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IGUSA INTEGRALS AND VOLUME ASYMPTOTICS IN ANALYTIC AND ADELIC GEOMETRY

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We establish asymptotic formulas for volumes of height balls in analytic varieties over local fields and in adelic points of algebraic varieties over number fields, relating the Mellin transforms of height functions to Igusa integrals and to global geometric invariants of the underlying variety. In the adelic setting, this involves the construction of general Tamagawa measures.

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1. Introduction

The study of rational and integral points on algebraic varieties defined over a number field often leads to considerations of volumes of real, p-adic or adelic spaces. A typical problem in arithmetic geometry is to establish asymptotic expansions, when $B \to \infty$, for the number $N_f(B)$ of solutions in rational integers smaller than B of a polynomial equation $f(\mathbf{x}) = 0$.

When applicable, the circle method gives an answer in terms of a "singular integral" and a "singular series", which itself can be viewed as a product of p-adic densities. The size condition is only reflected in a parameter in the singular integral, whose asymptotic expansion therefore governs that of $N_f(B)$.

More generally, one considers systems of polynomial equations, i.e. algebraic varieties over a number field or schemes of finite type over rings of integers, together with embedding into a projective or affine space. Such an embedding induces a *height function* (see, e.g. [34, 43, 30]) such that there are only finitely many solutions of bounded height, in a fixed number field (resp. ring of integers). A natural generalization of the problem above is to understand the asymptotic behavior of the number of such solutions, as well as their distribution in the ambient space for the local or adelic topologies, when the bound grows to infinity.

Apart from applications of the circle method, many other instances of this problem have been successfully investigated in recent years, in particular, in the context of linear algebraic groups and their homogeneous spaces. For such varieties, techniques from ergodic theory and harmonic analysis are very effective; for integral points, see [22–24, 7, 36, 28]; for rational points, see [3, 25, 38, 5, 13].

In most cases, the proof is subdivided into two parts:

(1) comparison of the point counting with a volume asymptotic;
(2) explicit computation of this volume asymptotic.

In this paper, we develop a general geometric framework for the second part, i.e. for the understanding of densities and volumes occurring in the counting problems above. We now explain the main results.

1.1. *Tamagawa measures for algebraic varieties*

Let F be a number field. Let $\mathrm{Val}(F)$ be the set of equivalence classes of absolute values of F. For $v \in \mathrm{Val}(F)$, we write $v \mid p$ if v defines the p-adic topology on $\mathbf{Q}$, and $v \mid \infty$ if it is archimedean. For $v \in \mathrm{Val}(F)$, let F_v be the corresponding completion of F and, if v is ultrametric, let $\mathfrak{o}_v$ be its ring of integers. We identify v with the specific absolute value $|\cdot|_v$ on F_v defined by the formula $\mu(a\Omega) = |a|_v \mu(\Omega)$, where μ is any Haar measure on the additive group F_v, $a \in F_v$ and Ω is a measurable subset of F_v of finite measure.

Let X be a smooth projective algebraic variety over F. Fix an adelic metric on its canonical line bundle K_X (see Sec. 2.2.3). For any $v \in \mathrm{Val}(F)$, the set $X(F_v)$ carries an analytic topology and the chosen v-adic metric on K_X induces a Radon measure $\tau_{X,v}$ on $X(F_v)$ (see Sec. 2.1.7).

Let $\mathscr{X}$ be a projective flat model of X over the ring of integers of F. Then, for almost all finite places v, the measure $\tau_{X,v}$ coincides with the measure on $\mathscr{X}(\mathfrak{o}_v) = X(F_v)$ defined by Weil in [47].

Let D be an effective divisor on X which, geometrically, has strict normal crossings and set $U = X \backslash D$. Let f_D denote the canonical section of the line bundle $\mathscr{O}_X(D)$ (corresponding to the regular function 1 when $\mathscr{O}_X(D)$ is viewed as a subsheaf of the sheaf of meromorphic functions); by construction, its divisor is D. Let us also fix an adelic metric on this line bundle. We let $\mathscr{D}$ be the Zariski closure of D in the model $\mathscr{X}$ and $\mathscr{U} = \mathscr{X} \backslash \mathscr{D}$ be its complement.

For any place $v \in \mathrm{Val}(F)$, we define a measure

$$\tau_{(X,D),v} = \frac{1}{\|\mathsf{f}_D\|_v} \tau_{X,v}$$

on $U(F_v)$. Note that it is still a Radon measure; however $U(F_v)$ has infinite volume. If U is an algebraic group, this construction allows one to recover the Haar measure of $U(F_v)$ (see Sec. 2.1.11).

Let $\mathbb{A}_F$ be the adele ring of F, that is, the restricted product of the fields F_v with respect to the subrings $\mathfrak{o}_v$.

A nonzero Radon measure τ on the adelic space $U(\mathbb{A}_F)$ induces measures τ_v on any of the sets $U(F_v)$, which are well-defined up to a factor. Conversely, we can recover the Radon measure τ as the product of such measures τ_v if the set of measures $(\tau_v)_{v\nmid\infty}$ satisfies the convergence condition: *the infinite product* $\prod_{v\nmid\infty}\tau_v(\mathscr{U}(\mathfrak{o}_v))$ *is absolutely convergent.*

A family of convergence factors for (τ_v) is a family $(\lambda_v)_{v\nmid\infty}$ of positive real numbers such that the family of measures $(\lambda_v\tau_v)$ satisfies the above convergence condition.

Our first result in this paper is a definition of a measure on $U(\mathbb{A}_F)$ via an appropriate choice of convergence factors.

Let $\bar{F}$ be an algebraic closure of F and let $\Gamma = \mathrm{Gal}(\bar{F}/F)$ be the absolute Galois group. Let M be a free $\mathbf{Z}$-module of finite rank endowed with a continuous action of Γ; we let $\mathrm{L}(s, M)$ be the corresponding Artin L-function, and, for all finite places $v \in \mathrm{Val}(F)$, $\mathrm{L}_v(s, M)$ its local factor at v. The function $s \mapsto \mathrm{L}(s, M)$ is holomorphic for $\mathrm{Re}(s) > 1$ and admits a meromorphic continuation to $\mathbf{C}$; let ρ be its order at $s = 1$ and define

$$\mathrm{L}^*(1, M) = \lim_{s\to 1}(s-1)^{-\rho}\mathrm{L}(s, M);$$

it is a positive real number.

Theorem 1.1. *Assume that* $\mathrm{H}^1(X, \mathscr{O}_X) = \mathrm{H}^2(X, \mathscr{O}_X) = 0$. *The abelian groups* $M^0 = \mathrm{H}^0(U_{\bar{F}}, \mathbf{G}_m)/\bar{F}^*$ *and* $M^1 = \mathrm{H}^1(U_{\bar{F}}, \mathbf{G}_m)/$*torsion are free* $\mathbf{Z}$*-modules of finite rank with a continuous action of* Γ. *Moreover, the family* (λ_v) *given by*

$$\lambda_v = \mathrm{L}_v(1, M^0)/\mathrm{L}_v(1, M^1)$$

is a family of convergence factors.

Assume that the hypotheses of the theorem hold. We then define a Radon measure on $U(\mathbb{A}_F)$ by the formula

$$\tau_{(X,D)} = \frac{\mathrm{L}^*(1, M^1)}{\mathrm{L}^*(1, M^0)}\prod_{v\in\mathrm{Val}(F)}(\lambda_v\tau_{(X,D),v}),$$

where λ_v is given by the theorem if $v \nmid \infty$ and $\lambda_v = 1$ else. We call it the *Tamagawa measure on* U, *or*, *more precisely*, *on* $U(\mathbb{A}_F)$. This generalizes the construction of a Tamagawa measure on an algebraic group, where there is no M^1 (see, e.g. [47]) or on a projective variety, where there is no M^0, in [38].

1.2. *Volume asymptotics in analytic geometry*

Let L be an effective divisor in X whose support contains the support of D; again let f_L be the canonical section of the line bundle $\mathscr{O}_X(L)$. Fix an adelic metric on $\mathscr{O}_X(L)$ and a place $v \in \mathrm{Val}(F)$. For any positive real number B, the set of all $x \in U(F_v)$ such that $\|\mathsf{f}_L(x)\|_v \geq 1/B$ is a compact subset in $U(F_v)$. It thus has finite volume $V(B)$ with respect to the measure $\tau_{(X,D),v}$.

Let us decompose the divisor $D_v = D_{F_v}$ as a sum of irreducible divisors:

$$D_v = \sum_{\alpha \in \mathscr{A}_v} d_{\alpha,v} D_{\alpha,v}.$$

For $\alpha \in \mathscr{A}_v$, let $\lambda_{\alpha,v}$ be the multiplicity of $D_{\alpha,v}$ in L_v; there exists an effective divisor E_v on X_{F_v} such that

$$L_v = E_v + \sum_{\alpha \in \mathscr{A}} \lambda_{\alpha,v} D_{\alpha,v}.$$

For any subset $A \subseteq \mathscr{A}_v$, we let $D_{A,v}$ be the intersections of the $D_{\alpha,v}$, for $\alpha \in A$.

Now let $a_v(L,D)$ be the least rational number such that for any $\alpha \in \mathscr{A}_v$, with $D_{\alpha,v}(F_v) \neq \varnothing$, one has $a_v(L,D)\lambda_{\alpha,v} \geq d_{\alpha,v} - 1$. Let $\mathscr{A}_v(L,D)$ be the set of those $\alpha \in \mathscr{A}_v$ where equality holds and $b_v(L,D)$ the maximal cardinality of subsets $A \in \mathscr{A}_v(L,D)$ such that $D_{A,v}(F_v) \neq \varnothing$. To organize the combinatorial structure of these subsets, we introduce variants of the simplicial complex considered, e.g. in [16] in the context of Hodge theory.

Theorem 1.2. *Assume that v is archimedean.*

If $a_v(L,D) > 0$, then $b_v(L,D) \geq 1$ and there exists a positive real number c such that

$$V(B) \sim cB^{a_v(L,D)}(\log B)^{b_v(L,D)-1}.$$

If $a_v(L,D) = 0$, then there exists a positive real number c such that

$$V(B) \sim cB^{a_v(L,D)}(\log B)^{b_v(L,D)}.$$

With the notation above, we also give an explicit formula for the constant c. It involves integrals over the sets $D_{A,v}(F_v)$ such that $\#(A) = b_v(L,D)$, with respect to measures induced from $\tau_{(X,D),v}$ via the adjunction formula.

To prove this theorem, we introduce the Mellin transform

$$Z(s) = \int_{U(F_v)} \|\mathsf{f}_L(x)\|_v^s \, \mathrm{d}\tau_{(X,D),v}(x)$$

and establish its analytic properties. We regard $Z(s)$ as an integral over the compact analytic manifold $X(F_v)$ of the function $\|\mathsf{f}_L\|_v^s$ with respect to a singular measure, connecting the study of such zeta functions with the theory of Igusa local zeta functions, see [31, 32]. In particular, we show that $Z(s)$ is holomorphic for $\mathrm{Re}(s) > a_v(L,D)$ and admits a meromorphic continuation to some half-plane $\{\mathrm{Re}(s) >$

$a_v(L,D)-\varepsilon\}$, with a pole at $s=a_v(L,D)$ of order $b_v(L,D)$. This part of the proof works over any local field.

If v is archimedean (and $\varepsilon>0$ is small enough), then $Z(s)$ has no other pole in this half-plane. Our volume estimate then follows from a standard Tauberian theorem. When v in non-archimedean, we can only deduce a weaker estimate, i.e. upper and lower bounds of the stated order of magnitude (Corollary 4.9).

1.3. *Asymptotics of adelic volumes*

Assume that $\mathrm{H}^1(X,\mathscr{O}_X)=\mathrm{H}^2(X,\mathscr{O}_X)=0$. Theorem 1.1 gives us Tamagawa measures $\tau_{(X,D)}$ and τ_X on the adelic spaces $U(\mathbb{A}_F)$ and $X(\mathbb{A}_F)$ respectively.

Suppose furthermore that the supports of the divisors L and D are equal. We then define a *height function* H_L on the adelic space $U(\mathbb{A}_F)$ by the formula

$$H_L((x_v)_v)=\prod_{v\in\mathrm{Val}(F)}\|\mathsf{f}_L(x_v)\|_v^{-1}.$$

This function $H_L\colon U(\mathbb{A}_F)\to\mathbf{R}_+$ is continuous and proper. In particular, for any real number B, the subset of $U(\mathbb{A}_F)$ defined by the inequality $H_L(\mathbf{x})\leq B$ is compact, hence has finite volume $V(B)$ with respect to $\tau_{(X,D)}$. We are interested in the asymptotic behavior of $V(B)$ as $B\to\infty$.

Let us decompose the divisors L and D as the sum of their irreducible components (over F). Since L and D have the same support, one can write

$$D=\sum_{\alpha\in\mathscr{A}}d_\alpha D_\alpha,\quad L=\sum_{\alpha\in\mathscr{A}}\lambda_\alpha D_\alpha$$

for some positive integers d_α and λ_α. Let $a(L,D)$ be the least positive rational number such that the $\mathbf{Q}$-divisor $E=a(L,D)L-D$ is effective; in other words,

$$a(L,D)=\max_{\alpha\in\mathscr{A}}d_\alpha/\lambda_\alpha.$$

Let moreover $b(L,D)$ be the number of $\alpha\in\mathscr{A}$ for which equality is achieved.

To the $\mathbf{Q}$-divisor E, we can also attach a height function on $X(\mathbb{A}_F)$ given by

$$H_E(\mathbf{x})=\prod_{v\in\mathrm{Val}(F)}\|\mathsf{f}_L(x_v)\|_v^{-a(L,D)}\|\mathsf{f}_D(x,v)\|_v$$

if $x_v\in U(F_v)$ for all v, and by $H_E(\mathbf{x})=+\infty$ else. The product can diverge to $+\infty$ but H_E has a positive lower bound, reflecting the effectivity of E. In fact, the function H_E^{-1} is continuous on $X(\mathbb{A}_F)$.

Theorem 1.3. *When $B\to\infty$, one has the following asymptotic expansion*

$$V(B)\sim\frac{1}{a(L,D)(b(L,D)-1)!}B^{a(L,D)}(\log B)^{b(L,D)-1}\int_{X(\mathbb{A}_F)}H_E(\mathbf{x})^{-1}\,\mathrm{d}\tau_X(\mathbf{x}).$$

As a particular case, let us take $L = D$. We see that $a(L, D) = 1$, $b(L, D)$ is the number of irreducible components of D and the integral in the theorem is equal to the Tamagawa volume $\tau_X(X(\mathbb{A}_F))$ defined by Peyre.

In both local and adelic situations, our techniques are valid for any metrization of the underlying line bundles. As was explained by Peyre in [38] in the context of rational points, this implies *equidistribution theorems*, see Corollary 4.8 in the local case, and Theorem 4.13 in the adelic case.

The referee suggested to consider local and global Mellin transforms involving more general quasi-characters. Indeed, such integrals do play a role in the study of height zeta functions, see, e.g. our recent paper [15]. For example, we were led to investigate integrals of the form

$$I(\sigma, \mathbf{s}, \lambda, \chi, \psi) = \int_{X_\sigma(F_v)} \|\mathbf{x}\|_v^{\mathbf{s}} \chi_v(\mathbf{x}_v) \psi_v({\mathbf{x}_v}^{\lambda}) \mathrm{d}x_v^*.$$

In that formula, X_σ is an *affine* toric variety with underlying torus T over a number field F, the groups of characters and cocharacter of which are denoted $\mathfrak{X}^*(T)$ and $\mathfrak{X}_*(T)$, σ is a cone in $\mathfrak{X}_*(T)_{\mathbf{R}}$ defining X_σ, χ_v is the v-adic component of an automorphic character $\chi \in (T(\mathbb{A}_F)/T(F))^*$, ψ_v is an additive character of F_v, and $\lambda \in \mathfrak{X}^*(T)$ the oscillatory phase. We proved that I is holomorphic for $\mathrm{Re}(\mathbf{s})$ contained in the interior of the cone spanned by the dual cone σ^* and λ, with poles on the boundary of this cone, and that it admits a meromorphic continuation. Geometric versions of such integrals may be important in other applications and we intend to return to these interesting questions in the future.

Roadmap of the paper

Section 2 is concerned with heights and measures on adelic spaces. We first recall notation and definitions for adeles, adelic metrics and measures on analytic manifolds. In Sec. 2.3, we then define height functions on adelic spaces and establish their basic properties. The construction of global Tamagawa measures is done in Sec. 2.4. We conclude this section by a general equidistribution theorem.

Section 3 is devoted to the theory of geometric analogues of Igusa integrals, both in the local and adelic settings. These integrals define holomorphic functions in several variables which admit meromorphic continuations. (In the adelic case, these meromorphic continuations may have natural boundaries.) To describe their first poles we introduce in Sec. 3.1 the geometric, algebraic and analytic Clemens complexes which encode the incidence properties of divisors involved in the definition of our geometric Igusa integrals. We then apply this theory in Secs. 4.2 and 4.4, where we establish Theorems 1.2 and 1.3 about volume asymptotics.

In Sec. 5, we make explicit the main results of our paper in the case of wonderful compactifications of semisimple groups. In particular, we explain how to recover the volume estimates established in [36] for Lie groups, and in [27, 45] for adelic groups.

2. Metrics, Heights, and Tamagawa Measures

2.1. Metrics and measures on local fields

2.1.1. *Haar measures and absolute values*

Let F be a local field of characteristic zero, i.e. either $\mathbf{R}$, $\mathbf{C}$, or a finite extension of the field $\mathbf{Q}_p$ of p-adic numbers. Fix a Haar measure μ on F. Its "modulus" is an absolute value $|\cdot|$ on F, defined by $\mu(a\Omega) = |a|\mu(\Omega)$ for any $a \in F$ and any measurable subset $\Omega \subset F$. For $F = \mathbf{R}$, this is the usual absolute value, for $F = \mathbf{C}$, it is its square. For $F = \mathbf{Q}_p$, it is given by $|p| = 1/p$ and if F'/F is a finite extension, one has $|a|_{F'} = |\mathrm{N}_{F'/F}(a)|_F$.

2.1.2. *Smooth functions*

Let f be a complex-valued function defined on an open subset of the n-dimensional affine space F^n. We say that f is *smooth* if it is C^∞ in the case where $F = \mathbf{R}$ or $\mathbf{C}$, and if it is locally constant when F is non-archimedean. This notion is local and extends to functions defined on open subsets of F-analytic manifolds. Smooth functions are continuous. Observe moreover that for any open subset U of F^n and any nonvanishing F-analytic function f on U, the function $x \mapsto |f(x)|$ is smooth.

On a compact F-analytic manifold X, a smooth function f has a sup-norm $\|f\| = \sup_{x\in X}|f(x)|$. In the archimedean case, using charts (so, non-canonically), we can also measure norms of derivatives and define norms $\|f\|_r$ (measuring the maximum of sup-norms of all derivatives of f of orders $\leq r$). In the ultrametric case, using a distance d, we can define a norm $\|f\|_1$ as follows:

$$\|f\|_1 = \|f\| \left(1 + \sup_{f(x)\neq f(y)} \frac{1}{d(x,y)}\right).$$

For $r > 1$, we define $\|f\|_r = \|f\|_1$.

