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Law of large numbers for superdiffusions: The non-ergodic case

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Abstract. In previous work of D. Turaev, A. Winter and the author, the Law of Large Numbers for the local mass of certain superdiffusions was proved under an ergodicity assumption. In this paper we go beyond ergodicity, that is we consider cases when the scaling for the expectation of the local mass is not purely exponential. *Inter alia*, we prove the analog of the Watanabe–Biggins LLN for super-Brownian motion.

Résumé. Dans un travail précédent, l'auteur, D. Turaev et A. Winter, ont prouvé la Loi des Grand Nombres pour la masse locale de certaines diffusions sous une hypothèse d'ergodicité. Dans cet article nous allons au delà de l'ergodicité, plus précisement nous considérons des cas où le scaling de l'espérance de la masse locale n'est pas purement exponentiel. Entre autres, nous prouvons l'analogue de la LGN de Watanabe–Biggins pour le super mouvement brownien.

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1. Introduction and statement of results

Since this paper extends some results of [5], it will be helpful if the reader has [5] at hand, especially regarding notation and the notion of H-transformed (or weighted) superprocesses.

Let $D \subseteq \mathbb{R}^d$ be a domain. We use the standard notation $\mathcal{M}_f(D)$, $\|\mu\|$, $\mu \in \mathcal{M}_f(D)$, $C_b^+(D)$, $C_c^+(D)$, $C^{k,\eta}(D)$ and $C^\eta(D)$ exactly as in [5]. For example, $C^{k,\eta}(D)$ is the usual space of functions whose kth order derivatives are η-Hölder. Furthermore, $A \subset\subset D$ means that the closure of the bounded domain A is in D. Let L be an elliptic operator on D of the form

$$L := \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla, \tag{1}$$

where $a_{i,j}, b_i \in C^{1,\eta}(D)$, $i, j = 1, \ldots, d$, for some $\eta \in (0, 1]$, and the matrix $a(x) := (a_{i,j}(x))$ is symmetric, and positive definite for all $x \in D$. In addition, let $\alpha, \beta \in C^\eta(D)$, and assume that α is positive and β is bounded from above. In this paper X denotes the $(L, \beta, \alpha; D)$-superdiffusion (for the definition, even with time-inhomogeneous coefficients, see [5]). Here α is the 'variance' or 'intensity' parameter and β is the 'mass creation' or 'growth bias' term. The corresponding probabilities will be denoted by $\{\mathbf{P}^\mu; \mu \in \mathcal{M}_f(D)\}$.

In order to understand what follows, we need a brief review on some concepts from the *criticality theory of second order elliptic operators*. Let

$$\lambda_c = \lambda_c(L + \beta, D) := \inf\{\lambda \in \mathbb{R}\colon \exists u > 0 \text{ satisfying } (L + \beta - \lambda)u = 0 \text{ in } D\}$$

denote the *generalized principal eigenvalue* for $L+\beta$ on D. By standard theory, $\lambda_c<\infty$ whenever β is upper bounded. The operator $L+\beta-\lambda_c$ is called *critical* if the associated space of positive harmonic functions is nonempty but the operator does not possess a (minimal positive) Green's function. In this case the space of positive harmonic functions is in fact one dimensional. Moreover, the space of positive harmonic functions of the adjoint of $L+\beta-\lambda_c$ is also one dimensional. Suppose that ϕ and $\widetilde{\phi}$ are representatives of these two spaces. We say that $L+\beta-\lambda_c$ is *product-critical*, if, in addition to criticality, $\int_D \phi\widetilde{\phi}\,\mathrm{d}x<\infty$ holds (in which case one usually picks ϕ and $\widetilde{\phi}$ with the normalization $\int_D \phi\widetilde{\phi}\,\mathrm{d}x=1$).

