Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2010, Vol. 46, No. 1, 159–189
DOI: 10.1214/09-AIHP204

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Large deviations for transient random walks in random environment on a Galton–Watson tree

Elie Aidékon

Laboratoire de Probabilités et Modèles Aléatoires, Université Paris VI, 4 Place Jussieu, F-75252 Paris Cedex 05, France.
E-mail: elie.aidekon@upmc.fr

Received 26 January 2008; revised 4 November 2008; accepted 14 January 2009

Abstract. Consider a random walk in random environment on a supercritical Galton–Watson tree, and let τ_n be the hitting time of generation n. The paper presents a large deviation principle for τ_n/n, both in quenched and annealed cases. Then we investigate the subexponential situation, revealing a polynomial regime similar to the one encountered in one dimension. The paper heavily relies on estimates on the tail distribution of the first regeneration time.

Résumé. Nous considérons une marche aléatoire en milieu aléatoire sur un arbre de Galton–Watson. Soit τ_n le temps d'atteinte du niveau n. Le papier présente un principe de grandes déviations pour τ_n/n, dans les cas quenched et annealed. Nous étudions ensuite le régime sous-exponentiel, qui fait apparaître un régime polynomial rappelant la dimension 1. Le papier repose principalement sur les estimations de la queue de distribution du premier temps de renouvellement.

MSC: 60K37; 60J80; 60F15; 60F10

Keywords: Random walk in random environment; Law of large numbers; Large deviations; Galton–Watson tree

1. Introduction

We consider a super-critical Galton–Watson tree $\mathbb{T}$ of root e and offspring distribution $(q_k, k \geq 0)$ with finite mean $m := \sum_{k\geq 0} kq_k > 1$. For any vertex x of $\mathbb{T}$, we call $|x|$ the generation of x, ($|e| = 0$) and $\nu(x)$ the number of children of x; we denote these children by x_i, $1 \leq i \leq \nu(x)$. We let $\nu_{\min}$ be the minimal integer such that $q_{\nu_{\min}} > 0$ and we suppose that $\nu_{\min} \geq 1$ (thus $q_0 = 0$). In particular, the tree survives almost surely. Following Pemantle and Peres [14], on each vertex x, we pick independently and with the same distribution a random variable $A(x)$, and we define:

- $\omega(x, x_i) := \frac{A(x_i)}{1+\sum_{i=1}^{\nu(x)} A(x_i)}$, $\forall 1 \leq i \leq \nu(x)$,
- $\omega(x, \overleftarrow{x}) := \frac{1}{1+\sum_{i=1}^{\nu(x)} A(x_i)}$.

To deal with the case $x = e$, we add a parent $\overleftarrow{e}$ to the root and we set $\omega(\overleftarrow{e}, e) = 1$. Once the environment built, we define the random walk $(X_n, n \geq 0)$ starting from $y \in \mathbb{T}$ by

$$P_\omega^y(X_0 = y) = 1,$$
$$P_\omega^y(X_{n+1} = z | X_n = x) = \omega(x, z).$$

The walk $(X_n, n \geq 0)$ is a $\mathbb{T}$-valued Random Walk in Random Environment (RWRE). To determine the transience or recurrence of the random walk, Lyons and Pemantle [11] provides us with the following criterion. Let A be a generic random variable having the distribution of $A(e)$.

Theorem A (Lyons and Pemantle [11]). *The walk* (X_n) *is transient if* $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m}$, *and is recurrent otherwise.*

In the transient case, let v denote the speed of the walk, which is the deterministic real $v \geq 0$ such that

$$\lim_{n\to\infty} \frac{|X_n|}{n} = v, \quad \text{a.s.}$$

Define

$$i := \operatorname{ess\,inf} A,$$

$$s := \operatorname{ess\,sup} A.$$

We make the hypothesis that $0 < i \leq s < \infty$. Under this assumption, we gave a criterion in [1] for the positivity of the speed v. Let

$$\Lambda := Leb\left\{t \in \mathbb{R}\colon\ \mathbf{E}\left[A^t\right] \leq \frac{1}{q_1}\right\} \quad (\Lambda = \infty \text{ if } q_1 = 0). \tag{1.1}$$

Theorem B ([1]). *Assume* $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m}$, *and let* Λ *be as in* (1.1).

(a) *If* $\Lambda < 1$, *the walk has zero speed.*
(b) *If* $\Lambda > 1$, *the walk has positive speed.*

When the speed is positive, we would like to have information on how hard it is for the walk to have atypical behaviours, which means to go a little faster or slower than its natural pace. Such questions have been discussed in the setting of biased random walks on Galton–Watson trees, by Dembo et al. in [5]. The authors exhibit a large deviation principle both in quenched and annealed cases. Besides, an uncertainty principle allows them to obtain the equality of the two rate functions. For the RWRE in dimensions one or more, we refer to Zeitouni [17] for a review of the subject. In our case, we consider a random walk which always avoids the parent $\overleftarrow{e}$ of the root, and we obtain a large deviation principle, which, following [5], has been divided into two parts.

We suppose in the rest of the paper that

$$\inf_{[0,1]} \mathbf{E}\left[A^t\right] > \frac{1}{m}, \tag{1.2}$$

$$\Lambda > 1, \tag{1.3}$$

which ensures that the walk is transient with positive speed. Before the statement of the results, let us introduce some notation. Define for any $n \geq 0$ and $x \in \mathbb{T}$,

$$\tau_n := \inf\left\{k \geq 0\colon\ |X_k| = n\right\},$$

$$D(x) := \inf\{k \geq 1\colon\ X_{k-1} = x, X_k = \overleftarrow{x}\}, \quad \inf\varnothing := \infty.$$

Let $\mathbf{P}$ denote the distribution of the environment ω conditionally on $\mathbb{T}$, and $\mathbf{Q} := \int \mathbf{P}(\cdot) GW(\mathrm{d}\mathbb{T})$. Similarly, we denote by $\mathbb{P}^x$ the distribution defined by $\mathbb{P}^x(\cdot) := \int P^x_\omega(\cdot)\mathbf{P}(\mathrm{d}\omega)$ and by $\mathbb{Q}^x$ the distribution

$$\mathbb{Q}^x(\cdot) := \int \mathbb{P}^x(\cdot) GW(\mathrm{d}\mathbb{T}).$$

Theorem 1.1 (Speed-up case). *There exist two continuous, convex and strictly decreasing functions* $I_a \leq I_q$ *from* $[1, 1/v]$ *to* $\mathbb{R}_+$, *such that* $I_a(1/v) = I_q(1/v) = 0$ *and for* $a < b$, $b \in [1, 1/v]$, *we have almost surely,*

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{Q}^e\left(\frac{\tau_n}{n} \in]a, b]\right) = -I_a(b), \tag{1.4}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln P^e_\omega\left(\frac{\tau_n}{n} \in]a, b]\right) = -I_q(b). \tag{1.5}$$

Theorem 1.2 (Slowdown case). *There exist two continuous, convex functions* $I_a \le I_q$ *from* $[1/v, +\infty[$ *to* $\mathbb{R}_+$, *such that* $I_a(1/v) = I_q(1/v) = 0$ *and for any* $1/v \le a < b$, *we have almost surely*,

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{Q}^e\left(\frac{\tau_n}{n} \in [a, b[\right) = -I_a(a), \tag{1.6}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln P_\omega^e\left(\frac{\tau_n}{n} \in [a, b[\right) = -I_q(a). \tag{1.7}$$

Besides, if $i > \nu_{\min}^{-1}$, *then* I_a *and* I_q *are strictly increasing on* $[1/v, +\infty[$. *When* $i \le \nu_{\min}^{-1}$, *we have* $I_a = I_q = 0$ *on the interval*.

As pointed by an anonymous referee, it would be interesting to know when I_a and I_q coincide. We do not know the answer in general. However, the computation of the value of the rate functions at $b = 1$ reveals situations where the rate functions differ. Let

$$\psi(\theta) := \ln\left(E_{\mathbf{Q}}\left[\sum_{i=1}^{\nu(e)} \omega(e, e_i)^\theta\right]\right).$$

Then $\psi(0) = \ln(m)$ and $\psi(1) = \ln(E_{\mathbf{Q}}[\sum_{i=1}^{\nu(e)} \omega(e, e_i)])$.

Proposition 1.3. *We have*

$$I_a(1) = -\psi(1), \tag{1.8}$$

$$I_q(1) = -\inf_{]0,1]} \frac{1}{\theta}\psi(\theta). \tag{1.9}$$

In particular, $I_a(1) = I_q(1)$ *if and only if* $\psi'(1) \le \psi(1)$. *Otherwise* $I_a(1) < I_q(1)$.

Quite surprisingly, we can exhibit elliptic environments on a regular tree for which the rate functions differ. This could hint that the uncertainty of the location of the first passage in [5] does not hold anymore for a random environment. Here is an explicit example. Consider a binary tree ($q_2 = 1$). Let A equal 0.01 with probability 0.8 and 500 with probability 0.2. Then we check that the walk is transient, but $\psi'(1) > \psi(1)$ so that $I_a(1) \ne I_q(1)$ on such an environment.

Theorem 1.2 exhibits a subexponential regime in the slowdown case when $i \le \nu_{\min}^{-1}$. The following theorem details this regime. Let

$$\mathbb{S}^e(\cdot) := \mathbb{Q}^e\big(\cdot | D(e) = \infty\big).$$

Theorem 1.4. *We place ourself in the case* $i < \nu_{\min}^{-1}$.

(i) *Suppose that either "*$i < \nu_{\min}^{-1}$ *and* $q_1 = 0$*" or "*$i < \nu_{\min}^{-1}$ *and* $s < 1$.*" There exist constants* $d_1, d_2 \in (0, 1)$ *such that for any* $a > 1/v$ *and n large enough*,

$$\mathrm{e}^{-n^{d_1}} < \mathbb{S}^e(\tau_n > an) < \mathrm{e}^{-n^{d_2}}. \tag{1.10}$$

(ii) *If* $q_1 > 0$ *and* $s > 1$ (*i.e. when* $\Lambda < \infty$), *the regime is polynomial and we have for any* $a > 1/v$,

$$\lim_{n\to\infty} \frac{1}{\ln(n)} \ln\big(\mathbb{S}^e(\tau_n > an)\big) = 1 - \Lambda. \tag{1.11}$$

We mention that in one dimension, which can be seen as a critical state of our model where $q_1 = 1$, such a polynomial regime is proved by Dembo et al. [6], our parameter Λ taking the place of the well-known κ of Kesten et al.

[9]. We did not deal with the critical case $i = \nu_{\min}^{-1}$. Furthermore, we do not have any conjecture on the optimal values of d_1 and d_2 and do not know if the two values are equal.

The rest of the paper is organized as follows. Section 2 describes the tail distribution of the first regeneration time, which is a preparatory step for the proof of the different theorems. Then we prove Theorems 1.1 and 1.2 in Section 3, which includes also the computation of the rate functions at speed 1 presented in Proposition 1.3. Section 4 is devoted to the subexponential regime with the proof of Theorem 1.4.

2. Moments of the first regeneration time

We define the first regeneration time

$$\Gamma_1 := \inf\bigl\{k > 0\colon\ \nu(X_k) \geq 2, D(X_k) = \infty, k = \tau_{|X_k|}\bigr\}$$

as the first time when the walk reaches a generation by a vertex having more than two children and never returns to its parent. We propose in this section to give information on the tail distribution of Γ_1 under $\mathbb{S}^e$. We first introduce some notation used throughout the paper. For any $x \in \mathbb{T}$, let

$$\begin{aligned} N(x) &:= \sum_{k\geq 0} \mathbb{1}_{\{X_k = x\}}, \qquad (2.1)\\ T_x &:= \inf\{k \geq 0\colon\ X_k = x\},\\ T_x^* &:= \inf\{k \geq 1\colon\ X_k = x\}. \end{aligned}$$

This permits to define

$$\begin{aligned} \beta(x) &:= P_\omega^x(T_{\overleftarrow{x}} = \infty),\\ \gamma(x) &:= P_\omega^x\bigl(T_{\overleftarrow{x}} = T_x^* = \infty\bigr). \qquad (2.2) \end{aligned}$$

The following fact can be found in [5] (Lemma 4.2) in the case of biased random walks, and is directly adaptable in our setting.

Fact A. *The first regeneration height $|X_{\Gamma_1}|$ admits exponential moments under the measure $\mathbb{S}^e(\cdot)$.*

2.1. *The case $i > \nu_{\min}^{-1}$*

This section is devoted to the case $i > \nu_{\min}^{-1}$, where Γ_1 is proved to have exponential moments.

Proposition 2.1. *Suppose that $i > \nu_{\min}^{-1}$. There exists $\theta > 0$ such that $E_{\mathbb{S}^e}[\mathrm{e}^{\theta\Gamma_1}] < \infty$.*

Proof. The proof follows the strategy of Proposition 1 of Piau [16]. We couple the distance of our RWRE to the root $(|X_n|)_{n\geq 0}$ with a biased random walk $(Y_n)_{n\geq 0}$ on $\mathbb{Z}$ as follows. Let $p := \frac{i\nu_{\min}}{1+i\nu_{\min}}$, and let $u_n, n \geq 0$, be a family of i.i.d. uniformly distributed [0, 1] random variables. We set $X_0 = e$ and $Y_0 = 0$. If X_k and Y_k are known, we construct

$$\begin{aligned} X_{k+1} &= x_i \quad \text{if } \sum_{\ell=1}^{i-1} \omega(x, x_\ell) \leq u_k < \sum_{\ell=1}^{i} \omega(x, x_\ell),\\ X_{k+1} &= \overleftarrow{x} \quad \text{otherwise},\\ Y_{k+1} &= y + 2\mathbb{1}_{\{u_k \leq p\}} - 1, \end{aligned}$$

where $x := X_k \in \mathbb{T}$ and $y := Y_k \in \mathbb{Z}$. Then $(X_n)_{n\geq 0}$ has the distribution of our $\mathbb{T}$-RWRE indeed, and $(Y_n)_{n\geq 0}$ is a random walk on $\mathbb{Z}$ which increases of one unit with probability $p > 1/2$ and decreases of the same value with probability $1-p$. Notice also that on the event $\{D(e)=\infty\}$, we have

$$|X_{k+1}| - |X_k| \geq Y_{k+1} - Y_k.$$

It implies that the first regeneration time $\mathcal{R}_1$ of $(Y_n)_{n\geq 0}$ defined by

$$\mathcal{R}_1 := \inf\{k > 0 \colon Y_\ell < Y_k \ \forall \ell < k, Y_m \geq Y_k \ \forall m > k\}$$

is necessarily a regeneration time for $(X_n, n \geq 0)$, which proves in turn that

$$\mathbb{S}^e(\Gamma_1 > n) \leq \mathbb{Q}^e(\mathcal{R}_1 > n).$$

To complete the proof, we must ensure that $\mathbb{Q}^e(\mathcal{R}_1 > n)$ is exponentially small, which is done in [6], Lemma 5.1. □

2.2. *The cases "$i < \nu_{\min}^{-1}$, $q_1 = 0$" and "$i < \nu_{\min}^{-1}$, $s < 1$"*

When $i < \nu_{\min}^{-1}$, if we assume also that $q_1 = 0$ or $s < 1$, we prove that Γ_1 has a subexponential tail. This situation covers, in particular, the case of RWRE on a regular tree.

Proposition 2.2. *Suppose that $i < \nu_{\min}^{-1}$ and $q_1 = 0$, then there exist $1 > \alpha_1 > \alpha_2 > 0$ such that for n large enough,*

$$\mathrm{e}^{-n^{\alpha_1}} < \mathbb{S}^e(\Gamma_1 > n) < \mathrm{e}^{-n^{\alpha_2}}. \tag{2.3}$$

The same relation holds with some $1 > \alpha_3 > \alpha_4 > 0$ in the case "$i < \nu_{\min}^{-1}$ and $s < 1$."

Proof of Proposition 2.2: lower bound. We only suppose that $i < \nu_{\min}^{-1}$, which allows us to deal with both cases of the lemma. Define for some $p' \in (0, 1/2)$ and $b \in \mathbb{N}$,

$$w_+ := \mathbf{Q}\left(\sum_{i=1}^{\nu} A(e_i) \geq \frac{1-p'}{p'}, \nu(e) \leq b\right),$$

$$w_- := \mathbf{Q}\left(\sum_{i=1}^{\nu} A(e_i) \leq \frac{p'}{1-p'}, \nu(e) \leq b\right).$$

By (1.2), $E_{\mathbf{Q}}[\sum_{i=1}^{\nu(e)} A(e_i)] > 1$ and therefore $\mathbf{Q}(\sum_{i=1}^{\nu(e)} A(e_i) > 1) > 0$. Since ess inf $A < \nu_{\min}^{-1}$, it guarantees that $\mathbf{Q}(\sum_{i=1}^{\nu(e)} A(e_i) < 1) > 0$. Consequently, by choosing p' close enough of $1/2$ and b large, we can take w_+ and w_- positive. Let $c := \frac{1}{6\ln(b)}$, and define $h_n := \lfloor c \ln(n) \rfloor$. A tree $\mathbb{T}$ is said to be n-good if:

- any vertex x of the h_n first generations verifies $\nu(x) \leq b$ and $\sum_{i=1}^{\nu(x)} A(x_i) \geq \frac{1-p'}{p'}$,
- any vertex x of the h_n following generations verifies $\nu(x) \leq b$ and $\sum_{i=1}^{\nu(x)} A(x_i) \leq \frac{p'}{1-p'}$.

