

Penalisation of a stable Lévy process involving its one-sided supremum

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Abstract. Penalisation involving the one-sided supremum for a stable Lévy process with index $\alpha \in (0, 2]$ is studied. We introduce the analogue of Azéma–Yor martingales for a stable Lévy process and give the law of the overall supremum under the penalised measure.

Résumé. On étudie des pénalisations d'un processus de Lévy stable d'indice $\alpha \in (0, 2]$ qui font intervenir son supremum unilatéral. On introduit pour un processus de Lévy stable, des martingales analogues aux martingales d'Azéma–Yor pour le mouvement brownien et son supremum; ceci permet d'obtenir la loi du supremum global relativement à la mesure pénalisée.

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1. Introduction

Royette–Vallois–Yor ([18–20] and [21]) have studied in depth the existence of limit laws:

$$\mathbb{W}_t^\Gamma := \frac{\Gamma_t}{\mathbb{W}[\Gamma_t]} \cdot \mathbb{W} \quad \text{as } t \rightarrow \infty, \quad (1.1)$$

where $X = (X_t, \mathcal{F}_t, \mathbb{W})$ denotes the canonical representation of a one-dimensional Brownian motion with $\mathbb{W}(X_0 = 0) = 1$ and (Γ_t) is a family of integrable, non-negative functionals. More precisely, the corresponding penalisation problem is to show that, for fixed s and any \mathcal{F}_s -measurable bounded functional F_s ,

$$\mathbb{W}_t^\Gamma[F_s] \longrightarrow \mathbb{W}_\infty^\Gamma[F_s] := \mathbb{W}[F_s M_s^\Gamma] \quad \text{as } t \rightarrow \infty, \quad (1.2)$$

for a certain $((\mathcal{F}_s), \mathbb{W})$ -martingale M_s^Γ . Among these studies, the three most “natural” ones are obtained from

- (i) supremum penalisation: $\Gamma_t = f(S_t)$ where $S_t = \sup_{s \leq t} X_s$;

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- (ii) local time penalisation: $\Gamma_t = f(L_t)$ where L_t denotes the local time at 0 of X ;
- (iii) Feynman–Kac penalisation: $\Gamma_t = \exp(-\int_0^t q(X_s) ds)$,

for non-negative Borel functions f and q with some integrability condition. The aim is to obtain new laws, close enough to \mathbb{W} , but for which, in (i): $S_\infty < \infty$; in (ii): $L_\infty < \infty$; and in (iii): $\int_0^\infty q(X_s) ds < \infty$ a.s. In other words, the penalisation procedure aims at giving some sense to the conditionings of \mathbb{W} with respect to: $(S_\infty < \infty)$; $(L_\infty < \infty)$; or $(\int_0^\infty q(X_s) ds < \infty)$. Such conditionings cannot be done in the traditional sense, and need some kind of Doob’s h -transform approach. This is precisely what the above penalisations are doing. As supremum penalisation (i) seems to be one of the most “direct” ones to perform, it seems of some interest to discuss the analogue of this particular penalisation problem for (possibly asymmetric) strictly α -stable Lévy processes. The authors have studied in [25] the other penalisations (ii) and (iii) for symmetric stable Lévy processes; we shall give a remark in Section 6 about comparison of the supremum penalisation (i) with (ii) and (iii).

Let us recall briefly supremum penalisation results in the Brownian case. Let f be a non-negative Borel function which satisfies $\int_0^\infty f(x) dx = 1$. Roynette–Vallois–Yor [19] have shown that, for any fixed s and any \mathcal{F}_s -measurable bounded functional F_s ,

$$\mathbb{W}_t^{(f)}[F_s] := \frac{\mathbb{W}[F_s f(S_t)]}{\mathbb{W}[f(S_t)]} \longrightarrow \mathbb{W}_\infty^{(f)}[F_s] := \mathbb{W}[F_s M_s^{(f)}] \quad \text{as } t \rightarrow \infty, \tag{1.3}$$

where

$$M_s^{(f)} = \lim_{t \rightarrow \infty} \frac{\mathbb{W}[f(S_t)|\mathcal{F}_s]}{\mathbb{W}[f(S_t)]} \quad \text{a.s. and in } L^1(\mathbb{W}). \tag{1.4}$$

The authors also found that $(M_s^{(f)}, s \geq 0)$ is a $(\mathcal{F}_s, \mathbb{W})$ -martingale which has the form

$$M_s^{(f)} = f(S_s)(S_s - X_s) + \int_{S_s}^\infty f(x) dx. \tag{1.5}$$

The class of these martingales $(M_s^{(f)})$ is known as (Brownian) Azéma–Yor martingales, and is helpful to solve the Skorokhod embedding problem in terms of (S_s, X_s) ; see Azéma–Yor [1] and [2]; see also [12–15] and [16].

In [19], there is furthermore the following description of the probability measure $\mathbb{W}_\infty^{(f)}$ featured in (1.3):

- (1) Under $\mathbb{W}_\infty^{(f)}$, the random variable S_∞ is distributed as $f(x) dx$.
- (2) Let $g = \sup\{u \geq 0: X_u = S_\infty\}$. Then $\mathbb{W}_\infty^{(f)}(g < \infty) = 1$, and under $\mathbb{W}_\infty^{(f)}$ we have the following:
 - (a) the processes $(X_u, u \leq g)$ and $(X_g - X_{u+g}, u \geq 0)$ are independent;
 - (b) $(X_g - X_{u+g}, u \geq 0)$ is distributed as a 3-dimensional Bessel process starting at 0;
 - (c) conditional on $S_\infty = x > 0$, the process $(X_u, u \leq g)$ is distributed as a Brownian motion started at 0 and stopped at its first hitting time of x .