2.1.3. *Metrics on line bundles*

Let X be an analytic variety over a locally compact valued field F and let $\mathscr{L}$ be a line bundle on X. We define a metric on $\mathscr{L}$ to be a collection of functions $\mathscr{L}(x) \to \mathbf{R}_+$, for all $x \in X$, denoted by $\ell \mapsto \|\ell\|$ such that

- for $\ell \in \mathscr{L}(x)\backslash\{0\}$, $\|\ell\| > 0$;
- for any $a \in F$, $x \in X$ and any $\ell \in \mathscr{L}(x)$, $\|a\ell\| = |a|\|\ell\|$;
- for any open subset $U \subset X$ and any section $\ell \in \Gamma(U, \mathscr{L})$, the function $x \mapsto \|\ell(x)\|$ is continuous on U.

We say that a metric on a line bundle $\mathscr{L}$ is smooth if for any nonvanishing local section $\ell \in \Gamma(U, \mathscr{L})$, the function $x \mapsto \|\ell(x)\|$ is smooth on U.

For a metric to be smooth, it suffices that there exists an open cover (U_i) of X, and, for each i, a nonvanishing section $\ell_i \in \Gamma(U_i, \mathscr{L})$ such that the function $x \mapsto \|\ell_i(x)\|$ is smooth on U_i. Indeed, let $\ell \in \Gamma(U, \mathscr{L})$ be a local nonvanishing section of $\mathscr{L}$; for each i there is a nonvanishing regular function $f_i \in \mathscr{O}_X(U_i \cap U)$ such that $\ell = f_i \ell_i$ on $U_i \cap U$, hence $\|\ell\| = |f_i| \|\ell_i\|$. Since the absolute value of a nonvanishing regular function is smooth, $\|\ell\|$ is a smooth function on $U_i \cap U$. Since this holds for all i, $\|\ell\|$ is smooth on U.

Any F-analytic manifold which is paracompact, a hypothesis which will always hold in this paper, admits smooth partitions of unity. Therefore, any line bundle on such a manifold can be endowed with a smooth metric.

There exist moreover natural constructions of metrics. For example, the trivial line bundle $\mathscr{O}_X$ admits a canonical (smooth) metric, defined by $\|1\| = 1$. If $\mathscr{L}$ and $\mathscr{M}$ are two (smooth) metrized line bundles on X, there are (smooth) metrics on $\mathscr{L} \otimes \mathscr{M}$ and on $\mathscr{L}^\vee$ defined by

$$\|\ell \otimes m\| = \|\ell\| \|m\|, \quad \|\varphi\| = |\varphi(\ell)| / \|\ell\|,$$

with $x \in X$, $\ell \in \mathscr{L}(x)$, $m \in \mathscr{M}(x)$ and $\varphi \in \mathscr{L}^\vee(x)$.

2.1.4. *Divisors, line bundles and metrics*

The theory of the preceding paragraph also applies if X is an analytic subspace of some F-analytic manifold, e.g. an algebraic variety, even if it possesses singularities. Recall that by definition, a function on such a space X is smooth if it extends to a smooth function in a neighborhood of X in the ambient space.

Let D be an effective Cartier divisor on X and let $\mathscr{O}_X(D)$ be the corresponding line bundle. It admits a canonical section f_D, whose divisor is equal to D. If $\mathscr{O}_X(D)$ is endowed with a metric, the function $\|\mathsf{f}_D\|$ is positive on X and vanishes along D.

More generally, let D be an effective $\mathbf{Q}$-divisor, that is, a linear combination of irreducible divisors with rational coefficients such that a multiple nD, for some positive integer n, is a Cartier divisor. By a metric on $\mathscr{O}_X(D)$ we mean a metric on $\mathscr{O}_X(nD)$. By $\|\mathsf{f}_D\|$, we mean the function $\|\mathsf{f}_{nD}\|^{1/n}$. It does not depend on the choice of n.

2.1.5. *Metrics defined by a model*

Here we assume that F is non-archimedean, and let $\mathfrak{o}_F$ be its ring of integers. Let X be a proper variety over F and L a line bundle on X. Choose a proper flat $\mathfrak{o}_F$-scheme $\mathscr{X}$ and a line bundle $\mathscr{L}$ on $\mathscr{X}$ extending X and L. These choices determine a metric on the line bundle defined by $\mathscr{L}$ on the analytic variety $X(F)$, by the following recipe: for $x \in X(F)$ let $\tilde{x} \colon \operatorname{Spec} \mathfrak{o}_F \to \mathscr{X}$ be the unique morphism extending x; by definition, the set of $\ell \in L(x)$ such that $\|\ell\| \leq 1$ is equal to $\tilde{x}^* \mathscr{L}$, which is a lattice in $L(x)$.

Let $\mathscr{U}$ be an open subset of $\mathscr{X}$ over which the line bundle $\mathscr{L}$ is trivial and let $\varepsilon \in \Gamma(\mathscr{U}, \mathscr{L})$ be a trivialization of $\mathscr{L}$ on $\mathscr{U}$. Then, for any point $x \in \mathscr{U}(F)$ such that x extends to a morphism $\tilde{x}\colon \operatorname{Spec}\mathfrak{o}_F \to \mathscr{U}$, one has $\|\varepsilon(x)\| = 1$. Indeed, $\tilde{x}^*\varepsilon$ is a basis of the free $\mathfrak{o}_F$-module $\tilde{x}^*\mathscr{L}$. Since $\mathscr{X}$ is proper, restriction to the generic fiber identifies the set $\mathscr{U}(\mathfrak{o}_F)$ with a compact open subset of $\mathscr{U}_F(F)$, still denoted by $\mathscr{U}(\mathfrak{o}_F)$. Observe that these compact open subsets cover $X(F)$.

Let F'/F be a finite field extension. A model of X determines a model of $X_{F'} = X \otimes_F F'$ over the ring $\mathfrak{o}_{F'}$, and thus metrics on the analytic variety $X(F')$. Conversely, note that a model of $X_{F'}$ determines metrics on $X(F')$ and hence, by restriction, also on $X(F)$.

2.1.6. *Example*: *Projective space*

Let F be a locally valued field and V a finite-dimensional vector space over F, let $V^\vee$ denote its dual space. Let $|\cdot|$ be a norm on V. Let $\mathbf{P}(V^\vee) = \operatorname{Proj}\operatorname{Sym}^\bullet V^\vee$ be the projective space of lines in V; denote by $[x]$ the point of $\mathbf{P}(V^\vee)$ associated to a nonzero element $x \in V$, i.e. the line it generates. This projective space carries a tautological ample line bundle, denoted $\mathscr{O}(1)$, whose space of global sections is precisely $V^\vee$. The formula

$$\|\ell([x])\| = \frac{|\ell(x)|}{\|x\|},$$

for $\ell \in V^\vee$ and $x \in V$, defines a norm on the F-vector space $\mathscr{O}(1)_{[x]}$. These norms define a metric on $\mathscr{O}(1)$.

Assume moreover that F is non-archimedean and let $\mathfrak{o}_F$ be its ring of integers. If the norm of V takes its values in the set $|F^\times|$, the unit ball $\mathscr{V}$ of V is an $\mathfrak{o}_F$-submodule of V which is free of rank $\dim V$ ([48], p. 28, Proposition 6 and p. 29). Then, the $\mathfrak{o}_F$-scheme $\mathbf{P}(\mathscr{V}^\vee)$ is a model of $\mathbf{P}(V^\vee)$ and the metric on $\mathscr{O}(1)$ is the one defined by this model and the canonical extension of the line bundle $\mathscr{O}(1)$ it carries.

2.1.7. *Volume forms*

Let X be an analytic manifold over a local field F. To simplify the exposition, assume that X is equidimensional and let n be its dimension. It is standard when $F = \mathbf{R}$ or $\mathbf{C}$, and also true in general, that any n-form ω on an open subset U of X defines a measure, usually denoted $|\omega|$, on U, through the following formula. Choose local coordinates $x_1, \dots, x_n$ on U and write

$$\omega = f(x_1, \dots, x_n)\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

These coordinates allow one to identify (a part of) U with an open subset of F^n which we endow with the product measure μ^n. (Recall that a Haar measure has been fixed on F in Sec. 2.1.1.) It pulls back to a measure which we denote

$$|\mathrm{d}x_1| \cdots |\mathrm{d}x_n|$$

and by definition, we let

$$|\omega| = |f(x_1, \dots, x_n)| \, |\mathrm{d}x_1| \cdots |\mathrm{d}x_n|. \tag{2.1}$$

By the change of variables formula in multiple integrals, this is independent of the choice of local coordinates.

2.1.8. *Metrics and measures*

Certain manifolds X possess a canonical (up to scalar) nonvanishing n-form, sometimes called a *gauge form*. Examples are analytic groups, or Calabi–Yau varieties. This property leads to the definition of a canonical measure on X (again, up to a scalar). In the case of groups, this has been studied by Weil in the context of Tamagawa numbers ([47]); in the case of Calabi–Yau varieties, this measure has been used by Batyrev to prove that smooth birational Calabi–Yau varieties have equal Betti numbers ([2]).

Even when X has no global n-forms, it is still possible to define a measure on X provided that the canonical line bundle $\omega_X = \bigwedge^n \Omega^1_X$ is endowed with a *metric*. Indeed, we may then attach to any local nonvanishing n-form ω the local measure $|\omega|/\|\omega\|$. It is immediate that these measures patch and define a measure τ_X on the whole of X. This is a Radon measure locally equivalent to any Lebesgue measure. In particular, if X is compact, its volume with respect to this measure is a positive real number. This construction is classical in differential geometry as well as in arithmetic geometry (see [42], at the end of p. 146); its introduction in the context of the counting problem of points of bounded height on algebraic varieties over number fields is due to E. Peyre ([38]).

2.1.9. *Singular measures*

For the study of integral points, we will have to consider several variants of this construction. We assume here that X is an algebraic variety over a local field F and that we are given an auxiliary effective Cartier $\mathbf{Q}$-divisor D in X. Let $U = X \setminus |D|$ be the complement of the support of D in X.

Let us first consider the case where D is a Cartier divisor and assume that the line bundle $\omega_X(D)$ is endowed with a *metric*. If f_D is the canonical section of $\mathscr{O}_X(D)$ and ω a local n-form, we can then consider the non-negative function $\|\mathsf{f}_D\omega\|$; it vanishes on $|D|$. In the general case, we follow the conventions of Sec. 2.1.4 and will freely talk of the function $\|\omega\mathsf{f}_D\|$, defined as $\|\omega^n\mathsf{f}_{nD}\|$ where n is any positive integer such that nD is a Cartier divisor, assuming the line bundle $\omega_X^{\otimes n}(nD)$ is endowed with a metric.

Then, the measures $|\omega|/\|\mathsf{f}_D\omega\|$, for ω any local nonvanishing n-form ω, patch and define a measure $\tau_{(X,D)}$ on the open submanifold $U(F)$. This measure is a Radon measure on $U(F)$, locally equivalent to the Lebesgue measure. When $U(F)$ is not compact, the volume of $U(F)$ may be infinite; this is in particular the case if D is an effective divisor with integer coefficients and the smooth locus of D has F-rational points.

We shall always identify this measure and the (generally not locally finite) measure on $X(F)$ obtained by push-out.

2.1.10. *Example*: *Gauge forms*

Let us show how the definition of a measure using a gauge form can be viewed as a particular case of this construction. Let ω be a meromorphic differential form on X; let D be the *opposite* of its divisor. One can write $D = \sum d_\alpha D_\alpha$, where the D_α are codimension 1 irreducible subvarieties in X and d_α are integers. Let $U = X \setminus |D|$ be the complement of the support of D. In other words, ω defines a trivialization of the line bundle $\omega_X(D)$. There is a unique metrization of $\omega_X(D)$ such that ω, viewed as a global section of $\omega_X(D)$, has norm 1 at every point in X. The measure on the manifold $U(F)$ defined by this metrization coincides with the measure $|\omega|$ defined by ω as a gauge form.

Moreover, if the line bundle $\mathscr{O}_X(D)$ is endowed with a metric, then so is the line bundle ω_X. In this case, the manifold $X(F)$ is endowed with three measures, τ_U, τ_X and $\tau_{(X,D)}$. Locally, one has:

- $\tau_X = |\omega|/\|\omega\|$;
- the measure τ_U is its restriction to U;
- $\tau_{(X,D)} = |\omega| = |\omega|/\|\mathsf{f}_D\omega\| = \|\mathsf{f}_D\|^{-1}\tau_X$.

Note that on each open, relatively compact subset of $X(F)\setminus|D|$, the measures τ_X and $\tau_{(X,D)}$ are equivalent.

2.1.11. *Example*: *Compactifications of algebraic groups*

We keep the notation of the previous paragraph, assuming moreover that U is an algebraic group G over F of which X is an equivariant compactification, and that the restriction to G of ω is invariant. If we consider ω as a gauge form, it then defines an invariant measure $|\omega|$ on $G(F)$, in other words, a Haar measure on this locally compact group, and also $\tau_{(X,D)}$.

2.1.12. *Residue measures*

Let us return to the case of a general manifold X. Let Z be a closed submanifold of X. To get a measure on Z, we need as before a metrization on ω_Z. However, the "adjunction formula"

$$\omega_X|_Z \simeq \omega_Z \otimes \det \mathscr{N}_Z(X)$$

implies that given a metric on ω_X, it suffices to endow the determinant of the normal bundle of Z in X with a metric. The case where $Z = D$ is a divisor (i.e. locally in charts, Z is defined by the vanishing of a coordinate) is especially interesting. In that case, one has

$$\omega_D = \omega_X(D)|_D$$

and a metric on ω_X plus a metric on $\mathscr{O}_X(D)$ automatically define a metric on ω_D, hence a measure on D.

Let us give an explicit formula for this measure. Let $\omega \in \Gamma(\omega_D)$ be a local $(n-1)$-form and $\tilde{\omega}$ any lift of ω to $\bigwedge^{n-1}\Omega^1_X$. If $u \in \mathscr{O}(-D)$ is a local equation for D, the image of ω in $\omega_X(D)|_D$ is nothing but the restriction to D of the differential form with logarithmic poles $\tilde{\omega} \wedge u^{-1}\mathrm{d}u$. Let f_D be the canonical section of $\mathscr{O}_X(D)$. Then,

$$\|\omega\| = \|\tilde{\omega} \wedge u^{-1}\mathrm{d}u\| = \|\tilde{\omega} \wedge \mathrm{d}u\| \, \|u^{-1}\|.$$

By definition, f_D corresponds locally to the function 1, hence, for $x \notin D$,

$$\|u^{-1}(x)\| = \frac{1}{|u(x)|}\|\mathsf{f}_D(x)\|$$

and this possesses a finite limit when x approaches D, by the definition of a metric. (As a section of $\mathscr{O}_X(D)$, f_D vanishes at order 1 on D). Moreover, the function $\lim \|\mathsf{f}_D\|/|u|$ on D defined by

$$x \mapsto \lim_{\substack{y\to x \\ y\notin D}} \|\mathsf{f}_D(y)\|/|u(y)|$$

is continuous and positive on D.

By induction, if Z is the transverse intersection of smooth divisors D_j $(1 \leq j \leq m)$, with metrizations on all $\mathscr{O}_X(D_j)$, we have a similar formula:

$$\|\omega\| = \|\tilde{\omega} \wedge \mathrm{d}u_1 \wedge \dots \wedge \mathrm{d}u_m\| \lim \frac{\|\mathsf{f}_{D_1}\|}{|u_1|} \cdots \lim \frac{\|\mathsf{f}_{D_m}\|}{|u_m|}, \tag{2.2}$$

where $u_1, \dots, u_m$ are local equations for the divisors $D_1, \dots, D_m$.

2.1.13. *Residue measures in an algebraic context*

Let X be a smooth algebraic variety over F; endow the canonical line bundle ω_X on $X(F)$ with a metric. Let F' be a finite Galois extension of F and let D_j, for $1 \leq j \leq m$, be smooth irreducible divisors on $X_{F'}$, whose union is a divisor D with strict normal crossings which is defined over F. Then the intersection $Z = \bigcap_{j=1}^m D_j$ is defined over F; if $Z(F) \neq \varnothing$, then $Z(F)$ is a smooth F-analytic submanifold of $X(F)$, of dimension $\dim Z = \dim X - m$. Moreover, the normal bundle of Z in $X_{F'}$ is isomorphic to the restriction to Z of the vector bundle $\bigoplus_{j=1}^m \mathscr{O}_{X'}(D_j)$, hence the isomorphism $\det \mathscr{N}_Z(X) \simeq \mathscr{O}_X(D)$, at least after extending the scalars to F'.

Endow the line bundles $\mathscr{O}_{X_{F'}}(D_j)$ on the F'-analytic manifold $X(F')$ with metrics. By the previous formulas, we obtain from the metrics on $\mathscr{O}_{X_{F'}}(D_j)$ a metric on the determinant of the normal bundle of $Z(F')$ in the manifold $X(F')$. The restriction of this metric to $Z(F)$ gives us a metric on the determinant of the normal bundle of $Z(F)$ in $X(F)$.

Accordingly, we obtain a positive Radon measure τ_Z on $Z(F)$ which is locally equivalent to any Lebesgue measure. In particular, one has $\tau_Z(Z(F)) = 0$ if and only if $Z(F) = \varnothing$.

2.2. *Adeles of number fields: Metrics and heights*

2.2.1. *Notation*

We specify some common notation concerning number fields and adeles that we will use throughout this text.

Let F be a number field and $\mathrm{Val}(F)$ the set of places of F. For $v \in \mathrm{Val}(F)$, we let F_v be the v-adic completion of F. This is a local field; its absolute value, defined as in Sec. 2.1.1, is denoted by $|\cdot|_v$. With these normalizations, the *product formula* holds. Namely, for all $a \in F^*$, one has $\prod_{v\in\mathrm{Val}(F)} |a|_v = 1$, where only finitely many factors differ from 1.

Let $v \in \mathrm{Val}(F)$. The absolute value $|\cdot|_v$ is archimedean when $F_v = \mathbf{R}$ or $F_v = \mathbf{C}$; we will say that v is infinite, or archimedean. Otherwise, the absolute value $|\cdot|_v$ is ultrametric and the place v is called finite, or non-archimedean.

Let v be a finite place of F. The set $\mathfrak{o}_v$ of all $a \in F_v$ such that $|a|_v \leq 1$ is a subring of F_v, called the ring of v-adic integers. Its subset $\mathfrak{m}_v$ consisting of all $a \in F_v$ such that $|a|_v < 1$ is its unique maximal ideal. The residue field $\mathfrak{o}_v/\mathfrak{m}_v$ is denoted by k_v. It is a finite field and we write q_v for its cardinality. The ideal $\mathfrak{m}_v$ is principal; a generator will be called a uniformizing element at v. For any such element ϖ, one has $|\varpi|_v = q_v^{-1}$.

Fix an algebraic closure $\bar{F}$ of F. The group Γ_F of all F-automorphisms of $\bar{F}$ is called the absolute Galois group of F. For any finite place $v \in \mathrm{Val}(F)$, fix an extension $|\cdot|_{\bar{v}}$ of the absolute value $|\cdot|_v$ to $\bar{F}$. The subgroup of Γ_F consisting of all $\gamma \in \Gamma_F$ such that $|\gamma(a)|_{\bar{v}} = |a|_{\bar{v}}$ for all $a \in \bar{F}$ is called the decomposition subgroup of Γ_F at v and is denoted by Γ_v.