We observe that $\mathbf{Q}(\mathbb{T} \text{ is } n\text{-good}) \geq w_+^{h_n b^{h_n}} w_-^{h_n b^{2h_n}} \geq \mathrm{e}^{-n^{1/3+\mathrm{o}(1)}}$ which is stretched exponential, i.e. behaving like $\mathrm{e}^{-n^{r+\mathrm{o}(1)}}$ for some $r \in (0,1)$. Define the events:

$$E_1 := \{\text{at time } \tau_{h_n} \text{ we cannot find an edge of level smaller than } h_n \text{ crossed only once}\} \cap \{D(e) > \tau_{h_n}\},$$

$$E_2 := \{\text{the walk visits the level } h_n\ n \text{ times before reaching the root or the level } 2h_n\},$$

$$E_3 := \{\text{after the } n\text{th visit of level } h_n, \text{ the walk reaches level } 2h_n \text{ before level } h_n\},$$

$$E_4 := \{\text{after time } \tau_{2h_n} \text{ the walk never comes back to level } 2h_n - 1\}.$$

Suppose that the tree is n-good. Since A is supposed bounded, there exists a constant $c_1 > 0$ such that for any x neighbour of y, we have

$$\omega(x, y) \geq \frac{c_1}{\nu(x)}. \tag{2.4}$$

It yields that $P^e_\omega(E_1)^{-1} = \mathrm{O}(n^K)$ for some $K > 0$ (where $\mathrm{O}(n^K)$ means that the function is bounded by a factor of $n \to n^K$). Combine (2.4) with the strong Markov property at time τ_{h_n} to see that

$$P^e_\omega(E_3 | E_1 \cap E_2)^{-1} = \mathrm{O}(n^K),$$

where K is taken large enough. We emphasize that the functions $\mathrm{O}(n^K)$ are deterministic. Still by Markov property,

$$P^e_\omega(E_1 \cap E_2 \cap E_3 \cap E_4) = E^e_\omega\big[\mathbb{1}_{E_1 \cap E_2 \cap E_3} \beta(X_{\tau_{2h_n}})\big]. \tag{2.5}$$

Let $(Y'_n)_{n\geq 0}$ be the random walk on $\mathbb{Z}$ starting from zero with

$$P_\omega\big(Y'_{n+1} = k+1 | Y'_n = k\big) = 1 - P_\omega\big(Y'_{n+1} = k-1 | Y'_n = k\big) = p'.$$

We introduce $T'_i := \inf\{k \geq 0\colon\ Y_k = i\}$, and p'_n the probability that $(Y'_n)_{n\geq 0}$ visits h_n before -1:

$$p'_n := P_\omega\big(T'_{-1} < T'_{h_n}\big).$$

By a coupling argument similar to that encountered in the proof of Proposition 2.1, we show that in an n-good tree,

$$P^e_\omega(E_1 \cap E_2) \geq P^e_\omega(E_1)(p'_n)^n = \mathrm{O}\big(n^K\big)^{-1}\big(p'_n\big)^n, \tag{2.6}$$

which gives

$$P^e_\omega(E_1 \cap E_2 \cap E_3) \geq \mathrm{O}\big(n^K\big)^{-1}\big(p'_n\big)^n. \tag{2.7}$$

Observing that $\mathbb{Q}^e(\Gamma_1 > n, D(e) = \infty) \geq E_{\mathbf{Q}}[\mathbb{1}_{\{\mathbb{T} \text{ is } n\text{-good}\}} \mathbb{1}_{E_1 \cap E_2 \cap E_3 \cap E_4}]$, we obtain by (2.5)

$$\begin{aligned}\mathbb{Q}^e\big(\Gamma_1 > n, D(e) = \infty\big) &\geq E_{\mathbb{Q}^e}\big[\mathbb{1}_{\{\mathbb{T} \text{ is } n\text{-good}\}} \mathbb{1}_{E_1 \cap E_2 \cap E_3} \beta(X_{\tau_{2h_n}})\big] \\ &= E_{\mathbb{Q}^e}\big[\mathbb{1}_{\{\mathbb{T} \text{ is } n\text{-good}\}} P^e_\omega(E_1 \cap E_2 \cap E_3)\big] E_{\mathbf{Q}}[\beta],\end{aligned}$$

by independence. By (2.7),

$$\mathbb{Q}^e\big(\Gamma_1 > n, D(e) = \infty\big) \geq \mathrm{O}(n^K)^{-1} \mathbf{Q}(\mathbb{T} \text{ is } n\text{-good})(p'_n)^n.$$

We already know that $\mathbf{Q}(\mathbb{T}$ is n-good) has a stretched exponential lower bound, and it remains to observe that the same holds for $(p'_n)^n$. But the method of gambler's ruin shows that $p'_n \geq 1 - (\frac{p'}{1-p'})^{h_n}$, which gives the required lower bound by our choice of h_n. $\square$

Let us turn to the upper bound. We divide the proof in two, depending on which case we deal with.

Proof of Proposition 2.2: upper bound in the case $q_1 = 0$. Assume that $q_1 = 0$ (the condition $i < \nu_{\min}^{-1}$ is not required in the proof). The proof of the following lemma is deferred. Recall the notation introduced in (2.2), $\gamma(e) := P^e_\omega(T_{\overleftarrow{e}} = T^*_e = \infty) \leq \beta(e)$.

Lemma 2.3. *When $q_1 = 0$, there exists a constant $c_2 \in (0, 1)$ such that for large n,*

$$E_{\mathbf{Q}}\big[\big(1 - \gamma(e)\big)^n\big] \leq \mathrm{e}^{-n^{c_2}}.$$

Denote by π_k the kth distinct site visited by the walk $(X_n, n \geq 0)$. We observe that

$$\begin{aligned}\mathbb{Q}^e(\Gamma_1 > n^3) \leq \mathbb{Q}^e(\Gamma_1 > \tau_n) + \mathbb{Q}^e\big(\text{more than } n^2 \text{ distinct sites are visited before } \tau_n\big) \\ + \mathbb{Q}^e\big(\exists k \leq n^2\colon\ N(\pi_k) > n\big).\end{aligned} \tag{2.8}$$

Since $\mathbb{Q}^e(\Gamma_1 > \tau_n) = \mathbb{Q}^e(|X_{\Gamma_1}| > n)$, it follows from Fact A that $\mathbb{Q}^e(\Gamma_1 > \tau_n)$ decays exponentially. For the second term of the right-hand side, beware that

$$\begin{aligned}&\mathbb{Q}^e\big(\text{more than } n^2 \text{ distinct sites are visited before } \tau_n\big) \\ &\quad \leq \sum_{k=1}^{n} \mathbb{Q}^e(\text{more than } n \text{ distinct sites are visited at level } k).\end{aligned}$$

If we denote by t_i^k the first time when the ith distinct site of level k is visited, we have, by the strong Markov property,

$$\begin{aligned}P_\omega^e(\text{more than } n \text{ sites are visited at level } k) &= P_\omega^e\big(t_n^k < \infty\big) \\ &\leq P_\omega^e\big(t_{n-1}^k < \infty, D(X_{t_{n-1}^k}) < \infty\big) \\ &= E_\omega^e\big[\mathbb{1}_{\{t_{n-1}^k < \infty\}}\big(1 - \beta(X_{t_{n-1}^k})\big)\big].\end{aligned}$$

The independence of the environments entails that

$$E_{\mathbb{Q}^e}\big[\mathbb{1}_{\{t_{n-1}^k < \infty\}}\big(1 - \beta(X_{t_{n-1}^k})\big)\big] = \mathbb{Q}^e\big(t_{n-1}^k < \infty\big) E_{\mathbf{Q}}[1 - \beta].$$

Consequently,

$$\begin{aligned}\mathbb{Q}^e\big(t_n^k < \infty\big) &\leq \mathbb{Q}^e\big(t_{n-1}^k < \infty\big) E_{\mathbf{Q}}[1 - \beta] \\ &\leq \big(E_{\mathbf{Q}}[1 - \beta]\big)^{n-1},\end{aligned} \tag{2.9}$$

which leads to

$$\mathbb{Q}^e\big(\text{more than } n^2 \text{ sites are visited before } \tau_n\big) \leq n\big(E_{\mathbf{Q}}[1 - \beta]\big)^{n-1}, \tag{2.10}$$

which is exponentially small. We remark, for later use, that Eq. (2.9) holds without the assumption $q_1 = 0$. For the last term of Eq. (2.9), we write

$$\mathbb{Q}^e\big(\exists k \leq n^2\colon\ N(\pi_k) > n\big) \leq \sum_{k=1}^{n^2} \mathbb{Q}^e\big(N(\pi_k) > n\big).$$

Let $U := \bigcup_{n \geq 0} (\mathbb{N}^*)^n$ be the set of words, where $(\mathbb{N})^0 := \{\varnothing\}$. Each vertex x of $\mathbb{T}$ is naturally associated with a word of U, and $\mathbb{T}$ is then a subset of U (see [13] for a more complete description). For any $k \geq 1$,

$$\begin{aligned}\mathbb{Q}^e\big(N(\pi_k) > n\big) &= \sum_{x \in U} \mathbb{Q}^e\big(x \in \mathbb{T}, N(x) > n, x = \pi_k\big) \\ &\leq \sum_{x \in U} E_{\mathbf{Q}}\big[\mathbb{1}_{\{x \in \mathbb{T}\}} P_\omega^e(x = \pi_k)\big(1 - \gamma(x)\big)^n\big],\end{aligned}$$

with the notation of (2.2). By independence,

$$\begin{aligned}\mathbb{Q}^e\big(N(\pi_k) > n\big) &\leq \sum_{x \in U} E_{\mathbf{Q}}\big[\mathbb{1}_{\{x \in \mathbb{T}\}} P_\omega^e(x = \pi_k)\big] E_{\mathbf{Q}}\big[\big(1 - \gamma(e)\big)^n\big] \\ &= E_{\mathbf{Q}}\big[\big(1 - \gamma(e)\big)^n\big].\end{aligned}$$

Apply Lemma 2.3 to complete the proof. □

Proof of Lemma 2.3. Let $\mu > 0$ be such that $q := \mathbf{Q}(\beta(e) > \mu) > 0$, and write

$$R := \inf\{k \geq 1: \exists |x| = k, \beta(x) \geq \mu\}.$$

Let x_R be such that $|x_R| = R$ and $\beta(x_R) \geq \mu$ and we suppose for simplicity that x_R is a descendant of e_1. We see that $\gamma(e) \geq \omega(e, e_1)\beta(e_1) \geq \frac{c_1}{\nu(e)}\beta(e_1)$ by Eq. (2.4). In turn, Eq. (2.1) of [1] implies that for any vertex x, we have

$$\frac{1}{\beta(x)} = 1 + \frac{1}{\sum_{i=1}^{\nu(x)} A(x_i)\beta(x_i)} \leq 1 + \frac{1}{\text{ess inf } A} \frac{1}{\beta(x_i)}$$

for any $1 \leq i \leq \nu(x)$. By recurrence on the path from e_1 to x_R, this leads to

$$\frac{1}{\beta(e_1)} \leq 1 + \frac{1}{\text{ess inf } A} + \cdots + \left(\frac{1}{\text{ess inf } A}\right)^{R-1} \frac{1}{\mu}.$$

We deduce the existence of constants $c_4, c_5 > 0$ such that

$$\gamma(e) \geq \frac{c_4}{\nu(e)} \mathrm{e}^{-c_5 R}. \tag{2.11}$$

It yields that

$$E_{\mathbf{Q}}\left[\left(1 - \gamma(e)\right)^n \mathbb{1}_{\{\nu(e) < \sqrt{n}\}}\right] \leq \mathbf{Q}\left(R > \frac{1}{4c_5} \ln(n)\right) + \mathrm{e}^{-n^{1/4+o(1)}}.$$

We observe that

$$\mathbf{Q}\left(R > \frac{1}{4c_5} \ln(n)\right) \leq \mathbf{Q}\left(\forall |x| = \frac{1}{4c_5} \ln(n), \beta(x) > \mu\right).$$

By assumption, $q_1 = 0$; thus $\#\{x \in \mathbb{T}: |x| = \frac{1}{4c_5} \ln(n)\} \geq 2^{1/4c_5 \ln(n)} =: n^{c_6}$. As a consequence, $\mathbf{Q}(\forall |x| = \frac{1}{4c_5} \ln(n), \beta(x) > \mu) \leq q^{n^{c_6}}$. Hence, the proof of our lemma is reduced to find a stretched exponential bound for $E_{\mathbf{Q}}[(1 - \gamma(e))^n \mathbb{1}_{\{\nu(e) \geq \sqrt{n}\}}]$. For any $x \in \mathbb{T}$, denote by V_x^μ the number of children x_i of x such that $\beta(x_i) > \mu$. For $\varepsilon \in (0, \mathbf{Q}(\beta(e) > \mu))$,

$$\begin{aligned}
&E_{\mathbf{Q}}\left[\left(1 - \gamma(e)\right)^n \mathbb{1}_{\{\nu(e) \geq \sqrt{n}\}}\right] \\
&\quad \leq \mathbb{Q}^e\left(\nu(e) \geq \sqrt{n}, V_e^\mu < \varepsilon \nu(e)\right) + E_{\mathbf{Q}}\left[\left(1 - \gamma(e)\right)^n \mathbb{1}_{\{V_e^\mu \geq \varepsilon \nu(e)\}}\right].
\end{aligned}$$

We apply Cramér's theorem to handle with the first term on the right-hand side. Turning to the second one, the bound is clear once we observe the general inequality,

$$\gamma(e) = \sum_{k=1}^{\nu(e)} \omega(e, e_k)\beta(e_k) \geq \frac{c_1}{\nu(e)} \sum_{k=1}^{\nu(e)} \beta(e_k) \geq \frac{c_1 \mu}{\nu(e)} V_e^\mu, \tag{2.12}$$

which is greater than $c_1 \mu \varepsilon$ on $\{V_e^\mu \geq \varepsilon \nu(e)\}$. □

Remark 2.3. *As a by-product, we obtain that* $E_{\mathbf{Q}}[(1 - \gamma(e))^n \mathbb{1}_{\{\nu(e) \geq \sqrt{n}\}}] \leq \mathrm{e}^{-n^{c_3}}$ *without the assumption* $q_1 = 0$.

Proof of Proposition 2.2: upper bound in the case *s < 1*. We follow the strategy of the case "$q_1 = 0$". The proof boils down to the estimate of

$$\begin{aligned}
&\mathbb{Q}^e\left(N(\pi_k) > n, D(e) = \infty\right) \\
&\quad = \mathbb{Q}^e\left(N(\pi_k) > n, \nu(\pi_k) < \sqrt{n}, D(e) = \infty\right) + \mathbb{Q}^e\left(N(\pi_k) > n, \nu(\pi_k) \geq \sqrt{n}, D(e) = \infty\right).
\end{aligned}$$

Let $x \in \mathbb{T}$ and consider the RWRE $(X_n, n \geq 0)$ when starting from $\overleftarrow{x}$. Inspired by Lyons et al. [12], we propose to couple it with a random walk $(Y''_n, n \geq 0)$ on $\mathbb{Z}$. We first define X''_n as the restriction of X_n on the path $[\![\overleftarrow{e}, x]\!]$. Beware that X''_n exists only up to a time T, which corresponds to the time when the walk $(X_n, n \geq 0)$ escapes the path $[\![\overleftarrow{e}, x]\!]$, id est leaves the path and never comes back to it. After this time, we set $X''_n = \Delta$ for some Δ in some space $\mathcal{E}$. Then $(X''_n)_{n\geq 0}$ is a random walk on $[\![\overleftarrow{e}, x]\!] \cup \{\Delta\}$, whose transition probabilities are, if $y \notin \{\overleftarrow{e}, x, \Delta\}$,

$$P_\omega^{\overleftarrow{x}}\big(X''_{n+1} = y_+ | X''_n = y\big) = \frac{\omega(y, y_+)}{\omega(y, y_+) + \omega(y, \overleftarrow{y}) + \sum_{y_k \neq y_+} \omega(y, y_k)\beta(y_k)},$$

$$P_\omega^{\overleftarrow{x}}\big(X''_{n+1} = \overleftarrow{y} | X''_n = y\big) = \frac{\omega(y, \overleftarrow{y})}{\omega(y, y_+) + \omega(y, \overleftarrow{y}) + \sum_{y_k \neq y_+} \omega(y, y_k)\beta(y_k)},$$

$$P_\omega^{\overleftarrow{x}}\big(X''_{n+1} = \Delta | X''_n = y\big) = \frac{\sum_{k=1}^{\nu(y)} \omega(y, y_k)\beta(y_k)}{\omega(y, y_+) + \omega(y, \overleftarrow{y}) + \sum_{y_k \neq y_+} \omega(y, y_k)\beta(y_k)},$$

where y_+ is the child of y which lies on the path $[\![\overleftarrow{e}, x]\!]$. Besides, the walk is absorbed on Δ and reflected on $\overleftarrow{e}$ and x. We recall that $s := \operatorname{ess\,sup} A$. We construct the adequate coupling with a biased random walk $(Y''_n)_{n\geq 0}$ on $\mathbb{Z}$, starting from $|x| - 1$, increasing with probability $s/(1+s)$, decreasing otherwise and such that $Y''_n \geq |X''_n|$ as long as $X''_n \neq \Delta$ (which is always possible since $P_\omega(X''_{n+1} = y_+ | X''_n = y) \leq \frac{s}{1+s}$). After time T, we let Y_n move independently. By coupling and then by gambler's ruin method, it leads to

$$P_\omega^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}}) \leq P_\omega^{|x|-1}\big(\exists n \geq 0\colon\ Y''_n = |x|\big) = s.$$