In the present paper we discuss analogues of these studies in the case of α -stable Lévy processes $X = (X_t, \mathcal{F}_t, \mathbb{P})$ with indices $\alpha \in (0, 2]$. In particular, we shall propose a certain class of martingales $(M_t^{(f)})$ similar to (1.5), study the law of S_∞ under $\mathbb{P}^{(f)}$ defined by $\mathbb{P}^{(f)}|_{\mathcal{F}_t} = M_t^{(f)} \cdot \mathbb{P}|_{\mathcal{F}_t}$, and then establish the convergence theorem analogous to (1.3).

The organization of the present paper is as follows. In Section 2 we recall some preliminary facts about Lévy processes, reflected Lévy processes and α -stable Lévy processes. In Section 3 we introduce a certain class of martingales $(M_t^{(f)})$ given by functions of X_t and S_t only, and present the measure $\mathbb{P}^{(f)}$, which is in fact the limit measure of our penalisation problem. Section 4 is devoted to the study of the law of the overall supremum S_∞ under $\mathbb{P}^{(f)}$. In Section 5 we establish the convergence theorem and give a partial solution of our penalisation problem. In Section 6 we compare the supremum penalisation with local time and Feynman–Kac penalisations.

2. Preliminaries

All facts presented in this section are well-known; see, e.g., [3,5] and [8]. We basically follow the notation of Doney [8].

2.1. *Basics: General framework for Lévy processes*

Let $\mathcal{D}([0, \infty))$ be the space of càdlàg paths $\omega : [0, \infty) \rightarrow \mathbb{R} \cup \{\delta\}$ with lifetime $\zeta(\omega) = \inf\{s : \omega(s) = \delta\}$ where δ is a cemetery point. Let (X_t) denote the coordinate process, $X_t(\omega) = \omega(t)$, and let (\mathcal{F}_t) denote its natural filtration with $\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t$.

Let \mathbb{P} be the law of a Lévy process $X = (X_t, t \geq 0)$ with $\mathbb{P}(X_0 = 0) = 1$ such that

$$\mathbb{P}[\exp\{i\lambda X_t\}] = e^{-t\Psi(\lambda)}, \quad t \geq 0, \lambda \in \mathbb{R}, \tag{2.1}$$

where

$$\Psi(\lambda) = i\gamma\lambda + \frac{\sigma^2\lambda^2}{2} + \int_{\mathbb{R}} (1 - e^{i\lambda x} + i\lambda x 1_{\{|x| < 1\}}) \nu(dx) \tag{2.2}$$

for some constants γ, σ , and Lévy measure ν on $\mathbb{R} \setminus \{0\}$ which satisfies

$$\int_{\mathbb{R}} (x^2 \wedge 1) \nu(dx) < \infty. \tag{2.3}$$

For every $x \in \mathbb{R}$, we denote by \mathbb{P}_x the law of $X + x$ under \mathbb{P} , that is, the law of the Lévy process starting at point x . Let T_A denote the entrance time of a set $A \subset \mathbb{R}$ of X :

$$T_A = T_A(X) = \inf\{s > 0 : X_s \in A\} \tag{2.4}$$

with the convention $\inf \emptyset = \infty$. When a set A contains only one point a , $T_{\{a\}}$ means the first hitting time of the point a .

Let S_t and I_t respectively denote the past supremum and the past infimum, that is, for all $t \geq 0$,

$$S_t = \sup\{0 \vee X_s : 0 \leq s \leq t\} \quad \text{and} \quad I_t = \inf\{0 \wedge X_s : 0 \leq s \leq t\}. \tag{2.5}$$

Let R denote the reflected process of X , that is,

$$R = S - X. \tag{2.6}$$

We recall that R is a strong Markov process (Bingham [4], see also [3]). If 0 is regular for $(0, \infty)$ with respect to X (i.e., $\mathbb{P}(T_{(0, \infty)} = 0) = 1$), we can define a local time at 0 of R which we write as $L = \{L_t, t \geq 0\}$. As is well known, L is unique up to a multiplicative constant. We note that L_t is continuous in t . Let τ be the right continuous inverse of L and let $H = X(\tau) = S(\tau)$. Then τ and H are both subordinators, and we call them the upwards ladder time and the upwards ladder height processes of X . We note that the pair (τ, H) is a bivariate subordinator. Replacing X by $X^* = -X$, we can consider the dual process $R^* = S^* - X^* = X - I$. If 0 is regular for $(-\infty, 0)$ with respect to X (i.e., 0 is regular for $(0, \infty)$ with respect to X^*), we define L^* the local time at 0 of R^* , and we get the downwards ladder time process τ^* and the downwards ladder height process H^* .

Assume that 0 is regular for $(-\infty, 0)$ with respect to X . Consider the function h^* defined by

$$h^*(x) := \mathbb{P} \left[\int_{[0, \infty)} 1_{\{I_t \geq -x\}} dL_t^* \right]. \tag{2.7}$$

Note that h^* is finite, continuous, increasing, and subadditive on $[0, \infty)$. We also note that $h^*(0) = 0$ by the regularity assumption for 0. For $x > 0$, we denote by \mathbb{Q}_x the law on $\mathcal{D}([0, \infty))$ of the killed process, i.e.,

$$\mathbb{Q}_x(\Lambda, t < \zeta) = \mathbb{P}_x(\Lambda, t < T_{(-\infty, 0)}), \quad t \geq 0, \Lambda \in \mathcal{F}_t. \tag{2.8}$$

The following lemma is due to Silverstein [22]; see also Chaumont [5], p. 41, Chaumont–Doney [7], Lemma 5 and [8], Lemma VIII 10.