These data determine an algebraic closure $\bar{k}_v$ of the residue field k_v, together with a surjective group homomorphism $\Gamma_v \to \mathrm{Gal}(\bar{k}_v/k_v)$. Its kernel Γ_v^0 is called the inertia subgroup of Γ_F at v. Any element in Γ_v mapping to the Frobenius automorphism $x \mapsto x^{q_v}$ in $\mathrm{Gal}(\bar{k}_v/k_v)$ is called an arithmetic Frobenius element at v; its inverse is called a geometric Frobenius element at v. The subgroups Γ_v and Γ_v^0, and Frobenius elements, depend on the choice of the chosen extension of the absolute value $|\cdot|_v$; another choice changes them by conjugation in Γ_F.

The ring of adeles $\mathbb{A}_F$ of the field F is the restricted product of all local fields F_v, for $v \in \mathrm{Val}(F)$, with respect to the subrings $\mathfrak{o}_v$ for finite places v. It is a locally compact topological ring and carries a Haar measure μ. The quotient space $\mathbb{A}_F/F$ is compact; we shall often assume that μ is normalized so that $\mu(\mathbb{A}_F/F) = 1$.

2.2.2. *The adelic space of an algebraic variety*

Let U be an algebraic variety over F. The space $U(\mathbb{A}_F)$ of adelic points of U has a natural locally compact topology which we recall now. Let $\mathscr{U}$ be a model of U over the integers of F, i.e. a scheme which is flat and of finite type over $\operatorname{Spec}\mathfrak{o}_F$ together with an isomorphism of U with $\mathscr{U}\otimes F$. The natural maps $\mathbb{A}_F\to F_v$ induce a map from $U(\mathbb{A}_F)$ to $\prod_{v\in\operatorname{Val}(F)}U(F_v)$. It is not surjective unless U is proper over F; indeed its image — usually called the restricted product — can be described as the set of all $(x_v)_v$ in the product such that $x_v\in\mathscr{U}(\mathfrak{o}_v)$ for almost all finite places v. Since two models are isomorphic over a dense open subset of $\operatorname{Spec}\mathfrak{o}_F$, observe also that this condition is independent of the choice of the specific model $\mathscr{U}$.

We endow each $U(F_v)$ with the natural v-adic topology; notice that it is locally compact. Moreover, for any finite place v, $\mathscr{U}(\mathfrak{o}_v)$ is open and compact in $U(F_v)$ for this topology.

We then endow $U(\mathbb{A}_F)$ with the "restricted product topology", a basis of which is given by products $\prod_v\Omega_v$, where for each $v\in\operatorname{Val}(F)$, Ω_v is an open subset of $U(F_v)$, subject to the additional condition that $\Omega_v=\mathscr{U}(\mathfrak{o}_v)$ for almost all finite places v. Let $\Omega\subset U(\mathbb{A}_F)$ be any subset of the form $\prod_v\Omega_v$, with $\Omega_v\subset U(F_v)$, such that $\Omega_v=\mathscr{U}(\mathfrak{o}_v)$ for almost all finite places v. Then Ω is compact if and only if each Ω_v is compact. It follows that the topology on $U(\mathbb{A}_F)$ is locally compact.

2.2.3. *Adelic metrics*

Let X be a proper variety over a number field F and L a line bundle on X. An adelic metric on L is a collection of v-adic metrics on the associated line bundles on the F_v-analytic varieties $X(F_v)$, for all places v in F, which, except for a finite number of them, are defined by a single model $(\mathscr{X},\mathscr{L})$ over the ring of integers of F.

By standard properties of schemes of finite presentation, any two flat proper models are isomorphic at almost all places and will therefore define the same v-adic metrics at these places.

2.2.4. *Example*: *Projective space*

For any valued field F, let us endow the vector-space F^{n+1} with the norm given by $|(x_0,\dots,x_n)|=\max(|x_0|,\dots,|x_n|)$. When F is a number field and $v\in\operatorname{Val}(F)$, the construction in Sec. 2.1.6 on the field F_v furnishes a metric on the line bundle $\mathscr{O}(1)$ on $\mathbf{P}^n$. This collection of metrics is an adelic metric on $\mathscr{O}(1)$, which we will call the *standard adelic metric*.

2.2.5. *Extension of the ground field*

If F' is any finite extension of F, observe that an adelic metric on the line bundle $L\otimes F'$ over $X\otimes F'$ induces by restriction an adelic metric on the line bundle L over X. We only need to check that the family of metrics defined by restriction are induced at almost all places by a model of X on F.

Indeed, fix a model $(\mathscr{X}', \mathscr{L}')$ of $(X \otimes F', \mathscr{L} \otimes F')$ over $\operatorname{Spec} \mathfrak{o}_{F'}$, as well as a model $(\mathscr{X}, \mathscr{L})$ over $\operatorname{Spec} \mathfrak{o}_F$. There is a nonzero integer N such that the identity map $X \otimes F' \to X \otimes F'$ extends to an isomorphism $\mathscr{X} \otimes \mathfrak{o}_{F'}[\frac{1}{N}] \to \mathscr{X}' \otimes \mathfrak{o}_{F'}[\frac{1}{N}]$. Consequently, at all finite places of F' which do not divide N, both models define the same metric on $L \otimes F'$. It follows that the metrics of L are defined by the model $(\mathscr{X}, \mathscr{L})$ at all but finitely many places of F.

2.2.6. *Heights*

Let X be a proper variety over a number field F and let $\mathscr{L}$ be a line bundle on X endowed with an adelic metric. Let $x \in X(F)$ and let ℓ be any nonzero element in $\mathscr{L}(x)$. For almost all places v, one has $\|\ell\|_v = 1$; consequently, the product $\prod_v \|\ell\|_v$ converges absolutely. It follows from the product formula that its value does not depend on the choice of ℓ. We denote it by $H_{\mathscr{L}}(x)$ and call it the (exponential) *height* of x with respect to the metrized line bundle $\mathscr{L}$.

As an example, assume that $X = \mathbf{P}^n$ and $\mathscr{L} = \mathscr{O}(1)$, endowed with its standard adelic metric see Sec. 2.2.4. Then for any point $x \in \mathbf{P}^n(F)$ with homogeneous coordinates $[x_0 \cdots : x_n]$ such that $x_0 \neq 0$, we can take the element ℓ to be the value at x of the global section X_0. Consequently, the definition of the standard adelic metric implies that

$$H_{\mathscr{L}}(x) = \prod_{v \in \operatorname{Val}(F)} \left(\frac{|x_0|_v}{\max(|x_0|_v, \ldots, |x_n|_v)} \right)^{-1} = \prod_{v \in \operatorname{Val}(F)} \max(|x_0|_v, \ldots, |x_n|_v),$$

in view of the product formula $\prod_{v \in \operatorname{Val}(F)} |x_0|_v = 1$. We thus recover the classical (exponential) height of the point x in the projective space.

In general, the function

$$\log H_{\mathscr{L}} \colon X(F) \to \mathbf{R}$$

is a representative of the class of height functions attached to the line bundle $\mathscr{L}$ using Weil's classical "height machine" (see [30], especially Part B, §10).

When $\mathscr{L}$ is ample, the set of points $x \in X(F)$ such that $H_{\mathscr{L}}(x) \leq B$ is finite for any real number B (Northcott's theorem).

2.3. *Heights on adelic spaces*

Let f be a nonzero global section of $\mathscr{L}$ and let Z denote its divisor. We extend the definition of $H_{\mathscr{L}}$ to the adelic space $X(\mathbb{A}_F)$ by defining

$$H_{\mathscr{L},\mathsf{f}}(\mathbf{x}) = \left(\prod_{v \in \operatorname{Val}(F)} \|\mathsf{f}\|_v(x) \right)^{-1} \tag{2.3}$$

for any $\mathbf{x} \in X(\mathbb{A}_F)$ such that the infinite product makes sense.

To give an explicit example, assume again that X is the projective space $\mathbf{P}^n$ and that $\mathscr{L}$ is the line bundle $\mathscr{O}_{\mathbf{P}}(1)$ endowed with its standard adelic metric;

let us choose $\mathsf{f} = x_0$. Then, $U = X \setminus \operatorname{div}(\mathsf{f})$ can be identified to the set of points $[1 : \cdots : x_n]$ of $\mathbf{P}^n$ whose first homogeneous coordinate is equal to 1, i.e. to the affine space $\mathbf{A}^n$. For any $\mathbf{x} = (x_v)_v \in U(\mathbb{A}_F)$, given by $\mathbf{x} = [1 : \mathbf{x}_1 : \cdots : \mathbf{x}_n]$, with $\mathbf{x}_i = (x_{i,v})_v \in \mathbb{A}_F$, one has

$$H_{\mathscr{L},\mathsf{f}}(x) = \prod_{v \in \operatorname{Val}(F)} \max(1, |x_{1,v}|_v, \dots, |x_{n,v}|_v).$$

Lemma 2.1. *Let U be any open subset of X.*

(i) *For any $\mathbf{x} \in X(\mathbb{A}_F)$, the infinite product defining $H_{\mathscr{L},\mathsf{f}}(\mathbf{x})$ converges to an element of $(0, +\infty]$.*
(ii) *The resulting function is lower semi-continuous and admits a positive lower bound on the adelic space $X(\mathbb{A}_F)$.*
(iii) *If f does not vanish on U, then the restriction of $H_{\mathscr{L},\mathsf{f}}$ to $U(\mathbb{A}_F)$ is continuous (for the topology of $U(\mathbb{A}_F)$).*
(iv) *Assume that $X = U \cup Z$. Then, for any real number B, the set of points $\mathbf{x} \in U(\mathbb{A}_F)$ such that $H_{\mathscr{L},\mathsf{f}}(\mathbf{x}) \leq B$ is a compact subset of $U(\mathbb{A}_F)$.*

Proof. For any place v, let us set $c_v = \max(1, \|\mathsf{f}\|_v)$. One has $c_v < \infty$ since $X(F_v)$ is compact and the function $\|\mathsf{f}\|_v$ on $X(F_v)$ is continuous. Moreover, it follows from the definition of an adelic metric that $\|\mathsf{f}\|_v = 1$ for almost all v, so that $c_v = 1$ for almost all places v.

Consequently, the infinite series $\sum_{v \in \operatorname{Val}(F)} \log \|\mathsf{f}_v\|^{-1}$ has (almost all of) its terms non-negative; it converges to a real number or to $+\infty$. This shows that the infinite product $H_{\mathscr{L},\mathsf{f}}(\mathbf{x})$ converges to an element in $(0, +\infty]$. Letting $C = 1/\prod_v c_v$, one has $H_{\mathscr{L},\mathsf{f}}(\mathbf{x}) \geq C$ for any $\mathbf{x} \in X(\mathbb{A}_F)$, which proves (i) and the second half of (ii).

Let V_1 be the set of places $v \in \operatorname{Val}(F)$ such that $c_v > 1$; one has

$$\log H_{\mathscr{L},\mathsf{f}}(\mathbf{x}) = \sup_{V_1 \subset V \subset \operatorname{Val}(F)} \sum_{v \in V} \log \|\mathsf{f}_v\|^{-1},$$

where V runs within the finite subsets of $\operatorname{Val}(F)$ containing V_1. This expression shows that the function $\mathbf{x} \mapsto \log H_{\mathscr{L},\mathsf{f}}(\mathbf{x})$ is a supremum of lower semi-continuous functions, hence is lower semi-continuous on $X(\mathbb{A}_F)$, hence the remaining part of Assertion (ii).

Let us now prove (iii). Let $\mathscr{X}$ be a flat projective model of X over $\mathfrak{o}_F$, let $\mathscr{Z}$ be the Zariski closure of Z and $\mathscr{V} = \mathscr{X} \setminus \mathscr{Z}$, let $\mathscr{D}$ be the Zariski closure of $X \setminus U$ and let $\mathscr{U} = \mathscr{X} \setminus \mathscr{D}$.

By definition of an adelic metric, for almost all finite places $v \in \operatorname{Val}(F)$, $\|f\|_v$ is identically equal to 1 on $\mathscr{U}(\mathfrak{o}_v)$. By definition of the topology of $U(\mathbb{A}_F)$, this implies that the function $H_{\mathscr{L},\mathsf{f}}$ is locally given by a finite product of continuous functions; it is therefore continuous.

Let us finally establish (iv). Consider integral models as above; since U contains V by assumption, then $\mathscr{U} \supset \mathscr{V}$.

For any $\mathbf{x} = (x_v)_v \in X(\mathbb{A}_F)$, one has

$$\|\mathsf{f}\|(x_v) = H_{\mathscr{L},\mathsf{f}}(\mathbf{x})^{-1} \prod_{w \neq v} \|\mathsf{f}\|(x_v)^{-1} \geq C^{-1} H_{\mathscr{L},\mathsf{f}}(\mathbf{x})^{-1}.$$

If moreover $H_{\mathscr{L},\mathsf{f}}(\mathbf{x}) \leq B$, then $\|\mathsf{f}\|(x_v) \geq (BC)^{-1}$. The set of points $x_v \in X(F_v)$ satisfying this inequality is a closed subset of the compact space $X(F_v)$, hence is compact. Moreover, this set is contained in $U(F_v)$ because, by assumption, f vanishes on $X\backslash U$. Consequently, this set is a compact subset of $U(F_v)$.

If the cardinality q_v of the residue field k_v is large enough, so that the v-adic metric on $\mathscr{L}$ is defined by the line bundle $\mathscr{O}(\mathscr{Z})$ at the place v, then $-\log \|\mathsf{f}\|_v(x_v)$ is a non-negative integer times $\log q_v$. The inequality $-\log \|\mathsf{f}\|_v(x_v) \leq \log(BC)$, then implies that $-\log \|\mathsf{f}\|_v(x_v) = 0$.

Let E_B denote the set of points $\mathbf{x} \in U(\mathbb{A}_F)$ such that $H_{\mathscr{L},\mathsf{f}}(\mathbf{x}) \leq B$. By what we proved, there exists a finite set of places V (depending on B) such that E_B is the product of a compact subset of $\prod_{v \in V} U(F_v)$ and of $\prod_{v \notin V} \mathscr{V}(\mathfrak{o}_v)$. Since $\mathscr{V}(\mathfrak{o}_v) \subset \mathscr{U}(\mathfrak{o}_v)$, we see that E_B is relatively compact in $U(\mathbb{A}_F)$. By lower semi-continuity, it is also closed in $U(\mathbb{A}_F)$, hence compact. $\square$

We observe that the hypotheses in (iii) and (iv) are actually necessary. They hold in the important case where $U = X\backslash Z$; then, $H_{\mathscr{L},\mathsf{f}}$ defines a continuous exhaustion of $U(\mathbb{A}_F)$ by compact subsets.

2.4. *Convergence factors and Tamagawa measures on adelic spaces*

2.4.1. *Volumes and local densities*

Let X be a smooth, proper, and geometrically integral algebraic variety over a number field F. Fix an adelic metric on the canonical line bundle ω_X; by the results recalled in Sec. 2.1.7, for any place v of F, this induces a measure $\tau_{X,v}$ on $X(F_v)$ and its restriction $\tau_{U,v}$ to $U(F_v)$.

Let U be a Zariski open subset of X and let $Z = X\backslash U$. Fix a model $\mathscr{X}$ of X, and let $\mathscr{Z}$ be the Zariski closure of Z in $\mathscr{X}$ and $\mathscr{U} = \mathscr{X}\backslash\mathscr{Z}$.

By a well-known formula going back to Weil ([47], see also [41], Corollary 2.15), the equality

$$\tau_{U,v}(\mathscr{U}(\mathfrak{o}_v)) = q_v^{-\dim X} \# \mathscr{U}(k_v) \tag{2.4}$$

holds for almost all non-archimedean places v.

2.4.2. *Definition of an* L*-function*

Definition 2.2. We define $\mathrm{EP}(U)$ to be the following virtual $\mathbf{Q}[\Gamma_F]$-module:

$$[\mathrm{H}^0(U_{\bar{F}}, \mathbf{G}_m)/\bar{F}^*]_{\mathbf{Q}} - [\mathrm{H}^1(U_{\bar{F}}, \mathbf{G}_m)]_{\mathbf{Q}}.$$

It is an object of the Grothendieck group of the category of finite dimensional $\mathbf{Q}$-vector spaces endowed with a continuous action of Γ_F, meaning that there is a finite extension F' of F such that the subgroup $\Gamma_{F'}$ acts trivially. Such a virtual Galois-module (we shall often skip the word “virtual”) has an Artin L-function, given by an Euler product

$$\mathrm{L}(s, \mathrm{EP}(U)) = \prod_{v \text{ finite}} \mathrm{L}_v(s, \mathrm{EP}(U)),$$

$$\mathrm{L}_v(s, \mathrm{EP}(U)) = \det(1 - q_v^{-s} \operatorname{Fr}_v \mid \mathrm{EP}(U)^{\Gamma_v^0})^{-1},$$

where Fr_v is a geometric Frobenius element and Γ_v^0 is an inertia subgroup at the place v.

Lemma 2.3. *For any finite place v of $F, \mathrm{L}_v(s, \mathrm{EP}(U))$ is a positive real number.*

Proof. We have $\mathrm{L}_v(s, \mathrm{EP}(U)) = q_v^{s \operatorname{rank} \mathrm{EP}(U)} / f(q_v^s)$, where the rational function

$$f(X) = \det(X - \operatorname{Fr}_v \mid \mathrm{EP}(U))$$

is the virtual characteristic polynomial of Fr_v acting on $\mathrm{EP}(U)$, with the obvious modification for ramified v. Since the action of Γ_F on $\mathrm{EP}(U)$ factorizes through a finite quotient, the rational function has only roots of unity for zeroes and poles.

By irreducibility of the cyclotomic polynomials Φ_n in $\mathbf{Q}[X]$, this implies the existence of rational integers $(a_n)_{n \geq 1}$, almost all zero, such that

$$\det(X - \operatorname{Fr}_v \mid \mathrm{EP}(U)) = \prod_{n \geq 1} \Phi_n(X)^{a_n}.$$

From the inductive definition of the cyclotomic polynomials Φ_n, namely $\prod_{d|n} \Phi_n(X) = X^n - 1$, we see that $\Phi_n(x) > 0$ for any real number $x > 1$ and any positive integer n. In particular, $f(x) > 0$ for any real number $x > 1$ and $\mathrm{L}_v(1, \mathrm{EP}(U)) > 0$, as claimed. □

(To simplify notation, we shall also put $\mathrm{L}_v = 1$ for any archimedean place v.) More generally, if S is an arbitrary set of places of F, we define

$$\mathrm{L}^S(s, \mathrm{EP}(U)) = \prod_{v \notin S} \mathrm{L}_v(s, \mathrm{EP}(U)).$$

This Euler product converges for $\operatorname{Re}(s) > 1$ to a holomorphic function of s in that domain; by Brauer’s theorem [8], it admits a meromorphic continuation to the whole complex plane. We denote by

$$\mathrm{L}_*^S(1, \mathrm{EP}(U)) = \lim_{s \to 1} \mathrm{L}^S(s, \mathrm{EP}(U))(s-1)^{-r}, \quad r = \operatorname{ord}_{s=1} \mathrm{L}^S(s, \mathrm{EP}(U)),$$

its “principal value” at $s = 1$.

From the previous lemma, we deduce the following corollary.

Corollary 2.4. *For any finite set of places S of $F, \mathrm{L}_*^S(1, \mathrm{EP}(U))$ is a positive real number.*

We now show how this L-function furnishes renormalization factors for the local measures $\tau_{U,v}$.

