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Characterization of unitary processes with independent and stationary increments[1]

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Abstract. This is a continuation of the earlier work (*Publ. Res. Inst. Math. Sci.* **45** (2009) 745–785) to characterize unitary stationary independent increment Gaussian processes. The earlier assumption of uniform continuity is replaced by weak continuity and with technical assumptions on the domain of the generator, unitary equivalence of the process to the solution of an appropriate Hudson–Parthasarathy equation is proved.

Résumé. Cet article poursuit la recherche initiée dans (*Publ. Res. Inst. Math. Sci.* **45** (2009) 745–785) pour caractériser les processus stationnaires unitaires gaussiens à incréments indépendants. L'hypothèse antérieure d'uniforme continuité est remplacée par de la continuité faible. Avec des conditions techniques sur le domaine du générateur, nous montrons que le processus est équivalent unitairement à la solution d'une équation de Hudson–Parthasarathy appropriée.

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1. Introduction

In [13,14], by a co-algebraic treatment, Schürmann has proved that any weakly continuous unitary stationary independent increment process on the Hilbert space $\mathbf{h}\otimes\mathcal{H}$ ($\mathbf{h}$ finite dimensional), is unitarily equivalent to the solution of a Hudson–Parthasarathy (HP) type quantum stochastic differential equation [6]

$$\mathrm{d}V_t=\sum_{\mu,\nu\geq 0} V_t L^{\mu}_{\nu}\Lambda^{\nu}_{\mu}(\mathrm{d}t),\qquad V_0=1_{\mathbf{h}\otimes\Gamma},\tag{1.1}$$

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where Λ_μ^ν are fundamental processes in the symmetric Fock space $\Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$ with respect to a fixed orthonormal basis (onb) of the noise space $\mathbf{k}$ and the coefficients $L_\nu^\mu : \mu, \nu \geq 0$ are operators in the initial Hilbert space $\mathbf{h}$ given by

$$L_\nu^\mu = \begin{cases} G & \text{for } (\mu,\nu) = (0,0), \\ L_j & \text{for } (\mu,\nu) = (j,0), \\ -\sum_{j\geq 1} L_j^* W_k^j & \text{for } (\mu,\nu) = (0,k), \\ W_k^j - \delta_k^j 1_{\mathbf{h}} & \text{for } (\mu,\nu) = (j,k) \end{cases} \tag{1.2}$$

(δ_k^j stands for Kronecker delta) for some operators G, L_j in $\mathbf{h}$ and a unitary operators W on $\mathbf{h} \otimes \mathbf{k}$.

For the characterization of Fock adapted unitary evolutions see [1,5] and references therein. In [7,8], by extended semigroup methods, Lindsay and Wills have studied such problems for Fock-adapted contractive operator cocycles and completely positive cocycles.

Recently in [12] authors have studied the case of a unitary stationary independent increment process on Hilbert space $\mathbf{h} \otimes \mathcal{H}$ ($\mathbf{h}$ a separable Hilbert space), with norm-continuous expectation semigroup and showed its unitary equivalence to a Hudson–Parthasarathy flow. Here we are interested in unitary processes with weakly continuous and not uniformly continuous expectation semigroup. Under certain assumptions on the domain of the unbounded generators, extending the ideas of [12] we are able to construct the noise space $\mathbf{k}$ and the operators (unbounded) $G, L_j :\geq 1$ (see Proposition 4.1 and Lemma 4.3) such that the Hudson–Parthasarathy flow equation (1.1) with coefficients (1.2) (with W being the identity operator), admits a unique unitary solution and the solution is unitarily equivalent to the unitary process we started with (see Theorem 5.2).

2. Notation and preliminaries

We assume that all the Hilbert spaces appearing in this article are complex separable with inner product anti-linear in the first variable. For any Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ we denote the Banach space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and trace class operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\mathcal{B}_1(\mathcal{H})$ respectively. For a linear map T we write its domain as $\mathcal{D}(T)$, denote the trace on $\mathcal{B}_1(\mathcal{H})$ by Tr.

For any $\xi \in \mathcal{H} \otimes \mathcal{K}, h \in \mathcal{H}$ the map

$$\mathcal{K} \ni k \mapsto \langle \xi, h \otimes k \rangle$$

defines a bounded linear functional on $\mathcal{K}$ and thus by Riesz's representation theorem there exists a unique vector $\langle\langle h, \xi \rangle\rangle$ in $\mathcal{K}$ such that

$$\big\langle \langle\langle h, \xi \rangle\rangle, k \big\rangle = \langle \xi, h \otimes k \rangle \quad \forall k \in \mathcal{K}. \tag{2.1}$$

In other words $\langle\langle h, \xi \rangle\rangle = F_h^* \xi$ where $F_h \in \mathcal{B}(\mathcal{K}, \mathcal{H} \otimes \mathcal{K})$ is given by $F_h k = h \otimes k$.

Let $\mathbf{h}$ and $\mathcal{H}$ be two Hilbert spaces with some orthonormal bases $\{e_j:\ j \geq 1\}$ and $\{\zeta_j:\ j \geq 1\}$ respectively. For $A \in \mathcal{B}(\mathbf{h} \otimes \mathcal{H})$ and $u, v \in \mathbf{h}$ we define a linear operator $A(u,v) \in \mathcal{B}(\mathcal{H})$ by

$$\big\langle \xi_1, A(u,v)\xi_2 \big\rangle = \langle u \otimes \xi_1, Av \otimes \xi_2 \rangle \quad \forall \xi_1, \xi_2 \in \mathcal{H}$$

and read off the following properties (for a proof see Lemma 2.1 in [12]):

Lemma 2.1. *Let $A, B \in \mathcal{B}(\mathbf{h} \otimes \mathcal{H})$ then for any u, v, u_i and $v_i (i = 1,2)$ in $\mathbf{h}$:*

(i) $A(\cdot,\cdot) : \mathbf{h} \times \mathbf{h} \mapsto \mathcal{B}(\mathbf{h})$ *is a continuous sesqui-linear map and if* $A(u,v) = B(u,v), \forall u, v \in \mathbf{h}$ *then* $A = B$,
(ii) $\|A(u,v)\| \leq \|A\| \|u\| \|v\|$ *and* $A(u,v)^* = A^*(v,u)$,
(iii) $A(u_1,v_1)B(u_2,v_2) = [A(|v_1 >< u_2| \otimes 1_{\mathcal{H}})B](u_1,v_2)$,
(iv) $AB(u,v) = \sum_{j\geq 1} A(u,e_j)B(e_j,v)$ (*the series converging strongly*),
(v) $0 \leq A(u,v)^* A(u,v) \leq \|u\|^2 A^* A(v,v)$,
(vi) $\langle A(u,v)\xi_1, B(p,w)\xi_2 \rangle = \sum_{j\geq 1} \langle p \otimes \zeta_j, [B(|w >< v| \otimes |\xi_2 >< \xi_1|)A^* u \otimes \zeta_j \rangle = \langle v \otimes \xi_1, [A^*(|u >< p| \otimes 1_{\mathcal{H}})Bw \otimes \xi_2 \rangle$ *for* $\xi_1, \xi_2 \in \mathcal{H}$.

We also need to introduce the partial trace $\mathrm{Tr}_{\mathcal{H}}$ which is a linear map from $\mathcal{B}_1(\mathbf{h}\otimes\mathcal{H})$ to $\mathcal{B}_1(\mathbf{h})$ defined by, for $B\in\mathcal{B}_1(\mathbf{h}\otimes\mathcal{H})$,

$$\big\langle u,\mathrm{Tr}_{\mathcal{H}}(B)v\big\rangle:=\sum_{j\geq 1}\langle u\otimes\xi_j, Bv\otimes\xi_j\rangle\quad\forall u,v\in\mathbf{h}.$$

In particular, for $B=B_1\otimes B_2$, $\mathrm{Tr}_{\mathcal{H}}(B)=\mathrm{Tr}(B_2)B_1$.

For $A\in\mathcal{B}(\mathbf{h}\otimes\mathcal{H})$, $\varepsilon\in\mathbb{Z}_2=\{0,1\}$ we define operators $A^{(\varepsilon)}\in\mathcal{B}(\mathbf{h}\otimes\mathcal{H})$ by $A^{(\varepsilon)}:=A$ if $\varepsilon=0$ and $A^{(\varepsilon)}:=A^*$ if $\varepsilon=1$. For $1\leq k\leq n$, we define a unitary exchange map $P_{k,n}:\mathbf{h}^{\otimes n}\otimes\mathcal{H}\to\mathbf{h}^{\otimes n}\otimes\mathcal{H}$ by putting

$$P_{k,n}(u_1\otimes\cdots\otimes u_n\otimes\xi):=u_1\otimes\cdots\otimes u_{k-1}\otimes u_{k+1}\cdots\otimes u_n\otimes u_k\otimes\xi$$

on product vectors. Let $\underline{\varepsilon}=(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)\in\mathbb{Z}_2^n$. Consider the ampliation of the operator $A^{(\varepsilon_k)}$ in $\mathcal{B}(\mathbf{h}^{\otimes n}\otimes\mathcal{H})$ given by

$$A^{(n,\varepsilon_k)}:=P_{k,n}^*\big(1_{\mathbf{h}^{\otimes n-1}}\otimes A^{(\varepsilon_k)}\big)P_{k,n}.$$

Now we define the operator $A^{(\underline{\varepsilon})}:=\prod_{k=1}^n A^{(n,\varepsilon_k)}:=A^{(1,\varepsilon_1)}\cdots A^{(n,\varepsilon_n)}$ in $\mathcal{B}(\mathbf{h}^{\otimes n}\otimes\mathcal{H})$. Note that as here, through out this article, the product symbol $\prod_{k=1}^n$ stands for product with the ordering $1,2$ to n. For product vectors $\underline{u},\underline{v}\in\mathbf{h}^{\otimes n}$ one can see that

$$\prod_{i=1}^m A^{(n,\varepsilon_i)}(\underline{u},\underline{v})=\prod_{i=1}^m A^{(\varepsilon_i)}(u_i,v_i)\prod_{i=m+1}^n\langle u_i,v_i\rangle\in\mathcal{B}(\mathcal{H}).\tag{2.2}$$

When $\underline{\varepsilon}=\underline{0}\in\mathbb{Z}_2^n$, for simplicity we shall write $A^{(n,k)}$ for $A^{(n,\varepsilon_k)}$ and $A^{(n)}$ for $A^{(\underline{\varepsilon})}$.

Symmetric fock space and quantum stochastic calculus

Let us briefly recall the fundamental integrator processes of quantum stochastic calculus and the quantum stochastic differential equation (qsde), introduced by Hudson and Parthasarathy [6,11]. Let us consider the symmetric Fock space $\varGamma=\varGamma(L^2(\mathbb{R}_+,\mathbf{k}))$ and the exponential vector in the Fock space, associated with a vector $f\in L^2(\mathbb{R}_+,\mathbf{k})$, given by

$$\mathbf{e}(f)=\bigoplus_{n\geq 0}\frac{1}{\sqrt{n!}}f^{(n)},$$

where $f^{(n)}=\underbrace{f\otimes f\otimes\cdots\otimes f}_{n\text{-copies}}$ for $n>0$ and by convention $f^{(0)}=1$; where $\mathbf{k}$ is a separable Hilbert space. The exponential vector $\mathbf{e}(0)$ is called the vacuum vector. For any subset M of $L^2(\mathbb{R}_+,\mathbf{k})$ we shall write $\mathcal{E}(M)$ for the subspace spanned by $\{\mathbf{e}(f)\colon f\in M\}$. For an interval $\varDelta$ of $\mathbb{R}_+$, let $\varGamma_\varDelta$ be the symmetric Fock space over the Hilbert space $L^2(\varDelta,\mathbf{k})$. For $0\leq s\leq t<\infty$, the Hilbert space $\varGamma$ decompose as $\varGamma_{s]}\otimes\varGamma_{(s,t]}\otimes\varGamma_{[t}$ respectively, here we have abbreviated $[0,s]$ by $s]$ and (t,∞) by $[t$, and for any $f\in L^2(\mathbb{R}_+,\mathbf{k})$ the exponential vector $\mathbf{e}(f)=\mathbf{e}(f_{s]})\otimes\mathbf{e}(f_{(s,t]})\otimes\mathbf{e}(f_{[t})$ where $f_\varDelta=1_\varDelta f$.