Our principal interest is in establishing a Weak Law of Large Numbers (WLLN) for the local mass of certain superdiffusions. Since Pinsky proved that X exhibits local extinction if and only if $\lambda_c\le 0$, we can only hope to have the WLLN if we assume that $\lambda_c>0$. In fact, the following has been shown in [5]:

Proposition 1 ([5], Theorem 1). *In addition to the assumption* $\lambda_c>0$, *also assume that* $L+\beta-\lambda_c$ *is product-critical, that* $\alpha\phi_c$ *is bounded and that* X *starts in a state* μ *with* $\langle\mu,\phi_c\rangle<\infty$. *Let* $f\in C_c^+(D)$. *If* $f\not\equiv 0$ *and* $\|\mu\|\ne 0$, *then there exists a nonnegative non-degenerate random variable denoted by* $\overline{W}_\infty$ *satisfying that*

$$\lim_{t\to\infty}\frac{\langle X_t,f\rangle}{\mathbf{E}^\mu\langle X_t,f\rangle}=\frac{\overline{W}_\infty}{\langle\mu,\phi_c\rangle},\quad \text{in } \mathbf{P}^\mu\text{-probability}. \tag{2}$$

(The precise definition of $\overline{W}_\infty$ is given in the paragraph preceding Theorem 1 of this paper.)

In fact, product-criticality is equivalent to the property that the semigroup corresponding to $L+\beta$ scales precisely exponentially (see again [5], or see [4]). For example a simple case of a superdiffusion is when $D=\mathbb{R}^d,\ d\ge 1,\ L=\frac{1}{2}\Delta$, with α,β positive constants (supercritical super-Brownian motion). Then $\lambda_c=\beta,\ \phi=\widetilde{\phi}\equiv 1$ and so *this case is not included in the setup* of [5]. On the other hand, the corresponding (Strong) LLN is well known for *discrete particle systems*. Based on ideas in [7], Biggins [1] proved the Strong LLN for the case when branching-Brownian motion is replaced by branching random walk (in discrete time).

The purpose of this paper is to prove the WLLN for a class of superprocesses that *includes supercritical super-Brownian motion*. Instead of trying to adapt the Watanabe–Biggins approach to our setting, our method will use some ingredients from [5] and some results from [3,6] too. Note that the Watanabe–Biggins result gives an a.s. limit; thus, it would be interesting to see if our main result (see Theorem 1 later) can be strengthened to an a.s. result.

Throughout the paper the following, fairly mild assumption will be in force.

Assumption 1. *Let* $S=\{S_t\}_{t\ge 0}$ *denote the semigroup corresponding to* $L+\beta-\lambda_c$ *on* D.

(A.1) *(Local survival)* $\lambda_c>0$.
(A.2) *(Scaling of linear semigroup)* There exist two functions $s:(0,\infty)\to(0,\infty)$, and $h:D\to(0,\infty),\ h\in C^{2,\eta}(D)$, and a locally finite measure $r(\mathrm{d}x)$ such that
  (a) $\log s_t=\mathcal{O}(\log t)$ (i.e. $\sup_{t>0}\frac{\log s_t}{\log t}<\infty$) and $\lim_{t\to\infty}(\log s_t)'=0$,
  (b) $\sup_D \alpha h<\infty$,
  (c) $\lim_{t\to\infty}\langle\mu(\mathrm{d}x),s_t\cdot S_t(f)(x)\rangle=\langle r,f\rangle\cdot\langle\mu,h\rangle$ for all $f\in C_c^+(D)$, and $\mu\in\frac{1}{h}\mathcal{M}_f(D)$.
(A.3) *(Spatial spread)* There exist a function, $\zeta:(0,\infty)\to(0,\infty)$ and a family of subdomains $\{D_t;t\ge 0\},\ D_t\subset D$ such that
  (a) $\lim_{t\to\infty}\zeta_t=\infty$,
  (b) $\log(t+\zeta_t)=\mathrm{o}(t),\ t\to\infty$,
  (c) $\lim_{t\to\infty}s_{t+\zeta_t}/s_{\zeta_t}=1$,
  (d) $\lim_{t\to\infty}P^\mu[X_t(D_t^{\complement})>0]=0$,
  (e) if $f\in C_c^+(D)$, then

$$\lim_{t\to\infty}\sup_{x\in D_t}\left|\frac{s_{\zeta_t}}{h(x)}\cdot S_{\zeta_t}(f)(x)-\langle r,f\rangle\right|=0.$$