Lemma 2.1 ([5,7,8,22]). *Suppose that X does not drift to $-\infty$ and 0 is regular for $(-\infty, 0)$ under \mathbb{P} . Then the function h^* defined by (2.7) is invariant for the process (X, \mathbb{Q}_x) , $x > 0$. That is,*

$$\mathbb{Q}_x[h^*(X_t) 1_{\{t < \zeta\}}] = h^*(x), \quad x > 0, t \geq 0. \tag{2.9}$$

2.2. Stable Lévy processes

A Lévy process X has the scaling property of order $1/\alpha$ if and only if, for fixed t , X_t and $t^{1/\alpha} X_1$ have the same law. In this case we call X a (strictly) stable Lévy process with index α .

Henceforth the range of the index α is $(0, 2]$. The trivial case when X is a pure drift is excluded. The cases $\alpha = 1$ and $\alpha = 2$ are special, and correspond to a symmetric Cauchy process with drift and a Brownian motion up to a multiplicative constant, respectively.

Let $X = ((X_t, t \geq 0), \mathbb{P})$ be a stable Lévy process with index $\alpha \in (0, 1) \cup (1, 2)$. The characteristic exponent has the form

$$\Psi(\lambda) = c|\lambda|^\alpha \left(1 - i\beta \operatorname{sgn}(\lambda) \tan \frac{\pi\alpha}{2} \right), \quad \lambda \in \mathbb{R}, \tag{2.10}$$

where $c > 0$ and $\beta \in [-1, 1]$. The Lévy measure ν is absolutely continuous with respect to the Lebesgue measure and equals

$$\nu(dx) = \begin{cases} c_+ x^{-\alpha-1} dx, & \text{if } x > 0, \\ c_- |x|^{-\alpha-1} dx, & \text{if } x < 0, \end{cases}$$

where c_+ and c_- are two non-negative real numbers such that

$$\beta = \frac{c_+ - c_-}{c_+ + c_-}. \tag{2.11}$$

We remark that, when $c_+ = 0$ (respectively, $c_- = 0$), or equivalently $\beta = -1$ (respectively, $\beta = 1$), the process has no positive (respectively, no negative) jumps. When $c_+ = c_-$, or equivalently $\beta = 0$, the process is symmetric.

When $\alpha = 2$, the characteristic exponent is $\Psi(\lambda) = c\lambda^2$ for some constant $c > 0$ and X is a Brownian motion up to a multiplicative constant. In that case X is continuous and the Lévy measure is identically 0.

When $\alpha = 1$, X is a symmetric Cauchy process with drift. The characteristic exponent is given by $\Psi(\lambda) = c|\lambda| + d i\lambda$ where $d \in (-\infty, \infty)$ is a drift constant and $c > 0$. The Lévy measure is proportional to $|x|^{-2} dx$.

We set

$$\rho = \mathbb{P}(X_1 \geq 0). \tag{2.12}$$

By the scaling property of X , $\mathbb{P}(X_t \geq 0) = \rho$ for all $t > 0$. ρ is called the positivity parameter. For $\alpha \neq 1, 2$, ρ is computed in terms of the parameter β :

$$\rho = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan \left(\beta \tan \frac{\pi\alpha}{2} \right). \tag{2.13}$$

We note that the range of ρ is $[0, 1]$ for $0 < \alpha < 1$. The boundary points $\rho = 1$ and $\rho = 0$ correspond to the cases when X is a subordinator and the negative of a subordinator, respectively. For $1 < \alpha < 2$, the range of ρ is $[1 - 1/\alpha, 1/\alpha]$. Obviously $\rho = 1/2$ when X is symmetric. When $\alpha = 2$, ρ necessarily equals $1/2$. When $\alpha = 1$, the range of ρ is the interval $(0, 1)$.

Lemma 2.2 ([3], Lemma VIII 1). *Suppose that $|X|$ is not a subordinator. The ladder time process τ is a stable subordinator of index ρ , and the ladder height process H is a stable process of index $\alpha\rho$.*

We remark that $|X|$ is not a subordinator if and only if $0 < \rho < 1$, so we will assume that $0 < \rho < 1$ throughout this paper. In that case, the origin is regular for itself (i.e., $\mathbb{P}(T_{\{0\}} = 0) = 1$) and is an instantaneous point (i.e., $\mathbb{P}(T_{\mathbb{R} \setminus \{0\}} = 0) = 1$). In other words, the origin is regular for $(-\infty, 0)$ and $(0, \infty)$ with respect to X . Therefore we can define the local times L, L^* etc. for the reflected and dual processes in this case.

In the previous setting, the function h^* is given as follows.

Proposition 2.3 (Chaumont [6], p. 384). *The function h^* has the form*

$$h^*(x) = Cx^{\alpha(1-\rho)}, \quad x \geq 0, \quad (2.14)$$

for some constant C .