It follows that

$$1 - P_\omega^x\big(T_x^* < T_{\overleftarrow{e}}\big) \geq \omega(x, \overleftarrow{x})\big(1 - P_\omega^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}})\big) \geq \frac{c_1(1-s)}{\nu(x)},$$

by Eq. (2.4). Hence,

$$\begin{aligned}
&\mathbb{Q}^e\big(N(\pi_k) > n, \nu(\pi_k) \leq \sqrt{n}, D(e) = \infty\big) \\
&\quad = \sum_{x \in U} E_{\mathbf{Q}}\big[\mathbb{1}_{\{\nu(x) \leq \sqrt{n}\}} P_\omega^e\big(x = \pi_k, D(e) > T_x\big) P_\omega^x\big(N(x) > n, D(e) = \infty\big)\big] \\
&\quad \leq \sum_{x \in U} E_{\mathbf{Q}}\bigg[P_\omega^e(x = \pi_k)\bigg(1 - \frac{c_1(1-s)}{\sqrt{n}}\bigg)^n\bigg] = \bigg(1 - \frac{c_1(1-s)}{\sqrt{n}}\bigg)^n,
\end{aligned}$$

which decays stretched exponentially. On the other hand,

$$\begin{aligned}
&\mathbb{Q}^e\big(N(\pi_k) > n, \nu(\pi_k) \geq \sqrt{n}, D(e) = \infty\big) \\
&\quad \leq \mathbb{Q}^e\big(\nu(\pi_k) \geq \sqrt{n}, V_{\pi_k}^\mu < \varepsilon\nu(\pi_k)\big) + \mathbb{Q}^e\big(N(\pi_k) > n, V_{\pi_k}^\mu \geq \varepsilon\nu(\pi_k)\big)
\end{aligned}$$

with the notation introduced in the proof of Lemma 2.3. We have

$$\mathbb{Q}^e\big(\nu(\pi_k) \geq \sqrt{n}, V_{\pi_k}^\mu < \varepsilon\nu(\pi_k)\big) = \mathbf{Q}\big(\nu(e) \geq \sqrt{n}, V_e^\mu < \varepsilon\nu(e)\big),$$

which is stretched exponential by Cramér's theorem. We also observe that

$$\begin{aligned}
\mathbb{Q}^e\big(N(\pi_k) > n, V_{\pi_k}^\mu \geq \varepsilon\nu(\pi_k)\big) &\leq E_{\mathbb{Q}^e}\big[\mathbb{1}_{\{V_{\pi_k}^\mu \geq \varepsilon\nu(x)\}}\big(1 - \gamma(\pi_k)\big)^n\big] \\
&= E_{\mathbf{Q}}\big[\mathbb{1}_{\{V_e^\mu \geq \varepsilon\nu(x)\}}\big(1 - \gamma(e)\big)^n\big] \leq (1 - c\mu\varepsilon)^n,
\end{aligned}$$

by Eq. (2.12). This completes the proof. □

2.3. *The case $\Lambda < \infty$*

In this part, we suppose that $\Lambda < \infty$, where Λ is defined by

$$\Lambda := Leb\left\{t \in \mathbb{R}\colon \mathbf{E}\left[A^t\right] \leq \frac{1}{q_1}\right\}.$$

We prove that the tail distribution of Γ_1 is polynomial.

Proposition 2.4. *If $\Lambda < \infty$, then*

$$\lim_{n\to\infty} \frac{1}{\ln(n)} \ln\bigl(\mathbb{S}^e(\Gamma_1 > n)\bigr) = -\Lambda. \tag{2.13}$$

Proof. Lemma 3.3 of [1] already gives

$$\liminf_{n\to\infty} \frac{1}{\ln(n)} \ln\bigl(\mathbb{S}^e(\Gamma_1 > n)\bigr) \geq -\Lambda.$$

Hence, the lower bound of (2.13) is known. The rest of the section is dedicated to the proof of the upper bound.

We start with three preliminary lemmas. We first prove an estimate for one-dimensional RWRE, that will be useful later on. Denote by $(R_n, n \geq 0)$ a generic RWRE on $\mathbb{Z}$ such that the random variables $A(i)$, $i \geq 0$ are independent and have the distribution of A, when we set for $i \geq 0$,

$$A(i) := \frac{\omega_R(i, i+1)}{\omega_R(i, i-1)}$$

with $\omega_R(y, z)$ the quenched probability to jump from y to z. We denote by $P^k_{\omega,R}$ the quenched distribution associated with $(R_n, n \geq 0)$ when starting from k, and by $\mathbf{P}_R$ the distribution of the environment ω_R. Let $c_7 \in (0, 1)$ be a constant whose value will be given later on. For any $k \geq \ell \geq 0$ and $n \geq 0$, we introduce the notation

$$p(\ell, k, n) := E_{\mathbf{P}_R}\bigl[\bigl(1 - c_7 P^\ell_{\omega,R}\bigl(T^*_\ell > T_0 \wedge T_k\bigr)\bigr)^n\bigr]. \tag{2.14}$$

Lemma 2.5. *Let $0 < r < 1$, and $\Lambda_r := Leb\{t \in \mathbb{R}\colon \mathbf{E}[A^t] \leq \frac{1}{r}\}$. Then, for any $\varepsilon > 0$, we have for n large enough,*

$$\sum_{k\geq\ell\geq 0} r^k p(\ell, k, n) \leq n^{-\Lambda_r+\varepsilon}.$$

Proof. The method used is very similar to that of Lemma 5.1 in [1]. We feel free to present a sketch of the proof. We consider the one-dimensional RWRE $(R_n)_{n\geq 0}$. We introduce for $k \geq \ell \geq 0$, the potential $V(0) = 0$ and

$$V(\ell) = -\sum_{i=0}^{\ell-1} \ln\bigl(A(i)\bigr),$$

$$H_1(\ell) = \max_{0\leq i\leq \ell} V(i) - V(\ell),$$

$$H_2(\ell, k) = \max_{\ell\leq i\leq k} V(i) - V(\ell).$$

We know (e.g. [17]) that

$$\frac{\mathrm{e}^{-H_2(\ell+1,k)}}{k+1} \leq P^{\ell+1}_{\omega,R}(T_k < T_\ell) \leq \mathrm{e}^{-H_2(\ell+1,k)}, \tag{2.15}$$

$$\frac{\mathrm{e}^{-H_1(\ell)}}{k+1} \leq P^{\ell-1}_{\omega,R}(T_{-1} < T_\ell) \leq \mathrm{e}^{-H_1(\ell)}. \tag{2.16}$$

It yields that

$$P^{\ell}_{\omega,R}\big(T^*_\ell > T_0 \wedge T_k\big) \geq \mathrm{e}^{-H_1(\ell)\wedge H_2(\ell,k)+\mathrm{O}(\ln k)},$$

where $\mathrm{O}(\ln k)$ is a deterministic function. Let $\eta \in (0,1)$.

$$\begin{aligned} p(\ell,k,n) &\leq \big(1-c_7 n^{-1+\eta}\big)^n + \mathbf{P}_R\big(H_1(\ell)\wedge H_2(\ell,k) - \mathrm{O}(\ln k) \geq (1-\eta)\ln(n)\big) \\ &\leq \mathrm{e}^{-c_8 n^{\eta}} + \mathbf{P}_R\big(H_1(\ell)\wedge H_2(\ell,k) - \mathrm{O}(\ln k) \geq (1-\eta)\ln(n)\big). \end{aligned}$$

In Section 8.1 of [1], we proved that for any $s \in (0,1)$, $E_{\mathbf{P}_R}[\mathrm{e}^{\Lambda_s(H_1(\ell)\wedge H_2(\ell,k))}] \leq \mathrm{e}^{k\ln(1/s)+\mathrm{o}_s(k)}$, where $\mathrm{o}_s(k)$ is such that $\mathrm{o}_s(k)/k$ tends to 0 at infinity. This implies that, defining $\widetilde{\mathrm{o}}_s(k) := \mathrm{o}_s(k) - \Lambda_s \mathrm{O}(\ln k)$,

$$\begin{aligned} & s^k \mathbf{P}_R\big(H_1(\ell)\wedge H_2(\ell,k) - \mathrm{O}(\ln k) \geq (1-\eta)\ln(n)\big) \\ &\quad \leq s^k\big(1 \wedge \mathrm{e}^{k\ln(1/s) - \Lambda_s(1-\eta)\ln(n) + \widetilde{\mathrm{o}}_s(k)}\big) \\ &\quad \leq n^{-\Lambda_s(1-\eta)} \exp\big(\big(k\ln(s) + \Lambda_s(1-\eta)\ln(n)\big) \wedge \widetilde{\mathrm{o}}_s(k)\big). \end{aligned}$$

Observe that there exists M_s such that for any k and any n, we have $(k\ln(s) + \Lambda_s(1-\eta)\ln(n)) \wedge \widetilde{\mathrm{o}}_s(k) \leq \sup_{i\leq M_s \ln(n)} \widetilde{\mathrm{o}}(i) + \eta \ln n$, and notice that $\sup_{i \leq M_s \ln(n)} \widetilde{\mathrm{o}}_s(i)$ is negligible towards $\ln(n)$. This leads to, for n large enough,

$$s^k p(\ell,k,n) \leq s^k \mathrm{e}^{-c_8 n^{\eta}} + n^{-\Lambda_s(1-\eta)+2\eta}.$$

Let $r \in (0,1)$ and $s > r$. We have

$$r^k p(\ell,k,n) \leq r^k \mathrm{e}^{-c_8 n^{\eta}} + \left(\frac{r}{s}\right)^k n^{-\Lambda_s(1-\eta)+2\eta}.$$

Lemma 2.5 follows by choosing η small enough and s close enough to r. $\square$

Let Z_n represent the size of the nth generation of the tree $\mathbb{T}$. We have the following result.

Lemma 2.6. *There exists a constant $c_9 > 0$ such that for any $H > 0$, $B > 0$ and n large enough*,

$$E_{\mathbf{Q}}\big[\big(1-\gamma(e)\big)^n \mathbb{1}_{\{Z_H > B\}}\big] \leq n^{-c_9 B}.$$

Proof. We have

$$\begin{aligned} E_{\mathbf{Q}}\big[\big(1-\gamma(e)\big)^n \mathbb{1}_{\{Z_H>B\}}\big] &\leq E_{\mathbf{Q}}\big[\big(1-\gamma(e)\big)^n \mathbb{1}_{\{\nu(e)\geq\sqrt{n}\}}\big] + E_{\mathbf{Q}}\big[\big(1-\gamma(e)\big)^n \mathbb{1}_{\{Z_H>B,\nu(e)\leq\sqrt{n}\}}\big] \\ &\leq e^{-n^{c_3}} + E_{\mathbf{Q}}\big[\big(1-\gamma(e)\big)^n \mathbb{1}_{\{Z_H>B,\nu(e)\leq\sqrt{n}\}}\big] \end{aligned}$$

by Remark 2.3. When $\nu(e) \leq \sqrt{n}$, we have, by (2.11),

$$\gamma(e) \geq \frac{c_4}{\sqrt{n}} \mathrm{e}^{-c_5 R},$$

with $R := \inf\{k \geq 1\colon \exists |x| = k, \beta(x) \geq \mu\}$ as before ($\mu > 0$ is such that $q := \mathbf{Q}(\beta(e) > \mu) > 0$). Thus,

$$E_{\mathbf{Q}}\big[\big(1-\gamma(e)\big)^n \mathbb{1}_{\{Z_H>B,\nu(e)\leq\sqrt{n}\}}\big] \leq \mathbf{Q}\left(R > \frac{1}{4c_5}\ln(n) + H, Z_H > B\right) + \mathrm{e}^{-n^{1/4+\mathrm{o}(1)}}.$$

By considering the Z_H subtrees rooted at each of the individuals in generation H, we see that

$$\begin{aligned} \mathbf{Q}\big(R > c_{10}\ln(n) + H, Z_H > B\big) &= E_{GW}\big[\mathbf{Q}\big(R > c_{10}\ln(n)\big)^{Z_H} \mathbb{1}_{\{Z_H>B\}}\big] \\ &\leq \mathbf{Q}\big(R > c_{10}\ln(n)\big)^B. \end{aligned}$$

If $R > c_{10}\ln(n)$, we have in particular $\beta(x) < \mu$ for each $|x| = c_{10}\ln(n)$ which implies that

$$\mathbf{Q}\big(R > c_{10}\ln(n) + H, Z_H > B\big) \le E_{GW}\big[q^{Z_{c_{10}\ln(n)}}\big]^B.$$

Let $t \in (q_1, 1)$. For n large enough, $E_{GW}[q^{Z_{c_{10}\ln(n)}}] \le t^{c_{10}\ln(n)} = n^{c_{10}\ln(t)}$, ($E_{GW}[q^{Z_n}]/q_1^n$ has a positive limit by Corollary 1, page 40 of [2]). The lemma follows. □

Let $r \in (q_1, 1)$, $\varepsilon > 0$, B be such that

$$c_9 B\varepsilon > 2\Lambda \tag{2.17}$$

and H large enough so that

$$GW(Z_H \le B) < r^H \frac{1}{B} < 1. \tag{2.18}$$

In particular, $c_{11} := GW(Z_H > B) > 0$.

Let $\nu(x,k)$ denote for any $x \in \mathbb{T}$ the number of descendants of x at generation $|x| + k$ ($\nu(x,1) = \nu(x)$), and let

$$\mathcal{S}_H := \big\{x \in \mathbb{T}\colon\ \nu(x,H) > B\big\}. \tag{2.19}$$

For any $x \in \mathbb{T}$, we call $F(x)$ the youngest ancestor of x which lies in $\mathcal{S}_H$, and $G(x)$ an oldest descendant of x in $\mathcal{S}_H$. For any $x, y \in \mathbb{T}$, we write $x \le y$ if y is a descendant of x and $x < y$ if besides $x \ne y$. We define for any $x \in \mathbb{T}$, $W(x)$ as the set of descendants y of x such that there exists no vertex z with $x < z \le y$ and $\nu(z,H) > B$. In other words, $W(x) = \{y\colon\ y \ge x, F(y) \le x\}$. We define also

$$\overset{\circ}{W}(x) := W(x)\backslash\{x\},$$
$$\partial W(x) := \big\{y\colon\ \overleftarrow{y} \in W(x), \nu(y,H) > B\big\}.$$

Finally, let $W_j(e) := \{x\colon\ |x| = j, x \in W(e)\}$.

Lemma 2.7. *Recall that $m := E_{GW}[\nu(e)]$ and r is a real belonging to $(q_1, 1)$. We also recall that H and B verify $GW(Z_H \le B) < r^H \frac{1}{B}$. We have for any $j \ge 0$,*

$$E_{GW}\big[W_j(e)\big] < m r^{j-1}.$$

Proof. We construct the subtree $\mathbb{T}_H$ of the tree $\mathbb{T}$ by retaining only the generations kH, $k \ge 0$ of the tree $\mathbb{T}$. Let

$$\mathbb{W} = \mathbb{W}(\mathbb{T}) := \big\{x \in \mathbb{T}_H\colon\ \forall y \in \mathbb{T}_H, (y < x) \Rightarrow \nu(y,H) \le B\big\}. \tag{2.20}$$

The tree $\mathbb{W}$ is a Galton–Watson tree whose offspring distribution is of mean $E_{GW}[Z_H \mathbb{1}_{\{Z_H \le B\}}] \le B \times GW(Z_H \le B) \le r^H$ by (2.18). Then for each child e_i of e (in the original tree $\mathbb{T}$), let $\mathbb{W}^i := \mathbb{W}(\mathbb{T}_{e_i})$ where $\mathbb{T}_{e_i}$ is the subtree rooted at e_i. We conclude by observing that $W_j(e) \le \sum_{i=1}^{\nu(e)} \#\{x \in \mathbb{W}^i\colon\ |x| = 1 + \lceil (j-1)/H \rceil \times H\}$ hence $E_{GW}[W_j(e)] \le E_{GW}[\nu(e)] r^{j-1}$. □

We still have $r \in (q_1, 1)$ and $\varepsilon > 0$. We prove that for n large enough, and r and ε close enough to q_1 and 0, we have

$$\mathbb{Q}^e\big(\Gamma_1 > n, D(e) = \infty\big) \le c_{12} n^{-(1-2\varepsilon)\Lambda_r + 3\varepsilon}, \tag{2.21}$$

where $\Lambda_r := Leb\{t \in \mathbb{R}\colon\ \mathbf{E}[A^t] \le \frac{1}{r}\}$ as in Lemma 2.5. This suffices to prove Proposition 2.4 since ε and Λ_r can be arbitrarily close to 0 and Λ, respectively. We recall that we defined B, H and $\mathcal{S}_H$ in (2.17)–(2.19).