Let $H(x,t) := \exp(-\lambda_c t)h(x)$, where λ_c and h are as in Assumption 1 and consider the weighted superprocess $(X^H, \widetilde{\mathbf{P}}^{h\mu})$ defined by

$$X_t^H(\mathrm{d}x) := H(t,x)X_t(\mathrm{d}x), \quad t \geq 0.$$

Let $\overline{X_t}^H := \|X_t^H\|$. Following [5], we first show that

$$\lim_{t\to\infty} \widetilde{\mathrm{Var}}^{h\mu}\big(\overline{X_t}^H\big) = \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-2\lambda_c s}\big\langle \mu, \mathcal{S}_s\big[\alpha h^2\big]\big\rangle < \infty, \tag{3}$$

and that if the diffusion process $\widehat{Y}$ corresponding to $L_0^h := L + a\frac{\nabla h}{h} \cdot \nabla$ on D is conservative (that is, it never leaves D with probability one), then $\overline{X}^H$ is a uniformly integrable (UI) $\widetilde{\mathbf{P}}^{h\mu}$-martingale, whereas in general, $\overline{X}^H$ is a (non-negative) $\widetilde{\mathbf{P}}^{h\mu}$-supermartingale.

Let $\mathfrak{S} = \{\mathfrak{S}_s\}_{s\geq 0}$ denote the semigroup corresponding to $\widehat{Y}$ (i.e. to L_0^h), that is, $\mathfrak{S} := S^h$. Clearly, $\mathfrak{S}_s 1 \leq 1$. (If $\widehat{Y}$ is conservative, then actually $\mathfrak{S}_s 1 = 1$.) If $\widehat{Y}$ is conservative, then consider the class

$$\mathfrak{C}(D) := \big\{ f \in C^2(D) : \exists \mho \subset D \text{ bounded s.t. } \overline{\mho} \subset D;\ f = \text{const on } D \setminus \mho \big\}.$$

By Theorem A2 in [2], we have that for all $f \in \mathfrak{C}(D)$,

$$\mathrm{d}\big\langle X_t^H, f\big\rangle = \big\langle X_t^H, L_0^h f\big\rangle \mathrm{d}t + \mathrm{d}M_t(f), \tag{4}$$

where $\{M_t(f)\}_{t\geq 0}$ is a square-integrable $\widetilde{\mathbf{P}}^{h\mu}$-martingale, and its quadratic variation $\langle M(f)\rangle$ is given by

$$\big\langle M(f)\big\rangle_t = \int_0^t \mathrm{d}s\, \mathrm{e}^{-\lambda_c s}\big\langle X_s^H, \alpha h f^2\big\rangle, \quad t \geq 0. \tag{5}$$

(The point is that one can take the function class $\mathfrak{C}(D)$ instead of just $C_c^2(D)$ when $\widehat{Y}$ is conservative.)

Applying (4) to the function $f \equiv 1$, it follows that $\overline{X}^H$ is a $\widetilde{\mathbf{P}}^{h\mu}$-martingale. Furthermore, by (5),

$$\widetilde{\mathbf{E}}^{h\mu}\big[\big\langle X_t^H, 1\big\rangle^2\big] = \langle \mu, h\rangle^2 + \int_0^t \mathrm{d}s\, \mathrm{e}^{-\lambda_c s}\big\langle h\mu, \mathfrak{S}_s[\alpha h]\big\rangle. \tag{6}$$

That is

$$\widetilde{\mathrm{Var}}^{h\mu}\big(\overline{X}_t^H\big) = \int_0^t \mathrm{d}s\, \mathrm{e}^{-\lambda_c s}\big\langle h\mu, \mathfrak{S}_s[\alpha h]\big\rangle = \int_0^t \mathrm{d}s\, \mathrm{e}^{-2\lambda_c s}\big\langle \mu, S_s\big[\alpha h^2\big]\big\rangle. \tag{7}$$