Proof. From (2.7), we obtain, by the obvious change of variables and Fubini's theorem,

$$\begin{aligned} h^*(x) &= \mathbb{P} \left[\int_{[0, \infty)} 1_{\{H_s^* \leq x\}} ds \right] \\ &= \int_{[0, \infty)} \mathbb{P}(H_s^* \leq x) ds. \end{aligned} \quad (2.15)$$

Since H_t^* is a stable subordinator of index $\alpha(1-\rho)$ by Lemma 2.2 we have, for some constant $C' > 0$,

$$\begin{aligned} \exp\{-C't\lambda^{\alpha(1-\rho)}\} &= \mathbb{P}[e^{-\lambda H_t^*}] \\ &= \lambda \int_{[0, \infty)} e^{-\lambda x} \mathbb{P}(H_t^* \leq x) dx. \end{aligned} \quad (2.16)$$

Integrating (2.16) with respect to t , we obtain

$$\begin{aligned} \frac{1}{C'\lambda^{\alpha(1-\rho)+1}} &= \int_0^\infty dt \int_{[0, \infty)} e^{-\lambda x} \mathbb{P}(H_t^* \leq x) dx \\ &= \int_{[0, \infty)} e^{-\lambda x} dx \int_0^\infty \mathbb{P}(H_t^* \leq x) dt. \end{aligned}$$

Hence, inverting the Laplace transform in λ , we obtain

$$\int_0^\infty \mathbb{P}(H_t^* \leq x) dt = Cx^{\alpha(1-\rho)}$$

for $C = 1/\{C'\Gamma(\alpha(1-\rho) + 1)\}$. This ends the proof of Proposition 2.3 thanks to (2.15). \square

Corollary 2.4. *For every $\alpha \in (0, 2]$ and $\rho \in (0, 1)$, it holds that*

$$\mathbb{P}_x[(X_t)^{\alpha(1-\rho)} 1_{\{T_{(-\infty, 0)} > t\}}] = x^{\alpha(1-\rho)}, \quad x > 0. \quad (2.17)$$

Equivalently, for every $\alpha \in (0, 2]$ and $\rho \in (0, 1)$, one has

$$\mathbb{P}_x^*[(X_t^*)^{\alpha\rho} 1_{\{T_{(-\infty, 0)}^* > t\}}] = x^{\alpha\rho}, \quad x > 0, \quad (2.18)$$

where \mathbb{P}_x^* indicates the law of X^* starting at x and $T_A^* = T_A(X^*)$.

Proof. (2.17) follows from Proposition 2.3, (2.9) and (2.8).

Moreover, (2.17) is equivalent to

$$\mathbb{P}[(X_t + x)^{\alpha(1-\rho)} 1_{\{T_{(-\infty, -x)} > t\}}] = x^{\alpha(1-\rho)}. \quad (2.19)$$

Applying (2.19) to X^* , we obtain

$$\mathbb{P}[(X_t^* + x)^{\alpha\rho} 1_{\{T_{(-\infty, -x)}^* > t\}}] = x^{\alpha\rho}, \quad (2.20)$$

or equivalently,

$$\mathbb{P}_x^*[(X_t^*)^{\alpha\rho} 1_{\{T_{(-\infty,0)}^* > t\}}] = x^{\alpha\rho}, \tag{2.21}$$

which proves Corollary 2.4. □

3. Martingales which are functions of X_t and S_t only

Let $X = ((X_t, t \geq 0), \mathbb{P})$ be a (α, ρ) -stable Lévy process with $\mathbb{P}(X_0 = 0) = 1$. Recall that $\alpha \in (0, 2]$ and $\rho \in (0, 1)$.

Let us consider the following conditions bearing upon a non-negative Borel function f on $[0, \infty)$:

(C1) $\int_0^\infty x^{\alpha\rho-1} f(x) dx < \infty$;

(C2) f is absolutely continuous and satisfies $\lim_{x \rightarrow \infty} f(x) = 0$ and $\int_0^\infty x^{\alpha\rho} |f'(x)| dx < \infty$.

It is easy to see that (C2) implies (C1).

For a non-negative Borel function f satisfying the condition (C1), we define $M_t^{(f)}$ as

$$M_t^{(f)} := f(S_t)(S_t - X_t)^{\alpha\rho} + \alpha\rho \int_{S_t}^\infty f(x)(x - X_t)^{\alpha\rho-1} dx. \tag{3.1}$$

If the function f satisfies the condition (C2), it follows from (3.1) by integration by parts that

$$M_t^{(f)} = - \int_{S_t}^\infty f'(x)(x - X_t)^{\alpha\rho} dx. \tag{3.2}$$

Theorem 3.1. *Let f be a non-negative Borel function satisfying the condition (C1). Then $(M_t^{(f)})$ is a $(\mathcal{F}_t, \mathbb{P})$ -martingale which converges to 0 a.s. as $t \rightarrow \infty$.*

Proof. (I) Let $f_a(x) = 1_{[0,a]}(x)$ for $0 < a < \infty$. We write $M_t^{(a)} := M_t^{(f_a)}$. In this case $M_t^{(a)}$ takes the following simple form (3.3):

$$\begin{aligned} M_t^{(a)} &= f_a(S_t)(S_t - X_t)^{\alpha\rho} + \alpha\rho \int_{S_t}^\infty f_a(x)(x - X_t)^{\alpha\rho-1} dx \\ &= 1_{\{S_t \leq a\}} \left\{ (S_t - X_t)^{\alpha\rho} + \alpha\rho \int_{S_t}^a (x - X_t)^{\alpha\rho-1} dx \right\} \\ &= 1_{\{S_t \leq a\}} (a - X_t)^{\alpha\rho}. \end{aligned} \tag{3.3}$$