The strategy is to divide the tree in subtrees in which vertices are constrained to have a small number of children (at most B children at generation H). With $B = H = 1$, we would have literally pipes. In general, the traps constructed are slightly larger than pipes. We then evaluate the time spent in such traps by comparison with a one-dimensional random walk. We define π_k^s as the kth distinct site visited in the set $\mathcal{S}_H$. We observe that

$$\begin{aligned}
&\mathbb{Q}^e\big(\Gamma_1 > n, D(e) = \infty\big) \\
&\quad \le \mathbb{Q}^e(\Gamma_1 > \tau_{\ln^2(n)}) + \mathbb{Q}^e\big(\text{more than } \ln^4(n) \text{ distinct sites are visited before } \tau_{\ln^2(n)}\big) \\
&\qquad + \mathbb{Q}^e\big(\exists k \le \ln^4(n), \exists x \in W\big(\pi_k^s\big), N(x) > n/\ln^4(n)\big) \\
&\qquad + \mathbb{Q}^e\big(\exists x \in W(e), N(x) > n/\ln^4(n), D(e) = \infty, Z_H \le B\big).
\end{aligned} \tag{2.22}$$

The first term on the right-hand side decays like $\mathrm{e}^{-\ln^2(n)}$ by Fact A, and so does the second term by equation (2.9). We proceed to estimate the third term on the right-hand side of (2.22). Since

$$\mathbb{Q}^e\big(\exists k \le \ln^4(n), \exists x \in W\big(\pi_k^s\big), N(x) > n/\ln^4(n)\big) \le \sum_{k=1}^{\ln^4(n)} \mathbb{Q}^e\big(\exists x \in W\big(\pi_k^s\big), N(x) > n/\ln^4(n)\big)$$

we look at the rate of decay of $\mathbb{Q}^e(\exists x \in W(\pi_k^s), N(x) > n/\ln^4(n))$ for any $k \ge 1$. We first show that the time spent at the frontier of $W(\pi_k^s)$ will be negligible. Precisely, we show

$$\mathbb{Q}^e\big(N\big(\pi_k^s\big) > n^\varepsilon\big) \le c_{14} n^{-2\Lambda}, \tag{2.23}$$

$$\mathbb{Q}^e\big(\exists z \in \partial W\big(\pi_k^s\big), N(z) > n^\varepsilon\big) \le c_{15} n^{-2\Lambda}. \tag{2.24}$$

As $P_\omega^y(N(y) > n^\varepsilon) \le (1 - \gamma(y))^{n^\varepsilon}$ for any $y \in \mathbb{T}$, we have,

$$\begin{aligned}
\mathbb{Q}^e\big(N\big(\pi_k^s\big) > n^\varepsilon\big) &= E_{\mathbf{Q}}\Big[\sum_{y \in \mathcal{S}_H} P_\omega^e\big(\pi_k^s = y\big) P_\omega^y\big(N(y) > n^\varepsilon\big)\Big] \\
&\le E_{\mathbf{Q}}\Big[\sum_{y \in \mathcal{S}_H} P_\omega^e\big(\pi_k^s = y\big)\big(1 - \gamma(y)\big)^{n^\varepsilon}\Big].
\end{aligned} \tag{2.25}$$

We would like to split the expectation $E_{\mathbf{Q}}[P_\omega^e(\pi_k^s = y)(1 - \gamma(y))^{n^\varepsilon}]$ in two. However the random variable $P_\omega^e(\pi_k^s = y)$ depends on the structure of the first H generations of the subtree rooted at y. Nevertheless, we are going to show that, for some $c_{14} > 0$,

$$E_{\mathbf{Q}}\big[P_\omega^e\big(\pi_k^s = y\big)\big(1 - \gamma(y)\big)^{n^\varepsilon}\big] \le c_{14} E_{\mathbf{Q}}\big[P_\omega^e\big(\pi_k^s = y\big)\big] E_{\mathbf{Q}}\big[\big(1 - \gamma(y)\big)^{n^\varepsilon} | \nu(y, H) > B\big].$$

Let $U := \bigcup_{n \ge 0} (\mathbb{N}^*)^n$ be, as before, the set of words. We have seen that U allows us to label the vertices of any tree (see [13]). Let $y \in U$ and let ω_y represent the restriction of the environment ω to the outside of the subtree rooted at y (when y belongs to the tree). For $1 \le L \le H$, we denote by y_L the ancestor of y such that $|y_L| = |y| - L$. We attach to each y_L the variable $\zeta(y_L) := \mathbb{1}_{\{\nu(y_L, H) > B\}}$. We notice that there exists a measurable function f such that $P_\omega^e(\pi_k^s = y) = f(\omega_y, \zeta)\mathbb{1}_{\{\nu(y,H) > B\}}$ where $\zeta := (\zeta(y_L))_{1 \le L \le H}$. Let $\mathcal{E}(\omega_y) := \{e \in \{0,1\}^H\colon\ \mathbf{Q}(\zeta = e | \omega_y) > 0\}$. We have

$$E_{\mathbf{Q}}\big[f(\omega_y, \zeta) | \omega_y\big] \ge \max_{e \in \mathcal{E}(\omega_y)} f(\omega_y, e)\mathbf{Q}(\zeta = e | \omega_y).$$

We claim that there exists a constant $c_{13} > 0$ such that for almost every ω and any $e \in \mathcal{E}(\omega_y)$,

$$\mathbf{Q}(\zeta = e | \omega_y) \ge c_{13}.$$

Let us prove the claim. If ω_y is such that $\nu(\overleftarrow{y}) > B$, then $\mathcal{E}(\omega_y) = \{(1, \dots, 1)\}$ and $\mathbf{Q}(\zeta = e|\omega_y) = 1$. Therefore suppose $\nu(\overleftarrow{y}) \le B$ and let $h := \max\{1 \le L \le H \colon \nu(y_L, L) \le B\}$. We observe that, for any $e \in \mathcal{E}(\omega_y)$, we necessarily have $e_L = 1$ for $h < L \le H$. We are reduced to the study of

$$\mathbf{Q}(\zeta = e|\omega_y) = \mathbf{Q}\biggl(\bigcap_{1 \le L \le h} \{\zeta(y_L) = e_L\} |\omega_y\biggr).$$

For any tree $\mathcal{T}$, we denote by $\mathcal{T}^j$ the restriction to the j first generations. Let also $\mathbb{T}_{y_h}$ designate the subtree rooted at y_h in $\mathbb{T}$. Since $\nu(y_h, h) \le B$, we observe that $\mathbb{T}_{y_h}^h$ belongs almost surely to a finite (deterministic) set in the space of all trees. We construct the set

$$\begin{aligned}\Psi\bigl(\mathbb{T}_{y_h}^h, e\bigr) := \bigl\{&\text{tree } \mathcal{T} : \mathcal{T}^h = \mathbb{T}_{y_h}^h, GW\bigl(\mathcal{T}^{h+H}\bigr) > 0, \forall |x| \le 2H, \nu_{\mathcal{T}}(x) \le B \\ &\forall 1 \le L \le h, \nu_{\mathcal{T}}(y_L, h) > B \text{ if and only if } e_L = 1\bigr\}.\end{aligned}$$

We observe that $\Psi(\mathbb{T}_{y_K}^K, e) \neq \varnothing$ as soon as $e \in \mathcal{E}(\omega_y)$. Let $\widetilde{\Psi}(\mathbb{T}_{y_K}^K, e) := \{\mathcal{T}^{h+H}, \mathcal{T} \in \Psi(\mathbb{T}_{y_h}^h, e)\}$ be the same set but where the trees are restricted to the first $h + H$ generations. Since $\widetilde{\Psi}(\mathbb{T}_{y_K}^K, e)$ is again included in a finite deterministic set in the space of trees, we deduce that there exists $c_{13} > 0$ such that, almost surely,

$$\inf\bigl\{GW\bigl(\mathcal{T}^{h+H} | \mathcal{T}^h\bigr), \mathcal{T} \in \Psi\bigl(\mathbb{T}_{y_h}^h, e\bigr), e \in \mathcal{E}(\omega_y)\bigr\} \ge c_{13}.$$

Consequently,

$$\mathbf{Q}(\zeta = e|\omega_y) \ge \mathbf{Q}\bigl(\mathbb{T}_{y_h}^{h+H} \in \widetilde{\Psi}\bigl(\mathbb{T}_{y_h}^h, e\bigr)|\omega_y\bigr) \ge c_{13},$$

as required. We get

$$E_{\mathbf{Q}}\bigl[f(\omega_y, \zeta)|\omega_y\bigr] \ge c_{13} \max_{e \in \mathcal{E}(\omega_y)} f(\omega_y, e) \ge c_{13} f(\omega_y, \zeta).$$

Finally we obtain, with $c_{14} := \frac{1}{c_{13}}$,

$$f(\omega_y, \zeta) \le c_{14} E_{\mathbf{Q}}\bigl[f(\omega_y, \zeta)|\omega_y\bigr].$$

By (2.26), it entails that

$$\begin{aligned}\mathbb{Q}^e\bigl(N\bigl(\pi_k^s\bigr) > n^\varepsilon\bigr) &\le c_{14} \sum_{y \in U} E_{\mathbf{Q}}\bigl[\mathbb{1}_{\{\nu(y,H) > B\}} E_{\mathbf{Q}}\bigl[f(\omega_y, \zeta)|\omega_y\bigr]\bigl(1 - \gamma(y)\bigr)^{n^\varepsilon}\bigr] \\ &= c_{14} \sum_{y \in U} E_{\mathbf{Q}}\bigl[f(\omega_y, \zeta)\bigr] E_{\mathbf{Q}}\bigl[\mathbb{1}_{\{\nu(e,H) > B\}}\bigl(1 - \gamma(e)\bigr)^{n^\varepsilon}\bigr] \\ &= c_{14} \sum_{y \in U} E_{\mathbf{Q}}\bigl[P_\omega^e\bigl(\pi_k^s = y\bigr)\bigr] E_{\mathbf{Q}}\bigl[\bigl(1 - \gamma(e)\bigr)^{n^\varepsilon} | \nu(e, H) > B\bigr].\end{aligned}$$

It implies that

$$\mathbb{Q}^e\bigl(N\bigl(\pi_k^s\bigr) > n^\varepsilon\bigr) \le c_{14} E_{\mathbf{Q}}\bigl[\bigl(1 - \gamma(e)\bigr)^{n^\varepsilon} | Z_H > B\bigr] \le c_{14} n^{-c_9 \varepsilon B},$$

by Lemma 2.6. Since $c_9 \varepsilon B > 2\Lambda$, this leads to, for n large,

$$\mathbb{Q}^e\bigl(N\bigl(\pi_k^s\bigr) > n^\varepsilon\bigr) \le c_{14} n^{-2\Lambda}$$

which is Eq. (2.23). Similarly, recalling that $\partial W(y)$ designates the set of vertices z such that $\overleftarrow{z} \in W(y)$ and $\nu(z,H) > B$, we have that

$$\begin{aligned}
&\mathbb{Q}^e\big(\exists y \in \partial W\big(\pi_k^s\big), N(y) > n^\varepsilon\big)\\
&\quad\le E_{\mathbf{Q}}\bigg[\sum_{y\in\mathcal{S}_H} P_\omega^e\big(\pi_k^s = y\big) \sum_{z\in\partial W(y)} \big(1-\gamma(z)\big)^{n^\varepsilon}\bigg]\\
&\quad\le c_{14} E_{\mathbf{Q}}\bigg[\sum_{y\in\mathcal{S}_H} P_\omega^e\big(\pi_k^s = y\big)\bigg] E_{GW}\big[\partial W(e)\big] E_{\mathbf{Q}}\big[\big(1-\gamma(e)\big)^{n^\varepsilon} \,|\, Z_H > B\big]\\
&\quad= c_{14} E_{GW}\big[\partial W(e)\big] E_{\mathbf{Q}}\big[\big(1-\gamma(e)\big)^{n^\varepsilon} \,|\, Z_H > B\big].
\end{aligned}$$

We notice that $E_{GW}[\partial W] \le E_{GW}[\sum_{x\in W(e)} \nu(x)] = m E_{GW}[W(e)]$ which is finite by Lemma 2.7. It yields, by Lemma 2.6,

$$\mathbb{Q}^e\big(\exists x \in W\big(\pi_k^s\big), N\big(G(x)\big) > n^\varepsilon\big) \le c_{15} n^{-2\Lambda}$$

thus proving (2.24). Our next step is then to find an upper bound to the probability to spend most of our time at a vertex x belonging to some $\mathring{W}(y)$. To this end, recall that $G(x)$ is an oldest descendant of x such that $\nu(x,H) > B$. We have just proved that the time spent at $y (= F(x))$ or $G(x)$ is negligible. Therefore, starting from x, the probability to spend much time in x is not far from the probability to spend the same time without reaching y neither $G(x)$. Then, this probability is bound by coupling with a one-dimensional random walk.

Define $\widetilde{T}_x^{(\ell)}$ as the ℓth time the walk visits x after visiting either $F(x)$ or $G(x)$, i.e. $\widetilde{T}_x^{(1)} = T_x$ and,

$$\widetilde{T}_x^{(\ell)} := \inf\big\{k > \widetilde{T}_x^{(\ell-1)}\colon\ X_k = x, \exists i \in \big(\widetilde{T}_x^{(\ell-1)}, k\big), X_i = F(x) \text{ or } G(x)\big\}.$$

Let also $N^{(\ell)}(x) = \sum_{k=\widetilde{T}^{(\ell)}(x)}^{\widetilde{T}^{(\ell+1)}(x)-1} \mathbb{1}_{\{X_k = x\}}$ be the time spent at x between $\widetilde{T}^{(\ell)}$ and $\widetilde{T}^{(\ell+1)}$. We observe that, for any $k \ge 1$,

$$\begin{aligned}
&\mathbb{Q}^e\big(\exists x \in W\big(\pi_k^s\big), N(x) > n/\ln^4(n)\big)\\
&\quad\le \mathbb{Q}^e\big(N\big(\pi_k^s\big) > n^\varepsilon\big) + \mathbb{Q}^e\big(\exists x \in W\big(\pi_k^s\big), N\big(G(x)\big) > n^\varepsilon\big)\\
&\qquad + \mathbb{Q}^e\big(\exists x \in \mathring{W}\big(\pi_k^s\big), \exists \ell \le 2n^\varepsilon, N^{(\ell)}(x) > n^{1-2\varepsilon}\big)\\
&\quad\le (c_{14}+c_{15}) n^{-2\Lambda} + \sum_{\ell\le 2n^\varepsilon} \mathbb{Q}^e\big(\exists x \in \mathring{W}\big(\pi_k^s\big), N^{(\ell)}(x) > n^{1-2\varepsilon}\big).
\end{aligned} \tag{2.26}$$

Since

$$\begin{aligned}
&\mathbb{Q}^e\big(\exists x \in W\big(\pi_k^s\big), N^{(\ell)}(x) > n^{1-2\varepsilon}\big)\\
&\quad\le E_{\mathbf{Q}}\bigg[\sum_{y\in\mathcal{S}_H} P_\omega^e\big(\pi_k^s = y\big) \sum_{x\in\mathring{W}(y)} P_\omega^x\big(N^{(\ell)}(x) > n^{1-2\varepsilon}\big)\bigg],
\end{aligned}$$

and by the strong Markov property at $\widetilde{T}_x^{(\ell)}$,

$$\begin{aligned}
P_\omega^x\big(N^{(\ell)}(x) > n^{1-2\varepsilon}\big) &= P_\omega^x\big(\widetilde{T}_x^{(\ell)} < \infty\big) P_\omega^x\big(N^{(1)}(x) > n^{1-2\varepsilon}\big)\\
&\le P_\omega^x\big(N^{(1)}(x) > n^{1-2\varepsilon}\big),
\end{aligned}$$

this yields

$$
\begin{aligned}
&\mathbb{Q}^e\big(\exists x \in W\big(\pi_k^s\big), N^{(\ell)}(x) > n^{1-2\varepsilon}\big) \\
&\quad \le E_{\mathbf{Q}}\bigg[\sum_{y\in\mathcal{S}_H} P_\omega^e\big(\pi_k^s = y\big) \sum_{x\in \overset{\circ}{W}(y)} P_\omega^x\big(N^{(1)}(x) > n^{1-2\varepsilon}\big)\bigg] \\
&\quad \le c_{14} E_{\mathbf{Q}}\bigg[\sum_{y\in\mathcal{S}_H} P_\omega^e\big(\pi_k^s = y\big)\bigg] E_{\mathbf{Q}}\bigg[\sum_{x\in \overset{\circ}{W}(e)} P_\omega^x\big(N^{(1)}(x) > n^{1-2\varepsilon}\big) | Z_H > B\bigg] \\
&\quad = c_{14} E_{\mathbf{Q}}\bigg[\sum_{x\in \overset{\circ}{W}(e)} P_\omega^x\big(N^{(1)}(x) > n^{1-2\varepsilon}\big) | Z_H > B\bigg].
\end{aligned} \tag{2.27}
$$