Let us consider the Hudson–Parthasarathy (HP) equation on $\mathbf{h}\otimes\varGamma(L^2(\mathbb{R}_+,\mathbf{k}))$:

$$V_{s,t}=1_{\mathbf{h}\otimes\varGamma}+\sum_{\mu,\nu\geq 0}\int_s^t V_{s,\tau}L_\nu^\mu\varLambda_\mu^\nu(\mathrm{d}\tau).\tag{2.3}$$

Here the coefficients $L_\nu^\mu\colon \mu,\nu\geq 0$ are operators in $\mathbf{h}$ (not necessarily bounded) and $\varLambda_\mu^\nu$ are fundamental processes with respect to a fixed orthonormal basis $\{E_j\colon j\geq 1\}$ of $\mathbf{k}$:

$$\varLambda_\nu^\mu(t)=\begin{cases} t1_{\mathbf{h}\otimes\varGamma} & \text{for } (\mu,\nu)=(0,0),\\ a(1_{[0,t]}\otimes E_j) & \text{for } (\mu,\nu)=(j,0),\\ a^\dagger(1_{[0,t]}\otimes E_k) & \text{for } (\mu,\nu)=(0,k),\\ \varLambda\big(1_{[0,t]}\otimes|E_k><E_j|\big) & \text{for } (\mu,\nu)=(j,k).\end{cases}\tag{2.4}$$

The fundamental processes $a, a^\dagger$ and Λ are called annihilation, creation and conservation respectively (for their definition and detail about quantum stochastic calculus see [4,11]).

3. Unitary processes with stationary and independent increments

Let $\{U_{s,t}\colon\ 0 \le s \le t < \infty\}$ be a family of unitary operators in $\mathcal{B}(\mathbf{h} \otimes \mathcal{H})$ and Ω be a fixed unit vector in $\mathcal{H}$. We shall write $U_t := U_{0,t}$ for simplicity. Let us consider the family of unitary operators $\{U_{s,t}^{(\varepsilon)}\}$ in $\mathcal{B}(\mathbf{h} \otimes \mathcal{H})$ for $\varepsilon \in \mathbb{Z}_2$ given by $U_{s,t}^{(\varepsilon)} = U_{s,t}$ if $\varepsilon = 0$, $U_{s,t}^{(\varepsilon)} = U_{s,t}^*$ if $\varepsilon = 1$. As in Section 2, for $n \ge 1$, $\underline{\varepsilon} \in \mathbb{Z}_2^n$ fixed and $1 \le k \le n$, we define the families of operators $\{U_{s,t}^{(n,\varepsilon_k)}\}$ and $\{U_{s,t}^{(\underline{\varepsilon})}\}$ in $\mathcal{B}(\mathbf{h}^{\otimes n} \otimes \mathcal{H})$. By identity (2.2) we have, for product vectors $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$ and $\underline{\varepsilon} \in \mathbb{Z}_2^n$,

$$U_{s,t}^{(\underline{\varepsilon})}(\underline{u}, \underline{v}) = \prod_{i=1}^n U_{s,t}^{(\varepsilon_i)}(u_i, v_i).$$

Furthermore, for $\underline{s} = (s_1, s_2, \ldots, s_n), \underline{t} = (t_1, t_2, \ldots, t_n)\colon\ 0 \le s_1 \le t_1 \le s_2 \le \cdots \le s_n \le t_n < \infty$, we define $U_{\underline{s},\underline{t}}^{(\underline{\varepsilon})} \in \mathcal{B}(\mathbf{h}^{\otimes n} \otimes \mathcal{H})$ by setting

$$U_{\underline{s},\underline{t}}^{(\underline{\varepsilon})} := \prod_{k=1}^n U_{s_k,t_k}^{(n,\varepsilon_k)}. \tag{3.1}$$

Then for $\underline{u} = \bigotimes_{k=1}^n u_k, \underline{v} = \bigotimes_{k=1}^n v_k \in \mathbf{h}^{\otimes n}$ we have

$$U_{\underline{s},\underline{t}}^{(\underline{\varepsilon})}(\underline{u}, \underline{v}) = \prod_{k=1}^n U_{s_k,t_k}^{(\varepsilon_k)}(u_k, v_k).$$

When $\underline{\varepsilon} = \underline{0}$, we write $U_{\underline{s},\underline{t}}$ for $U_{\underline{s},\underline{t}}^{(\underline{\varepsilon})}$. For $\alpha, \beta \ge 0$, $\underline{s} = (s_1, s_2, \ldots, s_n), \underline{t} = (t_1, t_2, \ldots, t_n)$ we write $\alpha \le \underline{s}, \underline{t} \le \beta$ if $\alpha \le s_1 \le t_1 \le s_2 \le \cdots \le s_n \le t_n \le \beta$.
We assume the following on the family of unitary $\{U_{s,t} \in \mathcal{B}(\mathbf{h} \otimes \mathcal{H})\}$.

Assumption A.

A1 (*Evolution*) *For any* $0 \le r \le s \le t < \infty$, $U_{r,s}U_{s,t} = U_{r,t}$.
A2 (*Independence of increments*) *For any* $0 \le s_i \le t_i < \infty\colon\ i = 1, 2$ *such that* $[s_1, t_1) \cap [s_2, t_2) = \varnothing$:

(i) $U_{s_1,t_1}(u_1, v_1)$ *commutes with* $U_{s_2,t_2}(u_2, v_2)$ *and* $U_{s_2,t_2}^*(u_2, v_2)$ *for every* $u_i, v_i \in \mathbf{h}$.
(ii) *For* $s_1 \le \underline{a}, \underline{b} \le t_1$, $s_2 \le \underline{q}, \underline{r} \le t_2$ *and* $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$, $\underline{p}, \underline{w} \in \mathbf{h}^{\otimes m}$, $\underline{\varepsilon} \in \mathbb{Z}_2^n$, $\underline{\varepsilon}' \in \mathbb{Z}_2^m$

$$\langle \Omega, U_{\underline{a},\underline{b}}^{(\underline{\varepsilon})}(\underline{u}, \underline{v}) U_{\underline{q},\underline{r}}^{(\underline{\varepsilon}')}(\underline{p}, \underline{w})\Omega\rangle = \langle \Omega, U_{\underline{a},\underline{b}}^{(\underline{\varepsilon})}(\underline{u}, \underline{v})\Omega\rangle\langle \Omega, U_{\underline{q},\underline{r}}^{(\underline{\varepsilon}')}(\underline{p}, \underline{w})\Omega\rangle.$$

A3 (*Stationarity of increments*) *For any* $0 \le s \le t < \infty$ *and* $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$, $\underline{\varepsilon} \in \mathbb{Z}_2^n$

$$\langle \Omega, U_{s,t}^{(\underline{\varepsilon})}(\underline{u}, \underline{v})\Omega\rangle = \langle \Omega, U_{t-s}^{(\underline{\varepsilon})}(\underline{u}, \underline{v})\Omega\rangle.$$

Assumption B′ (Weak/strong continuity).

$$\lim_{t \to 0}\langle \Omega, (U_t - 1)(u, v)\Omega\rangle = 0 \quad \forall u, v \in \mathbf{h}.$$

Remark 3.1. *The Assumption* B′ *is an weakening of the Assumption* B *in* [12].

As in [12] we also assume the following simplifying conditions.

Assumption C (Gaussian condition). *For any $u_i, v_i \in \mathbf{h}$, $\varepsilon_i \in \mathbb{Z}_2$: $i = 1, 2, 3$*

$$\lim_{t\to 0} \frac{1}{t}\big\langle \Omega, \big(U_t^{(\varepsilon_1)} - 1\big)(u_1, v_1)\big(U_t^{(\varepsilon_2)} - 1\big)(u_2, v_2)\big(U_t^{(\varepsilon_3)} - 1\big)(u_3, v_3)\Omega\big\rangle = 0. \tag{3.2}$$

Assumption D (Minimality). *The set $\mathcal{S}_0 = \{U_{\underline{s},\underline{t}}(\underline{u}, \underline{v})\Omega := U_{s_1,t_1}(u_1, v_1)\cdots U_{s_n,t_n}(u_n, v_n)\Omega\colon\ \underline{s} = (s_1, s_2, \dots, s_n)$, $\underline{t} = (t_1, t_2, \dots, t_n)$: $0 \le s_1 \le t_1 \le s_2 \dots, s_n \le t_n < \infty, n \ge 1, \underline{u} = \bigotimes_{i=1}^n u_i, \underline{v} = \bigotimes_{i=1}^n v_i$ with $u_i, v_i \in \mathbf{h}\}$ is total in $\mathcal{H}$.*

Remark 3.2. *The Assumption D is not really a restriction, one can as well work with replacing $\mathcal{H}$ by closure of span of $\mathcal{S}_0$.*

Remark 3.3. *Under the Assumption D, if we take $u_i, v_i \in \mathcal{D} \subseteq \mathbf{h}$ in the definition of $\mathcal{S}_0$, then the resulting $\mathcal{S}_0$ will continue to be total.*

3.1. *Expectation semigroups*

Let us look at the various semigroups associated with the evolution $\{U_{s,t}\}$. Define a family of operators $\{T_t\}$ on $\mathbf{h}$ by setting

$$\langle u, T_t v\rangle := \big\langle \Omega, U_t(u, v)\Omega\big\rangle \quad \forall u, v \in \mathbf{h}.$$

Then as in Lemma 6.1 [12] it can be seen that under the Assumption A and B′, $\{T_t\}$ is a weakly continuous, hence strongly continuous, contractive semigroup on $\mathbf{h}$. Let us denote the strong generator of the semigroup T_t by G. By a simple computation we have the following useful observation (see Eq. (6.2) in [12])

$$\sum_{k\ge 1}\big\|(U_t - 1)(\phi_k, w)\Omega\big\|^2 = \big\langle w, (1 - T_t)w\big\rangle + \big\langle (1 - T_t)w, w\big\rangle. \tag{3.3}$$

Lemma 3.4. *Under the Assumption C we have the following.*

(i) *For any $n \ge 3$, $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$, $\underline{\varepsilon} \in \mathbb{Z}_2^n$*

$$\lim_{t\to 0} \frac{1}{t}\big\langle \Omega, \big(U_t^{(\varepsilon_1)} - 1\big)(u_1, v_1)\cdots\big(U_t^{(\varepsilon_n)} - 1\big)(u_n, v_n)\Omega\big\rangle = 0. \tag{3.4}$$

(ii) *For vectors $u \in \mathbf{h}$, $v \in \mathcal{D}(G)$, product vectors $\underline{p}, \underline{w} \in \mathbf{h}^{\otimes n}$ and $\varepsilon \in \mathbb{Z}_2$, $\underline{\varepsilon}' \in \mathbb{Z}_2^n$*

$$\begin{aligned}
&\lim_{t\to 0} \frac{1}{t}\big\langle (U_t - 1)^{(\varepsilon)}(u, v)\Omega, \big(U_t^{(\underline{\varepsilon}')} - 1\big)(\underline{p}, \underline{w})\Omega\big\rangle \\
&\quad = (-1)^{\varepsilon} \lim_{t\to 0} \frac{1}{t}\big\langle (U_t - 1)(u, v)\Omega, \big(U_t^{(\underline{\varepsilon}')} - 1\big)(\underline{p}, \underline{w})\Omega\big\rangle.
\end{aligned} \tag{3.5}$$

Proof. (i) The proof is identical to that of Lemma 6.7 in [12].