Letting $t \to \infty$ we obtain (3). Replacing t by ∞ in the first of the integrals in (7), we have from the fact that $\mathfrak{S}_s 1 \leq 1$ and from our assumptions that

$$\widetilde{\mathrm{Var}}^{h\mu}\big(\overline{X}_t^H\big) \leq \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-\lambda_c s}\big\langle h\mu, \mathfrak{S}_s[\alpha h]\big\rangle \leq {\lambda_c}^{-1} \|\alpha h\|_\infty \langle \mu, h\rangle < \infty.$$

Hence, by (6), $\sup_{t\geq 0} \widetilde{\mathbf{E}}^{\mu h}(\overline{X}_t^H)^2 < \infty$, and consequently $\overline{X}^H$ is UI.

Abbreviate $W := X^H$ and let $\overline{W}$ denote the *total mass process*: $\overline{W} := \overline{X}^H = \|X^H\|$. Let $\mathfrak{D} := D \cup \{\Delta\}$ be the *one-point compactification* of D (when the underlying diffusion process $\widehat{Y}$ is non-conservative on D, Δ is the *cemetery state* for $\widehat{Y}$). Relaxing the assumption on the conservativeness of $\widehat{Y}$, the argument in [2], pp. 726–727 shows that, although one can *not* work directly with the function class $\mathfrak{C}(D)$ (only with its subclass $C_c^2(D)$), one can extend $(W, \widetilde{\mathbf{P}})$ appropriately and get $(\mathfrak{X}, \mathfrak{P})$ on $\mathfrak{D}$ making $\|\mathfrak{X}\|$ a $\mathfrak{P}^{h\mu}$-martingale. Since the mass on the cemetery state Δ is nondecreasing in time, $\overline{W}$ is a $\widetilde{\mathbf{P}}^{h\mu}$-*supermartingale*. (In the non-conservative case, intuitively, mass is 'lost' at the Euclidean boundary of D or at infinity.)

Now, if $\overline{W}_\infty := \lim_{t\to\infty} \overline{W}_t$, then one does not have $\mathbf{E}^\mu \overline{W}_\infty = \langle \mu, h\rangle$ in general. Nevertheless, as we have seen, when $\widehat{Y}$ is conservative on D, $\overline{W}$ is a UI martingale and then, of course, $\mathbf{E}^\mu \overline{W}_\infty = \langle \mu, h\rangle > 0$ for $\|\mu\| \neq 0$.

Let $\mathcal{M}_c(D) \subset \mathcal{M}_f(D)$ denote the subspace of finite measures with compact support in D. Our main result is as follows.

Theorem 1 (WLLN). *With the notations of Assumption* 1, *if* $0 \not\equiv f \in C_c^+(D)$ *and* $\mathbf{0} \not\equiv \mu \in \mathcal{M}_c(D)$, *then in* $\mathbf{P}^\mu$*-probability*,

$$\lim_{t\to\infty} \frac{\langle X_t, f\rangle}{\mathbf{E}^\mu \langle X_t, f\rangle} = \frac{\overline{W}_\infty}{\langle \mu, h\rangle}.$$

The limit is mean-one (*and thus*, *it is not identically zero*) *when* L_0^h *corresponds to a conservative diffusion.*

(The proof is given in Section 2.)

In order to give a simple condition for the limit to be mean-one, let us recall that the superprocess X possesses the *compact support property* if $\mathbf{P}^\mu(\bigcup_{0\le s\le t} \operatorname{supp}(X_s) \subset\subset D) = 1$, for all $\mu \in \mathcal{M}_c(D)$, $t \ge 0$. Since there are various conditions given in [2,3] for the compact support property to hold, the following result is useful. (The proof is given in Section 2.)

Theorem 2 (No loss of mass in the limit). *If the compact support property holds*, *then the diffusion process corresponding to* L_0^h *on* D *is conservative*, *and consequently*, *the limit appearing in Theorem* 1 *is mean-one.*

We close this section with examples for $D = \mathbb{R}^d$.