For any $x \in W(e)$, define, for any $y \in [\![e, G(x)]\!]$,

$$
\begin{aligned}
\widetilde{\omega}(y, y_+) &:= \frac{\omega(y, y_+)}{\omega(y, y_+) + \omega(y, \overleftarrow{y})}, \\
\widetilde{\omega}(y, \overleftarrow{y}) &:= \frac{\omega(y, \overleftarrow{y})}{\omega(y, y_+) + \omega(y, \overleftarrow{y})},
\end{aligned}
$$

where as before y_+ represents the child of y on the path. We let $(\widetilde{X}_n)_{n\ge 0}$ be the random walk on $[\![e, G(x)]\!]$ with the transition probabilities $\widetilde{\omega}$ and we denote by $\widetilde{P}_{\omega,x}(\cdot)$ the probability distribution of $(\widetilde{X}_n, n \ge 0)$. By Lemma 4.4 of [1], we have the following comparisons:

$$
\begin{aligned}
&P_\omega^{\overleftarrow{x}}(T_x < T_e) \le \widetilde{P}_{\omega,x}^{\overleftarrow{x}}(T_x < T_e), \\
&P_\omega^{x_+}(T_{G(x)} < T_x) \le \widetilde{P}_{\omega,x}^{x_+}(T_{G(x)} < T_x).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&P_\omega^x\big(T_x^* < T_e \wedge T_{G(x)}\big) \\
&\quad = \omega(x, \overleftarrow{x}) P_\omega^{\overleftarrow{x}}(T_x < T_e) + \omega(x, x_+) P_\omega^{x_+}(T_x < T_{G(x)}) + \sum_{i\le \nu(x): x_i \ne x^+} \omega(x, x_i)\big(1 - \beta(x_i)\big) \\
&\quad \le \omega(x, \overleftarrow{x}) \widetilde{P}_{\omega,x}^{\overleftarrow{x}}(T_x < T_e) + \omega(x, x_+) \widetilde{P}_{\omega,x}^{x_+}(T_x < T_{G(x)}) + \sum_{i\le \nu(x): x_i \ne x_+} \omega(x, x_i) \\
&\quad = 1 - \big(\omega(x, \overleftarrow{x}) + \omega(x, x_+)\big) \widetilde{P}_{\omega,x}^x\big(T_x^* > T_e \wedge T_{G(x)}\big).
\end{aligned}
$$

Since $\nu(x) \le B$ (for $x \in \overset{\circ}{W}(e)$), we find by (2.4) a constant $c_{16} \in (0, 1)$ such that $\omega(x, \overleftarrow{x}) + \omega(x, x_+) \ge c_{16}$. It yields that

$$
P_\omega^x\big(T_x^* < T_e \wedge T_{G(x)}\big) \le 1 - c_{16} \widetilde{P}_{\omega,x}^x\big(T_x^* > T_e \wedge T_{G(x)}\big).
$$

We observe that, for any $x \in W(e)$, with the notation of (2.14) and taking $c_7 := c_{16}$,

$$
E_{\mathbf{P}}\big[\big(1 - c_{16} \widetilde{P}_{\omega,x}^x\big(T_x^* > T_e \wedge T_{G(x)}\big)\big)^n\big] = p\big(|x|, \big|G(x)\big|, n\big).
$$

It follows that

$$
E_{GW}\bigg[\sum_{x\in \overset{\circ}{W}(e)} \mathbb{P}^x\big(N^{(1)}(x) > n^{1-2\varepsilon}\big)\bigg] \le E_{GW}\bigg[\sum_{x\in \overset{\circ}{W}(e)} p\big(|x|, \big|G(x)\big|, n^{1-2\varepsilon}\big)\bigg].
$$

On the other hand, $\sum_{x\in W(e)} p(|x|,|G(x)|,n^{1-2\varepsilon}) \le \sum_{y\in\partial W(e)}\sum_{x\le y} p(|x|,|y|,n^{1-2\varepsilon})$. It implies that

$$E_{GW}\Big[\sum_{x\in\overset{\circ}{W}(e)} \mathbb{P}^x\big(N^{(1)}(x)>n^{1-2\varepsilon}\big)\Big] \le \sum_{j\ge 0} E_{GW}\big[\#\{y\in\partial W(e),|y|=j\}\big]\Big(\sum_{i\le j} p\big(i,j,n^{1-2\varepsilon}\big)\Big)$$
$$\le m\sum_{j\ge 0} E_{GW}\big[W_{j-1}(e)\big]\Big(\sum_{i\le j} p\big(i,j,n^{1-2\varepsilon}\big)\Big).$$

By Lemmas 2.5 and 2.7, for n large enough,

$$E_{GW}\Big[\sum_{x\in\overset{\circ}{W}(e)} \mathbb{P}^x\big(N^{(1)}(x)>n^{1-2\varepsilon}\big)\Big] \le m^2\sum_{j\ge 0} r^{j-2}\Big(\sum_{i\le j} p\big(i,j,n^{1-2\varepsilon}\big)\Big) \le n^{-(1-2\varepsilon)\Lambda_r+\varepsilon}. \tag{2.28}$$

Supposing r and ε close enough to q_1 and 0, Eq. (2.28) combined with (2.27) and (2.28), shows that, for any $k\ge 1$,

$$\mathbb{Q}^e\big(\exists x\in W\big(\pi_k^s\big), N(x)>n/\ln^4(n)\big) \le c_{17}n^{-(1-2\varepsilon)\Lambda_r+2\varepsilon}.$$

We arrive at

$$\mathbb{Q}^e\big(\exists k\le \ln^4(n), \exists x\in W\big(\pi_k^s\big), N(x)>n/\ln^4(n)\big) \le c_{18}n^{-(1-2\varepsilon)\Lambda_r+3\varepsilon}. \tag{2.29}$$

Finally, the estimate of $\mathbb{Q}^e(\exists x\in W(e), N(x)>n/\ln^4(n), D(e)=\infty, Z_H\le B)$ in (2.22) is similar. Indeed,

$$\mathbb{Q}^e\big(\exists x\in W(e), N(x)>n/\ln^4(n), D(e)=\infty, Z_H\le B\big)$$
$$\le \mathbb{Q}^e\big(N(e)>n^\varepsilon, D(e)=\infty, \nu(e)\le B\big)+\mathbb{Q}^e\big(\exists x\in W(e), N\big(G(x)\big)>n^\varepsilon\big)$$
$$+\mathbb{Q}^e\big(\exists x\in W(e), \exists \ell\le 2n^\varepsilon, N^{(\ell)}(x)>n^{1-2\varepsilon}\big).$$

We have

$$\mathbb{Q}^e\big(N(e)>n^\varepsilon, D(e)=\infty, \nu(e)\le B\big) \le E_{\mathbf{Q}}\big[\big(1-\omega(e,\overleftarrow{e})\big)^{n^\varepsilon}\mathbb{1}_{\{\nu(e)\le B\}}\big]$$
$$\le (1-c_1/B)^{n^\varepsilon},$$

by (2.4). By Eq. (2.24),

$$\mathbb{Q}^e\big(\exists x\in W\big(\pi_k^s\big), N\big(G(x)\big)>n^\varepsilon\big) \le c_{15}n^{-2\Lambda}.$$

Finally,

$$\mathbb{Q}^e\big(\exists x\in \overset{\circ}{W}(e), \exists\ell\le 2n^\varepsilon, N^{(\ell)}(x)>n^{1-2\varepsilon}\big) \le \sum_{\ell\le 2n^\varepsilon}\mathbb{Q}^e\big(\exists x\in\overset{\circ}{W}(e), N^{(\ell)}(x)>n^{1-2\varepsilon}\big)$$
$$\le 2n^\varepsilon\mathbb{Q}^e\big(\exists x\in\overset{\circ}{W}(e), N^{(1)}(x)>n^{1-2\varepsilon}\big)$$
$$\le 2n^\varepsilon E_{GW}\Big[\sum_{x\in\overset{\circ}{W}(e)}\mathbb{P}^x\big(N^{(1)}(x)>n^{1-2\varepsilon}\big)\Big]$$
$$\le c_{17}n^{-(1-2\varepsilon)\Lambda_r+2\varepsilon},$$

by (2.28). We deduce that, for n large enough,

$$\mathbb{Q}^e\big(\exists x\in W(e), N(x)>n/\ln^4(n), D(e)=\infty, Z_H\le B\big) \le n^{-(1-2\varepsilon)\Lambda_r+3\varepsilon}. \tag{2.30}$$

In view of (2.22) combined with (2.29) and (2.30), Eq. (2.21) is proved, and Proposition 2.4 follows. □

3. Large deviations principles

We recall the definition of the first regeneration time

$$\Gamma_1 := \inf\big\{k > 0\colon\ \nu(X_k) \geq 2,\, D(X_k) = \infty,\, k = \tau_{|X_k|}\big\}.$$

We define by iteration

$$\Gamma_n := \inf\big\{k > \Gamma_{n-1}\colon\ \nu(X_k) \geq 2,\, D(X_k) = \infty,\, k = \tau_{|X_k|}\big\}$$

for any $n \geq 2$. We have the following fact (points (i) to (iii) are already discussed in [1]; point (iv) is shown in [8] in the case of regular trees and in [12] in the case of biased random walks, and is easily adaptable to our case).

Fact B.

(i) *For any $n \geq 1$, $\Gamma_n < \infty$ $\mathbb{Q}^e$-a.s.*
(ii) *Under $\mathbb{Q}^e$, $(\Gamma_{n+1} - \Gamma_n, |X_{\Gamma_{n+1}}| - |X_{\Gamma_n}|), n \geq 1$ are independent and distributed as $(\Gamma_1, |X_{\Gamma_1}|)$ under the distribution $\mathbb{S}^e$.*
(iii) *We have $E_{\mathbb{S}^e}[|X_{\Gamma_1}|] < \infty$.*
(iv) *The speed v verifies $v = \frac{E_{\mathbb{S}^e}[|X_{\Gamma_1}|]}{E_{\mathbb{S}^e}[\Gamma_1]}$.*

The rest of the section is devoted to the proof of Theorems 1.1 and 1.2. It is in fact easier to prove them when conditioning on never returning to the root. Our theorems become

Theorem 3.1 (Speed-up case). *There exist two continuous, convex and strictly decreasing functions $I_a \leq I_q$ from $[1, 1/v]$ to $\mathbb{R}_+$, such that $I_a(1/v) = I_q(1/v) = 0$ and for $a < b$, $b \in [1, 1/v]$, we have almost surely,*

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{Q}^e\left(\frac{\tau_n}{n} \in\,]a, b] \,\middle|\, D(e) = \infty\right)\right) = -I_a(b), \tag{3.1}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(P_\omega^e\left(\frac{\tau_n}{n} \in\,]a, b] \,\middle|\, D(e) = \infty\right)\right) = -I_q(b). \tag{3.2}$$

Theorem 3.2 (Slowdown case). *There exist two continuous, convex functions $I_a \leq I_q$ from $[1/v, +\infty[$ to $\mathbb{R}_+$, such that $I_a(1/v) = I_q(1/v) = 0$ and for any $1/v \leq a < b$, we have almost surely*

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{Q}^e\left(\frac{\tau_n}{n} \in [a, b[\, \middle|\, D(e) = \infty\right)\right) = -I_a(a), \tag{3.3}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(P_\omega^e\left(\frac{\tau_n}{n} \in [a, b[\, \middle|\, D(e) = \infty\right)\right) = -I_q(a). \tag{3.4}$$

If ess inf $A =: i > \nu_{\min}^{-1}$, then I_a and I_q are strictly increasing on $[1/v, +\infty[$. If $i \leq \nu_{\min}^{-1}$, then $I_a = I_q = 0$.

Theorems 1.1 and 1.2 follow from Theorems 3.1 and 3.2 and the following proposition.

Proposition 3.3. *We have, for $a < b \leq 1/v$,*

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{Q}^e\left(\frac{\tau_n}{n} \in\,]a, b]\right)\right) = \lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{Q}^e\left(\frac{\tau_n}{n} \in\,]a, b] \,\middle|\, D(e) = \infty\right)\right), \tag{3.5}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(P_\omega^e\left(\frac{\tau_n}{n} \in\,]a, b]\right)\right) = \lim_{n\to\infty} \frac{1}{n} \ln\left(P_\omega^e\left(\frac{\tau_n}{n} \in\,]a, b] \,\middle|\, D(e) = \infty\right)\right). \tag{3.6}$$

Similarly, in the slowdown case, we have for $1/v \le a < b$,

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(\mathbb{Q}^e\left(\frac{\tau_n}{n}\in[a,b[\right)\right)=\lim_{n\to\infty}\frac{1}{n}\ln\left(\mathbb{Q}^e\left(\frac{\tau_n}{n}\in[a,b[\,\middle|\,D(e)=\infty\right)\right), \tag{3.7}$$

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(P^e_\omega\left(\frac{\tau_n}{n}\in[a,b[\right)\right)=\lim_{n\to\infty}\frac{1}{n}\ln\left(P^e_\omega\left(\frac{\tau_n}{n}\in[a,b[\,\middle|\,D(e)=\infty\right)\right). \tag{3.8}$$

Theorems 3.1 and 3.2 are proved in two distinct parts for sake of clarity. Proposition 3.3 is proved in Section 3.3.

3.1. *Proof of Theorem* 3.1

For any real numbers $h \ge 0$ and $b \ge 1$, any integer $n \in \mathbb{N}$ and any vertex $x \in \mathbb{T}$ with $|x| = n$, define

$$A(h,b,x) := \left\{\omega\colon\ P^e_\omega(\tau_n = T_x, \tau_n \le bn, T_{\overleftarrow{e}} > \tau_n) \ge \mathrm{e}^{-hn}\right\},$$

$$e_n(h,b) := E_{\mathbf{Q}}\left[\sum_{|x|=n}\mathbb{1}_{A(h,b,x)}\right].$$

We define also for any $b \ge 1$

$$h_c(b) := \inf\left\{h\ge 0\colon\ \exists p\in\mathbb{N}, e_p(h,b)>0\right\}.$$

Lemma 3.4. *There exists for any* $b \ge 1$ *and* $h > h_c(b)$, *a real* $e(h,b) > 0$ *such that*

$$\lim_{n\to\infty}\frac{1}{n}\ln\bigl(e_n(h,b)\bigr)=\ln\bigl(e(h,b)\bigr).$$

Moreover, the function $(h,b) \to \ln(e(h,b))$ *from* $\{(h,b)\in\mathbb{R}_+\times[1,+\infty[\colon\ h>h_c(b)\}$ *to* $\mathbb{R}$ *is concave, is nondecreasing in* h *and in* b, *and*

$$\lim_{h\to\infty}\ln\bigl(e(h,b)\bigr)=\ln(m).$$

Proof. Let $x \le y$ be two vertices of $\mathbb{T}$ with $|x| = n$ and $|y| = n+m$. We observe that

$$\begin{aligned} A(h,b,y) &\supset A(h,b,x)\cap\left\{\omega\colon\ P^x_\omega(\tau_{n+m}=T_y,\tau_{n+m}\le bm, T_{\overleftarrow{x}}>\tau_{n+m})\ge \mathrm{e}^{-hm}\right\}\\ &=: A(h,b,x)\cap A_x(h,b,y). \end{aligned}$$

It yields that

$$\begin{aligned} e_{n+m}(h,b) &\ge E_{\mathbf{Q}}\left[\sum_{|x|=n}\mathbb{1}_{A(h,b,x)}\sum_{|y|=n+m,\,y\ge x}\mathbb{1}_{A_x(h,b,y)}\right]\\ &= E_{\mathbf{Q}}\left[\sum_{|x|=n}\mathbb{1}_{A(h,b,x)}\right]E_{\mathbf{Q}}\left[\sum_{|x|=m}\mathbb{1}_{A(h,b,x)}\right] \\ &= e_n(h,b)e_m(h,b). \end{aligned} \tag{3.9}$$

Let $h > h_c$ and p be such that $e_p(h_c,b) > 0$, where we write h_c for $h_c(b)$. Then $e_{np}(h_c,b) > 0$ for any $n \ge 1$. We want to show that $e_k(h,b) > 0$ for k large enough. By (2.4), $\omega(e,e_1) \ge c_1$ if $\nu(e) = 1$ so that $e_k(-\ln(c_1),b) \ge q_1^k$. Let n_c be such that $\mathrm{e}^{-h_c n_c}c_1 \ge \mathrm{e}^{-hn_c}$. We check as before that for any $n \ge n_c$, and any $r \le p$, we have indeed

$$\begin{aligned} e_{np+r}(h,b) &\ge e_{np}(h_c,b)e_r\bigl(-\ln(c_1),b\bigr)\\ &\ge e_{np}(h_c,b)q_1^r > 0. \end{aligned}$$

Thus (3.9) implies that

$$\lim_{n\to\infty}\frac{1}{n}\ln\bigl(e_n(h,b)\bigr)=\sup\left\{\frac{1}{k}\ln\bigl(e_k(h,b)\bigr),k\geq 1\right\}=:\ln\bigl(e(h,b)\bigr), \tag{3.10}$$

with $e(h,b)>0$. Similarly, we can check that

$$e_n\bigl(th_1+(1-t)h_2,tb_1+(1-t)b_2\bigr)\geq e_{nt}(h_1,b_1)e_{n(1-t)}(h_2,b_2),$$

which leads to

$$\ln\bigl(e\bigl(th_1+(1-t)h_2,tb_1+(1-t)b_2\bigr)\bigr)\geq t\ln\bigl(e(h_1,b_1)\bigr)+(1-t)\ln\bigl(e(h_2,b_2)\bigr),$$

hence the concavity of $(h,b)\to\ln(e(h,b))$. The fact that $e(h,b)$ is nondecreasing in h and in b is direct. Finally, $\limsup_{h\to\infty}\ln(e(h,b))\leq\ln(m)$ and $\liminf_{h\to\infty}\ln(e(h,b))\geq\liminf_{h\to\infty}\ln(e_1(h,b))=\ln(m)$ by dominated convergence. □

In the rest of the section, we extend $e(h,b)$ to $\mathbb{R}_+\times[1,+\infty[$ by taking $e(h,b)=0$ for $h\leq h_c(b)$.