Theorem 2.5. *In addition to the notation and assumptions of Sec.* 2.4.1, *suppose that*

$$\mathrm{H}^1(X, \mathscr{O}_X) = \mathrm{H}^2(X, \mathscr{O}_X) = 0.$$

Then $\mathrm{L}_v(1, \mathrm{EP}(U))\tau_{U,v}(\mathscr{U}(\mathfrak{o}_v)) = 1 + \mathrm{O}(q_v^{-3/2})$. *In particular, the infinite product*

$$\prod_{v \notin S} \mathrm{L}_v(1, \mathrm{EP}(U))\tau_{U,v}(\mathscr{U}(\mathfrak{o}_v))$$

converges absolutely.

Remark 2.6. For a smooth projective variety X and any integer i, the vector-space $\mathrm{H}^0(X, \Omega^i_X)$ defines a birational invariant of X. Let us sketch the proof. For any birational morphism $f\colon X \to X'$, there are open subsets $V \subset X$ and $V' \subset X'$ whose complementary subsets have codimension at least 2 such that f is defined on V and f^{-1} is defined on V'. The restriction morphism $\mathrm{H}^0(X, \Omega^i_X) \to \mathrm{H}^0(V, \Omega^i_X)$ is then injective since V is dense, and surjective by the Hartogs principle; similarly, the restriction morphism $\mathrm{H}^0(X', \Omega^i_{X'}) \to \mathrm{H}^0(V', \Omega^i_{X'})$ is an isomorphism. Consequently, the corresponding regular maps $g\colon V \to X'$ and $g'\colon V' \to X$ define morphisms $\mathrm{H}^0(X', \Omega^i_{X'}) \to \mathrm{H}^0(X, \Omega^i_X)$ and $\mathrm{H}^0(X, \Omega^i_X) \to \mathrm{H}^0(X', \Omega^i_{X'})$ both compositions of which equal the identity map, hence are isomorphisms.

When the ground field is the field of complex numbers, Hodge theory identifies these vector-spaces with the conjugates of the cohomology spaces $\mathrm{H}^i(X, \mathscr{O}_X)$. Consequently, these spaces define birational invariants of smooth complex projective varieties. Moreover, by the Lefschetz principle, this extends to smooth projective varieties defined over a field of characteristic 0. In particular, the assumptions of the theorem are therefore conditions on the variety U and do not depend on its smooth compactification X.

Proof. Blowing up the subscheme Z and resolving the singularities of the resulting scheme, we obtain a smooth variety X', with a proper morphism $\pi\colon X' \to X$, which is an isomorphism above U, such that $Z' = \pi^{-1}(Z)$ is a divisor.

By the previous remark, the variety X' satisfies $\mathrm{H}^1(X', \mathscr{O}_{X'}) = \mathrm{H}^2(X', \mathscr{O}_{X'}) = 0$. Let $U' = \pi^{-1}(U)$; since π is an isomorphism, any model $(\mathscr{X}', \mathscr{Z}')$ of (X', Z') will be such that $\mathscr{U}' = \mathscr{X}' \backslash \mathscr{Z}'$ is isomorphic to $\mathscr{U}$, at least over a dense open subset of $\operatorname{Spec} \mathfrak{o}_F$. In particular, the cardinalities of $\mathscr{U}'(k_v)$ and $\mathscr{U}(k_v)$ will be equal for almost all places v. In view of Weil's formula recalled as Eq. (2.4), this implies that $\tau_{U',v}(\mathscr{U}'(\mathfrak{o}_v)) = \tau_{U,v}(\mathscr{U}(\mathfrak{o}_v))$ for almost all finite places v of F.

Consequently, assuming that Z is a divisor does not reduce the generality of the argument which follows.

For almost all places v, one has

$$\tau_{U,v}(\mathscr{U}(\mathfrak{o}_v)) = q_v^{-\dim U} \# \mathscr{U}(k_v).$$

Evidently,

$$\#\,\mathscr{U}(k_v) = \#\,\mathscr{X}(k_v) - \#\,\mathscr{Z}(k_v).$$

Let ℓ be a prime number. By the smooth and proper base change theorems, and by Poincaré duality, the number of points of $\mathscr{X}(k_v)$ can be computed, for any finite place v of F, via the trace of a geometric Frobenius element $\mathrm{Fr}_v \in \Gamma_v$ on the ℓ-adic cohomology of $Z_{\bar F}$, as soon as $\mathscr{X}$ is smooth over the local ring of $\mathfrak{o}_F$ at v and the residual characteristic at v is not equal to ℓ. Specifically, for any such finite place v, one has the equality

$$q_v^{-\dim X}\,\#\,\mathscr{X}(k_v) = \sum_{i=0}^{2d} q_v^{-i/2}\,\mathrm{tr}(\mathrm{Fr}_v\,|\mathrm{H}^i(X_{\bar F}, \mathbf{Q}_\ell)).$$

By Deligne's proof [20] of the Weil conjectures (analogue of the Riemann hypothesis) the eigenvalues of Fr_v on $\mathrm{H}^i(X_{\bar F}, \mathbf{Q}_\ell)$ are algebraic numbers with archimedean value $q^{i/2}$. Therefore, the ith term of the sum above is an algebraic number whose archimedean absolute values are bounded by $\dim \mathrm{H}^i(X_{\bar F}, \mathbf{Q}_\ell) q^{-i/2}$. In particular, the sum of all terms corresponding to $i \geq 3$ is an algebraic number whose archimedean absolute values are $\mathrm{O}(q_v^{-3/2})$ when v varies through the set of finite places of F.

The term corresponding to $i = 0$ is equal to 1 since X is geometrically connected.

Moreover, it follows from Peyre's arguments ([38], proof of Lemma 2.1.1 and Proposition 2.2.2) that one has

$$\mathrm{H}^1(X_{\bar F}, \mathbf{Q}_\ell) = 0$$

and that the cycle map induces an isomorphism of $\mathbf{Q}_\ell[\Gamma_F]$-modules

$$\mathrm{Pic}(X_{\bar F}) \otimes \mathbf{Q}_\ell \xrightarrow{\sim} \mathrm{H}^2(X_{\bar F}, \mathbf{Q}_\ell(1)).$$

Let $\Pi(Z_{\bar F})$ denote the free $\mathbf{Z}$-module with basis the set of irreducible components of $Z_{\bar F}$, endowed with its Galois action.

Lemma 2.7. *For almost all v, one has the equality*

$$\#\,\mathscr{Z}(k_v) = n_v q_v^{\dim Z} + \mathrm{O}(q_v^{\dim Z - 1/2}),$$

where $n_v = \mathrm{tr}(\mathrm{Fr}_v\,|\Pi(Z_{\bar F}))$ is the number of irreducible components of $\mathscr{Z} \otimes k_v$ which are geometrically irreducible.

Proof.[a] In fact, we prove, for any scheme $\mathscr{Z}/\mathfrak{o}_F$ which is flat, of finite type, and whose generic fiber Z has dimension d, the following equality

$$\#\,\mathscr{Z}(k_v) = n_v q_v^d + \mathrm{O}(q_v^{d-1/2}),$$

[a]The proof below uses Deligne's proof of Weil's conjecture but the result can be deduced from the estimates of Lang–Weil in [35].

where n_v is the number of fixed points of Fr_v in the set of d-dimensional irreducible components of $Z_{\bar F}$.

First observe that there is a general upper bound

$$\#\,\mathscr{Z}(k_v) \leq C q_v^d,$$

for some integer $C > 0$. When $\mathscr{Z}$ is quasi-projective, this follows for example from Lemma 3.9 in [12], where C can be taken to be the degree of the closure of Z in a projective compactification; the general case follows from this since $\mathscr{Z}$ is assumed to be of finite type.

Consequently, adding or removing a subscheme of lower dimension gives an equivalent inequality, so that we may assume that $\mathscr{Z}$ is projective and equidimensional. By resolution of singularities, we may also assume that Z is smooth.

Let ℓ be a fixed prime number. As above, we can compute $\#\,\mathscr{Z}(k_v)$ using the ℓ-adic cohomology of $Z_{\bar F}$: one has an upper bound

$$q_v^{-d}\,\#\,\mathscr{Z}(k_v) = \mathrm{tr}(\mathrm{Fr}_v|\mathrm{H}^i(Z_{\bar F}, \mathbf{Q}_\ell)) + \mathrm{O}(q_v^{-1/2}).$$

Moreover, $\mathrm{H}^0(Z_{\bar F}, \mathbf{Q}_\ell) \simeq \mathbf{Q}_\ell^{\Pi(Z_{\bar F})}$, where $\Pi(Z_{\bar F})$ denotes the set of connected components of $Z_{\bar F}$, the Frobenius element Fr_v acting trivially on this ring. Consequently,

$$\mathrm{tr}(\mathrm{Fr}_v\,|\mathrm{H}^0(Z_{\bar F}, \mathbf{Q}_\ell)) = \mathrm{tr}(\mathrm{Fr}_v\,|\Pi(Z_{\bar F}))$$

and the lemma follows from this. ◻

Let us return to the proof of Theorem 2.5. By the preceding lemma, we have

$$q_v^{-\dim X}\,\#\,\mathscr{Z}(k_v) = q_v^{-1}\,\mathrm{tr}(\mathrm{Fr}_v\,|\Pi(Z_{\bar F})) + \mathrm{O}(q_v^{-3/2}),$$

hence

$$q_v^{-\dim X}\,\#\,\mathscr{U}(k_v) = 1 - \frac{1}{q_v}\,\mathrm{tr}\left(\mathrm{Fr}_v\middle|\Pi(Z_{\bar F})\otimes\mathbf{Q}_\ell - \mathrm{H}^2_{\text{ét}}(X_{\bar F}, \mathbf{Q}_\ell(1))\right) + \mathrm{O}(q_v^{-3/2}).$$

Let us show that the trace appearing in the previous formula is equal to $\mathrm{tr}(\mathrm{Fr}_v|\,\mathrm{EP}(U))$. The order of vanishing/pole along any irreducible component of $Z_{\bar F}$ defines an exact sequence of abelian sheaves for the étale site of $X_{\bar F}$:

$$1 \to \mathbf{G}_m \to i_*\mathbf{G}_m \xrightarrow{\mathrm{ord}} \oplus j_{\alpha *}\mathbf{Z} \to 0,$$

where $i\colon U_{\bar F} \to X_{\bar F}$ is the inclusion and $j_\alpha\colon Z_\alpha \to X_{\bar F}$ are the inclusions of the irreducible components of $Z_{\bar F}$. These sheaves are endowed with an action of Γ_F for which the above exact sequence is equivariant. Taking étale cohomology gives an exact sequence of Γ_F-modules:

$$\begin{aligned} 0 &\to \mathrm{H}^0(X_{\bar F}, \mathbf{G}_m) \to \mathrm{H}^0(U_{\bar F}, \mathbf{G}_m) \to \oplus\mathrm{H}^0(Z_\alpha, \mathbf{Z}) \\ &\to \mathrm{H}^1(X_{\bar F}, \mathbf{G}_m) \to \mathrm{H}^1(U_{\bar F}, \mathbf{G}_m). \end{aligned}$$

Moreover, this last map is surjective: it can be identified with the restriction map $\mathrm{Pic}(X_{\bar F}) \to \mathrm{Pic}(U_{\bar F})$ which is surjective because $X_{\bar F}$ is smooth and $U_{\bar F}$ open in $X_{\bar F}$. The scheme $X_{\bar F}$ is proper, smooth and connected hence $\mathrm{H}^0(X_{\bar F}, \mathbf{G}_m) = \bar F^*$. Moreover for any α, Z_α is connected which implies $\mathrm{H}^0(Z_\alpha, \mathbf{Z}) = \mathbf{Z}$, that is $\oplus \mathrm{H}^0(Z_\alpha, \mathbf{Z}) = \Pi(Z_{\bar F})$. Finally, one obtains an exact sequence of $\mathbf{Z}[\Gamma_F]$-modules

$$0 \to \mathrm{H}^0(U_{\bar F}, \mathbf{G}_m)/\bar F^* \to \Pi(Z_{\bar F}) \to \mathrm{Pic}(X_{\bar F}) \to \mathrm{Pic}(U_{\bar F}) \to 0,$$

which after tensoring with $\mathbf{Q}_\ell$ gives an equality of virtual representations

$$\mathrm{EP}(U) = \Pi(Z_{\bar F})_{\mathbf{Q}_\ell} - \mathrm{Pic}(X_{\bar F})_{\mathbf{Q}_\ell} = \Pi(Z_{\bar F})_{\mathbf{Q}_\ell} - \mathrm{H}^2_{\text{ét}}(X_{\bar F}, \mathbf{Q}_\ell(1)). \tag{2.5}$$

This implies that

$$q_v^{-\dim X} \# \mathscr{U}(k_v) = 1 - \frac{1}{q_v} \operatorname{tr}(\mathrm{Fr}_v \mid \mathrm{EP}(U)) + \mathrm{O}(q_v^{-3/2}).$$

Since the eigenvalues of Fr_v on the two Galois modules defining $\mathrm{EP}(U)$ are algebraic numbers whose absolute values are bounded by 1, one has

$$\det(1 - q_v^{-1} \mathrm{Fr}_v \mid \mathrm{EP}(U)) = 1 - \frac{1}{q_v} \operatorname{tr}(\mathrm{Fr}_v \mid \mathrm{EP}(U)) + \mathrm{O}(q_v^{-2}).$$

Now,

$$\begin{aligned} \mathrm{L}_v(1, \mathrm{EP}(U)) q_v^{-\dim X} \# \mathscr{U}(k_v) &= \det(1 - q_v^{-1} \mathrm{Fr}_v \mid \mathrm{EP}(U))^{-1} q_v^{-\dim X} \# \mathscr{U}(k_v) \\ &= 1 + \mathrm{O}(q_v^{-3/2}), \end{aligned}$$

and the asserted absolute convergence follows.

Definition 2.8. Let F be a number field; let X be a smooth proper, geometrically integral variety over F such that $\mathrm{H}^1(X, \mathscr{O}_X) = \mathrm{H}^2(X, \mathscr{O}_X) = 0$. Let Z be a Zariski closed subset in X, and let $U = X \setminus Z$. The Tamagawa measure on the adelic space $U(\mathbb{A}_F^S)$ is defined as the measure

$$\tau_U^S = \mathrm{L}_*^S(1, \mathrm{EP}(U))^{-1} \left(\prod_{v \notin S} \mathrm{L}_v(1, \mathrm{EP}(U)) \tau_{U,v} \right).$$

By Theorem 2.5, this infinite product of measures converges; moreover, non-empty, open and relatively compact, subsets of $U(\mathbb{A}_F^S)$ have a positive (finite) τ_U^S-measure. In particular, compact subsets of $U(\mathbb{A}_F^S)$ have a finite τ_U^S-measure and τ_U^S is a Radon measure on $U(\mathbb{A}_F^S)$.

Remark 2.9. In the literature, families (λ_v) of positive real numbers such that the product of measures $\prod_v (\lambda_v \tau_{U,v})$ converges absolutely are called sets of convergence factors (see, e.g. [47] or [41]). Our theorem can thus be stated as saying that for any smooth geometrically integral algebraic variety U having a smooth compactification X satisfying $\mathrm{H}^1(X, \mathscr{O}_X) = \mathrm{H}^2(X, \mathscr{O}_X) = 0$, the family $(\mathrm{L}_v(1, \mathrm{EP}(U)))$ is a set of convergence factors. Let us compare our construction with the previously known cases.

(a) The Picard group of an affine connected algebraic group U is finite; in that case, $\mathrm{EP}(U)$ is therefore the Galois module of characters of U and we recover Weil's definition ([47]). For semi-simple groups, there are no nontrivial characters and the product of the natural local densities converges absolutely, as is well known. This is also the case for homogeneous varieties G/H studied by Borovoi and Rudnick in [7], where $G \supset H$ are semi-simple algebraic groups without nontrivial rational characters.

(b) On the opposite side, integral projective varieties have no non-constant global functions, so that $\mathrm{EP}(U) = \mathrm{Pic}(U_{\bar F})$ if U is projective. We thus recover Peyre's definition of the Tamagawa measure of a projective variety [38].

(c) Salberger [41] has developed a theory of Tamagawa measures on the "universal torsors" introduced by Colliot-Thélène and Sansuc [18]. A universal torsor is a principal homogeneous space E over an algebraic variety V whose structure group is an algebraic torus T dual to the Picard group of V such that the canonical map $\mathrm{Hom}(T, \mathbf{G}_m) \to \mathrm{H}^1(V, \mathbf{G}_m)$, sending a character χ to the $\mathbf{G}_m$-torsor $\chi_*[E]$ deduced from E by push-out along χ, is an isomorphism. Such torsors have been successfully applied to the study of rational points on the variety V (Hasse principle, weak approximation, counting of rational points of bounded height); see [18, 40, 9, 46].

By a fundamental theorem of Colliot-Thélène and Sansuc,[b] universal torsors have no non-constant invertible global functions and their Picard group is trivial (at least up to torsion). It follows that the virtual Γ_F-module $\mathrm{EP}(E)$ is trivial. This gives a conceptual explanation for the discovery by Salberger in [41] that the Tamagawa measures on universal torsors could be defined by an absolutely convergent product of "naive" local measures, without any regularizing factors.

2.4.3. Assume moreover that Z is the support of a divisor D in X (more generally, of a Cartier $\mathbf{Q}$-divisor) and that $\omega_X(D)$ is endowed with an adelic metric; this induces a natural metric on $\mathscr{O}_X(D)$. For any place v, the line bundle $\omega_X(D)$ on $X(F_v)$ is metrized and this metric gives rise to a measure $\tau_{(X,D),v}$ on $U(F_v)$, which is related to the measure $\tau_{X,v}$ by the formula

$$\mathrm{d}\tau_{(X,D),v}(x) = \frac{1}{\|\mathsf{f}_D\|(x)}\,\mathrm{d}\tau_{X,v}(x),$$

where f_D is the canonical section of $\mathscr{O}_X(D)$. By definition of an adelic metric, the metric on $\mathscr{O}_X(D)$ is induced, for almost all finite places v, by the line bundle $\mathscr{O}_{\mathscr{X}}(\mathscr{D})$ on the integral model $\mathscr{X}$, where $\mathscr{D}$ is a Cartier divisor with generic fiber D. For such places v, one has $\|\mathsf{f}_D\|(x) = 1$ for any point $x \in \mathscr{U}(\mathfrak{o}_v)$, so that the measures $\tau_{(X,D),v}$ and $\tau_{X,v}$ coincide on $\mathscr{U}(\mathfrak{o}_v)$.

[b] Apply [18], Proposition 2.1.1 with $K = \bar k$, the map $\hat S \to \mathrm{Pic}(X_{\bar k})$ being an isomorphism by the very definition of a universal torsor.

This shows that the product

$$\tau^S_{(X,D)} = \mathrm{L}^S_*(1, \mathrm{EP}(U))^{-1} \left(\prod_{v \notin S} \mathrm{L}_v(1, \mathrm{EP}(U)) \tau_{(X,D),v} \right)$$

also defines a Radon measure on $U(\mathbb{A}^S_F)$. In fact, one has

$$\mathrm{d}\tau^S_{(X,D)}(x) = \left(\prod_v \|\mathsf{f}_D\|(x_v) \right)^{-1} \mathrm{d}\tau^S_U(x) = H_D(x)\, \mathrm{d}\tau^S_U(x),$$

and $H_D(x)$ is the height of x relative to the metrized line bundle $\mathscr{O}_X(D)$.