Example 1 (Supercritical SBM). The assumptions are satisfied for supercritical super-Brownian motion. Indeed, if $\beta(\cdot) \equiv \beta > 0$, then $\lambda_c = \beta$, because $\lambda_c(\Delta, \mathbb{R}^d) = 0$. Furthermore choose $h \equiv 1$, $r(\mathrm{d}x) := \mathrm{d}x$ and the Brownian scaling factor $s_t := t^{d/2}$. Finally, as far as the spatial spread of the process is concerned, D_t can be defined as $D_t := B_{(\sqrt{2\beta}+\varepsilon)t}$, $\varepsilon > 0$, where B_r denotes the ball of radius $r > 0$ centered at the origin (see [6]); thus ζ_t can be defined e.g. as $\zeta_t = t^m$ with $m > 2$. This setting satisfies conditions (A.1)–(A.3), as long as $0 < \alpha$ is bounded from above, and so Theorem 1 holds.

Consider now the simplest case of the previous example, the one when α is a positive constant. Then the non-degenerate random variable $\overline{W}_\infty$ can be thought of as the scaled limit of a one dimensional diffusion. Indeed, $Y := \|X\|$ is a diffusion corresponding to the operator $x(\alpha \frac{\partial^2}{\partial x^2} + \beta \frac{\partial}{\partial x})$ on $[0, \infty)$ with $Y_0 = \|\mu\|$, and $\overline{W}_\infty = \lim_{t\to\infty} \mathrm{e}^{-\beta t} Y_t$.

Next, take a smooth, positive, but otherwise arbitrary function $h : D \to \mathbb{R}$ and recall that the spatial h-transform is a particular case of the space–time H-transform with $H(t, x) = h(x)$, $t \ge 0$.

Let X be the supercritical super-Brownian motion of Example 1 ($\beta > 0$ is constant and $\alpha > 0$ is upper bounded). Since the WLLN holds true for X starting with any nonzero measure in $\mathcal{M}_c(D)$, therefore it is also true for X^h starting with any nonzero measure in $\mathcal{M}_c(D)$.

Then, different particular choices of h lead to different further examples as follows. Write $x = (x_1, x_2, \ldots, x_d)$ and $x^2 := |x|^2 = \sum_{i=1}^d x_i^2$. The supercritical SBM with drift, supercritical SBM with outward drift, supercritical Super Ornstein–Uhlenbeck process with quadratic β and supercritical outward SOU with quadratic β can all be treated by applying h-transforms (where $h(x) := \mathrm{e}^{cx_1}$, $h(x) := \mathrm{e}^{c|x|}$, $|x| \gg 1$, $h(x) := \mathrm{e}^{-cx^2}$ and $h(x) := \mathrm{e}^{cx^2}$, respectively and $c > 0$). The WLLN for these cases follows from h-transform invariance and the limiting random variable is always *non-degenerate*. Since $\alpha^h = \alpha h$, in each case α has to satisfy $\alpha(x) = \mathcal{O}(h(x))$, as $|x| \to \infty$.

2. Proofs

Proof of Theorem 1. We first claim that $(L+\beta)h = \lambda_c h$. To see, this note that (A.2)(c) with $\mu = \delta_x$ yields $\lim_{t\to\infty} s_t \cdot S_t(f)(x) = \langle r, f\rangle \cdot h(x)$ for $x \in D$ and $f \in C_c^+(D)$. Using the notation u' for time derivative and defining $\widehat{\beta}(t, x) := \beta(x) + (\log s_t)'$, the equation $(L+\beta)h = \lambda_c h$ follows from the fact that $u(t, x) := S_t(f)(x)$ solves $(L+\beta-\lambda_c)u = u'$ and therefore $v(t, x) := s_t \cdot u(t, x)$ solves $(L+\beta-\lambda_c)v = v' - (\log s_t)' v$, that is $(L+\widehat{\beta}-\lambda_c)v = v'$. By (A.2)(a),

