coupling property.

Theorem 3.1 of [14] establishes the coupling property for Lévy processes whose jump measure ν has a non-trivial absolutely continuous part. It seems to us that this condition is not directly comparable with (1.3). In fact, based on the

Lindvall–Rogers 'zero–two law' for random walks [8], Propsotion 1, we give in Section 4 a necessary and sufficient condition guaranteeing that a Lévy process has the coupling property. In this section we will also find the connection between (1.3) and the existence of a non-trivial absolutely continuous component of the Lévy measure. In particular, we obtain some extensions of Theorem 1.1 and [14], Theorem 3.1.

Once we know that a Lévy process admits the coupling property, many interesting new questions arise which are, however, beyond the scope of the present paper. For example, it would be interesting to construct explicitly the corresponding successful Markov coupling process and to determine its infinitesimal operator. There are a number of applications of optimal Markov processes and operators; we refer to [2,15] for background material and a more detailed account on diffusions and q-processes. We will discuss those topics for Lévy processes in a forthcoming paper [1].

2. The coupling property of compound Poisson processes

In this section, we consider the coupling property of compound Poisson processes. Let $(L_t)_{t\ge 0}$ be a compound Poisson process on $\mathbb{R}^d$ such that $L_0 = x$ and with Lévy measure ν. Then, L_t can be written as

$$L_t = x + \sum_{i=1}^{N_t} \xi_i, \quad t \ge 0,$$

where N_t is the Poisson process with rate $\lambda := \nu(\mathbb{R}^d)$ and $(\xi_i)_{i\ge 1}$ is a sequence of i.i.d. random variables on $\mathbb{R}^d$ with distribution $\nu(\cdot)/\lambda$; moreover, we assume that the ξ_i's are independent of N_t. As usual we use the convention that $\sum_{i=1}^0 \xi_i = 0$.

The transition semigroup for a compound Poisson process is explicitly known. This allows us to reduce the coupling problem for a compound Poisson processes to that of a random walk. Let P_t and L be the semigroup and the generator for L_t, respectively. Then, it is well known that for any $f \in B_b(\mathbb{R}^d)$,

$$\begin{aligned} Lf(x) &= \int \bigl(f(x+z) - f(x)\bigr)\nu(\mathrm{d}z) \\ &= \lambda \int \bigl(f(x+z) - f(x)\bigr)\nu_0(\mathrm{d}z) \end{aligned}$$

and

$$P_t = \mathrm{e}^{tL} = \sum_{n=0}^\infty \frac{t^n L^n}{n!} = \mathrm{e}^{-\lambda t} \sum_{n=0}^\infty \frac{(t\lambda)^n \nu_0^{*n}}{n!}, \quad t \ge 0; \tag{2.1}$$

here ν_0^{*n} is the n-fold convolution of ν_0 and $\nu_0^{*0} := \delta_0$.

The following result explains the relationship of transition probabilities of compound Poisson processes and of random walks.

Proposition 2.1. *Let $P_t(x,\cdot)$ be the transition probability of the compound Poisson process $L = (L_t)_{t\ge 0}$ and let $S = (S_n)_{n\ge 1}$, $S_n = \xi_1 + \cdots + \xi_n$, be a random walk where $(\xi_i)_{i\ge 1}$ are i.i.d. random variables with $\xi_1 \sim \nu_0 := \nu/\nu(\mathbb{R}^d)$. Then, for all $x, y \in \mathbb{R}^d$*

$$\begin{aligned} \bigl\| P_t(x,\cdot) - P_t(y,\cdot)\bigr\|_{\mathrm{Var}} &\le \mathrm{e}^{-\lambda t} \sum_{n=0}^\infty \frac{(\lambda t)^n \|\mathbb{P}(x+S_n\in\cdot) - \mathbb{P}(y+S_n\in\cdot)\|_{\mathrm{Var}}}{n!} \\ &= \mathrm{e}^{-\lambda t}\Biggl[2(1-\delta_{x,y}) + \sum_{n=1}^\infty \frac{(\lambda t)^n \|\mathbb{P}(x+S_n\in\cdot) - \mathbb{P}(y+S_n\in\cdot)\|_{\mathrm{Var}}}{n!}\Biggr], \end{aligned}$$

where $\delta_{x,y}$ *is a Kronecker delta function, i.e.* $\delta_{x,y}=1$ *if* $x=y$, *and* 0 *otherwise.*

Proof. Let P_t and P_n^S be the semigroups of the compound Poisson process L and the random walk S, respectively. Because of (2.1) we find

$$\begin{aligned}\left\|P_t(x,\cdot)-P_t(y,\cdot)\right\|_{\mathrm{Var}} &= \sup_{\|f\|_\infty\le 1}\left|P_tf(x)-P_tf(y)\right|\\ &= \mathrm{e}^{-\lambda t}\left|\sup_{\|f\|_\infty\le 1}\sum_{n=0}^\infty \frac{(\lambda t)^n(\delta_x*\nu_0^{*n}(f)-\delta_y*\nu_0^{*n}(f))}{n!}\right|\\ &\le \mathrm{e}^{-\lambda t}\left|\sup_{\|f\|_\infty\le 1}\sum_{n=0}^\infty \frac{(\lambda t)^n(P_n^Sf(x)-P_n^Sf(y))}{n!}\right|\\ &\le \mathrm{e}^{-\lambda t}\sum_{n=0}^\infty \frac{(\lambda t)^n\sup_{\|f\|_\infty\le 1}|P_n^Sf(x)-P_n^Sf(y)|}{n!}\\ &= \mathrm{e}^{-\lambda t}\sum_{n=0}^\infty \frac{(\lambda t)^n\|\mathbb{P}(x+S_n\in\cdot)-\mathbb{P}(y+S_n\in\cdot)\|_{\mathrm{Var}}}{n!},\end{aligned}$$

which proves the first assertion.