(II) We now prove the martingale property of $(M_t^{(a)})$. We have

$$\begin{aligned} \mathbb{P}[M_t^{(a)} | \mathcal{F}_s] &= 1_{\{S_s \leq a\}} \mathbb{P}[1_{\{\sup_{u \in (s,t)} X_u \leq a\}} (a - X_t)^{\alpha\rho} | \mathcal{F}_s] \\ &= 1_{\{S_s \leq a\}} \widehat{\mathbb{P}}[1_{\{X_s + \widehat{S}_{t-s} \leq a\}} (a - X_s - \widehat{X}_{t-s})^{\alpha\rho}] \end{aligned}$$

by the Markov property of X (where \widehat{X} , distributed as X , is independent from \mathcal{F}_s). It now suffices to prove that (without hats)

$$\mathbb{P}[1_{\{S_t \leq b\}} (b - X_t)^{\alpha\rho}] = b^{\alpha\rho}. \tag{3.4}$$

Our rationale for changing a into b is that we take $b = a - X_s$. The desired identity (3.4) may be written in terms of $X^* = -X$,

$$\mathbb{P}[1_{\{-I_t^* \leq b\}} (b + X_t^*)^{\alpha\rho}] = b^{\alpha\rho}. \tag{3.5}$$

By using the fact $(X_t^* + b, \mathbb{P}) \stackrel{\text{law}}{=} (X_t^*, \mathbb{P}_b^*)$, we get

$$\begin{aligned} \text{the LHS of (3.5)} &= \mathbb{P}_b^*[1_{\{-I_t^* \leq 0\}}(X_t^*)^{\alpha\rho}] \\ &= \mathbb{P}_b^*[1_{\{T_{(-\infty, 0)}^* > t\}}(X_t^*)^{\alpha\rho}], \end{aligned}$$

therefore (3.4) is equivalent to the following statement about X^* :

$$\mathbb{P}_b^*[1_{\{T_{(-\infty, 0)}^* > t\}}(X_t^*)^{\alpha\rho}] = b^{\alpha\rho}. \tag{3.6}$$

But this last identity (3.6) is nothing but the identity (2.18), thus we have derived the identity (3.4) from Corollary 2.4.

(III) The martingale property extends to general non-negative Borel f 's with the help of the monotone-class theorem. Therefore we obtain the desired conclusion and the integrability of $(M_t^{(f)})$ when

$$M_0^{(f)} = \alpha\rho \int_0^\infty x^{\alpha\rho-1} f(x) dx < \infty.$$

(IV) Remaining problem is to show that $M_t^{(f)}$ converges to 0 a.s., however, we postpone the proof to the end of the next section. □

Remark 3.2. For more about Σ_a , see Lemma 4.1 below.

Remark 3.3. Azéma–Yor [1,2] have introduced the following class of Brownian local martingales:

$$F(B_t, S_t) = f(S_t)(S_t - B_t) - \int_0^{S_t} f(x) dx = - \int_0^t f(S_u) dB_u \tag{3.7}$$

for locally integrable functions f where (B_t) is a one-dimensional Brownian motion and $S_t = \sup_{s \leq t} B_s$. They utilized this class in order to solve the Skorokhod embedding problem; see Obłój [14] for an excellent survey on this topic. Obłój [15] (see also Obłój–Yor [16]) proved that any Brownian local martingale of the form $F(B_t, S_t)$ for some Borel function F is given by (3.7) for some locally integrable function f . As a special case, if f is globally integrable and $\bar{f} := \int_0^\infty f(x) dx$, then we have

$$f(S_t)(S_t - B_t) + \int_{S_t}^\infty f(x) dx = \bar{f} + f(S_t)(S_t - B_t) - \int_0^{S_t} f(x) dx \tag{3.8}$$

$$= \bar{f} - \int_0^t f(S_u) dB_u. \tag{3.9}$$

This is nothing else but our $M_t^{(f)}$ as defined in (3.1) when $\alpha = 2$ and $\rho = 1/2$.

It is quite natural to ask now about its Lévy counterpart in greater generality, but we have not succeeded in solving this problem.

Remark 3.4. As a nice application of Theorem 3.1, the following inequality holds, which does not seem straightforward to obtain: If $0 < \alpha < 2$, then

$$\mathbb{P}(S_\tau \leq a) \geq \left(\frac{a}{\mathbb{P}[|a - X_\tau|]} \right)^{\alpha\rho/(1-\alpha\rho)} \tag{3.10}$$

for any bounded stopping time τ .

Proof of (3.10). If $f_a(x) = 1_{[0, a]}(x)$ for $0 < a < \infty$, then we have

$$M_t^{(a)} = M_t^{(f_a)} = 1_{\{S_t \leq a\}}(a - X_t)^{\alpha\rho}. \tag{3.11}$$

By the optimal stopping theorem and by Hölder’s inequality, we obtain

$$1 = \mathbb{P}[M_\tau^{(a)}] \leq \mathbb{P}(S_\tau \leq a)^{1/q} \cdot \mathbb{P}[|a - X_\tau|^{\alpha\rho p}]^{1/p}$$

where $p, q > 0$ and $(1/p) + (1/q) = 1$. As we choose $p = 1/\alpha\rho$ and $q = 1/(1 - \alpha\rho)$, we obtain (3.10). □

Now we introduce the probability measure $\mathbb{P}^{(f)}$ as follows: For a non-negative, non-zero Borel function f satisfying (C1), we define

$$\mathbb{P}^{(f)}|_{\mathcal{F}_t} = \frac{M_t^{(f)}}{M_0^{(f)}} \cdot \mathbb{P}|_{\mathcal{F}_t}. \tag{3.12}$$

Since $M_t^{(f)}$ is a positive martingale, the measure $\mathbb{P}^{(f)}$ is well-defined.