$\sup_{x\in D}|\widehat{\beta}(t,x)-\beta(x)|$ tends to zero as $t\uparrow\infty$. Then a standard argument (see [2], p. 708) together with the second relation in (A.2)(a) gives that $w(x):=\lim_{t\to\infty}v(t,x)=\langle f,r(\mathrm{d}y)\rangle\cdot h(x)$ belongs to $C^{2,\eta}(D)$ and solves the steady state equation $(L+\beta-\lambda_c)w=0$. Then, also

$$(L+\beta-\lambda_c)h=0, \tag{8}$$

and

$$L_0^h(u)=(L+\beta-\lambda_c)^h(u)=h^{-1}(L+\beta-\lambda_c)(hu)=H^{-1}(L+\beta+\partial_t)(Hu).$$

By (8), $(X^H,\widetilde{\mathbf{P}})$ is the $(L_0^h,0,\alpha h\mathrm{e}^{-\lambda_c t};D)$-superdiffusion. Recalling the definition of $\mathfrak{S}$ and defining $\nu:=h\mu,\ g:=f/h,\ q:=hr$, one can reformulate (A.2)(c):

$$\text{(A*.2)(c)}\quad \lim_{t\to\infty}\big\langle\nu(\mathrm{d}x),s_t\cdot\mathfrak{S}_t(g)(x)\big\rangle=\langle q,g\rangle\cdot\|\nu\|,\quad g\in C_c^+(D),\ \nu\in\mathcal{M}_f(D).$$

Similarly, (A.3)(d)–(A.3)(e) become

$$\text{(A*.3)(d)}\quad \lim_{t\to\infty}\widetilde{P}^{\nu}\big[W_t\big(D_t^{\complement}\big)>0\big]=0\quad\text{and}$$

$$\text{(A*.3)(e)}\quad \lim_{t\to\infty}\sup_{x\in D_t}\big|s_{\zeta_t}\cdot\mathfrak{S}_{\zeta_t}(g)(x)-\langle q,g\rangle\big|=0,\quad g\in C_c^+(D).$$

Finally, the theorem itself transforms into the following statement: if $0\not\equiv g\in C_c^+(D)$ and $\mathbf{0}\not\equiv\nu\in\mathcal{M}_c(D)$, then in $\widetilde{\mathbf{P}}^{\nu}$-probability,

$$\lim_{t\to\infty}\frac{\langle X_t^H,g\rangle}{\widetilde{\mathbf{E}}^{\nu}\langle X_t^H,g\rangle}=\frac{\overline{W}_\infty}{\|\nu\|}\quad\text{or, equivalently,}\quad \lim_{t\to\infty}s_t\langle W_t,g\rangle=\langle q,g\rangle\overline{W}_\infty. \tag{9}$$

(Recall (A*.2)(c) and note that $\mathfrak{S}$ is the *expectation* semigroup.)

In order to show (9), the main idea is to use the comparison with the deterministic flow as in [5], however, there is an essential difference. In [5] we argued that by considering some large time $t+T$ (both t and T are large), the changes of $\overline{X}^H$ are negligible after t, while the remaining time T is still long enough to distribute the produced mass according to the ergodic flow given by the H-transformed migration. We then let $T\uparrow\infty$ and then $t\uparrow\infty$.

Reading carefully the proof in [5] one can see that this method relied heavily on the ergodicity of the flow and would break down here. Hence, instead of letting first T and then t go to infinity, we now define $T:=T_t=\zeta_t$. Similarly to [5], the strategy is to first show that the total mass more or less stabilizes by time T_t, then to identify the limit of the *scaled flow* (starting from the state of the process at T_t) at time $t+T_t$, and finally to show that it agrees with the scaling limit of the process itself. Of course, the first part is simple: being a supermartingale, the *total mass converges*:

$$\lim_{t\to\infty}\|W_t\|=\overline{W}_\infty,\quad \widetilde{\mathbf{P}}^{\nu}\text{-a.s.} \tag{10}$$