Corollary 3.5. *Let $S:=\{h\geq 0\colon\ e(h,b)>1\}$ and $S':=\{h\geq 0\colon\ e(h,b)\geq 1\}$. We have*

$$\sup\bigl\{\mathrm{e}^{-h}e(h,b),h\in S\bigr\}=\sup\bigl\{\mathrm{e}^{-h}e(h,b),h\in S'\bigr\}.$$

Proof. Let $M:=\inf\{h\colon\ e(h,b)>1\}$. We claim that if $h<M$, then $e(h,b)<1$. Indeed, suppose that there exists $h_0<M$ such that $e(h_0,b)\geq 1$. Then $e(h_0,b)=1$ by definition of M, so that $e(h,b)$ is constant equal to 1 on $[h_0,M[$. By concavity, $\ln(e(h,b))$ is equal to 0 on $[h_0,+\infty[$, which is impossible since it tends to $\ln(m)$ at infinity. The corollary follows. □

We have the tools to prove Theorem 1.1.

Proof of Theorem 1.1. For $b\in[1,+\infty[$, let

$$J_a(b):=-\sup\bigl\{-h+\ln(e(h,b)),h\geq 0\bigr\},$$
$$J_q(b):=-\sup\bigl\{-h+\ln(e(h,b)),h\in S\bigr\}.$$

Define then for any $b\leq 1/v$,

$$I_a(b)=J_a(b),$$
$$I_q(b)=J_q(b).$$

We immediately see that $I_a\leq I_q$. The convexity of J_a and J_q stems from the convexity of the function $h-\ln(e(h,b))$. Indeed, let J represent either J_a or J_q and let $1\leq b_1\leq b_2$ and $t\in[0,1]$. Denote by h_1, h_2, b and h the reals that verify

$$J(b_1)=h_1-\ln\bigl(e(h_1,b_1)\bigr),$$
$$J(b_2)=h_2-\ln\bigl(e(h_2,b_2)\bigr),$$
$$h:=th_1+(1-t)h_2,$$
$$b:=tb_1+(1-t)b_2.$$

We observe that

$$
\begin{aligned}
J(b) &\le h - \ln\bigl(e(h,b)\bigr)\\
&\le t\bigl(h_1 - \ln\bigl(e(h_1,b_1)\bigr)\bigr) + (1-t)\bigl(h_2 - \ln\bigl(e(h_2,b_2)\bigr)\bigr) = tJ(b_1) + (1-t)J(b_2)
\end{aligned}
$$

which proves the convexity. We show now that, for any $b \ge 1$,

$$\lim_{n\to\infty} \frac{1}{n} \ln\bigl(\mathbb{Q}^e(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn)\bigr) = -J_a(b), \tag{3.11}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln\bigl(P^e_\omega(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn)\bigr) = -J_q(b). \tag{3.12}$$

We first prove (3.11). Since $\mathbb{Q}^e(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn) \ge \mathrm{e}^{-hn} e_n(h,b)$ for any $h \ge 0$, we have

$$\liminf_{n\to\infty} \frac{1}{n} \ln\bigl(\mathbb{Q}^e(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn)\bigr) \ge -I_a(b).$$

Turning to the upper bound, take a positive integer k. We observe that

$$
\begin{aligned}
\mathbb{Q}^e(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn) &\le \sum_{\ell=0}^{k-1} \mathrm{e}^{-n\ell/k} e_n\bigl((\ell+1)/k, b\bigr)\\
&\le k e^{n/k} \sup\bigl\{\mathrm{e}^{-hn} e_n(h,b), h \ge 0\bigr\}.
\end{aligned}
$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \ln\bigl(\mathbb{Q}^e(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn)\bigr) \le \frac{1}{k} - J_a(b).$$

Letting k tend to infinity gives the upper bound of (3.11).

To prove Eq. (3.12), let k be still a positive integer and $h \in S$. Denote by $V_{pk}(\mathbb{T})$ the set of vertices $|x| = pk$ such that $P^{x_{\ell-1}}_\omega(\tau_{\ell k} < T_{\overleftarrow{x}_{\ell-1}}, \tau_{\ell k} = T_{x_\ell} \le bk) \ge \mathrm{e}^{-hk}$ for any $\ell \le p$, where x_ℓ represents the ancestor of x at generation ℓk. Call $V(\mathbb{T}) := \bigcup_{p\ge 0} V_{pk}(\mathbb{T})$ the subtree thus obtained. We observe that V is a Galton–Watson tree of mean offspring $e_k(h,b)$. Let

$$\mathcal{T}_{k,h} := \{\mathbb{T}\colon\ V(\mathbb{T}) \text{ is infinite}\}.$$

Take $\mathbb{T} \in \mathcal{T}_{k,h}$. For any $x \in V_{pk}$, we have

$$
\begin{aligned}
&P^e_\omega(\tau_{pk} < T_{\overleftarrow{e}}, \tau_{pk} = T_x \le bpk)\\
&\quad\ge P^e_\omega(\tau_k < T_{\overleftarrow{e}}, \tau_k = T_{x_1} \le bk) \cdots P^{x_{k-1}}_\omega(\tau_{pk} < T_{\overleftarrow{x}_{k-1}}, \tau_{pk} = T_x \le bk) \ge \mathrm{e}^{-hpk}.
\end{aligned}
$$

It implies that

$$P^e_\omega(\tau_{pk} < T_{\overleftarrow{e}}, \tau_{pk} \le bpk) \ge \mathrm{e}^{-hpk} \# V_{pk}(\mathbb{T}).$$

By the Seneta–Heyde theorem (see [2], page 30, Theorem 3),

$$\lim_{p\to\infty} \frac{1}{p} \ln\bigl(\# V_{pk}(\mathbb{T})\bigr) = \ln\bigl(e_k(h,b)\bigr), \quad \mathbf{Q}\text{-a.s.}$$

It follows that, as long as $\mathbb{T} \in \mathcal{T}_{k,h}$,

$$\liminf_{p\to\infty} \frac{1}{pk} \ln\bigl(P^e_\omega(\tau_{pk} < T_{\overleftarrow{e}}, \tau_{pk} \le bpk)\bigr) \ge -h + \frac{1}{k} \ln\bigl(e_k(h,b)\bigr).$$

Notice that

$$P^e_\omega(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn) \ge P^e_\omega(\tau_{pk} < T_{\overleftarrow{e}}, \tau_{pk} \le bpk) \min_{|x|=pk} P^x_\omega\big(\tau_n < T_{\overleftarrow{x}}, \tau_n \le b(n-pk)\big),$$

where $p := \lfloor \frac{n}{k} \rfloor$. Since A is bounded, there exists $c_{17} > 0$ such that $\sum_{i=1}^{\nu(y)} \omega(y, y_i) \ge c_{17}$ $\forall y \in \mathbb{T}$. It yields that

$$\min_{|x|=pk} P^x_\omega\big(\tau_n < T_{\overleftarrow{x}}, \tau_n = (n-pk)\big) \ge c_{17}^k,$$

Hence,

$$\liminf_{n\to\infty} \frac{1}{n} \ln\big(P^e_\omega(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn)\big) \ge -h + \frac{1}{k} \ln\big(e_k(h,b)\big). \tag{3.13}$$

Take now a general tree $\mathbb{T}$. Notice that since $h \in S$, $\mathbf{Q}(\mathcal{T}_{k,h}) > 0$ for k large enough, and there exists almost surely a vertex $z \in \mathbb{T}$ such that the subtree rooted at it belongs to $\mathcal{T}_{k,h}$. It implies that for large k, (3.13) holds almost surely. Then letting k tend to infinity and taking the supremum over all $h \in S$ leads to

$$\liminf_{n\to\infty} \frac{1}{n} \ln\big(P^e_\omega(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn)\big) \ge -J_q(b).$$

For the upper bound in (3.12), we observe that, for any integer k,

$$P^e_\omega(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn) \le \sum_{\ell=0}^{k-1} \mathrm{e}^{-\ell n/k} \sum_{|x|=n} \mathbb{1}_{A((\ell+1)/k,b,x)}.$$

By Markov's inequality, we have

$$\mathbf{Q}\bigg(\sum_{|x|=n} \mathbb{1}_{A(h,b,x)} > \big(e(h,b)+1/k\big)^n\bigg) \le \frac{e_n(h,b)}{(e(h,b)+1/k)^n} \le \bigg(\frac{e(h,b)}{e(h,b)+1/k}\bigg)^n,$$

by (3.10). An application of the Borel–Cantelli lemma proves that $\sum_{|x|=n} \mathbb{1}_{A(h,b,x)} \le (e(h,b)+1/k)^n$ for all but a finite number of n, $\mathbf{Q}$-a.s. In particular, if $e(h,b)+1/k < 1$, then $\sum_{|x|=n} \mathbb{1}_{A(h,b,x)} = 0$ for n large enough. Consequently, for n large,

$$P^e_\omega(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn) \le \mathrm{e}^{n/k} k \sup\big\{\mathrm{e}^{-hn}(e(h,b)+1/k)^n, h\colon\ e(h,b)+1/k \ge 1\big\}.$$

We find that

$$\limsup_{n\to\infty} \frac{1}{n} \ln\big(P^e_\omega(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn)\big) \le 1/k + \sup\big\{-h + \ln\big(e(h,b)+1/k\big), h\colon\ e(h,b)+1/k \ge 1\big\}.$$

Let k tend to infinity and use Corollary 3.5 to complete the proof of (3.12).

We observe that

$$\begin{aligned} P^e_\omega(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn) - P^e_\omega(\tau_n < T_{\overleftarrow{e}} < \infty, \tau_n \le bn) &\le P^e_\omega(T_{\overleftarrow{e}} = \infty, \tau_n \le bn) \\ &\le P^e_\omega(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn). \end{aligned}$$

But $P^e_\omega(\tau_n < T_{\overleftarrow{e}} < \infty, \tau_n \le bn) \le P^e_\omega(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn) \max_{i=1,\dots,\nu(e)}(1-\beta(e_i))$. Since $\max_{i=1,\dots,\nu(e)}(1-\beta(e_i)) < 1$ almost surely, we obtain that

$$\lim_{n\to\infty} \frac{1}{n} \ln\big(P^e_\omega(\tau_n \le bn) | D(e) = \infty\big) = -J_q(b). \tag{3.14}$$

In the annealed case, notice that $\mathbb{S}^e(\tau_n < T_{\overleftarrow{e}} < \infty, \tau_n \le bn) = \mathbb{S}^e(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn) E_{\mathbf{P}}[1-\beta]$ which leads similarly to

$$\lim_{n\to\infty} \frac{1}{n} \ln\bigl(\mathbb{S}^e(\tau_n \le bn)\bigr) = -J_a(b). \tag{3.15}$$

We can now finish the proof of the theorem. The continuity has to be proved only at $b = 1$ (since J_a and J_q are convex on $[1, +\infty[$), which is directly done with the arguments of [5], Section 4. We let $b < 1/v = E_{\mathbb{S}^e}[\Gamma_1]/E_{\mathbb{S}^e}[|X_{\Gamma_1}|]$ and we observe that for any constant $c_{18} > 0$,

$$\mathbb{S}^e(\tau_n \le bn) \le \mathbb{S}^e(\tau_n < \Gamma_{c_{18}n}) + \mathbb{S}^e(\Gamma_{c_{18}n} \le bn).$$

Choose c_{18} such that $b(E_{\mathbb{S}^e}[\Gamma_1])^{-1} < c_{18} < (E_{\mathbb{S}^e}[|X_{\Gamma_1}|])^{-1}$. Use Cramér's theorem with Facts A and B to see that $\mathbb{S}^e(\tau_n < \Gamma_{c_{18}n})$ and $\mathbb{S}^e(\Gamma_{c_{18}n} \le bn)$ decrease exponentially. Then, $\mathbb{S}^e(\tau_n \le bn)$ has an exponential decay and, by (3.15), $I_a(b) > 0$ which leads to $I_q(b) > 0$ since $I_a \le I_q$. We deduce in particular that I_a and I_q are strictly decreasing. Furthermore, $P^e_\omega(\tau_n \le bn | D(e) = \infty)$ tends to 1 almost surely when $b > 1/v$, which in virtue of (3.14), implies that $J_q(b) = 0$. By continuity, $I_q(1/v) = 0$ and therefore $I_a(1/v) = 0$. Finally, let $a < b$, $b \in [1, 1/v]$.

$$P^e_\omega\bigl(an < \tau_n \le bn | D(e) = \infty\bigr) = P^e_\omega\bigl(\tau_n \le bn | D(e) = \infty\bigr) - P^e_\omega\bigl(\tau_n \le an | D(e) = \infty\bigr).$$

Equation (3.2) follows since I_q is strictly decreasing. The same argument proves (3.1). □

3.2. *Proof of Theorem* 3.2

The proof is the same as before by taking for $b \ge 1$,

$$\widetilde{A}(h, b, x) := \bigl\{\omega\colon\ P^e_\omega(\tau_n = T_x, T_{\overleftarrow{e}} > \tau_n \ge bn) \ge \mathrm{e}^{-hn}\bigr\},$$

$$\widetilde{e}_n(h, b) := E_{\mathbf{Q}}\Biggl[\sum_{|x|=n} \mathbb{1}_{\widetilde{A}(h,b,x)}\Biggr],$$

$$\widetilde{S} := \bigl\{h\colon\ \widetilde{e}(h, b) > 1\bigr\}.$$

Define also for any $b \ge 1$,

$$\widetilde{J}_a(b) := -\sup\bigl\{-h + \ln\bigl(\widetilde{e}(h, b)\bigr), h \ge 0\bigr\},$$

$$\widetilde{J}_q(b) := -\sup\bigl\{-h + \ln\bigl(\widetilde{e}(h, b)\bigr), h \in \widetilde{S}\bigr\},$$

and for any $b \ge 1/v$,

$$I_a(b) := \widetilde{J}_a(b),$$

$$I_q(b) := \widetilde{J}_q(b).$$

We verify that $I_a \le I_q$ and both functions are convex. We have then for any $b \ge 1$,

$$\lim_{n\to\infty} \frac{1}{n} \ln\bigl(\mathbb{Q}^e(T_{\overleftarrow{e}} > \tau_n \ge bn)\bigr) = -\widetilde{J}_a(b), \tag{3.16}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln\bigl(P^e_\omega(T_{\overleftarrow{e}} > \tau_n \ge bn)\bigr) = -\widetilde{J}_q(b). \tag{3.17}$$

As before, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \ln\bigl(\mathbb{S}^e(\tau_n \ge bn)\bigr) = -\widetilde{J}_a(b),$$

$$\lim_{n\to\infty} \frac{1}{n} \ln\bigl(P^e_\omega\bigl(\tau_n \ge bn | D(e) = \infty\bigr)\bigr) = -\widetilde{J}_q(b).$$

We have $\widetilde{J}_a = \widetilde{J}_q = 0$ on $[1, 1/v]$. In the case $i > \nu_{\min}^{-1}$, the positivity of I_a and I_q on $]1/v, +\infty[$ comes from Proposition 2.1 and Cramér's theorem, which implies that they are strictly increasing. Equations (3.3) and (3.4) follow in that case. In the case $i \le \nu_{\min}^{-1}$, we follow the strategy of [5]. Let $\eta > 0$. As in the proof of Proposition 2.2, we set $h_n := \lfloor \ln(n)/(6\ln(b)) \rfloor$, and for some $b \in \mathbb{N}$,

$$w_+ := \mathbf{Q}\left(\sum_{i=1}^{\nu} A(e_i) \ge 1 + \eta, \nu(e) \le b\right),$$

$$w_- := \mathbf{Q}\left(\sum_{i=1}^{\nu} A(e_i) \le \frac{1}{1+\eta}, \nu(e) \le b\right).$$

Taking b large enough, we have $w_+ > 0$ and $w_- > 0$. We say that $\mathbb{T}$ is a n-good tree if

- any vertex x of the h_n first generations verifies $\nu(x) \le b$ and $\sum_{i=1}^{\nu(x)} A(x_i) \ge 1 + \eta$,
- any vertex x of the h_n following generations verifies $\nu(x) \le b$ and $\sum_{i=1}^{\nu(x)} A(x_i) \le \frac{1}{1+\eta}$.