For $n=0$ we have $S_0=0$; thus

$$\left\|\mathbb{P}(x+S_0\in\cdot)-\mathbb{P}(y+S_0\in\cdot)\right\|_{\mathrm{Var}}=\|\delta_x-\delta_y\|_{\mathrm{Var}}=2(1-\delta_{x,y})$$

and the second assertion follows. □

An immediate of Proposition 2.1 is the following estimate for $\|P_t(x,\cdot)-P_t(y,\cdot)\|_{\mathrm{Var}}$ which is based on a similar inequality for $\|\mathbb{P}(x+S_n\in\cdot)-\mathbb{P}(y+S_n\in\cdot)\|_{\mathrm{Var}}$.

Proposition 2.2. *Assume that for all* $x,y\in\mathbb{R}^d$ *there is a constant* $C(x,y)>0$ *such that*

$$\left\|\mathbb{P}(x+S_n\in\cdot)-\mathbb{P}(y+S_n\in\cdot)\right\|_{\mathrm{Var}}\le\frac{C(x,y)}{\sqrt{n}}\quad\text{for } n\ge 1. \tag{2.2}$$

Then,

$$\left\|P_t(x,\cdot)-P_t(y,\cdot)\right\|_{\mathrm{Var}}\le 2\mathrm{e}^{-\lambda t}(1-\delta_{x,y})+\frac{\sqrt{2}C(x,y)(1-\mathrm{e}^{-\lambda t})}{\sqrt{\lambda t}}.$$

Proof. A combination of Proposition 2.1 and (2.2) yields for all $x,y\in\mathbb{R}^d$

$$\left\|P_t(x,\cdot)-P_t(y,\cdot)\right\|_{\mathrm{Var}}\le \mathrm{e}^{-\lambda t}\left[2(1-\delta_{x,y})+C(x,y)\sum_{n=1}^\infty\frac{(\lambda t)^n}{\sqrt{n}n!}\right].$$

Jensen's inequality for the concave function $x^{1/2}$ gives

$$\begin{aligned}\sum_{n=1}^\infty\sqrt{\frac{1}{n}}\frac{(\lambda t)^n}{n!}&\le\frac{\mathrm{e}^{\lambda t}-1}{(\mathrm{e}^{\lambda t}-1)^{1/2}}\left(\sum_{n=1}^\infty\frac{(\lambda t)^n}{n\cdot n!}\right)^{1/2}\\ &=\left(\frac{\mathrm{e}^{\lambda t}-1}{\lambda t}\right)^{1/2}\left(\sum_{n=1}^\infty\frac{(\lambda t)^{n+1}}{(n+1)!}\cdot\frac{n+1}{n}\right)^{1/2}\end{aligned}$$

$$\leq \left(\frac{e^{\lambda t}-1}{\lambda t}\right)^{1/2}\sqrt{2}\big(e^{\lambda t}-1-\lambda t\big)^{1/2}$$
$$\leq \frac{\sqrt{2}(e^{\lambda t}-1)}{\sqrt{\lambda t}}.$$

The required assertion follows form the estimates above. □

We will now show that L has the coupling property whenever S has.

Proposition 2.3. *Let $(L_t)_{t\geq 0}$ be the compound Poisson process with Lévy measure ν, and let $(S_n)_{n\geq 0}$, be a random walk where $S_n = S_0 + \xi_1 + \cdots + \xi_n$, and $(\xi_i)_{i\geq 1}$ are i.i.d. random variables with $\xi_1 \sim \nu_0 := \nu/\nu(\mathbb{R}^d)$. If S_n admits a successful coupling, so does L_t.*

Proof. For $x, y \in \mathbb{R}^d$, let L_t be a compound Poisson process starting from $x \in \mathbb{R}^d$. Then $L_t = \sum_{i=0}^{N_t} \xi_i$, where $\xi_0 = x$ and where $(\xi_i)_{i\geq 1}$ are i.i.d. random variables with common distribution $\nu_0 := \nu/\nu(\mathbb{R}^d)$; $(N_t)_{t\geq 0}$ is a Poisson process with rate $\lambda := \nu(\mathbb{R}^d)$. Moreover, $(N_t)_{t\geq 0}$ and $(\xi_i)_{i\geq 1}$ are independent.

Set $S_0 = x$ and $S_n = \sum_{i=0}^{n} \xi_i$ for $n \geq 1$. Since S has the coupling property, there exists another random walk $S'_n = \sum_{i=0}^{n} \xi'_i$ such that $S - S_0$ and $S' - S'_0$ have the same law and such that for any starting point $\xi'_0 = y$ of S' the coupling time

$$T^S_{x,y} := \inf\{k \geq 1:\ S_k = S'_k\} \quad \text{is a.s. finite.}$$

Without loss of generality, we can assume that $S_k = S'_k$ for $k \geq T^S_{x,y}$, and that $(\xi'_i)_{i\geq 1}$ is independent of $(N_t)_{t\geq 0}$. Define

$$L'_t = \sum_{i=0}^{N_t} \xi'_i, \quad t \geq 0.$$

Then $L' = (L'_t)_{t\geq 0}$ is also a compound Poisson process with Lévy measure ν and starting point $L'_0 = y$. In order to show that L has the coupling property, it is enough to verify that

$$T^L_{x,y} := \inf\{t > 0:\ L_t = L'_t\} < \infty. \tag{2.3}$$

We claim that

$$T^L_{x,y} = K_{x,y} \quad \text{with } K_{x,y} := \inf\{t > 0:\ N_t \geq T^S_{x,y}\}. \tag{2.4}$$

This implies (2.3). By assumption we know that for almost every ω, $T^S_{x,y}(\omega) < \infty$. Since the Poisson process N_t tends to infinity as $t \to \infty$, there exists $\tau_0(\omega) < \infty$ such that $N_t \geq T^S_{x,y}(\omega)$ for all $t \geq \tau_0(\omega)$. Therefore, (2.4) tells us that $T^L_{x,y} \leq \tau_0 < \infty$.