(ii) For $\varepsilon = 0$ nothing to prove. To see this for $\varepsilon = 1$ consider the following

$$\begin{aligned}
&\lim_{t\to 0} \frac{1}{t}\big\langle \big(U_t + U_t^* - 2\big)(u, v)\Omega, \big(U_t^{(\underline{\varepsilon}')} - 1\big)(\underline{p}, \underline{w})\Omega\big\rangle \\
&\quad = -\lim_{t\to 0} \frac{1}{t}\big\langle \big[\big(U_t^* - 1\big)(U_t - 1)\big](u, v)\Omega, \big(U_t^{(\underline{\varepsilon}')} - 1\big)(\underline{p}, \underline{w})\Omega\big\rangle \\
&\quad = -\lim_{t\to 0} \frac{1}{t}\sum_{m\ge 1}\big\langle (U_t - 1)(e_m, v)\Omega, (U_t - 1)(e_m, u)\big(U_t^{(\underline{\varepsilon}')} - 1\big)(\underline{p}, \underline{w})\Omega\big\rangle.
\end{aligned} \tag{3.6}$$

That this limit vanishes can be seen from the following

$$\left|\frac{1}{t}\sum_{m\geq 1}\left\langle (U_t-1)(e_m,v)\Omega,(U_t-1)(e_m,u)\big(U_t^{(\varepsilon')}-1\big)(\underline{p},\underline{w})\Omega\right\rangle\right|^2$$
$$\leq \sum_{m\geq 1}\frac{1}{t}\big\|(U_t-1)(e_m,v)\Omega\big\|^2\sum_{m\geq 1}\frac{1}{t}\big\|(U_t-1)(e_m,u)\big(U_t^{(\varepsilon')}-1\big)(\underline{p},\underline{w})\Omega\big\|^2.$$

By identity (3.3) and Lemma 2.1(iv) the above quantity is equal to

$$2Re\left\langle v,\frac{1-T_t}{t}v\right\rangle\frac{1}{t}\left\langle\big(U_t^{(\varepsilon')}-1\big)(\underline{p},\underline{w})\Omega,\big[\big(U_t^*-1\big)(U_t-1)\big](u,u)\big(U_t^{(\varepsilon')}-1\big)(\underline{p},\underline{w})\Omega\right\rangle$$
$$\leq 2Re\left\langle v,\frac{1-T_t}{t}v\right\rangle\frac{1}{t}\left\langle\big(U_t^{(\varepsilon')}-1\big)(\underline{p},\underline{w})\Omega,\big(2-U_t^*-U_t\big)(u,u)\big(U_t^{(\varepsilon')}-1\big)(\underline{p},\underline{w})\Omega\right\rangle.$$

Therefore, since $Re\langle v,\frac{1-T_t}{t}v\rangle$ is uniformly bounded in t as T_t is strongly continuous and $v\in\mathcal{D}(G)$, by Assumption C we get

$$\lim_{t\to 0}\frac{1}{t}\sum_{m\geq 1}\left\langle (U_t-1)(e_m,u)\Omega,(U_t-1)(e_m,v)\big(U_t^{(\varepsilon')}-1\big)(\underline{p},\underline{w})\Omega\right\rangle=0.$$

Thus (3.5) follows. □

For vectors $u,p\in\mathbf{h}$ and $v,w\in\mathcal{D}(G)$, the identity (3.5) gives

$$\lim_{t\to 0}\frac{1}{t}\left\langle (U_t-1)^{(\varepsilon)}(u,v)\Omega,(U_t-1)^{\varepsilon'}(p,w)\Omega\right\rangle$$
$$=(-1)^{\varepsilon+\varepsilon'}\lim_{t\to 0}\frac{1}{t}\left\langle (U_t-1)(u,v)\Omega,(U_t-1)(p,w)\Omega\right\rangle. \tag{3.7}$$

Now define a family of operators $\{Z_t\colon t\geq 0\}$ on the Banach space $\mathcal{B}_1(\mathbf{h})$ by

$$Z_t\rho=\mathrm{Tr}_{\mathcal{H}}\big[U_t\big(\rho\otimes|\Omega><\Omega|\big)U_t^*\big],\quad \rho\in\mathcal{B}_1(\mathbf{h}).$$

In particular for $u,v,p,w\in\mathbf{h}$

$$\big\langle p,Z_t\big(|w><v|\big)u\big\rangle:=\big\langle U_t(u,v)\Omega,U_t(p,w)\Omega\big\rangle. \tag{3.8}$$

Lemma 3.5. *Under the Assumptions A, B′ the family $\{Z_t\}$ is a C_0-semigroup of contractive maps on $\mathcal{B}_1(\mathbf{h})$.*

Proof. For $\rho\in\mathcal{B}_1(\mathbf{h})$ by definition of Z_t and trace norm (see [2], p. 47) we have

$$\begin{aligned}\|Z_t\rho\|_1&=\big\|\mathrm{Tr}_{\mathcal{H}}\big[U_t\big(\rho\otimes|\Omega><\Omega|\big)U_t^*\big]\big\|_1\\ &=\sup_{\phi,\psi\text{ on } b\text{ of }\mathbf{h}}\sum_{k\geq 1}\big|\big\langle\phi_k,\mathrm{Tr}_{\mathcal{H}}\big[U_t\big(\rho\otimes|\Omega><\Omega|\big)U_t^*\big]\psi_k\big\rangle\big|\\ &\leq\sup_{\phi,\psi}\sum_{j,k\geq 1}\big|\big\langle\phi_k\otimes\zeta_j,U_t\big(\rho\otimes|\Omega><\Omega|\big)U_t^*\phi_k\otimes\zeta_j\big\rangle\big|\\ &\leq\big\|U_t\big(\rho\otimes|\Omega><\Omega|\big)U_t^*\big\|_1.\end{aligned}$$

Since for any $\{U_t\}$ is a family of unitary operators

$$\|Z_t\rho\|_1\leq\big\|\rho\otimes|\Omega><\Omega|\big\|_1\leq\|\rho\|_1.$$

Proof of semigroup property of $\{Z_t\}$ is same as in Lemma 6.4 [12]. In order to prove strong continuity Z_t, it is suffices to prove the same for rank one operator $\rho = |w><v|$, $w, v \in \mathbf{h}$. We have

$$\begin{aligned}
&\big\|(Z_t-1)\big(|w><v|\big)\big\|_1\\
&\quad= \sup_{\phi \text{ on } b \text{ of } \mathbf{h}} \big|\big\langle\phi_k, (Z_t-1)\big(|w><v|\big)\phi_k\big\rangle\big|\\
&\quad= \sup_{\phi}\sum_{k\geq 1}\big|\big\langle U_t(\phi_k,v)\Omega, U_t(\phi_k,w)\Omega\big\rangle - \overline{\langle\phi_k,v\rangle}\langle\phi_k,w\rangle\big|\\
&\quad\leq \sup_{\phi}\sum_{k\geq 1}\big|\big\langle (U_t-1)(\phi_k,v)\Omega, U_t(\phi_k,w)\Omega\big\rangle\big|\\
&\qquad+\sup_{\phi}\sum_{k\geq 1}\big|\overline{\langle\phi_k,v\rangle}\big\langle\Omega,(U_t-1)(\phi_k,w)\Omega\big\rangle\big|\\
&\quad\leq \sup_{\phi}\bigg[\sum_{k\geq 1}\big\|(U_t-1)(\phi_k,v)\Omega\big\|^2\bigg]^{1/2}\bigg[\sum_{k\geq 1}\big\|U_t(\phi_k,w)\Omega\big\|^2\bigg]^{1/2}\\
&\qquad+\sup_{\phi}\bigg[\sum_{k\geq 1}\big|\langle\phi_k,v\rangle\big|^2\bigg]^{1/2}\bigg[\sum_{k\geq 1}\big\|(U_t-1)(\phi_k,w)\Omega\big\|^2\bigg]^{1/2}.
\end{aligned}$$

Hence by identity (3.3) we obtain

$$\begin{aligned}
&\big\|(Z_t-1)\big(|w<>v|\big)\big\|_1\\
&\quad\leq \|w\|\sqrt{2\big\|(T_t-1)v\big\|}+\|v\|\sqrt{2\big\|(T_t-1)w\big\|}.
\end{aligned}$$

Thus by strong continuity of the semigroup T_t and the density of the finite rank vectors in $\mathcal{B}_1(\mathbf{h})$ the contractive semigroup Z_t is a strongly continuous. □

We shall denote the generator of the semigroup Z_t by $\mathcal{L}$. Also we note the following which can be proved identically as in [12] .

Lemma 3.6. *The semigroup Z_t is a positive trace preserving semigroup.*

Let us define a family $\{Y_t\colon t\geq 0\}$ of positive contractions on $\mathcal{B}_1(\mathbf{h})$ by $Y_t(\rho) := T_t\rho T_t^*, \forall\rho\in\mathcal{B}_1(\mathbf{h})$. Since T_t is a C_0- semigroup of contraction operators on $\mathcal{B}(\mathbf{h})$ it can be seen that Y_t is a contractive C_0-semigroup on $\mathcal{B}_1(\mathbf{h})$. It can also be seen that [4] the generator $\widetilde{\mathcal{L}}$ of Y_t satisfy

$$\widetilde{\mathcal{L}}(\rho) = G^*\rho+\rho G \quad \forall\rho\in\mathcal{F}\equiv\big\{(1-G)^{-1}\sigma\big(1-G^*\big)^{-1}\colon \sigma\in\mathcal{B}_1(\mathbf{h})\big\}$$

and $\mathcal{F}$ is a core for $\widetilde{\mathcal{L}}$.

We also define a family of maps F_t on the Banach space $\mathcal{B}_1(\mathbf{h})$ by

$$F_t\rho = \mathrm{Tr}_{\mathcal{H}}\big[U_t^*\big(\rho\otimes|\Omega><\Omega|\big)U_t\big] \quad \forall\rho\in\mathcal{B}_1(\mathbf{h}). \tag{3.9}$$

In particular for $u, v, p, w\in\mathbf{h}$ we have that $\langle p, F_t(|w><v|)u\rangle = \langle U_t^*(u,v)\Omega, U_t^*(p,w)\Omega\rangle$. Then as for Z_t it can be proved that $\{F_t\colon t\geq 0\}$ is a strongly continuous contractive semigroup on $\mathcal{B}_1(\mathbf{h})$. We shall denote its generator by $\mathcal{L}'$.

4. Construction of noise space

Let $M_0 := \{(\underline{u}, \underline{v}, \underline{\varepsilon})\colon\ \underline{u} = \bigotimes_{i=1}^n u_i, \underline{v} = \bigotimes_{i=1}^n v_i, u_i \in \mathbf{h}, v_i \in \mathcal{D}(G), \underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{Z}_2^n, n \geq 1\}$ and consider the relation " $\sim$ " on M_0 as defined in [12]: $(\underline{u}, \underline{v}, \underline{\varepsilon}) \sim (\underline{p}, \underline{w}, \underline{\varepsilon}')$ if $\underline{\varepsilon} = \underline{\varepsilon}'$ and $|\underline{u} >< \underline{v}| = |\underline{p} >< \underline{w}| \in \mathcal{B}(\mathbf{h}^{\otimes n})$. Expanding the vectors in terms of the orthonormal basis $\{e_{\underline{j}} = e_{j_1} \otimes \cdots \otimes e_{j_n}\colon\ \underline{j} = (j_1, \dots, j_n), j_1, \dots, j_n \geq 1\}$ from $\mathcal{D}(G)$, the identity $|\underline{u} >< \underline{v}| = |\underline{p} >< \underline{w}|$ is equivalent to $\underline{u}_{\underline{j}} \overline{\underline{v}_{\underline{k}}} = \underline{p}_{\underline{j}} \overline{\underline{w}_{\underline{k}}}$ for each multi-indices $\underline{j}, \underline{k}$ which gives, $(\underline{u}, \underline{v}, \underline{\varepsilon}) \sim (\underline{p}, \underline{w}, \underline{\varepsilon}') \Leftrightarrow A^{(\underline{\varepsilon})}(\underline{u}, \underline{v}) = A^{(\underline{\varepsilon}')}(\underline{p}, \underline{w})$ for all bounded operator A and make " $\sim$ " a well defined equivalence relation. Now consider the algebra M generated by $M_0/\sim$ with multiplication structure given by $(\underline{u}, \underline{v}, \underline{\varepsilon}).(\underline{p}, \underline{w}, \underline{\varepsilon}') = (\underline{u} \otimes \underline{p}, \underline{v} \otimes \underline{w}, \underline{\varepsilon} \oplus \underline{\varepsilon}')$. We define a scalar valued map K on $M \times M$ by setting, for $(\underline{u}, \underline{v}, \underline{\varepsilon}), (\underline{p}, \underline{w}, \underline{\varepsilon}') \in M_0$,

$$K\big((\underline{u}, \underline{v}, \underline{\varepsilon}), (\underline{p}, \underline{w}, \underline{\varepsilon}')\big) := \lim_{t \to 0} \frac{1}{t} \big\langle \big(U_t^{(\underline{\varepsilon})} - 1\big)(\underline{u}, \underline{v})\Omega, \big(U_t^{\underline{\varepsilon}'} - 1\big)(\underline{p}, \underline{w})\Omega \big\rangle, \quad \text{if it exists.}$$

If we define the subspace $\mathcal{N}_0 \equiv Span\{|u >< v|, u, v \in \mathcal{D}(G)\}$ of $\mathcal{B}_1(\mathbf{h})$, then it is clear that $\mathcal{N}_0$ is dense in $\mathcal{B}_1(\mathbf{h})$ and contained in $\mathcal{F}$, and we have the following result.