4. The law of S_∞ under $\mathbb{P}^{(f)}$

Our aim in this section is to obtain the law of S_∞ under $\mathbb{P}^{(f)}$, i.e., to compute $\mathbb{P}^{(f)}(S_\infty > x)$. We set

$$\Sigma_a := S_{T(a,\infty)} = X_{T(a,\infty)} \tag{4.1}$$

for every $a > 0$. The law of Σ_a is given by:

Lemma 4.1.

$$\Sigma_a \stackrel{\text{law}}{=} a \Sigma_1 \stackrel{\text{law}}{=} \frac{a}{\beta_{\alpha\rho, 1-\alpha\rho}} \quad \text{under } \mathbb{P} \tag{4.2}$$

where $\beta_{p,q}$ is a beta random variable with parameters p and q .

Proof. According to Ray’s theorem [17], Eq. (1) (see also [3], p. 238), we have

$$\mathbb{P}(\Sigma_a > b) = \frac{1}{B(\alpha\rho, 1 - \alpha\rho)} \int_0^a x^{\alpha\rho-1} (b - x)^{-\alpha\rho} dx \tag{4.3}$$

for $0 < a < b$ where B is the beta function. By the obvious change of variables, we have

$$\begin{aligned} \text{the RHS of (4.3)} &= \frac{1}{B(\alpha\rho, 1 - \alpha\rho)} \int_0^{a/b} x^{\alpha\rho-1} (1 - x)^{-\alpha\rho} dx \\ &= \mathbb{P}\left(\beta_{\alpha\rho, 1-\alpha\rho} < \frac{a}{b}\right), \end{aligned}$$

which shows our claim (4.2). □

Theorem 4.2. *Let f be a non-negative Borel function satisfying the condition (C1). For every $a > 0$, it holds that*

$$\mathbb{P}^{(f)}(S_\infty > a) = \frac{\alpha\rho}{M_0^{(f)}} \int_a^\infty x^{\alpha\rho-1} f(x) dx. \tag{4.4}$$

Consequently,

$$\mathbb{P}^{(f)}(S_\infty \in dx) = \frac{\alpha\rho}{M_0^{(f)}} x^{\alpha\rho-1} f(x) dx. \tag{4.5}$$

Corollary 4.3. *It holds that $\mathbb{P}^{(f)}(S_\infty < \infty) = 1$, or equivalently, $\mathbb{P}^{(f)}(S_\infty = \infty) = 0$.*

Proof. We have

$$\begin{aligned} \mathbb{P}^{(f)}(S_\infty > a) &= \mathbb{P}^{(f)}(T_{(a, \infty)} < \infty) \\ &= \mathbb{P}\left[\frac{M_{T_{(a, \infty)}}^{(f)}}{M_0^{(f)}}\right] \quad (\text{by the definition of } \mathbb{P}^{(f)}) \\ &= \mathbb{P}\left[\frac{\alpha\rho}{M_0^{(f)}} \int_{\Sigma_a}^{\infty} f(x)(x - \Sigma_a)^{\alpha\rho-1} dx\right] \\ &= \frac{\alpha\rho}{M_0^{(f)}} \int_0^{\infty} f(x) dx \mathbb{P}[1_{\{\Sigma_a \leq x\}}(x - \Sigma_a)^{\alpha\rho-1}]. \end{aligned}$$

Lemma 4.1 implies that

$$\begin{aligned} &\mathbb{P}[1_{\{\Sigma_a \leq x\}}(x - \Sigma_a)^{\alpha\rho-1}] \\ &= \frac{1}{B(\alpha\rho, 1 - \alpha\rho)} \int_0^1 1_{\{a < x\}} 1_{\{a/y < x\}} \left(x - \frac{a}{y}\right)^{\alpha\rho-1} y^{\alpha\rho-1} (1-y)^{-\alpha\rho} dy \\ &= \frac{1_{\{a < x\}} x^{\alpha\rho-1}}{B(\alpha\rho, 1 - \alpha\rho)} \int_{a/x}^1 \left(y - \frac{a}{x}\right)^{\alpha\rho-1} (1-y)^{-\alpha\rho} dy \\ &= \frac{1_{\{a < x\}} x^{\alpha\rho-1}}{B(\alpha\rho, 1 - \alpha\rho)} \int_0^{1-a/x} y^{\alpha\rho-1} \left(1 - \frac{a}{x} - y\right)^{-\alpha\rho} dy \\ &= 1_{\{a < x\}} x^{\alpha\rho-1}. \end{aligned}$$

Therefore we obtain

$$\mathbb{P}^{(f)}(S_\infty > a) = \frac{\alpha\rho}{M_0^{(f)}} \int_a^{\infty} x^{\alpha\rho-1} f(x) dx,$$

which completes the proof. □

Example 4.4. For $f(x) = f_a(x) = 1_{(-\infty, a]}(x)$, we note $\mathbb{P}^{(a)}$ for $\mathbb{P}^{(f_a)}$, and we have

$$\mathbb{P}^{(a)}(S_\infty > a) = \frac{\alpha\rho}{M_0^{(a)}} \int_a^{\infty} x^{\alpha\rho-1} 1_{(-\infty, a]}(x) dx = 0,$$

hence

$$\mathbb{P}^{(a)}(S_\infty \leq a) = 1.$$

On the other hand, we have

$$\mathbb{P}^{(a)}(S_\infty \leq x) = \left(\frac{x}{a}\right)^{\alpha\rho}, \quad x \leq a,$$

by (4.5). That is,

$$\text{under } \mathbb{P}^{(a)}, \quad S_\infty \stackrel{\text{law}}{=} a\mathcal{U}^{1/(\alpha\rho)}$$

where \mathcal{U} is a uniform variable on $[0, 1]$.