Then we know that $Q_n := \mathbf{Q}(\mathbb{T} \text{ is } n\text{-good}) \ge \exp(-n^{1/3+o(1)})$. Let Y' be a random walk starting from zero which increases (resp. decreases) of 1 with probability $\frac{1+\eta}{2+\eta}$ (resp. $\frac{1}{2+\eta}$). We define p'_n as the probability that Y' reaches -1 before h_n. We show that (2.6) is still true (by the exactly same arguments), so that there exists a constant $K > 0$ and a deterministic function $O(n^K)$ bounded by a factor of $n \to n^K$, such that

$$P_\omega^e(T_{\overleftarrow{e}} > \tau_{2h_n} \ge n) \ge O(n^K)^{-1}(p'_n)^n. \tag{3.18}$$

We have, by gambler's ruin formula,

$$p'_n = 1 - \frac{1}{1 + (1/(1+\eta)) + \cdots + (1/(1+\eta))^{h_n}} \ge \frac{1}{1+\eta}.$$

Let $k_n := \lfloor n^d \rfloor$ with $d \in (1/3, 1/2)$ and let $f \in (d, 1-d)$. We call an n-slow tree a tree in which we can find a vertex $|x| = k_n$ such that $\mathbb{T}_x$ is n-good (where $\mathbb{T}_x$ is the subtree rooted at x), and for any $y \le x$, we have $\nu(y) \le \exp(n^f)$. We observe that if a tree is not n-slow, then either there exists a vertex before generation k_n with more than $\exp(n^f)$ children, or any subtree rooted at generation k_n is not n-good. This leads to

$$\begin{aligned}
\mathbf{Q}(\mathbb{T} \text{ is not } n\text{-slow}) &\le \sum_{\ell=1}^{k_n} E_{GW}[Z_\ell] GW\left(\nu > e^{n^f}\right) + E_{GW}\left[(1-Q_n)^{Z_{k_n}}\right] \\
&\le k_n m^{k_n} m e^{-n^f} + (1-Q_n)^{(1+\varepsilon)^{k_n}} + GW\left(Z_{k_n} \le (1+\varepsilon)^{k_n}\right).
\end{aligned}$$

We notice that $(1-Q_n)^{(1+\varepsilon)^{k_n}} \le \exp(-(1+\varepsilon)^{n^{d+o(1)}})$. Moreover,

$$GW\left(Z_{k_n} \le (1+\varepsilon)^{k_n}\right) \le (1+\varepsilon)^{k_n} E_{GW}\left[\frac{1}{Z_{k_n}}\right].$$

Observe that for any $k \ge 0$, $E_{GW}[\frac{1}{Z_{k+1}}] \le q_1 E_{GW}[\frac{1}{Z_k}] + (1-q_1) E_{GW}[\frac{1}{X_1+X_2}]$ where X_1 and X_2 are independent and distributed as Z_k. We then verify $E_{GW}[\frac{1}{X_1+X_2}] \le (u/2) \wedge v$ where $u := E_{GW}[\min(X_1, X_2)^{-1}]$ and $v := E_{GW}[\max(X_1, X_2)^{-1}]$. Since $u + v = E_{GW}[\frac{2}{Z_k}]$, we deduce that $E_{GW}[\frac{1}{X_1+X_2}] \le \frac{2}{3} E_{GW}[\frac{1}{Z_k}]$, leading to $E_{GW}[\frac{1}{Z_{k+1}}] \le (q_1 + \frac{2}{3}(1-q_1)) E_{GW}[\frac{1}{Z_k}] \le (q_1 + \frac{2}{3}(1-q_1))^{k+1}$. We get

$$GW\left(Z_{k_n} \le (1+\varepsilon)^{k_n}\right) \le \left((1+\varepsilon)\left(q_1 + \frac{2}{3}(1-q_1)\right)\right)^{k_n},$$

and, taking ε small enough,

$$\mathbf{Q}(\mathbb{T} \text{ is not } n\text{-slow}) \le \exp\bigl(-n^{d+o(1)}\bigr). \tag{3.19}$$

Let $1/v \le a < b$. We want to show that (under the hypothesis $i \le \nu_{\min}^{-1}$),

$$\liminf_{n\to\infty} \frac{1}{n} \ln P_\omega^e\left(\frac{\tau_n}{n} \in [a,b[, D(e) > \tau_n\right) = 0. \tag{3.20}$$

If this is proved, the Jensen's inequality gives

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mathbb{Q}^e\left(\frac{\tau_n}{n} \in [a,b[, D(e) > \tau_n\right) = 0. \tag{3.21}$$

Equations (3.3) and (3.4) follow. Therefore, we focus on the proof of (3.20).

Let $n_1 := n - k_n - 2h_n$, $\delta > 0$, and $G_k := \{|x| = k \text{ s.t. } \mathbb{T}_x \text{ is } n\text{-slow}\}$. We have

$$\left\{\frac{\tau_n}{n} \in [a,b[, \tau_{\overleftarrow{e}} > \tau_n\right\} \subset E_5 \cap E_6 \cap E_7,$$

with

$$E_5 := \left\{T_{\overleftarrow{e}} > \tau_{n_1}, \frac{\tau_{n_1}}{n_1} \in \left[\frac{1}{v} - \delta, \frac{1}{v} + \delta\right[\right\},$$

$$E_6 := \{X_{\tau_{n_1}} \in G_{n_1}\},$$

$$E_7 := \left\{D(X_{\tau_{n_1}}) > \tau_n, \frac{\tau_n}{n} \in \left(a - \frac{1}{v} + \delta, b - \frac{1}{v} - \delta\right)\right\}.$$

We look at the probability of the event E_7 conditioned on E_5 and E_6. Therefore, we suppose that $u := X_{\tau_{n_1}}$ is known, and that the subtree $\mathbb{T}_u$ rooted at u is a n-slow tree. There exists x_n at generation $n_1 + k_n$ such that $\mathbb{T}_{x_n}$ is a n-good tree and $\nu(y) \le e^{n^f}$ for any $u \le y < x_n$. Let also n be large enough so that $k_n \le \delta n$. It implies that

$$\begin{aligned}
&P_\omega^u\left(D(u) > \tau_n, \frac{\tau_n}{n} \in \left(a - \frac{1}{v} + \delta, b - \frac{1}{v} - \delta\right)\right)\\
&\quad \ge P_\omega^u\bigl(D(u) > T_{x_n} = k_n\bigr) P_\omega^{x_n}\left(D(x_n) > \tau_n, \frac{\tau_n}{n} \in \left(a - \frac{1}{v} + \delta, b - \frac{1}{v} - 2\delta\right)\right)\\
&\quad \ge \exp\bigl(-c_{21} n^{c_{22}}\bigr) P_\omega^{x_n}\left(D(x_n) > \tau_n, \frac{\tau_n}{n} \in \left(a - \frac{1}{v} + \delta, b - \frac{1}{v} - 2\delta\right)\right)
\end{aligned}$$

for some $c_{22} \in (0,1)$. By definition of a n-good tree, any vertex x descendant of x_n and such that $|x| \le n$ verifies $\nu(x) \le b$. Therefore, there exists a constant $c_{23} > 0$ such that $P_\omega^y(\tau_n \le 2h_n) \ge c_{23}^{2h_n}$ for any $y \ge x_n$, $|y| < n$. By the strong Markov property,

$$\begin{aligned}
&P_\omega^{x_n}\left(D(x_n) > \tau_n, \frac{\tau_n}{n} \in \left(a - \frac{1}{v} + \delta, b - \frac{1}{v} - 2\delta\right)\right)\\
&\quad \ge P_\omega^{x_n}\left(D(x_n) > \tau_n, \frac{\tau_n}{n} \ge a - \frac{1}{v} + \delta\right) c_{23}^{2h_n}.
\end{aligned}$$

Let $L := a - \frac{1}{v} + \delta$. By Eq. (3.18),

$$P_\omega^{x_n}\left(D(x_n) > \tau_n, \frac{\tau_n}{n} \ge a - \frac{1}{v} + \delta\right) \ge O\bigl(n^K\bigr)^{-1}\left(\frac{1}{1+\eta}\right)^{Ln}.$$

Hence, by the strong Markov property,

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n}\ln P_\omega^e(E_7|E_5\cap E_6) &= \liminf_{n\to\infty}\frac{1}{n}\ln P_\omega^u\left(D(u)>\tau_n,\ \frac{\tau_n}{n}\in\left(a-\frac{1}{v}+\delta, b-\frac{1}{v}-\delta\right)\right)\\ &\geq -L(1+\eta).\end{aligned}$$

This implies that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\ln P_\omega^e\left(\frac{\tau_n}{n}\in[a,b[,\ D(e)>\tau_n\right) &\geq \liminf_{n\to}\frac{1}{n}\ln P_\omega^e(E_5\cap E_6\cap E_7)\\ &\geq \liminf_{n\to\infty}\frac{1}{n}\ln P_\omega^e(E_5\cap E_6)-L\ln(1+\eta).\end{aligned}\tag{3.22}$$

Notice that

$$\begin{aligned}E_{\mathbf{Q}}\left[P_\omega^e\left(E_5\cap E_6^c\right)\right] &= E_{\mathbf{Q}}\left[P_\omega^e(E_5)-P_\omega^e(E_5\cap E_6)\right]\\ &= \mathbb{Q}(E_5)\left(1-\mathbf{Q}(\mathbb{T}\text{ is }n\text{-slow})\right)\\ &\leq \mathbb{Q}(E_5)\exp\left(-n^{d+\mathrm{o}(1)}\right),\end{aligned}$$

by Eq. (3.19). By Markov's inequality,

$$\mathbf{Q}\left(P_\omega^e\left(E_5\cap E_6^c\right)\geq\frac{1}{n^2}\right)\leq n^2\mathbb{Q}(E_5)\mathrm{e}^{-n^{d+\mathrm{o}(1)}}.$$

The Borel–Cantelli lemma implies that almost surely, for n large enough,

$$P_\omega^e(E_5\cap E_6)\geq P_\omega^e(E_5)-\frac{1}{n^2}.$$

We observe that $P_\omega^e(E_5)\to P_\omega^e(T_{\overleftarrow{e}}=\infty)$ when n goes to infinity. Therefore, Eq. (3.23) becomes

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(P_\omega^e\left(\frac{\tau_n}{n}\in[a,b[,\ D(e)>\tau_n\right)\right)\geq-\left(a-\frac{1}{v}+\delta\right)\ln(1+\eta).$$

We let η go to 0 to get

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(P_\omega^e\left(\frac{\tau_n}{n}\in[a,b[,\ D(e)>\tau_n\right)\right)=0$$

which proves (3.20).

3.3. *Proof of Proposition* 3.3

The speed-up case is quite immediate. Indeed, reasoning on the last visit to the root, we have

$$\mathbb{Q}^e\left(\tau_n\leq bn, D(e)=\infty\right)\leq\mathbb{Q}^e(\tau_n\leq bn)\leq bn\mathbb{Q}^e\left(\tau_n\leq bn, D(e)=\infty\right).$$

Therefore, by Theorem 3.1,

$$\lim_{n\to\infty}\frac{1}{n}\ln\mathbb{Q}^e(\tau_n\leq bn)=\lim_{n\to\infty}\frac{1}{n}\ln\mathbb{Q}^e\left(\tau_n\leq bn|D(e)=\infty\right).$$

It already gives (3.5) since I_a is strictly decreasing on $[1,1/v]$. We do exactly the same for the quenched inequality. Therefore, let us turn to the slowdown case, beginning with the annealed inequality (3.7). We follow the arguments of [5]. We still write $i = \operatorname{ess\,inf} A$. For technical reasons, we need to distinguish the cases where $\mathbf{P}(A = i)$ is null or positive. We feel free to deal only with the case $\mathbf{P}(A = i) = 0$, the other one following with nearly any change. Moreover, we suppose without loss of generality that $i > \nu_{\min}^{-1}$, since the two sides are equal to zero when $i \leq \nu_{\min}^{-1}$. Let $k \geq 1$. We write $\ell = k[2]$ to say that ℓ and k have the same parity. Following [5], we write for $b > a > 1/v$,

$$\begin{aligned}
&P_\omega^e(bn > \tau_n \geq an) \\
&\quad = \sum_{\ell = k[2]} \sum_{|x| = k} P_\omega^e\bigl(bn > \tau_n \geq an, \tau_n > \ell, X_\ell = x, |X_i| > k, \forall i = \ell + 1, \ldots, \tau_n\bigr) \\
&\quad = \sum_{\ell = k[2]} \sum_{|x| = k} P_\omega^e(\tau_n > \ell, X_\ell = x) P_\omega^x\bigl(bn - \ell > \tau_n > an - \ell, D(x) > \tau_n\bigr).
\end{aligned}$$

By coupling, we have, for $p := \nu_{\min} i > 1$,

$$\sup_{|x| = k} P_\omega^e(\tau_n > \ell, X_\ell = x) \leq P_\omega^e\bigl(|X_\ell| \leq k\bigr) \leq P\bigl(S_\ell^p \leq k\bigr),$$

where S_ℓ^p stands for a reflected biased random walk on the half line, which moves of $+1$ with probability $p/1 + p$ and of -1 with probability $1/1 + p$. From (and with the notation of) Lemma 5.2 of [5], we know that for all ℓ of the same parity as k,

$$P\bigl(S_\ell^p \leq k\bigr) \leq c_k (1 + \delta_k)^\ell P\bigl(S_\ell^p = k, 1 \leq S_i \leq k - 1, i = 1, \ldots, \ell - 1\bigr),$$

where $c_k < \infty$ and $\delta = (\delta_k)$ is a sequence independent of all the parameters and tending to zero. In particular, we stress that δ do not depend on p. Hence, $P_\omega^e(bn > \tau_n \geq an)$ is smaller than

$$c_k (1 + \delta_k)^{bn} \sum_{\ell = k[2]} \sum_{|x| = k} P\bigl(S_\ell^p = k, 1 \leq S_i \leq k - 1, i = 1, \ldots, \ell - 1\bigr) W_n(x, \ell),$$

where

$$W_n(x, \ell) := P_\omega^x\bigl(bn - \ell > \tau_n \geq an - \ell, D(x) > \tau_n\bigr).$$

We deduce that

$$\begin{aligned}
P_\omega^e(bn > \tau_n \geq an) &\leq c_k (1 + \delta_k)^{bn} \sum_{\ell = k[2]} \sum_{|x| = k} P_{\omega_p}^e\bigl(\tau_k = \ell, D(e) > \ell\bigr) W_n(x, \ell) \\
&= \nu_{\min}^k c_k (1 + \delta_k)^{bn} \sum_{\ell = k[2]} \sum_{|x| = k} P_{\omega_p}^e\bigl(\tau_k = \ell, D(e) > \ell, X_\ell = x\bigr) W_n(x, \ell),
\end{aligned} \tag{3.23}$$

where ω_p represents the environment of the biased random walk on the $\nu_{\min}$-ary tree such that for any vertex x, $P_{\omega_p}^x(X_1 = x_i) = \frac{p}{\nu_{\min}(1+p)}$ for each child x_i, and $P_\omega^x(X_1 = \overleftarrow{x}) = \frac{1}{1+p}$. Taking the expectations yields that

$$\mathbb{Q}^e(bn > \tau_n \geq an) \leq \nu_{\min}^k c_k (1 + \delta_k)^{bn} \sum_{\ell = k[2]} \sum_{|x| = k} P_{\omega_p}^e\bigl(\tau_k = \ell, D(e) > \ell, X_\ell = x\bigr) E_{\mathbf{Q}}\bigl[W_n(x, \ell)\bigr]. \tag{3.24}$$

Moreover, define for any $|x| = k$,

$$\mathcal{S}_{k,\ell}^+(\mathbb{T}, x) = \bigl\{\{s_i\}_{i=0}^\ell \colon |s_{i+1}| - |s_i| = 1, s_0 = 0, k - 1 \geq |s_i| > 0, s_\ell = x\bigr\}$$

the set of paths on $\mathbb{T}$ which ends at x in ℓ steps and stays between generation 1 and $k - 1$ before. We notice that, for any environment ω,

$$P_\omega^e\bigl(\tau_k = \ell, D(e) > \ell, X_\ell = x\bigr) = \sum_{\{s\} \in \mathcal{S}_{k,\ell}^+(\mathbb{T}, x)} \sum_{y \in \mathbb{T}} \omega(y, \overleftarrow{y})^{N(y, \overleftarrow{y})} \sum_{i=1}^{\nu(y)} \omega(y, y_i)^{N(y, y_i)}, \tag{3.25}$$

where for each path $\{s_i\}$, $N(z,y)$ stands for the number of passage from z to y. Let $\varepsilon>0$, and $\mathcal{G}_k$ denote for any k the set of trees such that any vertex x of generation less than k verifies $\nu(x)=\nu_{\min}$ and $A(x)\le \text{ess inf } A+\varepsilon$. Let $p':=\nu_{\min}(\text{ess inf A}+\varepsilon)$. We observe that

$$P^e_{\omega_p}\big(\tau_k=\ell, D(e)>\ell, X_\ell=x\big)=\sum_{\{s\}\in\mathcal{S}^+_{k,\ell}(\mathbb{T},x)}\sum_{y\in\mathbb{T}}\left(\frac{1}{1+p}\right)^{N(y,\overleftarrow{y})}\sum_{i=1}^{\nu(y)}\left(\frac{p}{\nu_{\min}(1+p)}\right)^{N(y,y_i)}.$$

Therefore, if $\mathbb{T}$ belongs to $\mathcal{G}_k$, we have by Eq. (3.25),

$$P^e_{\omega_p}\big(\tau_k=\ell, D(e)>\ell, X_\ell=x\big)\le\left(\frac{1+p'}{1+p}\right)^{\ell}P^e_{\omega}\big(\tau_k=\ell, D(e)>\ell, X_\ell=k\big).$$