Proposition 4.1. *If $\mathcal{N}_0 \subseteq \mathcal{D}(\mathcal{L})$ then we have the following.*

(i) *The map K is a well defined positive definite kernel on M.*

(ii) *Up to unitary equivalence there exists a unique separable Hilbert space $\mathbf{k}$, an embedding $\eta : M \to \mathbf{k}$ and a representation π of $M, \pi : M \to \mathcal{B}(\mathbf{k})$ such that*

$$\{\eta(\underline{u}, \underline{v}, \underline{\varepsilon})\colon\ (\underline{u}, \underline{v}, \underline{\varepsilon}) \in M_0\} \quad \textit{is total in } \mathbf{k}, \tag{4.1}$$

$$\big\langle \eta(\underline{u}, \underline{v}, \underline{\varepsilon}), \eta\big(\underline{p}, \underline{w}, \underline{\varepsilon}'\big)\big\rangle = K\big((\underline{u}, \underline{v}, \underline{\varepsilon}), \big(\underline{p}, \underline{w}, \underline{\varepsilon}'\big)\big) \tag{4.2}$$

and

$$\pi(\underline{u}, \underline{v}, \underline{\varepsilon})\eta\big(\underline{p}, \underline{w}, \underline{\varepsilon}'\big) = \eta\big(\underline{u} \otimes \underline{p}, \underline{v} \otimes \underline{w}, \underline{\varepsilon} \oplus \underline{\varepsilon}'\big) - \langle \underline{p}, \underline{w}\rangle \eta(\underline{u}, \underline{v}, \underline{\varepsilon}). \tag{4.3}$$

(iii) *For any $(\underline{u}, \underline{v}, \underline{\varepsilon}) \in M_0$, $\underline{u} = \bigotimes_{i=1}^n u_i$, $\underline{v} = \bigotimes_{i=1}^n v_i$ and $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$.*

$$\eta(\underline{u}, \underline{v}, \underline{\varepsilon}) = \sum_{i=1}^n \prod_{k \neq i} \langle u_k, v_k \rangle \eta(u_i, v_i, \varepsilon_i). \tag{4.4}$$

(iv) $\eta(u, v, 1) = -\eta(u, v, 0), \forall u \in \mathbf{h}, v \in \mathcal{D}(G)$.

(v) *Writing $\eta(u, v)$ for the vector $\eta(u, v, 0) \in \mathbf{k}$,*

$$\overline{Span}\big\{\eta(u, v)\colon\ u \in \mathbf{h}, v \in \mathcal{D}(G)\big\} = \mathbf{k}. \tag{4.5}$$

Proof. (i) The proof is exactly same as in [12] except the fact that $v \in \mathcal{D}$. By Lemma 3.4, for elements $(\underline{u}, \underline{v}, \underline{\varepsilon}), (\underline{p}, \underline{w}, \underline{\varepsilon}') \in M_0$, $\underline{\varepsilon} \in \mathbb{Z}_2^m$ and $\underline{\varepsilon}' \in \mathbb{Z}_2^n$, we have

$$\begin{aligned} &K\big((\underline{u}, \underline{v}, \underline{\varepsilon}), \big(\underline{p}, \underline{w}, \underline{\varepsilon}'\big)\big) \\ &\quad = \lim_{t \to 0} \frac{1}{t} \big\langle \big(U_t^{(\underline{\varepsilon})} - 1\big)(\underline{u}, \underline{v})\Omega, \big(U_t^{\underline{\varepsilon}'} - 1\big)(\underline{p}, \underline{w})\Omega \big\rangle \\ &\quad = \sum_{1 \leq i \leq m, 1 \leq j \leq n} \prod_{k \neq i} \overline{\langle u_k, v_k \rangle} \prod_{l \neq j} \langle p_l, w_l \rangle \lim_{t \to 0} \frac{1}{t} \big\langle (U_t - 1)^{(\varepsilon_i)}(u_i, v_i)\Omega, (U_t - 1)^{\varepsilon'_j}(p_j, w_j)\Omega \big\rangle. \end{aligned} \tag{4.6}$$

Since

$$\begin{aligned}
&\big\langle (U_t-1)(u,v)\Omega,(U_t-1)(p,w)\Omega\big\rangle\\
&\quad=\big\langle U_t(u,v)\Omega,U_t(p,w)\Omega\big\rangle-\overline{\langle u,v\rangle}\langle p,w\rangle\\
&\qquad-\overline{\langle u,v\rangle}\big\langle\Omega,\big[(U_t-1)(p,w)\big]\Omega\big\rangle\\
&\qquad-\overline{\big\langle\Omega,\big[(U_t-1)(u,v)\big]\Omega\big\rangle}\langle p,w\rangle\\
&\quad=\big\langle p,(Z_t-1)\big(|w><v|\big)u\big\rangle-\overline{\langle u,v\rangle}\big\langle p,\big[(T_t-1)w\big]\big\rangle-\overline{\big\langle u,(T_t-1)v\big\rangle}\langle p,w\rangle.
\end{aligned}$$

Thus existence of the limits on the right hand side of (4.6) follows from the identity (3.5) since the semigroups T_t on $\mathbf{h}$ and Z_t on $\mathcal{B}_1(\mathbf{h})$ are strongly continuous and $|w><v|$ is in $\mathcal{D}(\mathcal{L})$. Hence K is well defined on M_0. Now extend this to the algebra M sesqui-linearly. In particular we have

$$\begin{aligned}
&K\big((u,v,\varepsilon),\big(p,w,\varepsilon'\big)\big)\\
&\quad=(-1)^{\varepsilon+\varepsilon'}\lim_{t\to0}\bigg\{\bigg\langle p,\frac{Z_t-1}{t}\big(|w><v|\big)u\bigg\rangle-\overline{\langle u,v\rangle}\bigg\langle p,\frac{T_t-1}{t}w\bigg\rangle-\overline{\bigg\langle u,\frac{T_t-1}{t}v\bigg\rangle}\langle p,w\rangle\bigg\}\\
&\quad=(-1)^{\varepsilon+\varepsilon'}\big\{\big\langle p,\mathcal{L}\big(|w><v|\big)u\big\rangle-\overline{\langle u,v\rangle}\langle p,Gw\rangle-\overline{\langle u,Gv\rangle}\langle p,w\rangle\big\}.
\end{aligned}\tag{4.7}$$

(ii) The Kolmogorov's construction [11] to the pair (M,K) provides a separable Hilbert space $\mathbf{k}$ as closure of the span of $\{\eta(\underline{u},\underline{v},\underline{\varepsilon})\colon (\underline{u},\underline{v},\underline{\varepsilon})\in M_0\}$. Now defining π by (4.3) we obtain a representation of the algebra M in $\mathbf{k}$ (proof goes similarly as in Lemma 7.1 [12].

(iii) Again as in [12] for any $(\underline{p},\underline{w},\underline{\varepsilon}')\in M_0$, by lemma 3.4, we have

$$\begin{aligned}
\big\langle\eta(\underline{u},\underline{v},\underline{\varepsilon}),\eta\big(\underline{p},\underline{w},\underline{\varepsilon}'\big)\big\rangle&=K\big((\underline{u},\underline{v},\underline{\varepsilon}),\big(\underline{p},\underline{w},\underline{\varepsilon}'\big)\big)\\
&=\sum_{i=1}^{n}\prod_{k\neq i}\overline{\langle u_k,v_k\rangle}\big\langle\eta(u_i,v_i,\varepsilon_i),\eta\big(\underline{p},\underline{w},\underline{\varepsilon}'\big)\big\rangle.
\end{aligned}$$

Since $\{\eta(\underline{p},\underline{w},\underline{\varepsilon}')\colon (\underline{p},\underline{w},\underline{\varepsilon}')\in M_0\}$ is a total subset of $\mathbf{k}$, (4.4) follows.

(iv) By (3.5) we have

$$\big\langle\eta(u,v,1),\eta\big(\underline{p},\underline{w},\underline{\varepsilon}'\big)\big\rangle=\big\langle-\eta(u,v,0),\eta\big(\underline{p},\underline{w},\underline{\varepsilon}'\big)\big\rangle.$$

Since $\{\eta(\underline{p},\underline{w},\underline{\varepsilon}')\colon (\underline{p},\underline{w},\underline{\varepsilon}')\in M_0\}$ is a total subset of $\mathbf{k}$, $\eta(u,v,1)=-\eta(u,v,0)$.

(v) It follows immediately from parts (iii) and (iv). □

Remark 4.2. *The representation π of M in $\mathbf{k}$ is trivial*

$$\pi(\underline{u},\underline{v},\underline{\varepsilon})\eta\big(\underline{p},\underline{w},\underline{\varepsilon}'\big)=\langle\underline{u},\underline{v}\rangle\eta\big(\underline{p},\underline{w},\underline{\varepsilon}'\big).\tag{4.8}$$

If we redefine $\mathcal{M}$ to be generated by $\underline{u},\underline{v}\in\mathcal{D}(G)^{\otimes n}$, then $\mathcal{M}$ can be a $*$-algebra with involution: $(\underline{u},\underline{v},\underline{\varepsilon})^*=(\overleftarrow{\underline{u}},\overleftarrow{\underline{v}},\underline{\varepsilon}^*)$ (for notations see [12]) and it is obvious that π given by (4.8) is indeed a $*$-representation.

In the sequel, we fix an orthonormal basis $\{E_j\colon j\geq1\}$ of $\mathbf{k}$.

Lemma 4.3. *Under the hypothesis of Proposition 4.1 we have the followings.*

(i) *There exists a unique family of operators $\{L_j\colon j\geq1\}$ in $\mathbf{h}$ with $\mathcal{D}(L_j)\supseteq\mathcal{D}(G)$ such that $\langle u,L_jv\rangle=\eta_j(u,v):=\langle E_j,\eta(u,v)\rangle$, $\forall u\in\mathbf{h}$, $v\in\mathcal{D}(G)$ and $\sum_{j\geq1}\|L_jv\|^2=-2Re\langle v,Gv\rangle$, $\forall v\in\mathcal{D}(G)$.*

(ii) *The family of operators* $\{L_j\colon\ j\geq 1\}$ *satisfies* $\sum_{j\geq 1}\langle u, c_j L_j v\rangle = 0, \forall u\in\mathbf{h}, v\in\mathcal{D}(G)$ *for some* $c=(c_j)\in l^2(\mathbb{N})$ *implies* $c=0$.

(iii) *The generator* $\mathcal{L}$ *of strongly continuous semigroup* Z_t *satisfies*

$$\begin{aligned}\big\langle p, \mathcal{L}\big(|w><v|\big)u\big\rangle &= \big\langle p, |Gw><v|u\big\rangle + \big\langle p, |w><Gv|u\big\rangle\\ &\quad+\sum_{j\geq 1}\big\langle p, |L_j w><L_j v|u\big\rangle\end{aligned}\tag{4.9}$$

for all $u, p\in\mathbf{h}$ *and* $v, w\in\mathcal{D}(G)$. *Furthermore, the family of operators* $G, L_j\colon\ j\geq 1$ *satisfies*

$$\langle v, Gw\rangle + \langle Gv, w\rangle + \sum_{j\geq 1}\langle L_j v, L_j w\rangle = 0,\tag{4.10}$$

for all $v, w\in\mathcal{D}(G)$.