Let us finally prove (2.4). For this argument we assume that ω is fixed. Let $t > 0$ be such that $N_t \geq T^S_{x,y}$, i.e. $t \geq K_{x,y}$. Since $S_k = S'_k$ for $k \geq T^S_{x,y}$, $S_{N_t} = S'_{N_t}$ and, by construction, $L_t = L'_t$; thus, $T^L_{x,y} \leq t$ and since $t \geq K_{x,y}$ was arbitrary, we have $T^L_{x,y} \leq K_{x,y}$. On the other hand, assume that $K_{x,y} > 0$. Then, by the very definition of $K_{x,y}$, for any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that $t_\varepsilon > K_{x,y} - \varepsilon$ and $N_{t_\varepsilon} \leq T^S_{x,y} - 1$. Hence, $S_{N_{t_\varepsilon}} \neq S'_{N_{t_\varepsilon}}$, i.e. $L_{t_\varepsilon} \neq L'_{t_\varepsilon}$. Therefore, $T^L_{x,y} \geq t_\varepsilon > K_{x,y} - \varepsilon$. Letting $\varepsilon \to 0$, we get $T^L_{x,y} \geq K_{x,y}$ and the proof is complete. □

Note that the proof of Proposition 2.3 already gives an estimate for the rate at which coupling occurs: for all $t > 0$

$$\mathbb{P}\big(T^L_{x,y} > t\big) = \mathbb{P}\big(N_t < T^S_{x,y}\big) = e^{-\lambda t}\left[1 + \sum_{k=1}^{\infty} \mathbb{P}\big(T^S_{x,y} > k\big)\frac{(\lambda t)^k}{k!}\right].$$

What remains to be done is to get estimates for the coupling time of the random walk S, $\mathbb{P}(T^S_{x,y} > k), k \ge 1$. This requires concrete coupling constructions for S; the most interesting random walk couplings rely on a suitable coupling of the steps ξ_j and ξ'_j of S and S', respectively. In the next section we will, therefore, consider the Mineka and Lindvall–Rogers couplings.

We close this section with two comments on Proposition 2.3.

Remark 2.4. (1) *Theorem 4.3 below will show, that the converse of Proposition 2.3 is also true*: Let $L = (L_t)_{t\ge 0}$ be a compound Poisson process with Lévy measure ν, and $S = (S_n)_{n\ge 0}$, $S_n = S_0 + \xi_1 + \cdots + \xi_n$, be a random walk where $(\xi_i)_{i\ge 1}$ are i.i.d. random variables with $\xi_1 \sim \nu_0 := \nu/\nu(\mathbb{R}^d)$. If L has the coupling property so does S.

(2) *Proposition 2.3 can be easily generalized to more general settings, see also [1]. More precisely, let $(X_t)_{t\ge 0}$ be a Markov process on $\mathbb{R}^d$ and let $(S_t)_{t\ge 0}$ be a subordinator (i.e. an increasing Lévy process) which is independent of X_t. If X_t has the coupling property and if S_t tends to infinity as $t \to \infty$, then the subordinate process X_{S_t} also has the coupling property.*

3. The Mineka and Lindvall–Rogers couplings – A review

Let $S = (S_n)_{n\ge 1}$, $S_n = \xi_1 + \cdots + \xi_n$, be a random walk on $\mathbb{R}^d$ with i.i.d. steps $(\xi_i)_{i\ge 0}$ such that $\xi_1 \sim \nu_0$. The main result of this section is

Theorem 3.1. *Suppose that for some $\delta > 0$,*

$$\eta_0 = \eta_0(\delta) := \inf_{x\in\mathbb{R}^d, |x|\le\delta} \nu_0 \wedge (\delta_x * \nu_0)\big(\mathbb{R}^d\big) > 0. \tag{3.1}$$

Then there exists a constant $C := C(\delta, \eta_0) > 0$ such that for all $x, y \in \mathbb{R}^d$ and $n \ge 1$,

$$\big\|\mathbb{P}(x + S_n \in \cdot) - \mathbb{P}(y + S_n \in \cdot)\big\|_{\mathrm{Var}} \le \frac{C(1 + |x - y|)}{\sqrt{n}}. \tag{3.2}$$

The proof of Theorem 3.1 is mainly based on Mineka's coupling [9], see also [7], Chapter II, Section 14, pp. 44–47, and the coupling argument of the zero–two law for random walks proved in [8], Proposition 1, by Lindvall and Rogers. These papers do not contain an estimate as explicit as (3.2). Therefore we decided to include a detailed proof on our own which again highlights the role of the sufficient condition (3.1).

We begin with an auxiliary result which describes the total variation norm of a signed measure under a non-degenerate linear transformation.

Lemma 3.2. *Let μ be a probability measure μ on $\mathbb{R}^d$. Then we have for all $x, y \in \mathbb{R}^d$*

$$\|\delta_x * \mu - \delta_y * \mu\|_{\mathrm{Var}} = \|\delta_{x-y} * \mu - \mu\|_{\mathrm{Var}} = \big\|\delta_{|x-y|e_1} * \big(\mu \circ R^{-1}_{x-y}\big) - \mu \circ R^{-1}_{x-y}\big\|_{\mathrm{Var}},$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$ and R_a is a non-degenerate rotation such that $R_a a = |a|e_1$. In particular, for any $a \in \mathbb{R}^d$,

$$\|\delta_a * \mu - \mu\|_{\mathrm{Var}} = \|\delta_{-a} * \mu - \mu\|_{\mathrm{Var}}.$$