2.4.4. *Example*: *Compactifications of algebraic groups*

As in the case of local fields, we briefly explain the case of algebraic groups and show how our theory of measures interacts with the construction of a Haar measure on an adelic group. We keep the notation of the previous paragraph, and assume moreover that U is an algebraic group G over F of which X is an equivariant compactification. Let us fix an invariant differential form of maximal degree ω on G; viewed as a meromorphic global section of ω_X, it has poles on each component of $X \setminus G$ (see [13], Lemma 2.4); we therefore write $-D$ for its divisor.

For any place $v \in \mathrm{Val}(F)$, viewing ω as a gauge form on $G(F_v)$ defines a Haar measure $|\omega|_v$ on $G(F_v)$. For almost all finite places v, $G(\mathfrak{o}_v)$ is a well-defined compact subgroup of $G(F_v)$. We normalize the Haar measure $\mathrm{d}g_v$ on $G(F_v)$ by dividing the measure $|\omega|_v$ by the quantity $\# G(k_v) q_v^{-\dim G}$ for these places v, and by 1 at other places. By construction, for almost all v, $\mathrm{d}g_v$ assigns volume 1 to the compact open subgroup $G(\mathfrak{o}_v)$, hence the product $\prod_v \mathrm{d}g_v$ is a well-defined Haar measure on $G(\mathbb{A}_F)$.

By Theorem 2.5 we have the estimate

$$\mathrm{L}_v(1, \mathrm{EP}(G))\, \# G(k_v) q_v^{-\dim G} = 1 + \mathrm{O}(q_v^{-3/2}),$$

from which we deduce that their infinite product converges absolutely. Consequently, $\prod_v \mathrm{L}_v(1, \mathrm{EP}(G)) |\omega|_v$ is also a Haar measure on $G(\mathbb{A}_F)$.

Now, by the very definition of the adelic measures $\tau_{(X,D)}$ and τ_X on $G(\mathbb{A}_F)$, one has

$$\begin{aligned}
\mathrm{d}\tau_{(X,D)}(g) &= H_D(g)\, \mathrm{d}\tau_X(g) \\
&= \prod_v \|\omega(g)\|_v \times \mathrm{L}_*(1, \mathrm{EP}(G))^{-1} \prod_v \left(\mathrm{L}_v(1, \mathrm{EP}(G)) \frac{\mathrm{d}|\omega|_v(g)}{\|\omega(g)\|_v} \right) \\
&= \mathrm{L}_*(1, \mathrm{EP}(G))^{-1} \prod_v (\mathrm{L}_v(1, \mathrm{EP}(G)) \mathrm{d}|\omega|_v(g)).
\end{aligned}$$

This shows that, in the particular case of an equivariant compactification of an algebraic group, our general definition of the measure $\tau_{(X,D)}$ on $X \setminus |D|$ gives rise to a Haar measure on $G(\mathbb{A}_F)$.

2.5. *An abstract equidistribution theorem*

In some cases, it is possible to count integral (resp. rational) points of bounded height with respect to almost any normalization of the height, i.e. with respect to any metrization on a given line bundle. Analogously, we obtain below an asymptotic expansion for the volume of height balls for almost any normalization of the height.

In this subsection, we show how to extract from the obtained asymptotic behavior a measure-theoretic information on the distribution of points of bounded height or of height balls.

The following proposition is an abstract general version of a result going back to Peyre ([38], Proposition 3.3).

Proposition 2.10. *Let X be a compact topological space, U a subset of X and $H\colon U \to \mathbf{R}_+$ a function. Let ν be a positive measure on X such that for any real number B, the set $\{x \in U; H(x) \leq B\}$ has finite ν-measure.*

Let S be a dense subspace of the space $\mathrm{C}(X)$ of continuous functions on X, endowed with the sup. norm. Assume that there exist a function $\alpha\colon \mathbf{R}_+ \to \mathbf{R}_+^*$ and a Radon measure μ on X such that for any positive function $\theta \in \mathrm{S}$,

$$\nu(\{x \in U; H(x) \leq \theta(x)B\}) \sim \alpha(B) \int_X \theta(x)\, \mathrm{d}\mu(x)$$

for $B \to +\infty$. Then, for $B \to +\infty$, the measures

$$\nu_B = \frac{1}{\alpha(B)} \mathbf{1}_{\{H(x) \leq B\}} \mathrm{d}\nu(x)$$

on X converge vaguely to the measure μ. In other words,

(i) *for any continuous function $f \in \mathrm{C}(X)$,*

$$\frac{1}{\alpha(B)} \int_X \mathbf{1}_{\{H(x) \leq B\}} f(x) \mathrm{d}\nu(x) \to \int_X f(x)\, \mathrm{d}\mu(x), \quad \text{for } B \to +\infty;$$

(ii) *for every open set $\Omega \subset X$ which is μ-regular,*

$$\nu(\{x \in \Omega \cap U; H(x) \leq B\}) = \alpha(B)\mu(\Omega) + \mathrm{o}(\alpha(B)).$$

Before entering the proof, let us introduce one more notation. For $\theta \in \mathrm{C}(X)$, $\theta > 0$, let

$$N(\theta; B) = \nu(\{x \in U; H(x) \leq \theta(x)B\}).$$

For any open subset $\Omega \subset X$, let

$$N_\Omega(B) = \nu(\{x \in \Omega \cap U; H(x) \leq B\}).$$

Proof. First remark that these hypotheses imply that for any $\lambda > 0$, $\alpha(\lambda B)/\alpha(B) \to \lambda$ when $B \to +\infty$. Indeed, taking any positive $\theta \in S$,

$$\begin{aligned} \frac{\alpha(\lambda B)}{\alpha(B)} &= \frac{\alpha(\lambda B)\int_X \theta(x)\mathrm{d}\mu(x)}{\alpha(B)\int_X \theta(x)\mathrm{d}\mu(x)} \sim \frac{N(\theta;\lambda B)}{N(\theta;B)} \\ &\sim \frac{N(\lambda\theta;B)}{N(\theta;B)} \sim \frac{\alpha(B)\int_X \lambda\theta(x)\,\mathrm{d}\mu(x)}{\alpha(B)\int_X \theta(x)\,\mathrm{d}\mu(x)} = \lambda. \end{aligned}$$

Now, it is of course sufficient to prove that $N_\Omega(B)/\alpha(B) \to \mu(\Omega)$ for any μ-regular open set $\Omega \subset B$. Fix $\varepsilon > 0$. Since $\Omega \subset X$ is μ-regular, there exist continuous functions f and g on X such that

$$f \leq \mathbf{1}_\Omega \leq g$$

and such that $\int (g-f)\mathrm{d}\mu \leq \varepsilon$. Since S is dense in $\mathrm{C}(X)$, we may also assume that f and g belong to S and that they are non-negative.

Let $f_\varepsilon = \varepsilon + (1-\varepsilon)f$, $\chi_\varepsilon = \varepsilon + (1-\varepsilon)\mathbf{1}_\Omega$ and $g_\varepsilon = \varepsilon + (1-\varepsilon)g$. The inequality $f_\varepsilon \leq \chi_\varepsilon \leq g_\varepsilon$ implies that

$$N(f_\varepsilon;B) \leq N(\chi_\varepsilon;B) \leq N(g_\varepsilon;B).$$

Moreover, by definition,

$$0 \leq N(\chi_\varepsilon;B) - N_\Omega(B) \leq N_X(\varepsilon B).$$

Therefore,

$$\begin{aligned} \liminf N_\Omega(B)\alpha(B)^{-1} &\geq \liminf N(\chi_\varepsilon;B)\alpha(B)^{-1} - \limsup N_X(\varepsilon B)\alpha(B)^{-1} \\ &\geq \liminf N(f_\varepsilon;B)\alpha(B)^{-1} - \varepsilon \\ &\geq \int f_\varepsilon \mathrm{d}\mu - \varepsilon = \int f\,\mathrm{d}\mu + \mathrm{O}(\varepsilon). \end{aligned}$$

Similarly,

$$\limsup N_\Omega(B)\alpha(B)^{-1} \leq \int g\,\mathrm{d}\mu + \mathrm{O}(\varepsilon).$$

When $\varepsilon \to 0$, one thus finds

$$\lim N_\Omega(B)\alpha(B)^{-1} = \mu(\Omega),$$

as claimed. □

3. Geometric Igusa Integrals: Preliminaries

3.1. *Clemens complexes and variants*

In this section, we study analytic properties of certain integrals of Igusa-type attached to a divisor D in a variety X. This requires the introduction of a simplicial set which encodes the intersections of the various components of D, and which we call the *Clemens complex*. Such a simplicial set has been used by Clemens in [16] in his study of the Picard–Lefschetz transformation (see also [26]).

3.1.1. *Simplicial sets*

We recall classical definitions concerning simplicial sets. A partially ordered set, abridged as *poset*, is a set endowed with a binary relation $\prec$ which is transitive ($a \prec b$ and $b \prec c$ implies $a \prec c$), antisymmetric ($a \prec b$ and $b \prec a$ implies $a = b$) and reflexive ($a \prec a$). If $a \prec b$, we say that a is a face of b; in particular, the transitivity axiom for a poset means that a face of a face is a face.

There are obvious definitions for morphisms of posets, sub-posets, as well as actions of groups on posets.

As an example, if $\mathscr{A}$ is a set, the set $\mathscr{P}^*(\mathscr{A})$ of non-empty subsets of $\mathscr{A}$, together with the inclusion relation, is a poset. We shall interpret it geometrically as the simplex on the set of vertices $\mathscr{A}$ and denote it by $\mathscr{S}_{\mathscr{A}}$. More generally, a *simplicial complex* on a set $\mathscr{A}$ is a set of non-empty finite subsets of $\mathscr{A}$, called *simplices*, such that any non-empty subset of a simplex is itself a simplex. The dimension of a simplex is defined as one less its cardinality: points, edges, ... are simplexes of dimension $0, 1, \ldots$. The dimension of a simplicial complex is the supremum of the dimensions of its faces. The order relation endows a simplicial complex with the structure of poset. In fact, simplicial complexes are sub-posets of the simplex $\mathscr{S}_{\mathscr{A}}$.

Let S be a poset. The dimension of an element $s \in S$, denoted $\dim(s)$, is defined as the supremum of the lengths n of chains $s_0 \prec \cdots \prec s_n$, where the s_i are distinct with $s_n \prec s$. Similarly, the codimension $\operatorname{codim}(s)$ of an element $s \in S$ is defined as the supremum of the lengths of such chains with $s \prec s_0$. Elements of dimension $0, 1, \ldots,$ are called vertices, edges, ... The dimension $\dim(S)$ of S is the supremum of all dimensions of all of its elements.

It is important to observe that given a poset S and a sub-poset S', the dimension or the codimension of a face of S' may differ from their dimension or codimension as a face of S.[c]

Let S be a poset. Categorically, an action of a group Γ on S is just a morphism of groups from Γ to the set $\operatorname{Aut}(S)$ of automorphisms of S. In other words, it is the data, for any $\gamma \in \Gamma$, of a bijection γ_* of S such that $\gamma_* s \prec \gamma_* s'$ is $s \prec s'$, subject to the compatibilities $(\gamma\gamma')_* = \gamma_*\gamma'_*$ for any γ and $\gamma' \in \Gamma$.

Let there be given a poset S and an action of Γ on S. The set S^Γ of fixed points of Γ in S is a sub-poset of S. In the other direction, the set S/Γ of orbits of Γ in S, can be endowed with the binary relation deduced from $\prec$ by passing to the quotient. Namely, for s and s' in S, with orbits $[s]$ and $[s']$, we say that $[s] \prec [s']$ if there exists $\gamma \in \Gamma$ such that $s \prec \gamma_* s'$. (This condition does not depend on the actual choice of elements s and s' in their orbits $[s]$ and $[s']$.)

There are obvious morphisms of posets, $S^\Gamma \to S$ and $S \to S_\Gamma$; these morphisms are the universal morphisms of posets, respectively to S and from S, which commute with the action of Γ on S.

[c] This probably means that the terminology "sub-poset" is inappropriate; sub-posets for which the dimension notion is compatible are sometimes called ideals ....

As an example, assume that $\mathscr{S} = \mathscr{S}_{\mathscr{A}}$, the simplex with vertices in a set $\mathscr{A}$. Actions of a group Γ on $\mathscr{S}_{\mathscr{A}}$ correspond to actions of Γ on $\mathscr{A}$. The simplicial sets $\mathscr{S}^{\Gamma}$ and $\mathscr{S}_{\Gamma}$ are respectively the simplices with vertices in the fixed-points $\mathscr{A}^{\Gamma}$ and the orbits $\mathscr{A}/\Gamma$.

3.1.2. *Incidence complexes*

Let X be a (geometrically integral) variety over a perfect field F and let D be a divisor in X. Fix a separable closure $\bar{F}$ of F. Let $\bar{\mathscr{A}}$ be the set of irreducible components of $D_{\bar{F}}$; for $\alpha \in \bar{\mathscr{A}}$, we denote by D_α the corresponding component of $D_{\bar{F}}$. For any subset $A \subset \bar{\mathscr{A}}$, we let $D_A = \bigcap_{\alpha \in A} D_\alpha$; in particular, $D_\varnothing = X_{\bar{F}}$.

We shall always make the assumption that the divisor $D_{\bar{F}}$ has simple normal crossings: all irreducible components D_α of $D_{\bar{F}}$ are supposed to be smooth and to meet transversally.[d] In particular, for any subset $A \subset \bar{\mathscr{A}}$ such that $D_A \neq \varnothing$, D_A is a smooth subvariety of $X_{\bar{F}}$ of codimension $\# A$.

The closed subschemes D_A, for $A \subset \bar{\mathscr{A}}$, are the closed strata of a stratification $(D_A^\circ)_{A \subset \bar{\mathscr{A}}}$, where, for any subset $A \subset \bar{\mathscr{A}}$, D_A° is defined by the formula:

$$D_A^\circ = D_A \Big\backslash \bigcup_{B \supsetneq A} D_B.$$

There are in fact several natural posets that enter the picture, encoding in various ways the combinatorial data of whether or not, for a given subset A of $\mathscr{A}$, the intersection D_A is empty. The following observation is crucial for our definitions to make sense:

Proposition 3.1. *Let A and A' be two subsets of $\bar{\mathscr{A}}$ such that $A' \subset A$. For any irreducible component Z of D_A, there is a unique irreducible component Z' of $D_{A'}$ which contains Z.*

Proof. If an irreducible component Z were contained in two distinct irreducible components of $D_{A'}$, these two components would meet along Z, which contradicts the simple normal crossings assumption on $D_{\bar{F}}$. Indeed, D_A being smooth, its irreducible components must be disjoint. □

3.1.3. *The geometric Clemens complex*

The incidence complex $\mathscr{I}_{\bar{F}}(D)$ defined by D is the sub-poset of $\mathscr{P}^*(\bar{\mathscr{A}})$ consisting of non-empty subsets A of $\bar{\mathscr{A}}$ such that the intersection D_A is not empty. More precisely, we define the geometric Clemens complex $\mathscr{C}_{\bar{F}}(D)$ as the set of all pairs

[d]This is equivalent to the weaker condition that D has normal crossings together with the smoothness of the geometric irreducible components of D; the condition that D has strict normal crossings is however stronger since the smoothness of an irreducible component of D implies that its geometric irreducible components do not meet.

(A, Z), where $A \subset \bar{\mathscr{A}}$ is any non-empty subset, and Z is an irreducible component of the scheme D_A, together with the partial order relation defined by $(A, Z) \prec (A', Z')$ if $A \subset A'$ and $Z \supset Z'$. In other words, the set of vertices of $\mathscr{C}_{\bar{F}}(D)$ is $\bar{\mathscr{A}}$, there are edges corresponding to irreducible component of each intersection $D_\alpha \cap D_{\alpha'}$, etc.

Mapping (A, Z) to A induces a morphism of posets from the geometric Clemens complex to the incidence complex. In fact, we could have defined $\mathscr{C}_{\bar{F}}(D)$ without any reference to $\bar{\mathscr{A}}$. Indeed, thanks to the normal crossings condition, given an irreducible component Z of a scheme D_A, we may recover A as the set of $\alpha \in \bar{\mathscr{A}}$ such that D_α contains the generic point of Z.

Since X is defined over F, Γ_F acts naturally on the set of integral subschemes of $X_{\bar{F}}$. Since F is perfect, an integral subscheme Z of $X_{\bar{F}}$ is defined over F if and only if it is a fixed point for this action.

For any $\gamma \in \Gamma_F$ and any $\alpha \in \bar{\mathscr{A}}$, observe that $\gamma_* D_\alpha$ is an irreducible component of $D_{\bar{F}}$, because D is defined over F; this induces an action of Γ_F on $\bar{\mathscr{A}}$, defined by $\gamma_* D_\alpha = D_{\gamma\alpha}$, for $\alpha \in \bar{\mathscr{A}}$ and $\gamma \in \Gamma_F$. Moreover, if Z is an irreducible component of an intersection D_A, for $A \subset \bar{\mathscr{A}}$, then $\gamma_* Z$ is an irreducible component of $\gamma_* D_A = D_{\gamma_* A}$. Consequently, we have a natural action of Γ_F on the geometric Clemens complex $\mathscr{C}_{\bar{F}}(D)$, given by $\gamma_*(A, Z) = (\gamma_* A, \gamma_* Z)$ for any element (A, Z) of $\mathscr{C}_{\bar{F}}(D)$.

The natural morphism of posets $\mathscr{C}_{\bar{F}}(D) \to \mathscr{I}_{\bar{F}}(D)$ mapping (A, Z) to A is Γ_F-equivariant.

3.1.4. *Rational Clemens complexes*

Let us denote by $\mathscr{I}_F(D)$ and $\mathscr{C}_F(D)$ the sub-posets of $\mathscr{I}_{\bar{F}}(D)$ and $\mathscr{C}_{\bar{F}}(D)$ consisting of Γ_F-fixed faces. These posets correspond to the intersections of the divisors D_α and to the irreducible components of these intersections which are defined over the base field F. Alternatively, they can be defined without any reference to the algebraic closure $\bar{F}$ by considering irreducible components of the divisor D, their intersections and the irreducible components of these which are geometrically irreducible.

More generally, let E be any extension of F, together with an embedding of $\bar{F}$ in an algebraic closure $\bar{E}$. As an example, one may take any extension of F contained in $\bar{F}$. In our study below, F will be a number field and E will be the completion of F at a place v; the choice of an embedding $\bar{F} \hookrightarrow \bar{E}$ corresponds to the choice of a decomposition group at v. Under these conditions, there is a natural morphism of groups from the Galois group Γ_E of $\bar{E}/E$ to Γ_F. In particular, the posets $\mathscr{C}_{\bar{F}}(D)$ and $\mathscr{I}_{\bar{F}}(D)$ are endowed with an action of Γ_E.

Let us define the E-rational Clemens complex, $\mathscr{C}_E(D)$, as the sub-poset of $\mathscr{C}_{\bar{F}}(D)$ fixed by Γ_E. In particular, for any face (A, Z) of $\mathscr{C}_E(D)$, the subschemes $(D_A)_{\bar{E}}$ and $Z_{\bar{E}}$ are defined over E. We shall denote $(D_A)_E$ and Z_E, or even D_A and Z, the corresponding subschemes of X_E. Observe that Z_E is geometrically irreducible.