Unlike in [5], however, we do not know a priori, that the limit is non-zero, and moreover, one cannot proceed further without exploiting what is known about the radial speed of the process. Therefore we continue as follows. Let $\{Z_{W_t}(s)\}_{s\geq 0}$ denote the deterministic flow starting from the (random) measure W_t. Since given W_t, $\langle Z_{W_t}(\zeta_t),g\rangle=\langle W_t(\mathrm{d}x),\mathfrak{S}_{\zeta_t}(g)(x)\rangle$, one has

$$\begin{aligned}&\widetilde{\mathbf{P}}^{\nu}\big(\big|s_{\zeta_t}\big\langle Z_{W_t}(\zeta_t),g\big\rangle-\|W_t\|\langle q,g\rangle\big|>\varepsilon\big)\\&\quad\leq\widetilde{\mathbf{P}}^{\nu}\Big[\|W_t\|\sup_{x\in D_t}\big|s_{\zeta_t}\big(\mathfrak{S}_{\zeta_t}(g)\big)(x)-\langle q,g\rangle\big|>\varepsilon\Big]+\widetilde{\mathbf{P}}^{\nu}\big[W_t\big(D_t^{\complement}\big)>0\big].\end{aligned}$$

Let us call the two terms on the righthand side A_t and B_t. By (A*.3)(d and e), one has $\lim_{t\to\infty}A_t=\lim_{t\to\infty}B_t=0$. Hence,

$$\lim_{t\to\infty}\widetilde{\mathbf{P}}^{\nu}\big(\big|s_{\zeta_t}\big\langle Z_{W_t}(\zeta_t),g\big\rangle-\|W_t\|\langle q,g\rangle\big|>\varepsilon\big)=0. \tag{11}$$

From this, (A.3)(c) and (10), one obtains the *scaling limit of the flow*:

$$\lim_{t\to\infty} \widetilde{\mathbf{P}}^{\nu}\big(\big|s_{t+\zeta_t}\langle Z_{W_t}(\zeta_t), g\rangle - \overline{W}_\infty\langle q, g\rangle\big| > \varepsilon\big) = 0. \tag{12}$$

Our goal is therefore to show that *the scaling limit of the flow agrees with the scaling limit of the measure-valued process*. To achieve this, recall (A.2)(b). A computation using Chebysev and the supermartingale property (essentially the same computation as the one giving formula (28) in [5]) yields:

$$\widetilde{\mathbf{P}}^{\nu}\big(\big|s_{t+\zeta_t}\langle Z_{W_t}(\zeta_t), g\rangle - s_{t+\zeta_t}\langle W_{t+\zeta_t,g}\rangle\big| > \varepsilon\big) \le \frac{C\widetilde{\mathbf{E}}^{\nu}\|W_t\| s^2_{t+\zeta_t}}{\varepsilon^2\lambda_c e^{\lambda_c t}} \le \frac{C\|\nu\| s^2_{t+\zeta_t}}{\varepsilon^2\lambda_c e^{\lambda_c t}},$$

where $C = C(\|g\|, \|\alpha h\|) := 18\,\|\alpha h\|\cdot\|g\|^2$. Recall the abbreviation $T := \zeta_t$. To finish the estimate it is enough to show that $\lim_{t\to\infty} e^{-\lambda_c t} s^2_{t+T} = 0$. By the first condition in (A.2)(a) and by (A.3)(b), there is a $k > 0$ such that

$$\log\big(e^{-\lambda_c t} s^2_{t+T}\big) = -\lambda_c t + 2\log s_{t+T} \le -\lambda_c t + 2k\log(t+T)$$

and

$$\lim_{t\to\infty}\big[-\lambda_c t + 2k\log(t+T)\big] = \lim_{t\to\infty} t\left[-\lambda_c + \frac{2k\log(t+T)}{t}\right] = -\infty. \qquad \square$$

Proof of Theorem 2. Suppose that X possesses the compact support property but the diffusion corresponding to L_0^h is not conservative. Since the support of the superprocess is invariant under h-transforms, X^h possesses the compact support property too. However, by Theorem 3 in [3], if the diffusion process corresponding to L on D is not conservative and $\sup_{x\in D}\alpha(x) < \infty$ and $\inf_{x\in D}\beta(x) > -\infty$, then the compact support property does not hold. (In [3] the domain is $D = \mathbb{R}^d$, but the proof goes through for general D.) Since X^h is the $(L_0^h, \lambda_c, \alpha h; D)$-superprocess, L_0^h is not conservative, and αh is bounded from above, we got a contradiction. $\square$

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