Example 4.5. If the function f is given by

$$f(x) = f_\mu(x) = \frac{1}{\Gamma(\mu)} x^{\mu-1} e^{-\mu x}, \quad \mu > 0,$$

then we have

$$\text{under } \mathbb{P}^{(f_\mu)}, \quad S_\infty \stackrel{\text{law}}{=} \gamma_{\mu+\alpha\rho-1}$$

where γ_p is a gamma random variable with parameter p .

Now we are all set to prove the remaining of Theorem 3.1; $M_t^{(f)}$ converges to 0 a.s.

Proof of the convergence of $M_t^{(f)}$. Since $(M_t^{(f)})$ is a non-negative martingale, $M_t^{(f)}$ converges \mathbb{P} -almost surely to a non-negative \mathcal{F}_∞ -measurable functional $M_\infty^{(f)}$. For $0 < s < t \leq \infty$ and any $a > 0$, by Corollary 4.3 and Fatou's lemma, we have

$$\mathbb{P}[M_\infty^{(f)}] = \mathbb{P}[M_\infty^{(f)} 1_{\{S_\infty \geq a\}}] \quad (\text{since } \mathbb{P}(S_\infty \geq a) = 1) \tag{4.6}$$

$$\leq \liminf_{t \rightarrow \infty} \mathbb{P}[M_t^{(f)} 1_{\{S_t \geq a\}}] \quad (\text{by Fatou}) \tag{4.7}$$

$$= M_0^{(f)} \liminf_{t \rightarrow \infty} \mathbb{P}^{(f)}[1_{\{S_t \geq a\}}] \tag{4.8}$$

$$= M_0^{(f)} \mathbb{P}^{(f)}[1_{\{S_\infty \geq a\}}]. \tag{4.9}$$

Since $\mathbb{P}^{(f)}(S_\infty = \infty) = 0$, we have $\lim_{a \rightarrow \infty} \mathbb{P}^{(f)}(S_\infty \geq a) = 0$. Hence we obtain $\mathbb{P}[M_\infty^{(f)}] = 0$, which implies that $\mathbb{P}(M_\infty^{(f)} = 0) = 1$. Therefore we complete the proof of Theorem 3.1. \square

5. Convergence results

In this section we shall prove the following theorem.

Theorem 5.1. (i) For every $a > 0$ and $s \leq t$,

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}(S_t \leq a | \mathcal{F}_s)}{\mathbb{P}(S_t \leq a)} = 1_{\{S_s \leq a\}} \frac{(a - X_s)^{\alpha\rho}}{a^{\alpha\rho}} \quad \text{a.s. and in } L^1(\mathbb{P}). \tag{5.1}$$

(ii) Let f be a non-negative, non-zero Borel function satisfying the condition (C2). Then it holds that

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}[f(S_t) | \mathcal{F}_s]}{\mathbb{P}[f(S_t)]} = \frac{M_s^{(f)}}{M_0^{(f)}} \quad \text{a.s. and in } L^1(\mathbb{P}), \tag{5.2}$$

where $(M_s^{(f)})$ is given by (3.2).

Corollary 5.2. For fixed s and any \mathcal{F}_s -measurable bounded functional F_s , it holds that, for the particular function f in Theorem 5.1(ii),

$$\frac{\mathbb{P}[F_s f(S_t)]}{\mathbb{P}[f(S_t)]} \longrightarrow \frac{\mathbb{P}[F_s M_s^{(f)}]}{M_0^{(f)}} \quad \text{as } t \rightarrow \infty. \tag{5.3}$$

This result is the analogue of (1.3) for the particular class of functions f considered above. This gives a partial solution to the supremum penalisation problem for a stable Lévy process.

In our proof of Theorem 5.1 and Corollary 5.2, the following estimate will play an important role:

Proposition 5.3 ([3], Proposition VIII 2). *Let X be a (α, ρ) -stable Lévy process. If $\rho \in (0, 1)$, then there exists a constant $k > 0$ such that*

$$\varphi(x) := \mathbb{P}(S_1 < x) \sim kx^{\alpha\rho} \quad \text{as } x \rightarrow 0+. \tag{5.4}$$

For the asymptotic behavior of the density of S_1 , see [9] and [10].