Proof. Using the definition of the total variation norm we get

$$\begin{aligned}
\|\delta_x * \mu - \delta_y * \mu\|_{\mathrm{Var}} &= 2 \sup_{A\in\mathcal{B}(\mathbb{R}^d)} \big|\delta_x * \mu(A) - \delta_y * \mu(A)\big| \\
&= 2 \sup_{A\in\mathcal{B}(\mathbb{R}^d)} \big|\mu(A - x) - \mu(A - y)\big| \\
&= 2 \sup_{B\in\mathcal{B}(\mathbb{R}^d)} \big|\mu\big(B - (x - y)\big) - \mu(B)\big| \\
&= \|\delta_{x-y} * \mu - \mu\|_{\mathrm{Var}}.
\end{aligned}$$

Now let $a \in \mathbb{R}^d$ and denote by R_a the rotation such that $R_a a = |a|e_1$. Clearly,

$$\mu \circ R_a(A) = \mu\{R_a x \in \mathbb{R}^d:\ x \in A\} = \mu\{y \in \mathbb{R}^d:\ R_a^{-1}(y) \in A\}, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Then,

$$\begin{aligned}
\|\delta_a * \mu - \mu\|_{\mathrm{Var}} &= 2 \sup_{A\in\mathcal{B}(\mathbb{R}^d)} \big|\mu(A-a) - \mu(A)\big| \\
&= 2 \sup_{A\in\mathcal{B}(\mathbb{R}^d)} \big|\mu \circ R_a^{-1}\big(R_a(A-a)\big) - \mu \circ R_a^{-1}(R_a A)\big| \\
&= 2 \sup_{A\in\mathcal{B}(\mathbb{R}^d)} \big|\mu \circ R_a^{-1}\big(R_a A - |a|e_1\big) - \mu \circ R_a^{-1}(R_a A)\big| \\
&= 2 \sup_{B\in\mathcal{B}(\mathbb{R}^d)} \big|\mu \circ R_a^{-1}\big(B - |a|e_1\big) - \mu \circ R_a^{-1}(B)\big| \\
&= \big\|\delta_{|a|e_1} * \big(\mu \circ R_a^{-1}\big) - \mu \circ R_a^{-1}\big\|_{\mathrm{Var}}.
\end{aligned}$$

□

Proposition 3.3. *Under* (3.1), *there exists a constant* $C = C(\eta_0) > 0$ *such that*

$$\sup_{|x-y|\le\delta} \big\|\mathbb{P}(x + S_n \in \cdot) - \mathbb{P}(y + S_n \in \cdot)\big\|_{\mathrm{Var}} \le \frac{C}{\sqrt{n}}. \tag{3.3}$$

Proof. *Step 1.* It is easy to see that for any $a \in \mathbb{R}^d$ and any probability measure μ,

$$\mu^{*n} \circ R_a^{-1} = \big(\mu \circ R_a^{-1}\big)^{*n}.$$

Lemma 3.2 shows that (3.3) is equivalent to the following estimate

$$\sup_{|a|\le\delta} \big\|\mathbb{P}\big(|a|e_1 + S_{a,n} \in \cdot\big) - \mathbb{P}(S_{a,n} \in \cdot)\big\|_{\mathrm{Var}} \le \frac{C}{\sqrt{n}}. \tag{3.4}$$

Here, $S_{a,n}$ is a random walk in $\mathbb{R}^d$ with i.i.d. steps $\xi_{a,1}, \xi_{a,2}, \dots$ and $\xi_{a,1} \sim \nu_0 \circ R_a^{-1}$.

On the other hand, Lemma 3.2 also shows that

$$\begin{aligned}
&\big(\nu_0 \circ R_a^{-1}\big) \wedge \big(\delta_{|a|e_1} * \big(\nu_0 \circ R_a^{-1}\big)\big)\big(\mathbb{R}^d\big) \\
&\quad = 1 - \frac{1}{2}\big\|\nu_0 \circ R_a^{-1} - \big(\delta_{|a|e_1} * \big(\nu_0 \circ R_a^{-1}\big)\big)\big\|_{\mathrm{Var}} \\
&\quad = 1 - \frac{1}{2}\big\|\nu_0 - (\delta_a * \nu_0)\big\|_{\mathrm{Var}} \\
&\quad = \nu_0 \wedge (\delta_a * \nu_0)\big(\mathbb{R}^d\big).
\end{aligned}$$

Therefore, (3.1) implies that for any $a \in \mathbb{R}^d$

$$\inf_{|a|\le\delta}\big\{\big(\nu_0 \circ R_a^{-1}\big) \wedge \big(\delta_{|a|e_1} * \big(\nu_0 \circ R_a^{-1}\big)\big)\big(\mathbb{R}^d\big)\big\} = \inf_{|a|\le\delta}\big\{\nu_0 \wedge (\delta_a * \nu_0)\big(\mathbb{R}^d\big)\big\} > 0. \tag{3.5}$$

In order to simplify the notation, we use $\nu := \nu_0 \circ R_a^{-1}$ and $S_n := S_{a,n}$. With this notation (3.4) becomes

$$\sup_{|a|\le\delta} \big\|\mathbb{P}\big(|a|e_1 + S_n \in \cdot\big) - \mathbb{P}(S_n \in \cdot)\big\|_{\mathrm{Var}} \le \frac{C}{\sqrt{n}}. \tag{3.6}$$