Conversely, let A be any non-empty subset of $\bar{\mathscr{A}}$ which is Γ_E-invariant; then, $(D_A)_{\bar{E}}$ is defined over E and corresponds to some subscheme $(D_A)_E$ of X_E. Let Z be any irreducible component of $(D_A)_E$ which is geometrically irreducible. Then, $Z_{\bar{E}}$ is an irreducible component of $(D_A)_{\bar{E}}$. By EGA IV (4.5.1), the set of irreducible components does not change when one extends the ground field from an algebraically closed field to any extension; consequently, $Z_{\bar{E}}$ is defined over $\bar{F}$ and (A, Z) corresponds to a face of $\mathscr{C}_E(D)$.

3.1.5. *E-analytic Clemens complexes*

Let E be any perfect extension of F together with an embedding $\bar{F} \hookrightarrow \bar{E}$ as above. We define the *E-analytic Clemens complex*, denoted $\mathscr{C}_E^{\mathrm{an}}(D)$, as the sub-poset of $\mathscr{C}_E(D)$ whose faces are those faces (A, Z) such that $Z(E) \neq \varnothing$. (When we write $Z(E)$, we identify Z with the unique E-subscheme of X_E whose extension to $\bar{E}$ is $Z_{\bar{E}}$.)

This complex can also be defined as follows. First let A be any non-empty subset of $\bar{\mathscr{A}}$ which is Γ_E-invariant and let Z be an irreducible component of D_A. While $(D_A)_{\bar{E}}$ is defined over E, for the moment, Z is just a subscheme of $X_{\bar{F}}$. However, we can define $Z(E)$ as the intersection in $X(\bar{E})$ of $X(E)$ and of $Z(\bar{E})$; this is indeed what one would get if $Z_{\bar{E}}$ were defined over E.

Assume that $Z(E) \neq \varnothing$. Let Z' be the smallest subscheme of X_E such that $Z'_{\bar{E}}$ contains Z. It is irreducible: if Z' were a union $Z'_1 \cup Z'_2$, one of them, say Z'_1, would be such that $(Z'_1)_{\bar{E}}$ contains $Z_{\bar{E}}$, because $Z_{\bar{E}}$ is irreducible. Moreover, Z' is contained in $(D_A)_E$, since $(D_A)_{\bar{E}}$ contains $Z_{\bar{E}}$. It follows that Z' is an irreducible component of $(D_A)_E$. By assumption, $Z'(E) \neq \varnothing$ hence, by EGA IV (4.5.17), Z' is geometrically connected. Since D_A is smooth, so are $(D_A)_{\bar{E}}$ and $Z'_{\bar{E}}$, this last subscheme being a union of irreducible components of $(D_A)_{\bar{E}}$. But a connected smooth scheme is irreducible, hence Z' is irreducible and $Z'_{\bar{E}} = Z_{\bar{E}}$. It follows that $Z_{\bar{E}}$ is defined over E and (A, Z) corresponds to a face of $\mathscr{C}_E^{\mathrm{an}}(D)$.

Let us finally observe that the dimension of a face (A, Z) of $\mathscr{C}_E^{\mathrm{an}}(D)$ is equal to $\#(A/\Gamma_E) - 1$. Indeed, let $n = \#(A/\Gamma_E)$ and consider any sequence $A_1 \subset \cdots \subset A_n = A$ of Γ_E-invariants subsets of A. Then, $D_{A_1} \supset D_{A_2} \supset \cdots \supset D_{A_n}$ and each of them has a unique irreducible component Z_i containing Z. Since $Z(E) \neq \varnothing$, $Z_i(E)$ is non-empty either, and the (A_i, Z_i) define a maximal increasing sequence of faces of $\mathscr{C}_E^{\mathrm{an}}(D)$.

This notion will be of interest to us under the supplementary assumption that E is a locally compact-valued field. Indeed, such fields allow for a theory of analytic manifolds as well as an implicit function theorem. By the local description of X that will be explained below, for any face (A, Z) of $\mathscr{C}_E(D)$, $Z(E)$ is either empty, or is an E-analytic submanifold of $X(E)$ of codimension $\#(A)$ which is Zariski-dense in Z. However, as a face of $\mathscr{C}_E^{\mathrm{an}}(D)$, (A, Z) has dimension $\#(A/\Gamma_E) - 1$ and it is that invariant which will be relevant in our analysis below.

3.1.6. *Example*

Below we describe these Clemens complexes in special cases, which are essentially governed by combinatorial data, namely for toric varieties and equivariant compactifications of semi-simple groups. We want to make clear that for general varieties, the three types of Clemens complexes can be very different.

For example, let X be the blow-up of the projective space $\mathbf{P}^n$ along a smooth subvariety V defined over the ground field, let D be the exceptional divisor in X and let $\pi\colon X \to \mathbf{P}^n$ be the canonical morphism. The map π induces a bijection between the set of irreducible components of $D_{\bar{F}}$ and the set of irreducible components of $V_{\bar{F}}$, as well as a bijection between the set of irreducible components of D and the set of irreducible components of V. Moreover, a component Z of D is geometrically irreducible if and only if the corresponding component $\pi(Z)$ of V is; indeed, Z is isomorphic to the projectivized normal bundle of V in X. This description shows also that $Z(F) = \varnothing$ if and only if $\pi(Z)(F) = \varnothing$.

To give a specific example, take V to be the disjoint union of a geometrically irreducible smooth curve C_1, without rational points, of a geometrically irreducible smooth curve C_2 with rational points and of a smooth irreducible curve C_3 with two geometrically components C_3' and C_3''. Then, $\mathscr{C}(X, D)$ consists of four points corresponding to C_1, C_2, C_2', C_3' (there are no intersections), the F-rational Clemens complex $\mathscr{C}_F(X, D)$ consists of two points corresponding to C_1 and C_2, and the F-analytic Clemens complex $\mathscr{C}_F^{\mathrm{an}}(X, D)$ consists of the single point corresponding to C_1. We present further examples in Sec. 5.

3.2. *Local description of a pair (X, D)*

3.2.1. *Notation*

Let F be a perfect field, $\bar{F}$ an algebraic closure of F, X a smooth algebraic variety over F and $D \subset X$ a reduced divisor such that $D_{\bar{F}}$ has strict normal crossings in $X_{\bar{F}}$. Let us recall that this means that $D_{\bar{F}}$ is the union of irreducible smooth divisors in $X_{\bar{F}}$ which meet transversally. In other words, each point of $X_{\bar{F}}$ has a neighborhood U, together with an étale map $U \to \mathbf{A}^n_{\bar{F}}$ such that for each irreducible component Z of $D_{\bar{F}} \cap U$, $Z \cap U$ is the preimage of some coordinate hyperplane in $\mathbf{A}^n_{\bar{F}}$.

Let $\bar{\mathscr{A}}$ be the set of irreducible components of $D_{\bar{F}}$; for $\alpha \in \bar{\mathscr{A}}$, let D_α denote the corresponding divisor, so that $D_{\bar{F}} = \sum_{\alpha \in \mathscr{A}} D_\alpha$. For $\alpha \in \bar{\mathscr{A}}$, we denote by F_α the field of definition of D_α in $\bar{F}$; its Galois group Γ_{F_α} is the stabilizer of D_α in Γ_F. For $A \subset \bar{\mathscr{A}}$, we denote by D_A the intersection of the divisors D_α, for $\alpha \in A$; by convention, $D_\varnothing = X$.

Let $\alpha \in \bar{\mathscr{A}}$. The union $\bigcup_{a \in \Gamma_F \alpha} D_a$ of the conjugates of D_α is a divisor in $X_{\bar{F}}$ which is defined over F; the corresponding divisor of X will be denoted by Δ_α. Similarly, the intersection of the divisors D_a, for $a \in \Gamma_F \alpha$, is a smooth subscheme of $X_{\bar{F}}$ which is defined over X; the corresponding subscheme of X will be denoted by E_α. If $E_\alpha \neq \varnothing$, then its codimension r_α is equal to $[F_\alpha : F]$. Moreover, $\Delta_\alpha(F) = E_\alpha(F)$.

The choice of another element $\alpha' = \gamma_* \alpha$ in the orbit $\Gamma_F \alpha$ does not change Δ_α nor E_α. However, it changes the field F_α to the conjugate $F_{\alpha'} = \gamma_* F_\alpha$ and its stabilizer Γ_{F_α} to the conjugate subgroup $\Gamma_{F_{\alpha'}} = \gamma \Gamma_{F_\alpha} \gamma^{-1}$. Observe that $r_{\alpha'} = r_\alpha$.

3.2.2. *Local equations for the étale topology*

Let us now give local equations of D around each rational point of X. Fix a point $\xi \in X(F)$ and let $\alpha \in \bar{\mathscr{A}}$ be such that $\xi \in D_\alpha$ (it is a Γ_F-invariant subset of $\bar{\mathscr{A}}$).

When $F_\alpha = F$, $D_\alpha = \Delta_\alpha \times \bar{F}$ is defined over F and the choice of a local equation for D_α defines a smooth map from an open neighborhood U_ξ of ξ in X to the affine line which maps $U_\xi \cap D_\alpha$ to $\{0\} \subset \mathbf{A}^1$.

To explain the general case, let us introduce other notations. We write $\mathbf{A}_{F_\alpha}$ for the Weil restriction of scalars $\operatorname{Res}_{F_\alpha/F} \mathbf{A}^1$ from F_α to F of the affine line. It is endowed with a canonical morphism $\mathbf{A}^1_{F_\alpha} \to (\operatorname{Res}_{F_\alpha/F} \mathbf{A}^1) \times_F F_\alpha$ which induces, for any F-scheme V, a bijection

$$\operatorname{Hom}(V, \operatorname{Res}_{F_\alpha/F} \mathbf{A}^1) = \operatorname{Hom}(V \times_F F_\alpha, \mathbf{A}^1).$$

In particular, the F-rational points of $\mathbf{A}_{F_\alpha}$ are in canonical bijection with F_α.

The F-scheme $\mathbf{A}_{F_\alpha}$ is an affine space of dimension $r_\alpha = [F_\alpha : F]$. Let us indeed choose an F-linear basis $(u_1, \dots, u_{r_\alpha})$ of F_α. Then, the morphism $\mathbf{A}^{r_\alpha} \to \mathbf{A}_{F_\alpha}$ given by $(x_1, \dots, x_{r_\alpha}) \mapsto \sum x_i u_i$ is an isomorphism. In these coordinates, the norm-form $\mathrm{N}_{F_\alpha/F}$ of F_α is a homogeneous polynomial of degree $[F_\alpha : F]$ which defines a hypersurface in $\mathbf{A}_{F_\alpha}$; it has a single F-rational point 0.

Moreover, the r_α F-linear embeddings of F_α into $\bar{F}$ induce an isomorphism $F_\alpha \otimes_F \bar{F} \simeq \bar{F}^{r_\alpha}$, hence an identification of $\mathbf{A}_{F_\alpha} \times_F \bar{F}$ with the affine space $\mathbf{A}^{r_\alpha}_{\bar{F}}$. Under this identification, the divisor of the norm-form corresponds to the union of the coordinates hyperplanes.

Lemma 3.2. *There is a Zariski open neighborhood U_ξ of ξ in X and a smooth map*

$$x_\alpha \colon U_\xi \to \mathbf{A}_{F_\alpha}$$

which maps $\Delta_\alpha \cap U_\xi$ to the hypersurface of $\mathbf{A}_{F_\alpha}$ defined by the norm equation $\mathrm{N}_{F_\alpha/F}(x) = 0$, *such that there is a diagram*

$$\begin{tikzcd}
E_\alpha \arrow[r, hook] \arrow[d] & \Delta_\alpha \arrow[r, hook] \arrow[d] & U_\xi \subset X \arrow[d] \\
\{0\} \arrow[r, hook] & \{\mathrm{N}_{F_\alpha/F}(x) = 0\} \arrow[r, hook] & \mathbf{A}_{F\alpha}.
\end{tikzcd}$$

Proof. Choose an element f in the local ring $\mathscr{O}_{X_{\bar{F}},\xi}$ which is a generator of the ideal of D_α, so that $f = 0$ is a local equation of D_α. For any $\gamma \in \Gamma_F$, $f^\gamma = 0$ is a local equation of $\gamma_* D_\alpha$. Consequently, f^γ/f is a local unit at ξ, for each $\gamma \in \Gamma_{F_\alpha}$, and the map $\gamma \mapsto f^\gamma/f$ is a 1-cocycle for the Galois group Γ_{F_α} with values in the local ring $\mathscr{O}^*_{X_{\bar{F}},\xi}$. By Hilbert's Theorem 90 for the multiplicative group, this cocycle is a coboundary (see [37], Proposition 4.9 and Lemma 4.10 for a non-elementary proof in that context) and there exists a unit $u \in \mathscr{O}^*_{X_{\bar{F}},\xi}$ such that $f^\gamma/f = u^\gamma/u$ for any $\gamma \in \Gamma_{F_\alpha}$. Consequently, fu^{-1} is an element of $\mathscr{O}_{X_{\bar{F}},\xi}$, fixed by Γ_{F_α}, which generates the ideal of D_α. In other words, there is a Zariski open neighborhood V of ξ in X, and a function $g \in \Gamma(V \times_F F_\alpha, \mathscr{O}_{V \times F_\alpha})$ such that $g = 0$ is an equation of D_α in $V \times_F F_\alpha$. To the morphism $V \times_F F_\alpha \to \mathbf{A}^1_{F_\alpha}$ corresponds by the universal property of the Weil restriction of scalars a morphism, still denoted θ, $V \to \mathbf{A}_{F_\alpha}$.

Through the canonical identification, over $\bar{F}$, of $\mathbf{A}_{F_\alpha}$ with the affine space $\mathbf{A}^{r_\alpha}_{\bar{F}}$, the map g induces an isomorphism from D_α to one of the coordinate hyperplanes, from Δ_α to the union of the coordinate hyperplanes, while ξ maps to 0. At the level of F-rational points, the diagram $(V, \Delta_\alpha) \to (\mathbf{A}_{F_\alpha}, H)$ corresponds to the map $(V(F), D_\alpha(F)) \to (F_\alpha, 0)$. □

3.2.3. *Local charts*

Let $\xi \in X(F)$. More generally, let A_ξ be the set of $\alpha \in \mathscr{A}$ such that $\xi \in D_\alpha$. Let Z be the irreducible component of D_{A_ξ} which contains ξ. It is geometrically irreducible and (A_ξ, Z) is a face of $\mathscr{C}^{\mathrm{an}}_F(D)$. For a sufficiently small neighborhood V of ξ, there exists, for each $\alpha \in A_\xi$, a smooth map $x_\alpha \colon V \to \mathbf{A}_{F_\alpha}$ as in the previous paragraph. Since the divisors D_α meet transversally, the map $(x_\alpha) \colon V \to \prod_{\alpha \in A_\xi} \mathbf{A}_{F_\alpha}$ is smooth. By choosing additional local coordinates, and shrinking V if needed, we can extend it to an étale map $q_\xi \colon U_\xi \to \prod_{\alpha \in A_\xi} \mathbf{A}_{F_\alpha} \times \mathbf{A}^r$, with $r = \operatorname{codim}_\xi(D_{A_\xi}, X) = \#(A_\xi)$. We will also assume, as we may, that the open subset U_ξ is affine and that $U_\xi \cap D_\alpha = \varnothing$ if $\alpha \notin A_\xi$.

Assume moreover that F is a local field. It follows from the preceding discussion that for each $\alpha \in \mathscr{A}$, $\Delta_\alpha(F) = E_\alpha(F)$ is a smooth F-analytic subvariety of $X(F)$, either empty, or of codimension $r_\alpha = [F : F_\alpha]$. For any Γ_F-invariant subset A of $\mathscr{A}$, let us denote by Δ_A the intersection of all Δ_α, for $\alpha \in A$. Then, $\Delta_A(F) = E_A(F)$ is either empty, or a smooth F-analytic subvariety of $X(F)$ of codimension $r_A = \#(A)$.

3.2.4. *Partitions of unity*

In this section, we assume in addition that F is a local field and we consider the situation from the analytic point of view. Observe that the maps q_ξ induce analytic étale maps $U_\xi(F) \to \prod_{\alpha \in A_\xi} F_\alpha \times F^r$. The open sets $U_\xi(F)$ cover $X(F)$ which is compact, since X is projective. Consequently, there is a finite partition of unity (θ_ξ) subordinate to this covering: for any ξ, $\theta_\xi \colon X(F) \to \mathbf{R}$ is a smooth function whose support is contained in $U_\xi(F)$, the map q_ξ is one-to-one on that support, $\sum_{\xi\in X(F)} \theta_\xi = 1$ and only a finite number of θ_ξ are nonzero.

3.2.5. *Metrized line bundles, volume forms*

Let $\alpha \in \bar{\mathscr{A}}$ and let us consider the line bundle $\mathscr{O}_X(\Delta_\alpha)$. It has a canonical section f_α the divisor of which is Δ_α. Let us assume that F is a local field. On each open set U_ξ of X, we have constructed a regular map $x_\alpha\colon U_\xi \to \mathbf{A}_{F_\alpha}$. Composed with the norm $\mathbf{A}_{F_\alpha} \to \mathbf{A}^1$, it gives us a regular function $\mathrm{N}(x_\alpha) \in \Gamma(U_\xi, \mathscr{O}_X)$ which generates the ideal of Δ_α on U_ξ. This function therefore induces a trivialization of $\mathscr{O}_X(-\Delta_\alpha)_{|U_\xi}$. Consequently, a metric on $\mathscr{O}_X(\Delta_\alpha)_{|U_\xi}$ takes the form $\|\mathsf{f}_\alpha\|(x) = |x_\alpha|_{F_\alpha} h_\xi(x)$ on $U_\xi(F)$, where we view x_α as a local coordinate $U_\xi(F) \to F_\alpha$ and $|x_\alpha|_{F_\alpha} = |\mathrm{N}_{F_\alpha/F}(x_\alpha)|_F$, and where $h_\xi\colon U_\xi(F) \to \mathbf{R}_+^*$ is any continuous function.

Similarly, on such a chart $U_\xi(F)$, we have a measure in the Lebesgue class defined by $\mathrm{d}\mathbf{x} = \prod_{\alpha\in A_\xi} \mathrm{d}x_\alpha \times \mathrm{d}x_1 \cdots \mathrm{d}x_r$. Any measure on $X(F)$ which belongs to the Lebesgue class can therefore be expressed in a chart $U_\xi(F)$ as the product of this measure $\mathrm{d}\mathbf{x}$ with a positive locally integrable function. Let us also observe that there are canonical isomorphisms

$$\mathscr{O}(\Delta_\alpha)|_{E_\alpha} \simeq \mathscr{N}_{\Delta_\alpha}(X)|_{E_\alpha} \simeq \det \mathscr{N}_{E_\alpha}(X).$$

These isomorphisms allow us to define residue measures on $\Delta_\alpha(F) = E_\alpha(F)$, and on their intersections, hence on all faces of the F-analytic Clemens complex $\mathscr{C}_F^{\mathrm{an}}(D)$.