Proof. (i) By the identity (4.7), for any $u\in\mathbf{h}$, $v\in\mathcal{D}(G)$

$$\begin{aligned}&\|\eta(u,v)\|^2\\ &\quad=\big\langle u, \mathcal{L}\big(|v><v|\big)u\big\rangle - \overline{\langle u, v\rangle}\langle u, Gv\rangle - \overline{\langle u, Gv\rangle}\langle u, v\rangle\\ &\quad\leq\big\{\big\|\mathcal{L}\big(|v><v|\big)\big\|_1 + 2\|Gv\|\|v\|\big\}\|u\|^2.\end{aligned}\tag{4.11}$$

Thus the linear map $\mathbf{h}\ni u\mapsto\eta(u,v)\in\mathbf{k}$ is a bounded linear map. Hence by Riesz's representation theorem, there exists unique linear operator L from $\mathcal{D}(G)\supseteq\mathbf{h}$ to $\mathbf{h}\otimes\mathbf{k}$ such that $\langle\langle u, Lv\rangle\rangle = \eta(u,v)$ where the vector $\langle\langle u, Lv\rangle\rangle\in\mathbf{k}$ is defined as in (2.1). Equivalently, there exists a unique family of linear operator $\{L_j\colon\ j\geq 1\}$ from $\mathcal{D}(G)$ to $\mathbf{h}$ such that $Lu=\sum_{j\geq 1}L_j u\otimes E_j$ and $\langle u, L_j v\rangle = \eta_j(u,v) = \langle E_j, \eta(u,v)\rangle$. Now, for any $v\in\mathcal{D}(G)$

$$\begin{aligned}\|Lv\|^2 &= \sum_j\|L_j v\|^2 = \sum_{j,k}\big|\eta_j(e_k, v)\big|^2 = \sum_k\big\|\eta(e_k, v)\big\|^2\\ &= \sum_k\big[\big\langle e_k, \mathcal{L}\big(|v><v|\big)e_k\big\rangle - \overline{\langle e_k, v\rangle}\langle e_k, Gv\rangle - \overline{\langle e_k, Gv\rangle}\langle e_k, v\rangle\big]\\ &= \operatorname{Tr}\mathcal{L}\big(|v><v|\big) - \langle v, Gv\rangle - \overline{\langle v, Gv\rangle}.\end{aligned}$$

Since Z_t is trace preserving and $|v><v|\in\mathcal{D}(\mathcal{L})$ by hypothesis it follows that

$$\operatorname{Tr}\mathcal{L}\big(|v<v|\big) = 0$$

and therefore

$$\|Lv\|^2 = \sum_j\|L_j v\|^2 = -\langle v, Gv\rangle - \overline{\langle v, Gv\rangle} = -2Re\langle v, Gv\rangle.\tag{4.12}$$

Note that the term on right hand side is positive since G is the generator of a contractive semigroup.

(ii) For some $c=(c_j)\in l^2(\mathbb{N})$ let $\langle u, \sum_{j\geq 1}c_j L_j v\rangle = 0, \forall u\in\mathbf{h}, v\in\mathcal{D}(G)$. We have

$$0 = \Big\langle u, \sum_{j\geq 1}c_j L_j v\Big\rangle = \sum_{j\geq 1}c_j\langle u, L_j v\rangle = \Big\langle\sum_{j\geq 1}\overline{c}_j E_j, \eta(u,v)\Big\rangle.$$

Since $\overline{Span}\{\eta(u,v)\colon\ u\in\mathbf{h}, v\in\mathcal{D}(G)\} = \mathbf{k}$, it follows that $\sum_{j\geq 1}\overline{c}_j E_j = 0\in\mathbf{k}$ and hence $c_j = 0, \forall j$.

(iii) By part (i) and identity (4.7), for any $u, p \in \mathbf{h}$ and $v, w \in \mathcal{D}(G)$ we have

$$\sum_{j\geq 1} \overline{\langle u, L_j v\rangle}\langle p, L_j w\rangle = \big\langle \eta(u,v), \eta(p,w)\big\rangle$$

$$= \big\langle p, \mathcal{L}\big(|w><v|\big)u\big\rangle - \overline{\langle u, v\rangle}\langle p, Gw\rangle - \overline{\langle u, Gv\rangle}\langle p, w\rangle,$$

which leads to (4.9).

Since, for any $v, w \in \mathcal{D}(G)$, by Lemma 3.6, $\mathrm{Tr}[\mathcal{L}(|w><v|)] = 0$, from the above identity we get

$$\langle v, Gw\rangle + \langle Gv, w\rangle + \sum_{j\geq 1}\langle L_j v, L_j w\rangle = 0. \tag{4.13}$$

□

Remark 4.4. *If there exists a positive self adjoint operator A such that $\langle v, Av\rangle = -2Re\langle v, Gv\rangle, \forall v \in \mathcal{D}(G)$, then $\|Lv\|^2 = \sum_j \|L_j v\|^2 = \langle v, Av\rangle = \|A^{1/2}v\|^2, \forall v \in \mathcal{D}(G) \subseteq \mathcal{D}(A) \subseteq \mathcal{D}(A^{1/2})$ and hence L will be closable. Closability of $(L, \mathcal{D}(G))$ can be seen as follows. Suppose $\{v_n\} \subseteq \mathcal{D}(G)$ converges to 0 and $\{Lv_n\}$ is convergent. Since $\|L(v_n - v_m)\| = \|A^{1/2}(v_n - v_m)\|$, convergence of $\{Lv_n\}$ implies $\{A^{1/2}v_n\}$ is Cauchy, so convergent in $\mathbf{h}$. As $A^{1/2}$ is a closed operator we get that $A^{1/2}v_n$ converges to 0 which implies Lv_n converges to 0.*

This can happen e.g. when $\{T_t\}$ is a holomorphic semigroup of contractions.

Remark 4.5. *If we replace $\mathcal{D}(G)$ by any dense subset $\mathcal{D} \subseteq \mathcal{D}(G)$, such that $|u><v| \in \mathcal{D}(\mathcal{L})$ for all $u, v \in \mathcal{D}$, then above Proposition 4.1 and Lemma 4.3 hold with the tensor algebra $\mathcal{M}$ modified so as to be generated by $(\bigotimes_{i=1}^n u_i, \bigotimes_{i=1}^n v_i)$: $u_i \in \mathbf{h}$ and $v_i \in \mathcal{D}$.*

5. Hudson–Parthasarathy (HP) flows and equivalence

In order to set up the Hudson–Parthasarathy (HP) equation and proceed further we shall work under the following extra assumption.

Assumption E. *There exists a dense set $\mathcal{D} \subseteq \mathcal{D}(G) \cap \mathcal{D}(G^*)$ such that $\mathcal{D}$ is a core of G in $\mathbf{h}$ and*:

E1. *$\mathcal{D} \subseteq \mathcal{D}(L_j^*)$ for every $j \geq 1$,*
E2. *$\mathcal{N} = Span\{|u><v|: u, v \in \mathcal{D}\}$ is a core for the generator $\mathcal{L}$ and $\mathcal{L}'$ of the semigroup Z_t and F_t on $\mathcal{B}_1(\mathbf{h})$ respectively,*
E3. *L_j maps $\mathcal{D}$ into itself and for any $v \in \mathcal{D}$, $\sum_{j\geq 1}\|GL_j v\|^2 < \infty$.*

Since $\mathcal{D}$ is dense in $\mathbf{h}$ one can see, by a simple approximation argument, that $\mathcal{N}$ is dense in $\mathcal{B}_1(\mathbf{h})$. Recall from the Remark 4.5 *that under the Assumption* E2, *replacing $\mathcal{D}(G)$ by the core $\mathcal{D}$ in Proposition* 4.1 *and Lemma* 4.3, *we get a separable Hilbert space $\mathbf{k}$ generated by $\{\eta(u,v): u \in \mathbf{h}, v \in \mathcal{D}\}$ and linear operators $\{L_j: j \geq 1\}$ defined on $\mathcal{D}$.*

Remark 5.1. *The Assumption* E1 *is needed for setting up an HP equation with coefficients G and $L_j: j \geq 1$, Assumption* E2 *is to assure the existence of unique unitary HP flow. The Assumption* E3 *will be necessary for proving the minimality of the associated HP flow which will be needed to establish unitary equivalence of the HP flow and unitary process U_t, we started with.*

Now let us state the main result of this article.

Theorem 5.2. *Assume A, B, C, D and E. Then we have the following.*

(i) *The HP equation*

$$V_t = 1_{\mathbf{h}\otimes\Gamma} + \sum_{\mu,\nu\geq 0}\int_0^t V_r L_\nu^\mu \Lambda_\mu^\nu(\mathrm{d}r) \tag{5.1}$$

on $\mathcal{D}\otimes\mathcal{E}(L^2(\mathbb{R}_+,\mathbf{k}))$ with coefficients L^μ_ν given by

$$L^\mu_\nu = \begin{cases} G & \text{for } (\mu,\nu)=(0,0),\\ L_j & \text{for } (\mu,\nu)=(j,0),\\ -L_k^* & \text{for } (\mu,\nu)=(0,k),\\ 0 & \text{for}(\mu,\nu)=(j,k), \end{cases} \tag{5.2}$$

admit a unique unitary solution V_t.

(ii) *There exists a unitary isomorphism $\widetilde{\Xi}:\mathbf{h}\otimes\mathcal{H}\to\mathbf{h}\otimes\Gamma$ such that*

$$U_t=\widetilde{\Xi}^*V_t\widetilde{\Xi}\quad \forall t\geq 0. \tag{5.3}$$

Here we shall sketch the prove of part (i) of the theorem and postpone the proof of (ii) to the next two subsections. In order to prove the part (i) we need the following definition and lemmas. For $\lambda>0,$ we define the Feller set $\beta_\lambda\subseteq\mathcal{B}(\mathbf{h})$ by

$$\Big\{x\geq 0\colon\ \langle v,xGw\rangle+\langle Gv,xw\rangle+\sum_{j\geq 1}\langle L_jv,xL_jw\rangle=\lambda\langle v,xw\rangle,\forall v,w\in\mathcal{D}\Big\}.$$

Similarly we define the Feller set $\widetilde{\beta_\lambda}$ for coefficients $\widetilde{L}^\mu_\nu\equiv(L^\nu_\mu)^*$.

Lemma 5.3. *Under the Assumption* E2, *the Feller condition*: $\beta_\lambda=\{0\}$ *as well as* $\widetilde{\beta_\lambda}=\{0\}$ *for some* $\lambda>0$ *hold.*

Proof. For any $x\geq 0$ in $\mathcal{B}(\mathbf{h})$, $v,w\in\mathcal{D}$ we have

$$\begin{aligned}\sum_{j\geq 1}\langle L_jv,xL_jw\rangle&=\big\langle Lv,(x\otimes 1_{\mathbf{k}})Lw\big\rangle\\ &=\sum_{m\geq 1}\big\langle\langle\langle x^{1/2}e_m,Lv\rangle\rangle,\langle\langle x^{1/2}e_m,Lw\rangle\rangle\big\rangle=\sum_{m\geq 1}\big\langle\eta\big(x^{1/2}e_m,v\big),\eta\big(x^{1/2}e_m,w\big)\big\rangle\\ &=\sum_{j\geq 1}\langle L_jv,xL_jw\rangle=\sum_{m\geq 1}\big\langle\eta\big(x^{1/2}e_m,v\big),\eta\big(x^{1/2}e_m,w\big)\big\rangle\\ &=\sum_{m\geq 1}\big\{\big\langle x^{1/2}e_m,\mathcal{L}\big(|w><v|\big)x^{1/2}e_m\big\rangle-\overline{\big\langle x^{1/2}e_m,Gv\big\rangle}\big\langle x^{1/2}e_m,w\big\rangle-\overline{\big\langle x^{1/2}e_m,v\big\rangle}\big\langle x^{1/2}e_m,Gw\big\rangle\big\}\\ &=\mathrm{Tr}\big[x\mathcal{L}\big(|w><v|\big)\big]-\langle v,xGw\rangle-\langle Gv,xw\rangle,\end{aligned}$$

where we have used (4.7). Thus

$$\langle v,xGw\rangle+\langle Gv,xw\rangle+\sum_{j\geq 1}\langle L_jv,xL_jw\rangle=\mathrm{Tr}\big[x\mathcal{L}\big(|w><v|\big)\big] \tag{5.4}$$

and for any $x\in\beta_\lambda$,

$$\mathrm{Tr}\big[x\mathcal{L}\big(|w><v|\big)\big]=\lambda\langle v,xw\rangle=\lambda\,\mathrm{Tr}\big(x|w><v|\big)\quad\forall v,w\in\mathcal{D}. \tag{5.5}$$

By Assumption E2 the subspace $\mathcal{N}=\mathit{Span}\{|w><v|\colon\ v,w\in\mathcal{D}\}$ is a core for $\mathcal{L}$ and hence the identity (5.5) extends to $\mathrm{Tr}[x\mathcal{L}(\rho)]=\lambda\,\mathrm{Tr}(x\rho),\forall\rho\in\mathcal{D}(\mathcal{L})$. It is also clear that for $x\in\beta_\lambda$ the scalar map $\phi_x:\mathcal{D}(\mathcal{L})\ni\rho\mapsto\mathrm{Tr}[x\mathcal{L}(\rho)]=\lambda\,\mathrm{Tr}(x\rho)$ extends to a bounded linear functional on $\mathcal{B}_1(\mathbf{h})$. Hence x is in the domain of $\mathcal{L}^*$ and we get $\mathrm{Tr}[(|w><v|)(\mathcal{L}^*-\lambda)x]=0$ which implies that $\langle v,(\mathcal{L}^*-\lambda)xw\rangle=0$ for $v,w\in\mathcal{D}$ which in its turn yield that $(\mathcal{L}^*-\lambda)x=0$. Since $\mathcal{L}^*$ is the generator of a C_0-semigroup $\{Z_t^*\}$ of contraction maps on $\mathcal{B}(\mathbf{h})$, for $\lambda>0,\mathcal{L}^*-\lambda$ is invertible and hence $x=0$.