Proofs of Theorem 5.1 and Corollary 5.2. (I) Using the Markov property of X and the function φ defined in (5.4), we have

$$\begin{aligned} \mathbb{P}(S_t \leq a | \mathcal{F}_s) &= 1_{\{S_s \leq a\}} \mathbb{P}\left(\sup_{u \in [s, t]} X_u \leq a | \mathcal{F}_s\right) \\ &= 1_{\{S_s \leq a\}} \widehat{\mathbb{P}}(\widehat{S}_{t-s} \leq a - X_s) \\ &= 1_{\{S_s \leq a\}} \varphi\left(\frac{a - X_s}{(t - s)^{1/\alpha}}\right) \\ &\sim k 1_{\{S_s \leq a\}} \left(\frac{a - X_s}{(t - s)^{1/\alpha}}\right)^{\alpha\rho} \quad \text{as } t \rightarrow \infty. \end{aligned}$$

That is, we have

$$t^\rho \mathbb{P}(S_t \leq a | \mathcal{F}_s) \longrightarrow k 1_{\{S_s \leq a\}} (a - X_s)^{\alpha\rho} \quad \text{as } t \rightarrow \infty$$

almost surely. In particular, for $s = 0$,

$$t^\rho \mathbb{P}(S_t \leq a) \longrightarrow ka^{\alpha\rho} \quad \text{as } t \rightarrow \infty,$$

thus we obtain

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}(S_t \leq a | \mathcal{F}_s)}{\mathbb{P}(S_t \leq a)} = 1_{\{S_s \leq a\}} \frac{(a - X_s)^{\alpha\rho}}{a^{\alpha\rho}} \quad \text{a.s.} \tag{5.5}$$

Thanks to Scheffé’s lemma, we get the L^1 -convergence once we know the a.s. convergence because both sides of (5.5) have expectation 1. The proof of (i) is now completed.

(II) We write, for $s < t$,

$$\begin{aligned} \mathbb{P}[f(S_t) | \mathcal{F}_s] &= f(S_s) + \mathbb{P}\left[\int_{S_s}^{S_t} f'(x) \, dx \mid \mathcal{F}_s\right] \\ &= f(S_s) + \int_{S_s}^{\infty} dx f'(x) \mathbb{P}(S_t > x | \mathcal{F}_s) \\ &= - \int_{S_s}^{\infty} dx f'(x) \mathbb{P}(S_t \leq x | \mathcal{F}_s). \end{aligned}$$

Now we note that there exists a constant $K > 0$ such that

$$\frac{\mathbb{P}(S_t \leq a | \mathcal{F}_s)}{k(a - X_s)^{\alpha\rho}} (t - s)^\rho \leq K 1_{\{S_s \leq a\}}$$

since $\varphi(x)/kx^{\alpha\rho}$ is bounded, which follows from (5.4). Hence we have

$$\begin{aligned} |f'(x)| t^\rho \mathbb{P}(S_t \leq x | \mathcal{F}_s) &= |f'(x)| \frac{t^\rho}{(t - s)^\rho} (t - s)^\rho \frac{\mathbb{P}(S_t \leq x | \mathcal{F}_s)}{k(x - X_s)^{\alpha\rho}} k(x - X_s)^{\alpha\rho} \\ &\leq K k 1_{\{S_s \leq x\}} |f'(x)| (x - X_s)^{\alpha\rho}. \end{aligned} \tag{5.6}$$

We note that (5.6) is integrable by the condition (C2).

On the other hand, we have already shown (5.5), and hence, by the dominated convergence theorem, we have

$$\begin{aligned} t^\rho \mathbb{P}[f(S_t)|\mathcal{F}_s] &= - \int_{S_s}^\infty dx f'(x) t^\rho \mathbb{P}(S_t \leq x | \mathcal{F}_s) \\ &\longrightarrow - \int_{S_s}^\infty dx f'(x) k 1_{\{S_s \leq x\}} (x - X_s)^{\alpha\rho} = -k \int_{S_s}^\infty dx f'(x) (x - X_s)^{\alpha\rho} \end{aligned}$$

as $t \rightarrow \infty$. Thus we obtain

$$\frac{\mathbb{P}[f(S_t)|\mathcal{F}_s]}{\mathbb{P}[f(S_t)]} \longrightarrow \frac{M_s^{(f)}}{M_0^{(f)}} \quad \text{a.s.} \tag{5.7}$$

(III) Since we have already shown that $\mathbb{P}[M_s^{(f)}] = M_0^{(f)}$, we now deduce from Scheffé’s lemma that the convergence (5.7) also holds in $L^1(\mathbb{P})$ which easily yields the conclusion of Corollary 5.2. \square

6. Comparison with other kinds of penalisations

In [25], we have developed two kinds of penalisation problems (local time penalisation and Feynman–Kac penalisation) in the case of any symmetric stable Lévy process on index $\alpha \in (1, 2]$. We have introduced the universal σ -finite measure \mathcal{P} defined as follows to solve this penalisation problem:

$$\mathcal{P} = \frac{\Gamma(1/\alpha)}{\alpha\pi} \int_0^\infty \frac{du}{u^{1/\alpha}} (Q^{(u)} \bullet P_0^h), \tag{6.1}$$

where $Q^{(u)}$ stands for the law of the bridge from 0 to 0 of length u and P_0^h for the h -path process with respect to the function $|x|^{\alpha-1}$ of the process killed at the first hitting time of 0 (see [23,24] and [25]). This is the analogue of the measure \mathcal{W} introduced by Najnudel–Roynette–Yor [11] in the Brownian case. We should remark that the measure \mathcal{W} also unifies the supremum penalisation problem with other penalisations in the Brownian case, while \mathcal{P} cannot be used this way in our case; in fact, $S_\infty = \infty$ \mathcal{P} -a.e. since $S_\infty = \infty$ P_0^h -a.s. (see Corollary 1.4 in [23]). This means that supremum penalisation is essentially different from local time and Feynman–Kac penalisations.

In a future paper [26], the second author describes more completely the penalised laws $\mathbb{P}^{(f)}$, in particular the analogue of the result (2) in Section 1. She introduces a certain σ -finite measure \mathcal{P}_{sup} which is different from \mathcal{P} , and gives another approach to supremum penalisation.

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