It entails that

$$\begin{aligned}
&\mathbb{1}_{\{\mathbb{T}\in\mathcal{G}_k\}}\sum_{\ell=k[2]}\sum_{|x|=k}P^e_{\omega_p}\big(\tau_k=\ell, D(e)>\ell, X_\ell=x\big)W_n(x,\ell)\\
&\quad\le\mathbb{1}_{\{\mathbb{T}\in\mathcal{G}_k\}}\left(\frac{1+p'}{1+p}\right)^{bn}\sum_{\ell=k[2]}\sum_{|x|=k}P^e_{\omega}\big(\tau_k=\ell, D(e)>\ell, X_\ell=x\big)W_n(x,\ell)\\
&\quad=\mathbb{1}_{\{\mathbb{T}\in\mathcal{G}_k\}}\left(\frac{1+p'}{1+p}\right)^{bn}P^e_{\omega}\big(bn>\tau_n\ge an, D(e)>\tau_n\big)\\
&\quad\le\left(\frac{1+p'}{1+p}\right)^{bn}P^e_{\omega}\big(bn>\tau_n\ge an, D(e)>\tau_n\big).
\end{aligned}\tag{3.26}$$

Taking expectations gives

$$\begin{aligned}
&\mathbf{Q}(\mathbb{T}\in\mathcal{G}_k)\sum_{\ell=k[2]}\sum_{|x|=k}P^e_{\omega_p}(\tau_k=\ell, X_\ell=x)E_{\mathbf{Q}}[W_n(x,\ell)]\\
&\quad\le\left(\frac{1+p'}{1+p}\right)^{bn}\mathbb{Q}^e\big(bn>\tau_n\ge an, D(e)>\tau_n\big).
\end{aligned}\tag{3.27}$$

As before,

$$\begin{aligned}
&\mathbb{Q}^e\big(bn>\tau_n\ge an, D(e)=\infty\big)+\mathbb{Q}^e\big(bn>\tau_n\ge an, \infty>D(e)>\tau_n\big)\\
&\quad=\mathbb{Q}^e\big(bn>\tau_n\ge an, D(e)>\tau_n\big)\\
&\quad\ge\mathbb{Q}^e\big(bn>\tau_n\ge an, D(e)=\infty\big).
\end{aligned}$$

Since $\mathbb{Q}^e(bn>\tau_n\ge an, \infty>D(e)>\tau_n)\le\mathbb{Q}^e(bn>\tau_n\ge an, D(e)>\tau_n)E_{\mathbf{Q}}[1-\beta]$, we get

$$\lim_{n\to\infty}\frac{1}{n}\ln\mathbb{Q}^e\big(bn>\tau_n\ge an, D(e)>\tau_n\big)=\lim_{n\to\infty}\frac{1}{n}\ln\mathbb{Q}^e\big(bn>\tau_n\ge an|D(e)=\infty\big).$$

Consequently, we have by (3.24) and (3.27)

$$\limsup_{n\to\infty}\mathbb{Q}^e(bn>\tau_n\ge an)\le b\ln\left(\frac{1+p'}{1+p}(1+\delta_k)\right)+\lim_{n\to\infty}\frac{1}{n}\ln\mathbb{Q}^e\big(bn>\tau_n\ge an|D(e)=\infty\big).$$

Since $\mathbb{Q}^e(cn>\tau_n>bn)\ge\mathbb{Q}^e(cn>\tau_n>bn, D(e)=\infty)$, we prove Eq. (3.7) by taking p' arbitrarily close to p, and letting k tend to infinity.

We prove now the quenched equality (3.8). For any environment ω, construct the environment $f_p(\omega)$ by setting $A(x)=i$ $(:=\text{ess inf } A)$ for any $|x|\le k$. We construct also for $p'>p$, an environment $f_{p'}(\omega)$ by picking independently

$A(x)$ in $[i, p'/\nu_{\min}]$ for any $x \le k$, such that $A(x)$ has the distribution of A conditioned on $A \in [i, p'/\nu_{\min}]$. By Eq. (3.23), we have almost surely

$$\limsup_{n\to\infty} \frac{1}{n} \ln P^e_\omega(bn > \tau_n \ge an) \le \limsup_{n\to\infty} \frac{1}{n} P^e_{f_p(\omega)}\big(bn > \tau_n \ge an, D(e) > \tau_n\big) + b\ln(1+\delta_k).$$

Equation (3.26) applied to the environment $f_{p'}(\omega)$, together with Theorem 3.2 shows that

$$\limsup_{n\to\infty} \frac{1}{n} \ln P^e_{f_p(\omega)}\big(bn > \tau_n \ge an, D(e) > \tau_n\big) \le -I_q(b) + b\ln\frac{1+p'}{1+p}.$$

Let p' tend to p to get that almost surely,

$$\limsup_{n\to\infty} \frac{1}{n} \ln P_{f_p(\omega)}\big(bn > \tau_n \ge an, D(e) > \tau_n\big) \le -I_q(b).$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \ln P^e_\omega(bn > \tau_n \ge an) \le -I_q(b) + b\ln(1+\delta_k).$$

When k goes to infinity, we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \ln P^e_\omega(bn > \tau_n > an) \le -I_q(b),$$

which gives Eq. (3.8).

3.4. *Proof of Proposition* 1.3

Recall that, for any $\theta \in \mathbb{R}$,

$$\psi(\theta) := \ln\left(E_{\mathbf{Q}}\left[\sum_{i=1}^{\nu(e)} \omega(e, e_i)^\theta\right]\right).$$

Obviously, for any $n \in \mathbb{N}$,

$$\frac{1}{n} \ln\big(\mathbb{Q}^e(\tau_n = n)\big) = \ln\left(E_{\mathbf{Q}}\left[\sum_{i=1}^{\nu(e)} \omega(e, e_i)\right]\right) = \psi(1).$$

This proves (1.8). For the quenched case, we have that

$$P^e_\omega(\tau_n = n) = \sum_{|x|=n} \prod_{k=0}^{n-1} \omega(x_k, x_{k+1}),$$

where x_k is the ancestor of the vertex x at generation k. We observe that we are reduced to the study of a generalized multiplicative cascade, as studied in [10]. The following lemma is well known in the case of a regular tree (see [7] and [4]). We extend it easily to a Galton–Watson tree.

Lemma 3.6. *We have* $\lim_{n\to\infty} \frac{1}{n} \ln(\sum_{|x|=n} \prod_{k=0}^{n-1} \omega(x_k, x_{k+1})) = \inf_{]0,1]} \frac{1}{\theta}\psi(\theta)$.

Proof. When $\psi'(1) < \psi(1)$, Biggins [3] shows that $\lim_{n\to\infty} \frac{1}{n}\ln(\sum_{|x|=n}\prod_{k=0}^{n-1}\omega(x_k,x_{k+1})) = \psi(1) = \inf_{]0,1]}\frac{1}{\theta}\times\psi(\theta)$. Therefore, let us assume that $\psi'(1)\ge\psi(1)$. By the argument of [7], we obtain,

$$\liminf_{n\to\infty}\frac{1}{n}\ln\left(\sum_{|x|=n}\prod_{k=0}^{n-1}\omega(x_k,x_{k+1})\right)\ge\inf_{]0,1]}\frac{1}{\theta}\psi(\theta).$$

Finally, let $\theta\in]0,\theta_c[$ where $\psi(\theta_c)=\inf_{]0,1]}\frac{1}{\theta}\psi(\theta)$. Since $(\sum_i a_i)^\theta\le\sum_i a_i^\theta$ for any $(a_i)_i$ with $a_i\ge 0$, it yields that

$$\limsup_{n\to\infty}\frac{1}{n}\ln\left(\sum_{|x|=n}\prod_{k=0}^{n-1}\omega(x_k,x_{k+1})\right)\le\frac{1}{\theta}\limsup_{n\to\infty}\frac{1}{n}\ln\left(\sum_{|x|=n}\prod_{k=0}^{n-1}\omega(x_k,x_{k+1})^\theta\right).$$

We see that (still by [3]) $\lim_{n\to\infty}\frac{1}{n}\ln(\sum_{|x|=n}\prod_{k=0}^{n-1}\omega(x_k,x_{k+1})^\theta)=\psi(\theta)$. It remains to let θ tend to θ_c. $\square$

4. The subexponential regime: Theorem 1.4

We prove (1.10) and (1.11) separately. We recall that the speed v of the walk verifies $v=\frac{E_{\mathbb{S}^e}[|X_{\Gamma_1}|]}{E_{\mathbb{S}^e}[\Gamma_1]}$.

Proof of Theorem 1.4: Eq. (1.10). Suppose that either "$i<\nu_{\min}^{-1}$ and $q_1=0$" or "$i<\nu_{\min}^{-1}$ and $s<1$." Let $a>1/v$ and $c_{24}>0$ such that $c_{24}<(E_{\mathbb{S}^e}[X_{\Gamma_1}])^{-1}$. We have

$$\mathbb{S}^e(\tau_n\ge an)\ge\mathbb{S}^e(\Gamma_{nc_{24}}\ge an)-\mathbb{S}^e(\Gamma_{nc_{24}}>\tau_n).$$

The second term on the right-hand side decays exponentially by Cramér's theorem applied to the random walk $(|X_{\Gamma_n}|,n\ge 0)$ (recall that $|X_{\Gamma_1}|$ has exponential moments by Fact A). The simple inequality $\mathbb{S}^e(\Gamma_{nc_{24}}\ge an)\ge\mathbb{S}^e(\Gamma_1\ge an)$ thus implies by Proposition 2.2 the lower bound of (1.10). Hence, we turn to the upper bound of (1.10). Part (i) of Lemma 6.3 of [5] states:

Lemma A (Dembo et al. [5]). *Let $Y_1,Y_2,\dots$ be an i.i.d. sequence with $E(Y_1^2)<\infty$. If $P(Y_1\ge x)\le\exp(-cx^\gamma)$ for some $0<\gamma<1$, $c>0$ and all x large enough, then for all $t>E[Y_1]$,*

$$\limsup_{n\to\infty}n^{-\gamma}\ln P\left(\frac{1}{n}\sum_{j=1}^{n}Y_j\ge t\right)\le -c\big(t-E[Y_1]\big)^\gamma.$$

By Proposition 2.2, $Y_1=\Gamma_1$ meets the conditions of the lemma. Therefore, take in Lemma A, $Y_i=\Gamma_i-\Gamma_{i-1}$ and $t=a/c_{25}$ where c_{25} is such that

$$\big(E_{\mathbb{S}^e}\big[|X_{\Gamma_1}|\big]\big)^{-1}<c_{25}<a\big(E_{\mathbb{S}^e}[\Gamma_1]\big)^{-1}.$$

In particular, we have $t>E_{\mathbb{S}^e}[\Gamma_1]$. As a result, $\mathbb{S}^e(\Gamma_n>tn)$ is stretched exponential. We also know that $\mathbb{S}^e(|X_{\Gamma_{nc_{25}}}|\le n)$ is exponentially small by Cramér's theorem ($1/c_{25}<E_{\mathbb{S}^e}[|X_{\Gamma_1}|]$). The relation $\mathbb{S}^e(\tau_n\ge an)\le\mathbb{S}^e(\Gamma_{nc_{25}}\ge an)+\mathbb{S}^e(|X_{\Gamma_{nc_{25}}}|\le n)$ thus completes the proof. $\square$

We finish with the case "$\Lambda<\infty$."

Proof of Theorem 1.4: Eq. (1.11). Suppose that $\Lambda<\infty$ and let a, c_{24} and c_{25} be as before. We write

$$\begin{aligned}\mathbb{S}^e(\Gamma_{nc_{24}}\ge an)&\ge\sum_{k=1}^{nc_{24}}\mathbb{S}^e\big(\{\Gamma_k-\Gamma_{k-1}\ge an\}\cap\{\Gamma_\ell-\Gamma_{\ell-1}<an,\forall\ell\ne k\}\big)\\&=nc_{24}\mathbb{S}^e(\Gamma_1\ge an)\mathbb{S}^e(\Gamma_1<an)^{nc_{24}-1}.\end{aligned}$$

By Proposition 2.4, $\mathbb{S}^e(\Gamma_1 \geq an) = n^{-\Lambda+o(1)}$. Therefore, $\mathbb{S}^e(\Gamma_1 < an)^{nc_{24}-1}$ tends to 1 (since $\Lambda > 1$). Consequently,

$$\mathbb{S}^e(\Gamma_{nc_{24}} \geq an) \geq n^{1-\Lambda+o(1)},$$

which gives the lower bound of (1.11), by the inequality $\mathbb{S}^e(\tau_n \geq an) \geq \mathbb{S}^e(\Gamma_{nc_{24}} \geq an) - \mathbb{S}^e(\Gamma_{nc_{24}} > \tau_n)$. Turning, to the upper bound, write as before $\mathbb{S}^e(\tau_n \geq an) \leq \mathbb{S}^e(\Gamma_{nc_{25}} \geq an) + \mathbb{S}^e(|X_{\Gamma_{nc_{25}}}| \leq n)$. We already know that $\mathbb{S}^e(|X_{\Gamma_{nc_{25}}}| \leq n)$ is exponentially small. Let $H_n := \Gamma_n - E_{\mathbb{S}^e}[\Gamma_1]n$. When $E[H_1^p] < \infty$, Example 2.6.5 of [15] says that if $p \geq 2$,

$$P(H_n > x) \leq (1 + 2/p)^p n E\big[H_1^p\big] x^{-p} + \exp\big(-2(p+2)^{-2} e^{-p} x^2 / \big(n E\big[H_1^2\big]\big)\big)$$

and Example 2.6.20 of [15], combined with Chebyshev's inequality, shows that if $1 \leq p \leq 2$,

$$P(H_n > x) \leq (2 - 1/n) n E\big[H_1^p\big] x^{-p}.$$

By Proposition 2.4, $E[H_1^p] < \infty$, for any $p < \Lambda$. We take $x = (\frac{a}{c_{25}} E_{\mathbb{S}^e}[|X_{\Gamma_1}|] - E_{\mathbb{S}^e}[\Gamma_1])n$ to see that $\mathbb{S}^e(\Gamma_{nc_{25}} \geq an) \leq c(p) n^{1-p}$ for any $p < \Lambda$. Let p tend to Λ in order to complete the proof of Eq. (1.11). □

Acknowledgments

I am very grateful to Zhan Shi for his guidance in the redaction of this work. I thank also the anonymous referee for suggesting me several improvements.

References

[1] E. Aidékon. Transient random walks in random environment on a Galton–Watson tree. *Probab. Theory Related Fields* **142** (2008) 525–559. MR2438700
[2] K. B. Athreya and P. E. Ney. *Branching Processes*. Springer, New York, 1972. MR0373040
[3] J. D. Biggins. Martingale convergence in the branching random walk. *J. Appl. Probab.* **14** (1977) 25–37. MR0433619
[4] F. Comets and V. Vargas. Majorizing multiplicative cascades for directed polymers in random media. *ALEA* **2** (2006) 267–277. MR2249671
[5] A. Dembo, N. Gantert, Y. Peres and O. Zeitouni. Large deviations for random walks on Galton–Watson trees: Averaging and uncertainty. *Probab. Theory Related Fields* **122** (2002) 241–288. MR1894069
[6] A. Dembo, Y. Peres and O. Zeitouni. Tail estimates for one-dimensional random walk in random environment. *Comm. Math. Phys.* **181** (1996) 667–683. MR1414305
[7] J. Franchi. Chaos multiplicatif: Un traitement simple et complet de la fonction de partition. In *Séminaire de Probabilités, XXIX* 194–201. *Lecture Notes in Math.* **1613**. Springer, Berlin, 1995. MR1459460
[8] T. Gross. Marche aléatoire en milieu aléatoire sur un arbre. Ph.D. thesis, 2004.
[9] H. Kesten, M. V. Kozlov and F. Spitzer. A limit law for random walk in a random environment. *Compos. Math.* **30** (1975) 145–168. MR0380998
[10] Q. Liu. On generalized multiplicative cascades. *Stochastic Process. Appl.* **86** (2000) 263–286. MR1741808
[11] R. Lyons and R. Pemantle. Random walk in a random environment and first-passage percolation on trees. *Ann. Probab.* **20** (1992) 125–136. MR1143414
[12] R. Lyons, R. Pemantle and Y. Peres. Biased random walks on Galton–Watson trees. *Probab. Theory Related Fields* **106** (1996) 249–264. MR1410689
[13] J. Neveu. Arbres et processus de Galton–Watson. *Ann. Inst. H. Poincaré Probab. Statist.* **22** (1986) 199–207. MR0850756
[14] R. Pemantle and Y. Peres. Critical random walk in random environment on trees. *Ann. Probab.* **23** (1995) 105–140. MR1330763
[15] V. V. Petrov. *Sums of Independent Random Variables*. Springer, New York, 1975. (Translated from the Russian by A. A. Brown, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 82.) MR0388499
[16] D. Piau. Théorème central limite fonctionnel pour une marche au hasard en environment aléatoire. *Ann. Probab.* **26** (1998) 1016–1040. MR1634413
[17] O. Zeitouni. Random walks in random environment. In *Lectures on Probability Theory and Statistics* 189–312. *Lecture Notes in Math.* **1837**. Springer, Berlin, 2004. MR2071631