To prove $\widetilde{\beta_\lambda}=\{0\}$ let us consider the following. By identity (3.7) for vectors $u,p\in\mathbf{h}$ and $v,w\in\mathcal{D}$

$$\begin{aligned}
&\big\langle\eta(u,v),\eta(p,w)\big\rangle\\
&\quad=\lim_{t\to 0}\frac{1}{t}\big\langle(U_t-1)(u,v)\Omega,(U_t-1)(p,w)\Omega\big\rangle\\
&\quad=\lim_{t\to 0}\frac{1}{t}\big\langle\big(U_t^*-1\big)(u,v)\Omega,\big(U_t^*-1\big)(p,w)\Omega\big\rangle\\
&\quad=\lim_{t\to 0}\frac{1}{t}\big\{\big\langle U_t^*(u,v)\Omega,U_t^*(p,w)\Omega\big\rangle-\overline{\langle u,v\rangle}\langle p,w\rangle\\
&\qquad-\overline{\langle u,v\rangle}\big\langle\Omega,\big[\big(U_t^*-1\big)(p,w)\big]\Omega\big\rangle\\
&\qquad-\overline{\big\langle\Omega,\big[\big(U_t^*-1\big)(u,v)\big]\Omega\big\rangle}\langle p,w\rangle\big\}\\
&\quad=\lim_{t\to 0}\frac{1}{t}\big\{\big\langle p,(F_t-1)\big(|w><v|\big)u\big\rangle-\overline{\langle u,v\rangle}\big\langle p,\big(T_t^*-1\big)w\big\rangle-\overline{\big\langle u,\big(T_t^*-1\big)v\big\rangle}\langle p,w\rangle\big\}.
\end{aligned}$$

Since by E2, $v,w\in\mathcal{D}\subseteq\mathcal{D}(G^*)$ and $|w><v|\in\mathcal{D}(\mathcal{L}')$, we get that

$$\big\langle\eta(u,v),\eta(p,w)\big\rangle=\big\langle p,\mathcal{L}'\big(|w><v|\big)u\big\rangle-\overline{\langle u,v\rangle}\big\langle p,G^*w\big\rangle-\overline{\big\langle u,G^*v\big\rangle}\langle p,w\rangle. \tag{5.6}$$

Thus by (5.6) we have

$$\begin{aligned}
\sum_{j\ge 1}\langle L_jv,xL_jw\rangle&=\sum_{m\ge 1}\big\langle\eta\big(x^{1/2}e_m,v\big),\eta\big(x^{1/2}e_m,w\big)\big\rangle\\
&=\sum_{m\ge 1}\big\{\big\langle x^{1/2}e_m,\mathcal{L}'\big(|w><v|\big)x^{1/2}e_m\big\rangle-\overline{\big\langle x^{1/2}e_m,v\big\rangle}\big\langle x^{1/2}e_m,G^*w\big\rangle\\
&\qquad-\overline{\big\langle x^{1/2}e_m,G^*v\big\rangle}\big\langle x^{1/2}e_m,w\big\rangle\big\}\\
&=\mathrm{Tr}\big[x\mathcal{L}'\big(|w><v|\big)\big]-\big\langle G^*v,xw\big\rangle-\big\langle v,xG^*w\big\rangle.
\end{aligned}$$

Thus

$$\big\langle v,xG^*w\big\rangle+\big\langle G^*v,xw\big\rangle+\sum_{j\ge 1}\langle L_jv,xL_jw\rangle=\mathrm{Tr}\big[x\mathcal{L}'\big(|w><v|\big)\big] \tag{5.7}$$

and for any $x\in\widetilde{\beta_\lambda}$,

$$\mathrm{Tr}\big[x\mathcal{L}'\big(|w><v|\big)\big]=\lambda\langle v,xw\rangle=\lambda\,\mathrm{Tr}\big(x|w><v|\big)\quad\forall v,w\in\mathcal{D}. \tag{5.8}$$

Since the subspace $\mathcal{N}=Span\{|w><v|:\ v,w\in\mathcal{D}\}$ is a core for $\mathcal{L}'$ by the Assumption E2, a similar argument as above will give that $\widetilde{\beta_\lambda}=\{0\}$. □

Remark 5.4. *By (5.4) and (5.7) formally $(\mathcal{L}'-\mathcal{L})\rho=[G^*-G,\rho],\forall\rho\in\mathcal{N}$. Denoting the imaginary part of G by H consider the derivation $\delta_H(\rho)=-2i[H,\rho]$. If δ_H is bounded then the hypothesis that the subspace $\mathcal{N}$ is a core for $\mathcal{L}$ implies that it is a core for $\mathcal{L}'$ and no extra assumption is needed.*

Remark 5.5. *If $\{T_t\}$ is a holomorphic semigroup of contractions then the hypotheses on domains of G^* and $\mathcal{L}'$ will hold automatically.*

Lemma 5.6. *Assume the hypotheses* E1 *and* E2. *For $n\ge 1$, setting $L_j(n)=nL_j(n1_{\mathbf{h}}-G)^{-1}$ and $G(n)=n^2(n1_{\mathbf{h}}-G^*)^{-1}G(n1_{\mathbf{h}}-G)^{-1}$, we have*:

(i) *The operators* $L_j(n), G(n) \in \mathcal{B}(\mathbf{h})$ *and* $\sum_j \|L_j(n)v\|^2 = -2Re\langle v, G(n)v\rangle$.
(ii) *For* $v \in \mathcal{D}$, $\lim_{n\to\infty} L_j(n)v = L_j v$, $\lim_{n\to\infty} L_j(n)^* v = L_j^* v$ *and* $\lim_{n\to\infty} G(n)v = Gv$.

Proof. (i) For any $v \in \mathbf{h}$,

$$\begin{aligned}\sum_j \big\|L_j(n)v\big\|^2 &= \sum_j n^2 \big\|L_j(n1_{\mathbf{h}} - G)^{-1}v\big\|^2 \\ &= -2Re\, n^2\big\langle (n1_{\mathbf{h}} - G)^{-1}v, G(n1_{\mathbf{h}} - G)^{-1}v\big\rangle \\ &= -2Re\big\langle v, G(n)v\big\rangle.\end{aligned}$$

(ii) Since the sequences of bounded operators $\{nL_j(n1_{\mathbf{h}} - G)^{-1}\}$ and $\{nL_j(n1_{\mathbf{h}} - G^*)^{-1}\}$ are uniformly norm bounded and converge strongly to identity, the requirements follows. □

Sketch of the proof of part (i) of Theorem 5.2. For each $n \ge 1$ we consider the family of operators,

$$L_\nu^\mu(n) = \begin{cases} G(n) = n^2(n1_{\mathbf{h}} - G^*)^{-1}G(n1_{\mathbf{h}} - G)^{-1} & \text{for } (\mu,\nu) = (0,0), \\ L_j(n) = nL_j(n1_{\mathbf{h}} - G)^{-1} & \text{for } (\mu,\nu) = (j,0), \\ -L_k(n)^* & \text{for } (\mu,\nu) = (0,k), \\ 0 & \text{for } (\mu,\nu) = (j,k). \end{cases} \tag{5.9}$$

By hypothesis E1, we have that $\lim_{n\to\infty} L_\nu^\mu(n)v = L_\nu^\mu v, \forall v \in \mathcal{D}$ and hence there exist unique contractive solution $\{V_t\}$ for the HP equation (5.1) (see [3,4,9]). Under the Assumption E the first part of Lemma 5.3, on using Proposition 3.1 in [10] (also see [3,9]) and Theorem 7.2.3 in [4], yields that the solution $\{V_t\}$ of HP equation (5.1) is isometric. We shall conclude the unitarity of the process V_t by employing time reversal operator and the results in [4,10]. As V_t satisfies Eq. (5.1), V_t^* satisfies the HP equation on $\mathcal{D} \otimes \mathcal{E}(\mathcal{K})$, since $\mathcal{D} \subseteq \mathcal{D}(G^*)$ by E2,

$$V_t^* = 1_{\mathbf{h}\otimes\Gamma} + \sum_{\mu,\nu\ge 0} \int_0^t (L_\nu^\mu)^* V_r^* \Lambda_\nu^\mu(\mathrm{d}r). \tag{5.10}$$

Let us define $\widetilde{V}_t := [1_{\mathbf{h}} \otimes \Gamma(R_t)]V_t^*[1_{\mathbf{h}} \otimes \Gamma(R_t)]$, where R_t is the time reversal operator on $L^2(\mathbb{R}_+, \mathbf{k})$:

$$R_t f(x) = \begin{cases} f(t-x) & \text{if } x \le t, \\ f(x) & \text{if } x > t, \end{cases}$$

and $\Gamma(A)$ denote the second quantization of operator A: $\Gamma(A)\mathbf{e}(f) = \mathbf{e}(Af)$. Then it can be seen that the process $\{\widetilde{V}_t\}$ satisfies the HP equation on $\mathcal{D} \otimes \mathcal{E}(\mathcal{K})$,

$$\widetilde{V}_t = 1_{\mathbf{h}\otimes\Gamma} + \sum_{\mu,\nu\ge 0} \int_0^t \widetilde{V}_r \widetilde{L}_\nu^\mu \Lambda_\mu^\nu(\mathrm{d}r). \tag{5.11}$$

Since the Feller condition $\widetilde{\beta}_\lambda = \{0\}$ for $\widetilde{L}_\nu^\mu$ holds by the second part of Lemma 5.3 (see again [4]), the solution $\widetilde{V}_t$ and hence V_t^* is isometric or equivalently V_t is co-isometric and therefore V_t is a strongly continuous unitary process. □

Remark 5.7. *Using identity* (4.13) *one constructs the minimal semigroup* $\hat{Z}_t$ *with generator* $\hat{\mathcal{L}}$ *such that restrictions of* $\mathcal{L}$ *and* $\hat{\mathcal{L}}$ *to* $\mathcal{N}$ *are same* (*see* [4,9,10,15]). *Therefore, for any* $\lambda > 0$, *the closure* $\overline{(\lambda - \hat{\mathcal{L}})\mathcal{N}} = \overline{(\lambda - \mathcal{L})\mathcal{N}} = (\lambda - \mathcal{L})\mathcal{D}(\mathcal{L})$ *since by hypothesis* E2 *the subspace* $\mathcal{N}$ *is a core for* $\mathcal{L}$. *As* $\mathcal{L}$ *is the generator of a* C_0*-semigroup of contractions on* $\mathcal{B}_1(\mathbf{h})$ *the subspace* $(\lambda - \mathcal{L})\mathcal{D}(\mathcal{L}) = \mathcal{B}_1(\mathbf{h})$ *and hence* $\overline{(\lambda - \hat{\mathcal{L}})\mathcal{N}} = \mathcal{B}_1(\mathbf{h})$. *Thus by Theorem* 3.2.16(ii) *and* (iii) *in* [4] *we have that* $\mathrm{Tr}(\hat{Z}_t\rho) = \mathrm{Tr}(\rho)$, *i.e the minimal semigroup* $\hat{Z}_t$ *is conservative which also implies that the Feller condition is satisfied. We also have* $\overline{(\lambda - \hat{\mathcal{L}})\mathcal{N}} = \mathcal{B}_1(\mathbf{h}) = (\lambda - \hat{\mathcal{L}})\mathcal{D}(\hat{\mathcal{L}})$ *which implies* $\mathcal{N}$ *is a core for* $\hat{\mathcal{L}}$ *as well and hence* $\mathcal{L} = \hat{\mathcal{L}}$. *Thus* Z_t *is the minimal semigroup.*

For any $0 \le s \le t < \infty$, we define a unitary operator $V_{s,t} := 1_{\Gamma_{s]}} \otimes [\Gamma(\theta_s) V_{t-s} \Gamma(\theta_s^*)]$, where θ_s is the right shift operator on $L^2(\mathbb{R}_+, \mathbf{k})$:

$$\theta_s f(x) = \begin{cases} f(x-s) & \text{if } x \ge s, \\ f(x) & \text{if } x < s. \end{cases}$$

It has been shown in Lemma 7.1.3 of [4] that the solution of (5.1) under the condition given above is a unitary cocycle, i.e. $V_{0,t} = V_{0,s} V_{s,t} = V_{0,s}[\Gamma(\theta_s) V_{t-s} \Gamma(\theta_s^*)]$ and satisfies the HP equation

$$V_{s,t} = 1_{\mathbf{h} \otimes \Gamma} + \sum_{\mu,\nu \ge 0} \int_s^t V_{s,r} L_\nu^\mu \Lambda_\mu^\nu(\mathrm{d}r). \tag{5.12}$$

As for the family of unitary operators $\{U_{s,t}\}$ on $\mathbf{h} \otimes \mathcal{H}$, for $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \mathbb{Z}_2^n$ we define $V_{s,t}^{(\varepsilon)} \in \mathcal{B}(\mathbf{h}^{\otimes n} \otimes \Gamma)$ by setting $V_{s,t}^{(\varepsilon)} \in \mathcal{B}(\mathbf{h} \otimes \Gamma)$ by

$$V_{s,t}^{(\varepsilon)} = \begin{cases} V_{s,t} & \text{for } \varepsilon = 0, \\ V_{s,t}^* & \text{for } \varepsilon = 1. \end{cases}$$

The next result verifies the properties of Assumption A for the family $V_{s,t}$ with $\mathbf{e}(0) \in \Gamma$ replacing $\Omega \in \mathcal{H}$.