Step 2. Assume that $|a| \in (0, \delta]$ and set $\nu_{|a|} = \delta_{|a|e_1} * \nu$ and $\nu_{-|a|} = \delta_{-|a|e_1} * \nu$. Let $(\xi, \Delta\xi) \in \mathbb{R}^d \times \mathbb{R}^d$ be a pair of random variables with the following distribution

$$\mathbb{P}\big((\xi, \Delta\xi) \in A \times D\big) = \begin{cases} \frac{1}{2}(\nu \wedge \nu_{-|a|})(A), & D = \big\{|a|e_1\big\}; \\ \frac{1}{2}(\nu \wedge \nu_{|a|})(A), & D = \big\{-|a|e_1\big\}; \\ \big(\nu - \frac{1}{2}(\nu \wedge \nu_{-|a|} + \nu \wedge \nu_{|a|})\big)(A), & D = \{0\}; \end{cases}$$

where $A \in \mathcal{B}(\mathbb{R}^d)$ and $D \in \{\{-|a|e_1\}, \{0\}, \{|a|e_1\}\}$. We see from (3.5) that

$$\mathbb{P}\big(\Delta\xi = |a|e_1\big) = \frac{1}{2}\big(\nu \wedge (\delta_{-|a|e_1} * \nu)\big)\big(\mathbb{R}^d\big) \ge \frac{1}{2} \inf_{|a| \le \delta} \nu_0 \wedge (\delta_a * \nu_0)\big(\mathbb{R}^d\big).$$

By Lemma 3.2,

$$\begin{aligned} \mathbb{P}\big(\Delta\xi = -|a|e_1\big) &= \frac{1}{2}\big(\nu \wedge (\delta_{|a|e_1} * \nu)\big)\big(\mathbb{R}^d\big) \\ &= \frac{1}{2}\big(\nu \wedge (\delta_{-|a|e_1} * \nu)\big)\big(\mathbb{R}^d\big) \\ &= \mathbb{P}\big(\Delta\xi = |a|e_1\big). \end{aligned}$$

It is clear that the distribution of ξ is ν. Let $\xi' = \xi + \Delta\xi$. We claim that the distribution of ξ' is also ν. Indeed, for any $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned} \mathbb{P}\big(\xi' \in A\big) &= \mathbb{P}\big(\xi - |a|e_1 \in A, \Delta\xi = -|a|e_1\big) \\ &\quad + \mathbb{P}\big(\xi + |a|e_1 \in A, \Delta\xi = |a|e_1\big) + \mathbb{P}(\xi \in A, \Delta\xi = 0) \\ &= \frac{\delta_{-|a|e_1} * (\nu \wedge \nu_{|a|})(A)}{2} + \frac{\delta_{|a|e_1} * (\nu \wedge \nu_{-|a|})(A)}{2} + \left(\nu - \frac{\nu \wedge \nu_{-|a|} + \nu \wedge \nu_{|a|}}{2}\right)(A) \\ &= \mu(A), \end{aligned}$$

where we have used that $\delta_{-|a|e_1} * (\nu \wedge \nu_{|a|}) = \nu \wedge \nu_{-|a|}$ and $\delta_{|a|e_1} * (\nu \wedge \nu_{-|a|}) = \nu \wedge \nu_{|a|}$. Now we construct the coupling (S_n, S_n') of S_n with the i.i.d. pairs (ξ_i, ξ_i'), $i \ge 1$, where $(\xi_1, \xi_1') \sim (\xi, \xi')$. Since $\xi_i' - \xi_i = \Delta\xi$, we know that $\xi_i - \xi_i'$ is, for all $i \ge 1$, symmetrically distributed, takes only the values $-|a|e_1$, 0 and $|a|e_1$. Because of (3.5), we have

$$\begin{aligned} P(a) &:= \mathbb{P}\big(\xi_i^{1\prime} - \xi_i^1 = 0\big) \\ &= \left(\nu - \frac{1}{2}(\nu \wedge \nu_{-|a|} + \nu \wedge \nu_{|a|})\right)\big(\mathbb{R}^d\big) \\ &= 1 - \nu \wedge \nu_{-|a|}\big(\mathbb{R}^d\big) \\ &\le 1 - \inf_{|a| \le \delta} \nu_0 \wedge (\delta_a * \nu_0)\big(\mathbb{R}^d\big) \\ &=: \gamma(\delta) < 1, \end{aligned}$$

where $\xi_i = (\xi_i^1, \xi_i^2, \dots, \xi_i^d)$ and $\xi_i' = (\xi_i^{1\prime}, \xi_i^{2\prime}, \dots, \xi_i^{d\prime})$. Set $S_n^j = \sum_{i=1}^n \xi_i^j$ and $S_n^{j\prime} = \sum_{i=1}^n \xi_i^{j\prime}$. We observe that $(S^1 - S^{1\prime})$ is a random walk, whose step sizes are $-|a|$, 0 and $|a|$ with probability $\frac{1}{2}(1 - P(a))$, $P(a)$ and $\frac{1}{2}(1 - P(a))$, respectively. Since $S_n^j = S_n^{j\prime}$ for $2 \le j \le d$, we get

$$\big\|\mathbb{P}\big(|a|e_1 + S_n \in \cdot\big) - \mathbb{P}(S_n \in \cdot)\big\|_{\mathrm{Var}} \le 2\mathbb{P}\big(T^S > n\big), \tag{3.7}$$

where

$$T^S = \inf\big\{k \ge 1: \ S_k^1 = S_k^{1\prime} + |a|\big\}.$$

Step 3. We will now estimate $\mathbb{P}(T^S > n)$. Let $V_1, V_2, \dots$ be i.i.d. symmetric random variables, whose common distribution is given by

$$\mathbb{P}(V_i = x) = \begin{cases} \frac{1}{2}\big(1 - P(a)\big), & x = -|a|; \\ \frac{1}{2}\big(1 - P(a)\big), & x = |a|; \\ P(a), & x = 0. \end{cases}$$

Define $Z_n = \sum_{i=1}^n V_i$. We have seen in Step 2 that $T^S = \inf\{n \ge 1\colon\ Z_n = |a|\}$. Then, by the reflection principle,

$$\mathbb{P}\big(T^S > n\big) = \mathbb{P}\Big(\max_{k \le n} Z_k < |a|\Big) \le 2\mathbb{P}\big(0 \le Z_n \le |a|\big).$$