3.3. *Mellin transformation over local fields*

3.3.1. *Local zeta functions*

Let F be a local field of characteristic zero, with normalized absolute value $|\cdot|_F$. Define a constant c_F as follows:

$$\mathrm{c}_F = \begin{cases} \mu([-1;1]) & \text{if } F = \mathbf{R};\\ \mu(B(0;1)) & \text{if } F = \mathbf{C};\\ (1-q^{-1})(\log q)^{-1}\mu(\mathfrak{o}_F) & \text{if } F \supset \mathbf{Q}_p, q = |\varpi_F|_F^{-1}. \end{cases} \tag{3.1}$$

(In the last formula, ϖ_F denotes a generator of the maximal ideal of $\mathfrak{o}_F$.)

For any locally integrable function $\varphi\colon F \to \mathbf{C}$, we let

$$\mathscr{M}_F(\varphi)(s) = \int_F \varphi(x)|x|_F^{s-1}\,\mathrm{d}\mu(x),$$

for any complex parameter s such that the integral converges absolutely. We call it the Mellin transform of φ; to shorten the notation and when no confusion can arise, we will often write $\hat{\varphi}$ instead of $\mathscr{M}_F(\varphi)$.

We write ζ_F for the Mellin transform $\mathscr{M}_F(\varphi_0)$ of φ_0, the characteristic function of the unit ball in F. The integral defining $\zeta_F(s)$ converges if and only if $\mathrm{Res}(s) > 0$, and one has

$$\begin{aligned}
\zeta_F(s) &= \mu([-1,1]) \int_0^1 x^{s-1}\,\mathrm{d}x = \frac{\mathrm{c}_F}{s} && \text{for } F = \mathbf{R} \\
&= \frac{\mu(B(0;1))}{\pi} 2\pi \int_0^1 r^{2s-1}\,\mathrm{d}r = \frac{\mathrm{c}_F}{s} && \text{for } F = \mathbf{C} \\
&= \sum_{n=0}^{\infty} q^{-n(s-1)} \mu(\varpi_F^n \mathfrak{o}_F^*) = \mu(\mathfrak{o}_F)\left(1 - \frac{1}{q}\right) \sum_{n=0}^{\infty} q^{-ns} \\
&= \mu(\mathfrak{o}_F) \frac{1-q^{-1}}{1-q^{-s}} && \text{otherwise.}
\end{aligned}$$

In all cases, the function $s \mapsto \zeta_F(s)$ is holomorphic on the half-plane defined by $\mathrm{Re}(s) > 0$ and extends to a meromorphic function on $\mathbf{C}$. It has a simple pole at $s = 0$, with residue c_F. In the archimedean case, note that $s = 0$ is the only pole of ζ_F. However, in the p-adic case, ζ_F is $(2i\pi/\log q)$-periodic so that $s = 2in\pi/\log q$ is also a pole for any integer n; this will play a role in the use of Tauberian arguments later.

3.3.2. *Mellin transform of smooth functions*

One of our goals in this paper is to establish the meromorphic continuation, together with growth estimates, for integrals on varieties that are defined locally like Mellin transforms. To explain our arguments, we begin with the one-dimensional toy example.

We say that a function $\varphi\colon F \to \mathbf{C}$ is smooth if it is either C^∞ when $F = \mathbf{R}$ or $\mathbf{C}$, or locally constant otherwise. If a and b are real numbers, we define $\mathsf{T}_{>a}$, resp. $\mathsf{T}_{(a,b)}$, as the set of $s \in \mathbf{C}$ such that $a < \mathrm{Re}(s)$ (resp. $a < \mathrm{Re}(s) < b$).

Lemma 3.3. *Let F be a local field of characteristic zero, let φ be a measurable, bounded and compactly supported function on F. Then, the Mellin transform*

$$\mathscr{M}_F(\varphi)(s) = \int_F |x|_F^{s-1} \varphi(x)\mathrm{d}x$$

converges for $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 0$ and defines a holomorphic function on $\mathsf{T}_{>0}$. Moreover, for any real numbers a and b such that $0 < a < b$, the function $\mathscr{M}_F(\varphi(s))$ is bounded in $\mathsf{T}_{(a,b)}$.

Assume that φ is smooth. Then there exists a holomorphic function φ_1 on $\mathsf{T}_{>-1/2}$ such that $\varphi_1(0) = \varphi(0)$ and $\mathscr{M}_F(\varphi)(s) = \zeta_F(s)\varphi_1(s)$. Moreover, for any real numbers a and b such that $-\frac{1}{2} < a < b, \varphi_1$ is bounded in $\mathsf{T}_{(a,b)}$ if F is ultrametric, while there is an upper bound

$$|\varphi_1(s)| \ll 1 + |\mathrm{Im}(s)|, \quad s \in \mathsf{T}_{(a,b)},$$

if F is archimedean. In particular, the function

$$s \mapsto \mathscr{M}_F(\varphi)(s) - \varphi(0)\zeta_F(s)$$

is holomorphic and

$$\lim_{s\to 0} s\mathscr{M}_F(\varphi)(s) = \mathrm{c}_F\varphi(0).$$

Proof. The absolute convergence of $\mathscr{M}_F(\varphi)(s)$ for $\mathrm{Re}(s) > 0$ and its holomorphy in that domain immediately follow from the absolute convergence of ζ_F and the fact that φ is bounded and has compact support. So does also its boundedness in vertical strips $\mathsf{T}_{(a,b)}$ for $0 < a < b$.

Assume that φ is smooth. Let us prove the stated meromorphic continuation. Again, let φ_0 denote the characteristic function of the unit ball in F. Replacing φ by $\varphi - \varphi(0)\varphi_0$, it suffices to show that $\mathscr{M}_F(\varphi)$ is holomorphic on the half-plane $\mathrm{Re}(s) > -\frac{1}{2}$ whenever φ is a compactly supported function on F which is smooth in a neighborhood of 0 and satisfies $\varphi(0) = 0$.

In the real case, one can then write $|\varphi|(x) = |x|_F\psi(x)$ for some function ψ which is continuous at 0. Similarly, in the complex case, one can write $|\varphi(z)| = |z|_F^{1/2}\psi(z)$. The required holomorphy of $\mathscr{M}_F(\varphi)(s)$ on the domain $\mathsf{T}_{>-\frac{1}{2}}$ follows. In the ultrametric case, φ vanishes in a neighborhood of 0 and $\mathscr{M}_F(\varphi)$ even extends to a holomorphic function on $\mathbf{C}$.

In all cases, observe that $\mathscr{M}_F(\varphi)(s)$ is bounded in vertical strips $\mathsf{T}_{(\alpha,\beta)}$, if α and β are real numbers such that $-\frac{1}{2} < \alpha < \beta$. Now, the existence of the function φ_1 and the asserted bound on this function follow from the fact that ζ_F^{-1} is holomorphic and satisfies this bound in these vertical strips. □

3.3.3. *Higher-dimensional Mellin transforms*

We consider a finite family $(F_j)_{1\leq j\leq n}$ of local fields. For simplicity, we assume these to be finite extensions of a single completion of $\mathbf{Q}$. Let us denote by V the space $\prod F_j$. Adopting the terminology introduced for functions of one variable, we say that a function on V is smooth if it is either C^∞, in the case where the F_j are archimedean, or locally constant, when the F_j are ultrametric. If $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ are families of real numbers, we define the sets $\mathsf{T}_{>\alpha}$ and $\mathsf{T}_{(\alpha,\beta)}$ as the open subsets of $\mathbf{C}^n$ consisting of those $s \in \mathbf{C}^n$ such that $\alpha_j < \mathrm{Re}(s_j)$ (resp. $\alpha_j < \mathrm{Re}(s_j) < \beta_j$) for $j = 1, \dots, n$. When all α_j are equal to a single one α, and similarly for the β_j, we will also write $\mathsf{T}^n_{>\alpha}$ and $\mathsf{T}^n_{(\alpha,\beta)}$, or even $\mathsf{T}_{>\alpha}$ and $\mathsf{T}_{(\alpha,\beta)}$ when the dimension n is clear from the context. Let $\mathscr{F}$ be the vector space generated by functions on $V \times \mathbf{C}^n$ of the form $(x; s) \mapsto u(x)v_1(x)^{s_1}\cdots v_n(x)^{s_n}$, where u is smooth, compactly supported on V, and $v_1, \dots, v_n$ are smooth, positive, and equal to 1 outside of a compact subset of V. For any function $\varphi \in \mathscr{F}$ and $s \in \mathbf{C}^n$, we let

$$\mathscr{M}_V(\varphi)(s) = \int_V |x_1|^{s_1-1}\cdots|x_n|^{s_n-1}\varphi(x; s)\mathrm{d}x_1\cdots\mathrm{d}x_n,$$

whenever the integral converges.

Proposition 3.4. *The integral $\mathscr{M}_V(\varphi)(s)$ converges for any $s \in \mathbf{C}^n$ such that $\operatorname{Re}(s_j) > 0$ for all j, and defines an holomorphic function in $\mathsf{T}_{>0}$. For any positive real numbers α and β, $\mathscr{M}_V(\varphi)$ is bounded on $\mathsf{T}^n_{(\alpha,\beta)}$.*

Moreover, there exists a unique holomorphic function φ_1 on the domain $\mathsf{T}^n_{>-1/2}$ such that

$$\mathscr{M}_V(\varphi)(s) = \prod_{j=1}^n \zeta_{F_j}(s_j)\varphi_1(s)$$

for $s \in \mathsf{T}^n_{>0}$. Let α and β be real numbers such that $-\frac{1}{2} < \alpha < \beta$. Then, φ_1 is bounded on $\mathsf{T}^n_{(\alpha,\beta)}$ if the F_j are ultrametric, while there exists a real number c such that

$$|\varphi_1(s)| \leq c\prod_{j=1}^n (1+|s_j|), \quad s \in \mathsf{T}_{(\alpha,\beta)},$$

if the F_j are archimedean. Moreover, $\varphi_1(0) = \varphi(0;0)$.

4. Geometric Igusa Integrals and Volume Estimates

4.1. *Igusa integrals over local fields*

We return to our geometric situation: X is a smooth quasi-projective variety over a local field F, D is a divisor on X such that $D_{\bar F}$ has simple normal crossings. We let $\bar{\mathscr{A}}$ be the set of irreducible components of $D_{\bar F}$. For $\alpha \in \bar{\mathscr{A}}$, D_α denotes the corresponding component, while Δ_α and E_α are respectively the sum and the intersection of the conjugates of D_α.

For $\alpha \in \bar{\mathscr{A}}$, we denote by f_α the canonical section of the line bundle $\mathscr{O}(\Delta_\alpha)$. Endow the line bundles $\mathscr{O}(\Delta_\alpha)$, as well as the canonical bundle ω_X with smooth metrics. Let τ_X the corresponding measure on X. Following the definitions of Sec. 2.1.12, the analytic variety $D_A(F)$, for any Γ_F-invariant subset $A \subset \bar{\mathscr{A}}$ supports a natural "residue measure". To simplify explicit formulas below, we define the measure τ_{D_A} as the residue measure multiplied by $\prod_{\alpha\in A} \mathrm{c}_{F_\alpha}$, where the constants c_{F_α} are defined in Eq. (3.1).

The integrals we are interested in take the form

$$\mathscr{I}(\Phi; (s_\alpha)_{\alpha\in\mathscr{A}}) = \int_{X(F)} \prod_{\alpha\in\mathscr{A}} \|\mathsf{f}_{D_\alpha}\|(x)^{s_\alpha-1}\Phi(x)\,\mathrm{d}\tau_X(x),$$

where Φ is a smooth, compactly supported function on $X(F)$ and the s_α are complex parameters. Letting Φ vary, it is convenient to view these integrals as distributions (of order 0) on $X(F)$.

For any subset A of the F-analytic Clemens complex $\mathscr{C}^{\mathrm{an}}_F(D)$, define also

$$\mathscr{I}_A(\Phi; (s_\alpha)_{\alpha\notin A}) = \int_{D_A(F)} \prod_{\alpha\notin A} \|\mathsf{f}_{D_\alpha}\|(x)^{s_\alpha-1}\Phi(x)\,\mathrm{d}\tau_{D_A}(x).$$

Lemma 4.1. *The integral $\mathscr{I}(\Phi;(s_\alpha)_{\alpha\in\mathscr{A}})$ converges for $s \in \mathsf{T}^{\mathscr{A}}_{>0}$ and the map*

$$(s_\alpha)_{\alpha\in\mathscr{A}} \mapsto \mathscr{I}(\Phi;(s_\alpha)_{\alpha\in\mathscr{A}})$$

is holomorphic on $\mathsf{T}^{\mathscr{A}}_{>0}$.

Similarly, for any $A \subset \mathscr{A}$, the integral $\mathscr{I}_A(\Phi;(s_\alpha))$ converges for $s \in \mathsf{T}^{\mathscr{A}\setminus A}_{>0}$ and the function $s \mapsto \mathscr{I}_A(\Phi;s)$ is holomorphic on that domain.

We shall therefore call such integrals "*holomorphic distributions on X*".

Proof. For $\xi \in X(F)$, let $\mathscr{A}_\xi$ be the set of $\alpha \in \mathscr{A}$ such that $\xi \in D_\alpha$ and let

$$q\colon U_\xi \to \prod_{\alpha\in\mathscr{A}_\xi} \mathbf{A}_{F_\alpha} \times \mathbf{A}^r$$

be an étale chart around ξ adapted to D, as in Sec. 3.2.3. Let $x_\alpha\colon U_\xi \to \mathbf{A}_{F_\alpha}$ be the composite of q followed by the projection to $\mathbf{A}_{F_\alpha}$, and $y\colon U_\xi \to \mathbf{A}^r$ be the composite of q with the projection to $\mathbf{A}^r$.

These maps induce local coordinates in a neighborhood of U_ξ, valued in $\prod_{\alpha\in\mathscr{A}_\xi} F_\alpha \times F^r$ in which the measure τ takes the form $\kappa((x_\alpha),y)\prod \mathrm{d}x_\alpha\,\mathrm{d}y$.

By definition of a smooth metric, there is, for any $\alpha \in \mathscr{A}$, a smooth non-vanishing function u_α on $U_\xi(F)$ such that $\|\mathsf{f}_{D_\alpha}\| = |x_\alpha|_{F_\alpha} u_\alpha$ if $\alpha \in \mathscr{A}_\xi$, and $\|\mathsf{f}_{D_\alpha}\| = u_\alpha$ otherwise.

Further, introducing a partition of unity (see Sec. 3.2.4), we see that it suffices to study integrals of the form

$$\int_{\prod F_\alpha\times F^d} \prod_{\alpha\in\mathscr{A}_\xi} |x_\alpha|_{F_\alpha}^{s_\alpha-1}\Phi((x_\alpha),y) \prod_{\alpha\in\mathscr{A}} u_\alpha^{s_\alpha-1}\kappa((x_\alpha),y)\prod \mathrm{d}x_\alpha\,\mathrm{d}y,$$

where Φ is a smooth function with compact support on $\prod F_\alpha\times F^r$. The holomorphy of such integrals is precisely the object of Proposition 3.4 above.

The case of the integrals $\mathscr{I}_A(\Phi;(s_\alpha))$ is analogous. □

By the same arguments, but using the meromorphic continuation of Mellin transforms and the estimate of their growth in vertical strips, we obtain the following result.

Proposition 4.2. *The holomorphic function*

$$s \mapsto \prod_{\alpha\in\mathscr{A}} \zeta_{F_\alpha}(s_\alpha)^{-1}\mathscr{I}(\Phi;(s_\alpha))$$

on $\mathsf{T}^{\mathscr{A}}_{>0}$ extends to a holomorphic function $\mathscr{M}(\Phi;\cdot)$ on $\mathsf{T}^{\mathscr{A}}_{>-1/2}$. Moreover, for any real numbers a and b and for any function Φ, there is a real number c such that

$$\mathscr{M}(\Phi;s) \le c\prod_{\alpha\in\mathscr{A}}(1+|s_\alpha|) \quad \text{for any } s\in\mathsf{T}^{\mathscr{A}}_{(a,b)}.$$

4.1.1. *Leading terms*

We will need to understand the leading terms at the poles of these integrals after restricting the parameter (s_α) to an affine line in $\mathbf{C}^{\mathscr{A}}$. For $\alpha \in \mathscr{A}$, let $s \mapsto s_\alpha = -\rho_\alpha + \lambda_\alpha s$ be an increasing affine function with real coefficients. To shorten notation, we write $\mathscr{I}(\Phi; s)$ instead of $\mathscr{I}(\Phi; (s_\alpha))$, and similarly for the integrals $\mathscr{I}_A$. Define

$$a(\lambda, \rho) = \max_{\alpha \in \mathscr{A}} \frac{\rho_\alpha}{\lambda_\alpha}$$

and let $\mathscr{A}(\lambda, \rho)$ denote the set of all $\alpha \in \mathscr{A}$ where the maximum is obtained, i.e. such that $a(\lambda, \rho) = \rho_\alpha/\lambda_\alpha$. By the preceding lemma, $\mathscr{I}(\Phi; s) = \mathscr{I}(\Phi, (-\rho_\alpha + \lambda_\alpha s))$ converges for $\mathrm{Re}(s) > a(\lambda, \rho)$ and defines a holomorphic function there. Similarly, letting

$$a_A(\lambda, \rho) = \max_{\alpha \notin A} \frac{\rho_\alpha}{\lambda_\alpha},$$

the function $s \mapsto \mathscr{I}_A(\Phi; s) := \mathscr{I}_A(\Phi, (-\rho_\alpha + \lambda_\alpha s))$ converges for $\mathrm{Re}(s) > a_A(\lambda, \rho)$. Notice that $a_A(\lambda, \rho) \leq a(\lambda, \rho)$ in general, and that the inequality may be strict, e.g. if $\mathscr{A}(\rho, \lambda) \subseteq A$.

Let $\mathscr{C}^{\mathrm{an}}_{F,(\lambda,\rho)}(D)$ be the intersection of the Clemens complex $\mathscr{C}^{\mathrm{an}}_F(D)$ with the simplicial subset $\mathscr{P}^+(\mathscr{A}(\lambda, \rho))$ of $\mathscr{P}^+(\mathscr{A})$. In other words, we remove from $\mathscr{C}^{\mathrm{an}}_F(D)$ all faces containing a vertex α such that $\rho_\alpha < a(\lambda, \rho)\lambda_\alpha$.

Proposition 4.3. *With the above notation, there exists a positive real number δ, and, for any face A of $\mathscr{C}^{\mathrm{an}}_{F,(\lambda,\rho)}(D)$ of maximal dimension, a holomorphic function $\mathscr{J}_A$ defined on $\mathsf{T}_{>a(\lambda,\rho)-\delta}$ with polynomial growth in vertical strips such that*

$$\mathscr{J}_A(\Phi; a(\lambda, \rho)) = \mathscr{I}_A(\Phi; (a(\lambda, \rho)\lambda_\alpha - \rho_\alpha)_{\alpha \notin A})$$

and such that

$$\mathscr{I}(\Phi; s) = \sum_A \mathscr{J}_A(\Phi; s) \prod_{\alpha \in \mathscr{A}} \zeta_{F_\alpha}(s - a(\lambda, \rho)),$$

where the sum is restricted to the faces A of $\mathscr{C}^{\mathrm{an}}_{F,(\lambda,\rho)}(D)$ of maximal dimension. In particular,

$$\lim_{s \to a(\lambda,\rho)} \mathscr{I}(\Phi; s)(s - a(\lambda, \rho))^{\dim \mathscr{C}^{\mathrm{an}}_{F,(\lambda,\rho)}}$$

$$= \sum_A \mathscr{I}_A(\Phi; (a(\lambda, \rho)\lambda_\alpha - \rho_\alpha)_{\alpha \notin A}) \prod_{\alpha \in A} \frac{1}{\lambda_\alpha}.$$

Proof. As in the proof of Lemma 4.1, we shall use a partition of unity and local coordinates.