Lemma 5.8. *The family of unitary operators $\{V_{s,t}\}$ satisfy*:

(i) *For any $0 \le r \le s \le t < \infty$, $V_{r,t} = V_{r,s} V_{s,t}$.*
(ii) *For $[q,r) \cap [s,t) = \varnothing$, $V_{q,r}(u,v)$ commute with $V_{s,t}(p,w)$ and $V_{s,t}(p,w)^*$ for every $u, v, p, w \in \mathbf{h}$.*
(iii) *For any $0 \le s \le t < \infty$,*

$$\langle \boldsymbol{e}(0), V_{s,t}(u,v)\boldsymbol{e}(0)\rangle = \langle \boldsymbol{e}(0), V_{t-s}(u,v)\boldsymbol{e}(0)\rangle = \langle u, T_{t-s} v\rangle \quad \forall u, v \in \mathbf{h}.$$

Proof. (i) For fixed $0 \le r \le s \le t < \infty$, we set $W_{r,t} = V_{r,s} V_{s,t}$. Then by (5.1) we have

$$\begin{aligned} W_{r,t} &= V_{r,s} + \sum_{\mu,\nu \ge 0} \int_s^t V_{r,s} V_{s,q} L_\nu^\mu \Lambda_\mu^\nu(\mathrm{d}q) \\ &= W_{r,s} + \sum_{\mu,\nu \ge 0} \int_s^t W_{r,q} L_\nu^\mu \Lambda_\mu^\nu(\mathrm{d}q). \end{aligned}$$

Thus the family of unitary operators $\{W_{r,t}\}$ also satisfies the HP equation (5.12). Hence by uniqueness of the solution of this quantum stochastic differential equation, $W_{r,t} = V_{r,t}, \forall t \ge s$ and the result follows.

(ii) For any $0 \le s \le t < \infty$, $V_{s,t} \in \mathcal{B}(\mathbf{h} \otimes \Gamma_{[s,t]})$. So for $p, w \in \mathbf{h}$, $V_{s,t}(p,w) \in \mathcal{B}(\Gamma_{[s,t]})$ and the statement follows.

The proof of (iii) follows from (5.12) and the cocycle property of $V_{s,t}$. □

Consider the family of maps $\widetilde{Z}_{s,t}$ defined by

$$\widetilde{Z}_{s,t}\rho = \mathrm{Tr}_{\mathcal{H}}\big[V_{s,t}\big(\rho \otimes \big|\mathbf{e}(0) >< \mathbf{e}(0)\big|\big) V_{s,t}^*\big] \quad \forall \rho \in \mathcal{B}_1(\mathbf{h}).$$

As for Z_t, it can be seen that $\widetilde{Z}_{s,t}$ is a contractive family of maps on $\mathcal{B}_1(\mathbf{h})$ and in particular, for any $u, v, p, w \in \mathbf{h}$

$$\langle p, \widetilde{Z}_{s,t}\big(|w >< v|\big)u\rangle = \langle V_{s,t}(u,v)\mathbf{e}(0), V_{s,t}(p,w)\mathbf{e}(0)\rangle.$$

Lemma 5.9. *The family $\widetilde{Z}_t := \widetilde{Z}_{0,t}$ is a C_0-semigroup of contraction on $\mathcal{B}_1(\mathbf{h})$ and $\widetilde{Z}_{s,t} = \widetilde{Z}_{t-s} = Z_{t-s}$.*

Proof. By (5.12) and Ito's formula for $u, v, p, w \in \mathcal{D}$

$$
\begin{aligned}
&\big\langle p, [\widetilde{Z}_{s,t} - 1]\big(|w><v|\big)u\big\rangle \\
&\quad = \big\langle V_{s,t}(u,v)\mathbf{e}(0), V_{s,t}(p,w)\mathbf{e}(0)\big\rangle - \overline{\langle u, v\rangle}\langle p, w\rangle \\
&\quad = \int_s^t \big\langle V_{s,\tau}(u,v)\mathbf{e}(0), V_{s,\tau}(p,Gw)\mathbf{e}(0)\big\rangle \mathrm{d}\tau + \int_s^t \big\langle V_{s,\tau}(u,Gv)\mathbf{e}(0), V_{s,\tau}(p,w)\mathbf{e}(0)\big\rangle \mathrm{d}\tau \\
&\qquad + \int_s^t \big\langle V_{s,\tau}(u,L_j v)\mathbf{e}(0), V_{s,\tau}(p,L_j w)\mathbf{e}(0)\big\rangle \mathrm{d}\tau \\
&\quad = \int_s^t \big\langle p, \widetilde{Z}_{s,\tau}\big(|Gw><v|\big)u\big\rangle \mathrm{d}\tau + \int_s^t \big\langle p, \widetilde{Z}_{s,\tau}\big(|w><Gv|\big)u\big\rangle \mathrm{d}\tau \\
&\qquad + \sum_{j\ge 1}\int_s^t \big\langle p, \widetilde{Z}_{s,\tau}\big(|L_j w><L_j v|\big)u\big\rangle \mathrm{d}\tau.
\end{aligned}
$$

Thus

$$
\big\langle p, [\widetilde{Z}_{s,t} - 1](\rho)u\big\rangle = \int_s^t \big\langle p, \widetilde{Z}_{s,\tau}\mathcal{L}(\rho)u\big\rangle \mathrm{d}\tau, \tag{5.13}
$$

where $\rho = |w><v|$. Since $\mathcal{D}$ is dense in $\mathbf{h}, \mathcal{N}$ is a core for $\mathcal{L}$ and $\widetilde{Z}_{s,\tau}$ is a contractive family Eq. (5.13) extends to $u, p \in \mathbf{h}$ and $\rho \in \mathcal{D}(\mathcal{L})$. Thus the family $\widetilde{Z}_{s,t}$ satisfies the differential equation

$$
\widetilde{Z}_{s,t}(\rho) = \rho + \int_s^t \widetilde{Z}_{s,\tau}\mathcal{L}(\rho)\,\mathrm{d}\tau, \quad \rho \in \mathcal{D}(\mathcal{L}).
$$

Since $\mathcal{L}$ is the generator of C_0-semigroup Z_t, it follows that $\widetilde{Z}_{s,t} = \widetilde{Z}_{t-s} = Z_{t-s}$. □

5.1. *Minimality of HP flows*

In this section we shall show the minimality of the HP flow $V_{s,t}$ discussed above which will be needed to prove Theorem 5.2(ii), i.e, to establish unitary equivalence of U_t and V_t. We shall prove here that the subset $\mathcal{S}' := \{\zeta = V_{\underline{s},\underline{t}}(\underline{u},\underline{v})\mathbf{e}(0) := V_{s_1,t_1}(u_1,v_1)\cdots V_{s_n,t_n}(u_n,v_n)\mathbf{e}(0)\colon \underline{s} = (s_1, s_2, \ldots, s_n), \underline{t} = (t_1, t_2, \ldots, t_n)\colon 0 \le s_1 \le t_1 \le \cdots \le s_n \le t_n < \infty, n \ge 1, \underline{u} = \bigotimes_{i=1}^n u_i \in \mathbf{h}^{\otimes n}, \underline{v} = \bigotimes_{i=1}^n v_i \in \mathcal{D}^{\otimes n}\}$ is total in the symmetric Fock space $\Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$.

Since $\mathcal{D}$ is dense in $\mathbf{h}$, by Remark 3.3 the subset $\mathcal{S} := \{\zeta = U_{\underline{s},\underline{t}}(\underline{u},\underline{v})\Omega := U_{s_1,t_1}(u_1,v_1)\cdots U_{s_n,t_n}(u_n,v_n)\Omega\colon \underline{s} = (s_1, s_2, \ldots, s_n), \underline{t} = (t_1, t_2, \ldots, t_n)\colon 0 \le s_1 \le t_1 \le s_2 \le \cdots \le s_n \le t_n < \infty, n \ge 1, \underline{u} = \bigotimes_{i=1}^n u_i \in \mathbf{h}^{\otimes n}, \underline{v} = \bigotimes_{i=1}^n v_i \in \mathcal{D}^{\otimes n}\}$ is total in $\mathcal{H}$.

Lemma 5.10. *Under the Assumption* E3, *for any* $v \in \mathcal{D}$, $\sum_{i,j\ge 1}\|L_i L_j v\|^2 < \infty$.

Proof. For any $j \ge 1$, $L_j v \in \mathcal{D}$ and by Lemma 4.3(i),

$$
\sum_{i\ge 1}\|L_i L_j v\|^2 = -2Re\sum_{j\ge 1}\langle L_j v, GL_j v\rangle \le 2\bigg[\sum_{j\ge 1}\|L_j v\|^2\bigg]^{1/2}\bigg[\sum_{j\ge 1}\|GL_j v\|^2\bigg]^{1/2} < \infty.
$$

□

Let $\tau \ge 0$ be fixed and as in (Ref. [12]), we note that for any $0 \le s < t \le \tau, u \in \mathbf{h}, v \in \mathcal{D}$

$$
\begin{aligned}
&\frac{1}{t-s}[V_{s,t} - 1](u,v)\mathbf{e}(0) \\
&\quad = \gamma(s,t,u,v) + \langle u, Gv\rangle\mathbf{e}(0) + \zeta(s,t,u,v) + \varsigma(s,t,u,v),
\end{aligned} \tag{5.14}
$$

where these vectors in the Fock space Γ are given by

$$\gamma(s,t,u,v) := \frac{1}{t-s}\sum_{j\geq 1}\langle u, L_j v\rangle a_j^{\dagger}\big([s,t]\big)\mathbf{e}(0),$$

$$\zeta(s,t,u,v) := \frac{1}{t-s}\sum_{j\geq 1}\int_s^t (V_{s,\lambda}-1)(u, L_j v)a_j^{\dagger}(\mathrm{d}\lambda)\mathbf{e}(0),$$

$$\varsigma(s,t,u,v) := \frac{1}{t-s}\int_s^t (V_{s,\lambda}-1)(u, Gv)\,\mathrm{d}\lambda\mathbf{e}(0).$$

Note that any $\xi\in\Gamma$ can be written as $\xi=\xi^{(0)}\mathbf{e}(0)\oplus\xi^{(1)}\oplus\cdots$, $\xi^{(n)}$ in the n-fold symmetric tensor product $L^2(\mathbb{R}_+,\mathbf{k})^{\otimes n}\equiv L^2(\Sigma_n)\otimes\mathbf{k}^{\otimes n}$ where Σ_n is the n-simplex $\{\underline{t}=(t_1,t_2,\ldots,t_n)\colon 0\leq t_1<t_2\cdots<t_n<\infty\}$.