Since Z is the sum of i.i.d. random variables with mean 0 and variance $\sigma^2 = |a|^2(1 - P(a))$, we can use the central limit theorem to deduce, for sufficiently large values of n,

$$\begin{aligned} \mathbb{P}\big(T^S > n\big) &= 2\mathbb{P}\bigg(0 \le \frac{Z_n}{|a|\sqrt{1 - P(a)}\sqrt{n}} \le \frac{1}{\sqrt{1 - P(a)}\sqrt{n}}\bigg) \\ &\le 2\mathbb{P}\bigg(0 \le \frac{Z_n}{|a|\sqrt{1 - P(a)}\sqrt{n}} \le \frac{1}{\sqrt{1 - \gamma(\delta)}\sqrt{n}}\bigg) \\ &\le \frac{C}{\sqrt{2\pi}} \int_0^{1/\sqrt{1-\gamma(\delta)}\sqrt{n}} \mathrm{e}^{-x^2/2}\,\mathrm{d}x \\ &\le \frac{C_{1,\gamma(\delta)}}{\sqrt{n}}. \end{aligned}$$

In the first inequality above we have used the fact that $1 - P(a) \ge 1 - \gamma(\delta) > 0$. Therefore we find from (3.7) for all large $n \ge 1$

$$\big\|\mathbb{P}\big(|a|e_1 + S_n \in \cdot\big) - \mathbb{P}(S_n \in \cdot)\big\|_{\mathrm{Var}} \le \frac{2C_{2,\gamma(\delta)}}{\sqrt{n}}.$$

Since the right-hand side is bounded by 2, this estimate actually holds for all $n \ge 1$.

We can now use Lemma 3.2 to get

$$\big\|\mathbb{P}\big(\pm|a|e_1 + S_n \in \cdot\big) - \mathbb{P}(S_n \in \cdot)\big\|_{\mathrm{Var}} \le \frac{2C_{2,\gamma(\delta)}}{\sqrt{n}}$$

which immediately yields (3.6), since $|a| \le \delta$ was arbitrary. □

We close this section with the proofs of Theorems 3.1 and 1.1.

Proof of Theorem 3.1. For any $x, y \in \mathbb{R}^d$, set $k = [\frac{|x-y|}{\delta}] + 1$. Pick $x_0, x_1, \dots, x_k \in \mathbb{R}^d$ such that $x_0 = x$, $x_k = y$ and $|x_i - x_{i-1}| \le \delta$ for $1 \le i \le k$. By Proposition 3.3,

$$\begin{aligned} \big\|\mathbb{P}(x + S_n \in \cdot) - \mathbb{P}(y + S_n \in \cdot)\big\|_{\mathrm{Var}} &\le \sum_{i=1}^k \big\|\mathbb{P}(x_i + S_n \in \cdot) - \mathbb{P}(x_{i-1} + S_n \in \cdot)\big\|_{\mathrm{Var}} \\ &\le \frac{C(\delta, \eta_0)(1 + |x - y|)}{\sqrt{n}}, \end{aligned}$$

which is what we claimed. □

Proof of Theorem 1.1. *Step 1.* Assume first that the Lévy triplet is of the form $(0, 0, \nu)$ and that the Lévy measure satisfies $\lambda = \nu(\mathbb{R}^d) < \infty$. This means that X_t is compound Poisson process. We use the notations from Section 2. For all $x \in \mathbb{R}^d$,

$$\nu \wedge (\delta_x * \nu)\big(\mathbb{R}^d\big) = \lambda\big[\nu_0 \wedge (\delta_x * \nu_0)\big(\mathbb{R}^d\big)\big] > 0.$$

Therefore, we can apply Proposition 2.2 and Theorem 3.1 to get Theorem 1.1 in this case.

Step 2. If $(X_t)_{t\geq 0}$ is a general Lévy process with Lévy triplet (b, Q, ν), we split X_t into two independent parts

$$X_t = X_t' + X_t'',$$

where X_t' is the compound Poisson process with Lévy triplet $(0, 0, \nu_\varepsilon)$ – ν_ε is defined in (1.2) – and X_t'' is the Lévy process with triplet $(b, Q, \nu - \nu_\varepsilon)$. Denote by P_t', P_t'', $P_t'(x, \mathrm{d}y)$, $P_t''(x, \mathrm{d}y)$ the transition semigroups and transition functions of the processes X_t' and X_t'', respectively. Then, $P_t = P_t' P_t''$. Observe that P_t'' is a contraction semigroup, i.e. $\|P_t'' f\|_\infty \leq 1$ whenever $\|f\|_\infty \leq 1$. Therefore,

$$\begin{aligned}
\big\|P_t(x,\cdot) - P_t(y,\cdot)\big\|_{\mathrm{Var}} &= \sup_{\|f\|_\infty \leq 1} \big|P_t f(x) - P_t f(y)\big| \\
&= \sup_{\|f\|_\infty \leq 1} \big|P_t' P_t'' f(x) - P_t' P_t'' f(y)\big| \\
&\leq \sup_{\|h\|_\infty \leq 1} \big|P_t' h(x) - P_t' h(y)\big| \\
&= \big\|P_t'(x,\cdot) - P_t'(y,\cdot)\big\|_{\mathrm{Var}},
\end{aligned}$$

which reduces the general case to the compound Poisson setting considered in the first part. □

4. Extensions: The Lindvall–Rogers 'zero–two' law

Motivated by Lindvall–Rogers's zero–two law for random walks [8], Propsotion 1, we present a necessary and sufficient condition for the coupling property of a Lévy process. We will add a few simple sufficient criteria in terms of the Lévy measure which are easy to verify. Throughout this section we assume that $(X_t)_{t\geq 0}$ is a d-dimensional Lévy process with Lévy measure $\nu \not\equiv 0$; as usual, $X_0 = 0$. By $P_t(x, \cdot)$ and P_t we denote the transition probability and transition semigroup, respectively.

Theorem 4.1 (Criterion for successful couplings). *The following statements are equivalent*:

(1) *The Lévy process $(X_t)_{t\geq 0}$ has the coupling property.*
(2) *There exists $t_0 > 0$ such that for any $t \geq t_0$, the transition probability $P_t(x, \cdot)$ has* (*with respect to Lebesgue measure*) *an absolutely continuous component.*

In either case, for every $x, y \in \mathbb{R}^d$, there exists a constant $C(x, y) > 0$ such that

$$\big\|P_t(x,\cdot) - P_t(y,\cdot)\big\|_{\mathrm{Var}} \leq \frac{C(x,y)}{\sqrt{t}}, \quad t > 0. \tag{4.1}$$

If the Lévy process has the strong Feller property, i.e. the corresponding semigroup maps $B_b(\mathbb{R}^d)$ into $C_b(\mathbb{R}^d)$, Theorem 4.1 becomes particularly simple.

Corollary 4.2. *Suppose that there exists some $t_0 > 0$ such that the semigroup P_{t_0} maps $B_b(\mathbb{R}^d)$ into $C_b(\mathbb{R}^d)$. Then, the Lévy process X_t has the coupling property. In particular, every Lévy process which enjoys the strong Feller property has the coupling property.*

Proof. By assumption P_{t_0} is a convolution operator which maps $B_b(\mathbb{R}^d)$ into $C_b(\mathbb{R}^d)$. Due to a result by Hawkes, cf. [5] or [6], Lemma 4.8.20, P_{t_0} and all P_t with $t \geq t_0$ are of the form $P_t(x) = p_t * f(x)$, where $p_t(x)$ is the transition density of the process. Therefore condition (2) of Theorem 4.1 is satisfied. □

Now, we turn to the proof of Theorem 4.1.

Proof of Theorem 4.1. As mentioned in Section 1, the coupling property is equivalent to (1.1). Observe that

$$\begin{aligned}\|\mu_1 P_t - \mu_2 P_t\|_{\mathrm{Var}} &\leq \|\mu_1 P_t - P_t\|_{\mathrm{Var}} + \|\mu_2 P_t - P_t\|_{\mathrm{Var}} \\ &\leq \int \|\delta_x * P_t - \delta_0 * P_t\|_{\mathrm{Var}} \mu_1(\mathrm{d}x) + \int \|\delta_x * P_t - \delta_0 * P_t\|_{\mathrm{Var}} \mu_2(\mathrm{d}x).\end{aligned}$$

Thus, if

$$\lim_{t\to\infty} \|\delta_x * P_t - \delta_0 * P_t\|_{\mathrm{Var}} = 0 \quad \text{for } x \in \mathbb{R}^d, \tag{4.2}$$

then we can use the dominated convergence theorem to see that (1.1) holds. Therefore, the assertions (1.1) and (4.2) are equivalent.

Since $\|\delta_x * P_t - \delta_0 * P_t\|_{\mathrm{Var}}$ is decreasing in t, we see that (4.2) is also equivalent to

$$\lim_{n\to\infty} \|\delta_x * P_n - \delta_0 * P_n\|_{\mathrm{Var}} = 0 \quad \text{for } x \in \mathbb{R}^d. \tag{4.3}$$

Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables on $\mathbb{R}^d$ with $\xi_1 \sim \mu_1 := \mathbb{P}(X_1 \in \cdot)$ and set $S_n = \sum_{i=1}^n \xi_i$ for $n \geq 1$ and $S_0 = 0$. Since the increments of a Lévy process are independent and stationary, (4.3) is the same as

$$\lim_{n\to\infty} \big\|\mathbb{P}(x + S_n \in \cdot) - \mathbb{P}(S_n \in \cdot)\big\|_{\mathrm{Var}} = 0 \quad \text{for } x \in \mathbb{R}^d. \tag{4.4}$$

According to [8], Remark (ii), pp. 124–125, or [10], Chapter 3, Section 3, Theorem 3.9, (4.4) holds if, and only if, μ_1 is spread out, i.e. for some $m \geq 1$, $\mu_1^{*m} = P_m(0, \cdot) := \mathbb{P}(X_m \in \cdot)$ has an absolutely continuous component. Since the semigroup of a Lévy process is a convolution semigroup, it is easy to see that for every $t \geq m$, the transition probability $P_t(x, \cdot)$ has an absolutely continuous part. Combining all the assertions above, we have proved that the statements (1) and (2) are equivalent.

Moreover, the arguments used in the proof of Proposition 3.3 together with [8], Proposition 1, show that

$$\big\|\mathbb{P}(x + S_n \in \cdot) - \mathbb{P}(S_n \in \cdot)\big\|_{\mathrm{Var}} = \mathrm{O}\big(n^{-1/2}\big) \quad \text{as } n \to \infty$$

whenever the random walk S_n has the coupling property. Therefore, (4.1) follows from the arguments used in the first part of the proof, in particular since $t \mapsto \|\delta_x * P_t - \delta_0 * P_t\|_{\mathrm{Var}}$ is decreasing and

$$\big\|\mathbb{P}(x + S_n \in \cdot) - \mathbb{P}(S_n \in \cdot)\big\|_{\mathrm{Var}} = \|\delta_x * P_n - \delta_0 * P_n\|_{\mathrm{Var}}.$$ □

Let us finally derive some sufficient conditions in terms of the Lévy measure, which extend Theorem 1.1 and [14], Theorem 3.1.