Lemma 5.11. *For any $u\in\mathbf{h}$, $v\in\mathcal{D}$, $0\leq s\leq t\leq\tau$ there exist positive constants $C_{\tau,u,v}, C'_{\tau,u,v}$ given by*

$$C_\tau = 2\mathrm{e}^{\tau},\qquad C_{\tau,u,v} = 2\|u\|^2\bigg[C_\tau\sum_{j\geq 1}\|L_j v\|^2+\tau\|Gv\|^2\bigg]$$

and

$$C'_{\tau,u,v} = 2C_\tau\|u\|^2\bigg[C_\tau\sum_{i,j\geq 1}\|L_jL_i v\|^2+\tau\sum_{i\geq 1}\|GL_i v\|^2\bigg]$$

such that

(i) $\|\sum_{j\geq 1}\int_s^t V_{s,\lambda}(u, L_j v)a_j^{\dagger}(\mathrm{d}\lambda)\boldsymbol{e}(0)\|^2\leq C_\tau(t-s)\|u\|^2\sum_{j\geq 1}\|L_j v\|^2$,
(ii) $\|(V_{s,t}-1)(u,v)\boldsymbol{e}(0)\|^2\leq C_{\tau,u,v}(t-s)$,
(iii) $\|\zeta(s,t,u,v)\|^2\leq C'_{\tau,u,v}$ *and* $\|\varsigma(s,t,u,v)\|\leq C_{\tau,u,v}\sqrt{t-s}$, $\forall 0\leq s<t\leq\tau$,
(iv) *For any $\xi\in\Gamma(L^2(\mathbb{R}_+,\mathbf{k}))$, $\lim_{s\to t}\langle\xi,\zeta(s,t,u,v)\rangle=0$ and*

$$\lim_{s\to t}\big\langle\xi,\gamma(s,t,u,v)\big\rangle=\sum_{j\geq 1}\langle u, L_j v\rangle\overline{\xi_j^{(1)}}(t)=\big\langle\xi^{(1)}(t),\eta(u,v)\big\rangle,\quad \textit{a.e. } t\geq 0.$$

Proof. We give the proof of (iii) only since the proof of (i), (ii) and (iv) are identical to that in [12] except for the observation that in the estimates the constants will depend on the norms of the vectors like $\|L_j v\|, \|Gv\|$ instead of $\|v\|$ for $v\in\mathcal{D}$. (iii) Using the standard estimate of a stochastic integral [11] and Assumption E3 we get that

$$\begin{aligned}
&\big\|\zeta(s,t,u,v)\big\|^2\\
&\quad\leq\frac{\|u\|^2}{(t-s)^2}\bigg\|\sum_{j\geq 1}\int_s^t (V_{s,\lambda}-1)L_j v a_j^{\dagger}(\mathrm{d}\lambda)\mathbf{e}(0)\bigg\|^2\\
&\quad\leq\frac{C_\tau\|u\|^2}{(t-s)^2}\sum_{j\geq 1}\int_s^t\big\|(V_{s,\lambda}-1)L_j v\mathbf{e}(0)\big\|^2\,\mathrm{d}\lambda\\
&\quad\leq 2\frac{C_\tau\|u\|^2}{(t-s)^2}\sum_{j\geq 1}(t-s)\bigg[C_\tau(t-s)\sum_{i\geq 1}\|L_iL_j v\|^2+(t-s)^2\big\|GL_j v\big\|^2\bigg]\\
&\quad\leq 2C_\tau\|u\|^2\sum_{j\geq 1}\bigg[C_\tau\sum_{i\geq 1}\|L_iL_j v\|^2+\tau\|GL_j v\|^2\bigg]=C'_{\tau,u,v}.
\end{aligned}$$

□

Lemma 5.12. *For $n \geq 1, \underline{t} \in \Sigma_n$ and $u_k \in \mathbf{h}, v_k \in \mathcal{D}: k = 1, \ldots, n, \xi \in \Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$ and $[s_k, t_k)$'s are disjoint.*

(i) $\lim_{\underline{s} \to \underline{t}} \langle \xi, \prod_{k=1}^n M(s_k, t_k, u_k, v_k)\boldsymbol{e}(0)\rangle = 0$, *where* $M(s_k, t_k, u_k, v_k) = \frac{(V_{s_k,t_k} - 1)}{t_k - s_k}(u_k, v_k) - \langle u_k, G v_k\rangle - \gamma(s_k, t_k, u_k, v_k)$ *and* $\lim_{\underline{s} \to \underline{t}}$ *means* $s_k \to t_k$ *for each* k.

(ii) $\lim_{\underline{s} \to \underline{t}} \langle \xi, \bigotimes_{k=1}^n \gamma(s_k, t_k, u_k, v_k)\rangle = \langle \xi^{(n)}(t_1, \ldots, t_n), \eta(u_1, v_1) \otimes \cdots \otimes \eta(u_n, v_n)\rangle$.

Proof. The proof is omitted since it is identical to that in [12] once we note that $v_k \in \mathcal{D} \subseteq \mathcal{D}(G)$, and part (ii) can be proved using Lemma 5.11 (iv). $\square$

The next lemma extends the lemma 8.4 of [12] to $\mathcal{S}'$ defined in term of $\underline{u} \in \mathbf{h}^{\otimes n}$, $\underline{v} \in \mathcal{D}^{\otimes n}$ instead of both in $\mathbf{h}^{\otimes n}$ as in [12]. The proof proceed along identical lines and hence omitted.

Lemma 5.13. *Let $\xi \in \Gamma$ be such that*

$$\langle \xi, \zeta\rangle = 0 \quad \forall \zeta \in \mathcal{S}', \tag{5.15}$$

then

(i) $\xi^{(0)} = 0$ *and* $\xi^{(1)}(t) = 0$ *for a.e.* $t \in [0, \tau]$.

(ii) *For any* $n \geq 0, \xi^{(n)}(\underline{t}) = 0$ *for a.e.* $\underline{t} \in \Sigma_n\colon t_i \leq \tau$.

(iii) *The set* $\mathcal{S}'$ *is total in the Fock space* Γ.

We shall now prove the part (ii) of the Theorem 5.2 that the unitary evolution $\{U_t\}$ on $\mathbf{h} \otimes \mathcal{H}$ is unitarily equivalent to the unitary solution $\{V_t\}$ of HP equation (5.1). To prove this we need the following two results. Let us recall that the subset $\mathcal{S} = \{\xi = U_{\underline{s},\underline{t}}(\underline{u}, \underline{v})\Omega := U_{s_1,t_1}(u_1, v_1) \cdots U_{s_n,t_n}(u_n, v_n)\Omega\colon \underline{s} = (s_1, s_2, \ldots, s_n), \underline{t} = (t_1, t_2, \ldots, t_n)\colon 0 \leq s_1 \leq t_1 \leq \cdots \leq s_n \leq t_n < \infty, n \geq 1, \underline{u} = \bigotimes_{i=1}^n u_i \in \mathbf{h}^{\otimes n}, \underline{v} = \bigotimes_{i=1}^n v_i \in \mathcal{D}^{\otimes n}\}$ is total in $\mathcal{H}$ and the subset $\mathcal{S}' := \{\zeta = V_{\underline{s},\underline{t}}(\underline{u}, \underline{v})\mathbf{e}(0) := V_{s_1,t_1}(u_1, v_1) \cdots V_{s_n,t_n}(u_n, v_n)\mathbf{e}(0)\colon \underline{u} = \bigotimes_{i=1}^n u_i \in \mathbf{h}^{\bigotimes n}, \underline{v} = \bigotimes_{i=1}^n v_i \in \mathcal{D}^{\otimes n}, \underline{s} = (s_1, s_2, \ldots, s_n), \underline{t} = (t_1, t_2, \ldots, t_n)\}$ is total in Γ.

Lemma 5.14. *Let $U_{\underline{s},\underline{t}}(\underline{u}, \underline{v})\Omega, U_{\underline{s}',\underline{t}'}(\underline{p}, \underline{w})\Omega \in \mathcal{S}$, where $\underline{v}, \underline{w} \in \mathcal{D}^{\otimes n}$. Then there exist an integer $m \geq 1, \underline{a} = (a_1, a_2, \ldots, a_m), \underline{b} = (b_1, b_2, \ldots, b_m)\colon 0 \leq a_1 \leq b_1 \leq \cdots \leq a_m \leq b_m < \infty$, partition $R_1 \cup R_2 \cup R_3 = \{1, \ldots, m\}$ with $|R_i| = m_i$, family of vectors $x_{k_l}, g_{k_i} \in \mathbf{h}$ and $y_{k_l}, h_{k_i} \in \mathcal{D}\colon l \in R_1 \cup R_2, i \in R_2 \cup R_3$ such that*

$$U_{\underline{s},\underline{t}}(\underline{u}, \underline{v}) = \sum_{\underline{k}} \prod_{l \in R_1 \cup R_2} U_{a_l,b_l}(x_{k_l}, y_{k_l}), \tag{5.16}$$

$$U_{\underline{s}',\underline{t}'}(\underline{p}, \underline{w}) = \sum_{\underline{k}} \prod_{l \in R_2 \cup R_3} U_{a_l,b_l}(g_{k_l}, h_{k_l}). \tag{5.17}$$

Proof. It follows from the evolution hypothesis of the family of unitary operators $\{U_{s,t}\}$ as for $r \in [s, t]$ and orthonormal basis $\{f_j\} \subseteq \mathcal{D}$ of $\mathbf{h}$ we can write $U_{s,t}(u, v) = \sum_{j \geq 1} U_{s,r}(u, f_j)U_{r,t}(f_j, v)$. $\square$

Remark 5.15. *Since the family of unitaries $\{V_{s,t}\}$ on $\mathbf{h} \otimes \Gamma$ enjoy all the properties satisfy by family of unitaries $\{U_{s,t}\}$ on $\mathbf{h} \otimes \mathcal{H}$ the above lemma also hold if we replace $U_{s,t}$ by $V_{s,t}$.*

Lemma 5.16. *For $U_{\underline{s},\underline{t}}(\underline{u}, \underline{v})\Omega, U_{\underline{s}',\underline{t}'}(\underline{p}, \underline{w})\Omega \in \mathcal{S}$.*

$$\left\langle U_{\underline{s},\underline{t}}(\underline{u}, \underline{v})\Omega, U_{\underline{s}',\underline{t}'}(\underline{p}, \underline{w})\Omega\right\rangle = \left\langle V_{\underline{s},\underline{t}}(\underline{u}, \underline{v})\mathbf{e}(0), V_{\underline{s}',\underline{t}'}(\underline{p}, \underline{w})\mathbf{e}(0)\right\rangle. \tag{5.18}$$

The proof of Lemma 5.18 is as in [12] and hence the part (ii) of Theorem 5.2 follows identically as in [12].

Remark 5.17. *The Assumption* C *is ruling out the presence of conservation* (*Poisson*) *terms in the associated HP equation as the representation* π, *we obtained*, *is trivial* (*see Remark* 4.2). *Without this Assumption* C, *the problem is not yet settled. In the absence of Assumption* C *the representation* π *shall be non trivial which in general will give rise to a unitary* (*different from identity*) *operator* W *on* $\mathbf{h}\otimes\mathbf{k}$ *and associated HP equation* (5.1) *will contain conservation terms with coefficients* $\{L^{\mu}_{\nu}\}$ *described as in* (2.4).

Remark 5.18. *The Assumption* E2, *i.e. there exists* $\mathcal{D}$, *core for* G *such that* $\mathcal{D}\subseteq\mathcal{D}(L_j^*)$ *for every* $j\geq 1$, *is a strong assumption. But this is necessary one in order that the quantum stochastic differential equation for* V_t *makes sense. Only way one can do away with this assumption is to abandon the quantum stochastic differential equation for* V_t *and just deal with* V_t *as a left cocycle described by the associated four semigroups* [8]. *This programme is not yet complete.*

Remark 5.19. *The Hypothesis* E3, *i.e. for any* $v\in\mathcal{D}$, $\sum_{j\geq 1}\|GL_jv\|^2<\infty$. *This holds trivially when* $[G,L_j]=0$. *Condition* $[G,L_j]=0$, *in particular holds for classical Brownian motion on* $\mathbb{R}^n$ *and for the Casimir operator* G *on the Lie algebra of a locally compact Lie group* $\mathcal{G}$ *with* $L_j=X_j$ *represented on the Hilbert space* $\mathbf{h}=L^2(\mathcal{G})$, *where* $\{X_j\}_{j=1}^n$ *a basis for the Lie algebra. The commutator* $[G,L_j]$ *also vanish in case of Quantum Brownian motion on non-commutative Torus*, *Quantum Heisenberg manifold and Quantum Plane* [4] .

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