Theorem 4.3 (Sufficient criteria for successful couplings). *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ with Lévy measure $\nu \not\equiv 0$ and define $\varepsilon > 0$ as in (1.2), i.e. for $B \in \mathcal{B}(\mathbb{R}^d)$*

$$\nu_\varepsilon(B) = \begin{cases} \nu(B), & \nu\big(\mathbb{R}^d\big) < \infty; \\ \nu\big\{z \in B\colon |z| \geq \varepsilon\big\}, & \nu\big(\mathbb{R}^d\big) = \infty. \end{cases}$$

If there exists some $\varepsilon > 0$ such that one of the following conditions is satisfied

(1) *For some $l \geq 1$, ν_ε^{*l} has an absolutely continuous component.*

(2) *There exist* $\delta > 0$ *and* $l \geq 1$ *such that* $\inf_{x\in\mathbb{R}^d,|x|\leq\delta} \nu_\varepsilon^{*l} \wedge (\delta_x * \nu_\varepsilon^{*l})(\mathbb{R}^d) > 0$.

Then the process X_t *has the coupling property.*

Conversely, assume that $\nu(\mathbb{R}^d) < \infty$ *and* X_t *is a compound Poisson process with Lévy measure* ν. *If* X_t *has the coupling property, then there is some* $l \geq 0$ *such that* ν^{*l} *has an absolutely continuous component.*

Proof. *Step 1.* The argument used in the proof of Theorem 1.1 shows that we only have to consider the coupling property for a compound Poisson process, whose Lévy measure is of the form ν_ε. Let $S = (S_n)_{n\geq 1}$, $S_n = \xi_1 + \cdots + \xi_n$, be a random walk on $\mathbb{R}^d$ with i.i.d. steps $\xi_1, \xi_2, \ldots$ such that $\xi_1 \sim \nu_\varepsilon/\nu_\varepsilon(\mathbb{R}^d)$. Because of Proposition 2.3, it is sufficient to show that, under the assumptions stated in the theorem, S has the coupling property.

As we have pointed out in the proof of Theorem 4.1, [8], Remark (ii), pp. 124–125, shows that S has the coupling property if, and only if, condition (1) holds. Again by [8], Remark (ii), pp. 124–125, condition (1) is equivalent to saying that for any $x \in \mathbb{R}^d$, there exists $l \geq 0$ such that $\nu_\varepsilon^{*l} \wedge (\delta_x * \nu_\varepsilon^{*l})(\mathbb{R}^d) > 0$. Clearly, such a condition is hard to check in applications.

Let $Z = (Z_n)_{n\geq 1}$, $Z_n = \zeta_1 + \cdots + \zeta_n$, be a random walk on $\mathbb{R}^d$ with i.i.d. steps $\zeta_1, \zeta_2, \ldots$ such that $\zeta_1 \sim \nu_\varepsilon^{*l}/\nu_\varepsilon^l(\mathbb{R}^d)$. That is, $\zeta_i = \sum_{k=(i-1)l+1}^{il} \xi_k$, where ξ_i is the step of the random walk S from the paragraph above. If (2) holds, Theorem 3.1 shows that Z, hence S, has the coupling property.

Step 2. Let $(X_t)_{t\geq 0}$ be a compound Poisson process with Lévy measure ν and suppose that X_t has the coupling property. Moreover, assume that none of the measures ν^{*l}, $l \geq 1$, has an absolutely continuous component. By the Lebesgue decomposition, each measure ν^{*l} is mutually singular with respect to Lebesgue measure Leb. Thus, for every $l \geq 1$, there exists some set $A_l \in \mathscr{B}(\mathbb{R}^d)$ such that $\mathrm{Leb}(A_l) = \nu^{*l}(A_l^c) = 0$. For $A := \bigcup_{i=1}^\infty A_i$ we have $\mathrm{Leb}(A) = \nu^{*l}(A^c) = 0$ for each $l \geq 1$. Therefore, by (2.1), for every $x \in \mathbb{R}^d$ and $t > 0$, the transition probability $P_t(x, \cdot)$ of X_t is singular with respect to Lebesgue measure Leb. This shows that condition (2) of Theorem 4.1 cannot hold, i.e. X_t does not have the coupling property. Since this contradicts our assumption, the proof is finished. □

Theorem 4.3 immediately yields that

Corollary 4.4. *Any Lévy process whose Lévy measure possesses an absolutely continuous component has the coupling property.*

The coupling property of a Lévy process is intimately connected with the choice of state space. According to Theorem 4.3, a Poisson process on $\mathbb{R}$ does not have the coupling property, see also the discussion in Remark 1.2. If, however, the process is considered on $\mathbb{Z}$, the situation changes.

Proposition 4.5. *A Poisson process* X_t *with state space* $\mathbb{Z}$ *has the coupling property.*

Proof. We use the coupling and shift coupling properties proved in [4]. Shift coupling is a slightly weaker notion than coupling. A Markov process $(X_t)_{t\geq 0}$ is said to have the shift coupling property, if for any two initial distributions μ, ν, there exists a coupling (X_t, Y_t) with marginal processes such that

- $X_0 \sim \mu$ and $Y_0 \sim \nu$;
- there are finite stopping times T_1, T_2 such that $X_{T_1} = Y_{T_2}$.

Let λ be the intensity of the Poisson process X_t. Then the generator is given by $Lf(i) = \lambda(f(i+1) - f(i))$ for $i \in \mathbb{Z}$. Thus, all harmonic functions $f : \mathbb{Z} \to \mathbb{R}$ are constant and, by [3], Theorem 1 and its second remark, or [4], Theorem 2, the process has the shift coupling property.

Similar to the proof of [14], Proposition 3.3, for any $s, t > 0$, $i \in \mathbb{Z}$ and $f \in B_b(\mathbb{Z})$ with $f \geq 0$,

$$P_{t+s}(i) = \mathbb{E}f(i + X_{t+s}) \geq \mathbb{E}f\big((i + X_t)\mathbb{1}_{\{X_{t+s}-X_t=0\}}\big) = \mathrm{e}^{-\lambda s}\mathbb{E}f(i + X_t) = \mathrm{e}^{-\lambda s} P_t f(i),$$

which shows that X_t has the coupling property, cf. [4], Theorem 5. □

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