Let $\xi \in X(F)$, let $\mathscr{A}_\xi$ be the set of $\alpha \in \mathscr{A}$ such that $\xi \in D_\alpha$. In a neighborhood Ω of ξ, we have local coordinates $x_\alpha \in F_\alpha$, for $\alpha \in \mathscr{A}_\xi$, and other coordinates $(y_1, \ldots, y_r)$ for some integer r. There exist smooth functions u_α on Ω such that $\|\mathsf{f}_{D_\alpha}\| = |x_\alpha|_{F_\alpha} u_\alpha$ if $\alpha \in \mathscr{A}_\xi$, and $\|\mathsf{f}_{D_\alpha}\| = u_\alpha$ otherwise. We may assume, after shrinking Ω, that $D_\alpha \cap \Omega = \varnothing$ if $\alpha \notin \mathscr{A}_\xi$; then, u_α does not vanish on Ω, for any $\alpha \in \mathscr{A}$. Finally, the restriction to Ω of the measure τ_X can be written as $\kappa \mathrm{d}y_1 \cdots \mathrm{d}y_r \prod_{\alpha \in \mathscr{A}_x} \mathrm{d}x_\alpha$, for some smooth function κ on Ω. Let θ_ξ be a smooth function with compact support in Ω.

The integral

$$\mathscr{I}_\xi(s) = \int_\Omega \prod_{\alpha \in \mathscr{A}} \|\mathsf{f}_{D_\alpha}\|(x)^{s_\alpha - 1} \Phi(x) \theta_\xi(x) \, \mathrm{d}\tau_X(x)$$

can be rewritten as

$$\mathscr{I}_\xi(s) = \int_\Omega \prod_{\alpha \in \mathscr{A}_\xi} |x_\alpha|_{F_\alpha}^{-\rho_\alpha + s\lambda_\alpha - 1} \times \prod_{\alpha \in \mathscr{A}} u_\alpha^{s_\alpha - 1}(x) \Phi(x) \theta_\xi(x) \kappa(x) \prod_{\alpha \in \mathscr{A}_\xi} \mathrm{d}x_\alpha \times \mathrm{d}y.$$

Let $a = a(\lambda, \rho)$; when $s \to a$, only the variables s_α such that $\rho_\alpha = a\lambda_\alpha$ and $\alpha \in \mathscr{A}_\xi$ contribute a pole. Write $A = \mathscr{A}_\xi \cap \mathscr{A}(\lambda, \rho)$ for this subset. Applying the regularization procedure which led to Proposition 4.2, but only to the variables x_α with $\alpha \in A$, furnishes an expression of the form

$$\mathscr{I}_\xi(s) = \mathscr{J}_\xi(s) \prod_{\alpha \in A} \zeta_{F_\alpha}(-\rho_\alpha + s\lambda_\alpha)$$

for the integral $\mathscr{I}_\xi(s)$, where $\mathscr{J}_\xi$ is holomorphic for $\operatorname{Re}(s) > a(\lambda, \rho) - \delta$ and has polynomial growth in vertical strips. Moreover,

$$\begin{aligned}
\lim_{s \to a} \mathscr{J}_\xi(s) &= \lim_{s \to a} \mathscr{I}_\xi(s) \prod_{\alpha \in A} \zeta_{F_\alpha}(-\rho_\alpha + s\lambda_\alpha)^{-1} \\
&= \int_{\Omega \cap D_A(F)} \prod_{\alpha \in \mathscr{A}} u_\alpha^{-\rho_\alpha + a\lambda_\alpha - 1} \prod_{\alpha \in \mathscr{A}_\xi \setminus A} |x_\alpha|_{F_\alpha}^{-\rho_\alpha + a\lambda_\alpha - 1} \theta_\xi(x) \Phi(x) \kappa(x) \\
&\qquad \times \prod_{\alpha \in \mathscr{A}_\xi \setminus A} \mathrm{d}x_\alpha \, \mathrm{d}y.
\end{aligned}$$

By the definition of the residue measure on $D_A(F)$ (Sec. 2.1.12) and its normalization used here, one thus has

$$\begin{aligned}
&\lim_{s \to a} \mathscr{I}_\xi(s) \prod_{\alpha \in A} \zeta_{F_\alpha}(-\rho_\alpha + s\lambda_\alpha)^{-1} \\
&\qquad = \prod_{\alpha \in A} \frac{1}{\mathrm{c}_{F_\alpha}} \int_{D_A(F)} \prod_{\alpha \notin A} \|\mathsf{f}_{D_\alpha}\|^{-\rho_\alpha + a\lambda_\alpha - 1} \theta_\xi(x) \Phi(x) \, \mathrm{d}\tau_{D_A}(x)
\end{aligned}$$

so that

$$\lim_{s \to a} (s - a)^{\# A} \mathscr{I}(\theta_\xi \Phi; s) = \mathscr{I}_A(\theta_\xi \Phi; (-\rho_\alpha + a\lambda_\alpha)) \prod_{\alpha \in A} \lambda_\alpha^{-1}.$$

Observe that A is a maximal face of $\mathscr{C}^{\mathrm{an}}_{F,(\lambda,\rho)}$, though maybe not one of maximal dimension.

Now choose the functions θ_ξ so that they form a finite partition of unity, i.e. $\sum \theta_\xi = 1$, and only finitely many θ_ξ are not zero. Then $\mathscr{I}(\Phi; s) = \sum \mathscr{I}_\xi(s)$;

regrouping the nonzero terms according to the minimal face of the Clemens complex $\mathscr{C}^{\mathrm{an}}_{F,(\lambda,\rho)}$ to which they correspond furnishes the desired expression for $\mathscr{I}(\Phi;s)$.

Let b be the dimension of this complex and let $\mathscr{C}^{\mathrm{an,max}}_{F,(\lambda,\rho)}$ be the set of its faces of dimension b. Granted the previous limits computations, one has

$$\begin{aligned}\lim_{s\to a}(s-a)^b\mathscr{I}(\Phi;s) &= \sum_{\xi}\lim_{s\to a}(s-a)^b\mathscr{I}(\theta_\xi\Phi;s)\\ &= \sum_{A\in\mathscr{C}^{\mathrm{an,max}}_{F,(\lambda,\rho)}}\prod_{\alpha\in A}\lambda_\alpha^{-1}\mathscr{I}_A(\Phi;(-\rho_\alpha+a\lambda_\alpha)),\end{aligned}$$

as claimed. ∎

Corollary 4.4. *If $\Phi\equiv 1$ or, more generally, if the restriction of Φ to $D(F)$ is not identically* 0, *then the order of the pole of $\mathscr{I}(\Phi;s)$ at $s=a(\lambda,\rho)$ is equal to*

$$1+\dim\mathscr{C}^{\mathrm{an}}_{F,(\lambda,\rho)}(D).$$

4.1.2. *The case of good reduction*

We recast in this geometric context a formula of J. Denef ([21], Theorem 3.1; see also Theorem 9.1 and Theorem 11.2 of [13]).

Assume that F is non-archimedean and that our situation comes from a smooth model $\mathscr{X}$ over $\mathfrak{o}_F$, that the divisors D_α extend to divisors $\mathscr{D}_\alpha$ on $\mathscr{X}$ whose sum becomes a relative Cartier divisor with strict normal crossings after base change to a finite étale extension of $\mathfrak{o}_F$, and that all metrics are defined by this model. The residue field of F is denoted by k, its cardinality by q. For $\alpha\in\mathscr{A}$, the extension $F\subset F_\alpha$ is unramified by the good reduction hypothesis; we denote by f_α its degree; let $\mathfrak{o}_{F_\alpha}$ be the ring of integers of F_α and $\mathfrak{m}_\alpha$ its maximal ideal.

The Zariski closure of the scheme E_α is a smooth subscheme $\mathscr{E}_\alpha$ of $\mathscr{X}$, of relative codimension d_α, such that $\mathscr{E}_\alpha(\mathfrak{o}_F)=\mathscr{D}_\alpha(\mathfrak{o}_F)$. Similarly, the Zariski closure of E_A is a smooth geometrically connected subscheme $\mathscr{E}_A$ of $\mathscr{X}$ of codimension d_A and $\mathscr{E}_A(\mathfrak{o}_F)=\mathscr{D}_A(\mathfrak{o}_F)$. For any subset $A\subset\mathscr{A}$, we let τ_{D_A} denote the Tamagawa measure on $D_A(F)=E_A(F)$. Since the extensions $F\subset F_\alpha$ are unramified, one also has $\mathscr{D}_\alpha(k)=\mathscr{E}_\alpha(k)$ for any $\alpha\in\mathscr{A}$, and $\mathscr{D}_A(k)=\mathscr{E}_A(k)$ for any $A\subset\mathscr{A}$.

Assume also that the function Φ is constant on residue classes; the induced function on $\mathscr{X}(k)$ will still be denoted by Φ.

Proposition 4.5. *Under the above conditions, one has*

$$\mathscr{I}(\Phi;(s_\alpha))=\sum_{A\subset\mathscr{A}}(q^{-1}\mu(\mathfrak{o}_F))^{\dim X}\prod_{\alpha\in A}\frac{q^{f_\alpha}-1}{q^{f_\alpha s_\alpha}-1}\left(\sum_{\tilde\xi\in\mathscr{D}_A^\circ(k)}\Phi(\tilde\xi)\right).$$

In particular, for $\Phi=1$, one has

$$\mathscr{I}(1;(s_\alpha))=\sum_{A\subset\mathscr{A}}(q^{-1}\mu(\mathfrak{o}_F))^{\dim X}\prod_{\alpha\in A}\frac{q^{f_\alpha}-1}{q^{f_\alpha s_\alpha}-1}\#(\mathscr{D}_A^\circ(k)).$$

Proof. Let $\tilde{\xi} \in \mathscr{X}(k)$, let $\mathscr{A}_{\tilde{\xi}} = \{\alpha \in \mathscr{A}; \tilde{\xi} \in \mathscr{D}_\alpha(k)\}$, so that $\tilde{\xi}$ belongs to the open stratum $\mathscr{D}^\circ_{\mathscr{A}_{\tilde{\xi}}}$. By the good reduction hypothesis, we can introduce local (étale) coordinates $x_\alpha \in \mathfrak{m}_\alpha$ (for $\alpha \in \mathscr{A}_{\tilde{\xi}}$) and $y_\beta \in \mathfrak{m}$ (for β in a set $\mathscr{B}_{\tilde{\xi}}$ of cardinality $\dim X - \sum_{\alpha \in \mathscr{A}_{\tilde{\xi}}} f_\alpha$) on the residue class $\Omega_{\tilde{\xi}}$ of $\tilde{\xi}$, such that $\mathscr{D}_\alpha$ is defined by the equation $x_\alpha = 0$ on $\Omega_{\tilde{\xi}}$. Then the local Tamagawa measure identifies with the measure $\prod_{\alpha \in \mathscr{A}_{\tilde{\xi}}} \mathrm{d}x_\alpha \times \prod_{\beta \in \mathscr{B}_{\tilde{\xi}}} \mathrm{d}y_\beta$ on $\prod_{\alpha \in \mathscr{A}_{\tilde{\xi}}} \mathfrak{m}_\alpha \times \prod_{\beta \in \mathscr{B}_{\tilde{\xi}}} \mathfrak{m}$.

Recall also (see Sec. 3.3.1) that for any ultrametric local field F, with ring of integers $\mathfrak{o}_F$ and maximal ideal $\mathfrak{m}$, and any complex number s such that $\mathrm{Re}(s) > 0$, one has

$$q \int_{\mathfrak{m}} |x|_F^{s-1} \,\mathrm{d}x = \frac{q-1}{q^s-1}\mu(\mathfrak{o}_F),$$

where q is the cardinality of the residue field.

These formulas, applied to the fields F_α, and the decomposition of the integral $\mathscr{I}$ as a sum of similar integrals over the residue classes $\tilde{\xi} \in \mathscr{X}(k)$, give us

$$\begin{aligned} \mathscr{I}(\Phi; (s_\alpha)) &= \sum_{\tilde{\xi} \in \mathscr{X}(k)} \Phi(\tilde{\xi}) (q^{-1}\mu(\mathfrak{o}_F))^{\# \mathscr{B}_{\tilde{\xi}}} \prod_{\alpha \in \mathscr{A}_{\tilde{\xi}}} \int_{\mathfrak{m}_\alpha} |x_\alpha|_{F_\alpha}^{s_\alpha - 1} \,\mathrm{d}x_\alpha \\ &= (q^{-1}\mu(\mathfrak{o}_F))^{\dim X} \sum_{\tilde{\xi} \in \mathscr{X}(k)} \Phi(\tilde{\xi}) \prod_{\alpha \in \mathscr{A}_{\tilde{\xi}}} \frac{q^{f_\alpha} - 1}{q^{f_\alpha s_\alpha} - 1}, \end{aligned}$$

since the residue field of F_α has cardinality q^{f_α} and $\mu(\mathfrak{o}_F) = \mu(\mathfrak{o}_{F_\alpha})$. Let us interchange the order of summation: one gets

$$\mathscr{I}(\Phi; (s_\alpha)) = (q^{-1}\mu(\mathfrak{o}_F))^{\dim X} \sum_{A \subset \mathscr{A}} \prod_{\alpha \in A} \frac{q^{f_\alpha} - 1}{q^{f_\alpha s_\alpha} - 1} \sum_{\tilde{\xi} \in \mathscr{D}^\circ_A(k)} \Phi(\tilde{\xi}).$$

In particular, if Φ is the constant function 1, one has

$$\mathscr{I}(1, (s_\alpha)) = \sum_{A \subset \mathscr{A}} (q^{-1}\mu(\mathfrak{o}_F))^{\dim X} \prod_{\alpha \in A} \frac{q^{f_\alpha} - 1}{q^{f_\alpha s_\alpha} - 1} \#(\mathscr{D}^\circ_A(k)). \qquad \square$$

By Weil's formula (cf. Eq. (2.4)), one then has

$$\tau_{D_A}(\mathscr{E}^\circ_A(\mathfrak{o}_F)) = \left((q^{-1}\mu(\mathfrak{o}_F))^{d_A}\right) \# \mathscr{E}^\circ_A(k) = \left((q^{-1}\mu(\mathfrak{o}_F))^{d_A}\right) \# \mathscr{D}^\circ_A(k).$$

Since moreover $D_A(F) = E_A(F)$, the last formula of the proposition can be rewritten as

$$\mathscr{I}(1, (s_\alpha)) = \sum_{A \subset \mathscr{A}} (q^{-1}\mu(\mathfrak{o}_F))^{d_A} \prod_{\alpha \in A} \frac{q^{f_\alpha} - 1}{q^{f_\alpha s_\alpha} - 1} \tau_{D_A}(\mathscr{D}^\circ_A(\mathfrak{o}_F)).$$

4.2. *Volume asymptotics over local fields*

Let X be a smooth projective variety over a local field F. Assume that X is purely of dimension n.

Let D be an effective divisor in X, denote by $\mathscr{A}$ the set of its irreducible components, by D_α the component corresponding to some $\alpha \in \mathscr{A}$ and by d_α its multiplicity. We have $d_\alpha > 0$ for all α and $D = \sum d_\alpha D_\alpha$.

Let $U = X \backslash D$, and assume that $\omega_X(D)$ is equipped with a metrization. Let $\tau_{(X,D)}$ denote the corresponding measure on $U(F)$ (see Sec. 2.1.8). Endow the line bundles $\mathscr{O}_X(D)$ and $\mathscr{O}_X(D_\alpha)$, for $\alpha \in \mathscr{A}$, with metrics, in such a way that the natural isomorphism $\mathscr{O}_X(D) \simeq \bigotimes \mathscr{O}_X(D_\alpha)^{d_\alpha}$ is an isometry. For $\alpha \in \mathscr{A}$, denote by f_α the canonical section of $\mathscr{O}_X(D_\alpha)$; denote by f_D the canonical section of $\mathscr{O}_X(D)$. One has $\mathsf{f}_D = \prod_{\alpha \in \mathscr{A}} \mathsf{f}_\alpha^{d_\alpha}$.

We also endow ω_X with the metric which makes the isomorphism $\omega_X(D) \simeq \omega_X \otimes \mathscr{O}_X(D)$ an isometry. Letting τ_X be the Tamagawa measure on $X(F)$ defined by the metrized line bundle ω_X, we have the following equalities:

$$\mathrm{d}\tau_{(X,D)}(x) = \|\mathsf{f}_D(x)\|^{-1}\,\mathrm{d}\tau_X(x) = \prod_\alpha \|\mathsf{f}_\alpha(x)\|^{-d_\alpha}\,\mathrm{d}\tau_X(x).$$

Let L be an effective divisor in X whose support contains the support of D; assume that the corresponding line bundle $\mathscr{O}_X(L)$ is endowed with a metric. The norm of its canonical section f_L vanishes on L, hence on D. Consequently, for any positive real number B, the set of all $x \in X(F)$ such $\|\mathsf{f}_L(x)\| \geq 1/B$ is a closed subset of $X(F)$, which is contained in $U(F)$, hence is compact in $U(F)$. Consequently, its volume with respect to the measure $\tau_{(X,D)}$,

$$V(B) = \int_{\|\mathsf{f}_L(x)\| \geq 1/B} \mathrm{d}\tau_{(X,D)}(x), \tag{4.1}$$

is finite for any $B > 0$. We are interested in its asymptotic behavior when $B \to \infty$. Let us introduce the Mellin transform of the function $\|\mathsf{f}_L\|$ with respect to the measure $\tau_{(X,D)}$, namely:

$$Z(s) = \int_{U(F)} \|\mathsf{f}_L(x)\|^s \mathrm{d}\tau_{(X,D)}(x). \tag{4.2}$$

The analytic properties of $Z(s)$ and $V(B)$ strongly depend on the geometry of the pair (X, D). We will assume throughout that over the algebraic closure $\bar{F}$, the divisor D has strict normal crossings in X; in that case we will see that the answer can be stated in terms of the analytic Clemens complex of D. In principle, using resolution of singularities, we can reduce to this situation, even if it may be difficult in explicit examples (see [29] for a specific computation related to the asymptotic behavior of integral points of bounded height established in [22]).

For $\alpha \in \mathscr{A}$, let λ_α be the multiplicity of D_α in L; the divisor $\Delta = L - \sum \lambda_\alpha D_\alpha$ is effective and all of its irreducible components meet U. We denote by f_Δ the canonical section of the line bundle $\mathscr{O}_X(\Delta)$; we endow this line bundle with a metric so that $\|\mathsf{f}_L\| = \|\mathsf{f}_\Delta\| \prod \|\mathsf{f}_\alpha\|^{\lambda_\alpha}$.

Following the conventions of Sec. 2.1.4, the results extend to the case where D and L are $\mathbf{Q}$-Cartier divisors; in that case, the coefficients λ_α and d_α are rational numbers.

Let $\sigma = \max(d_\alpha - 1)/\lambda_\alpha$, the maximum being over all $\alpha \in \mathscr{A}$ such that $D_\alpha(F) \neq \varnothing$. If there is no such α, we let $\sigma = -\infty$ by convention; this means precisely that $U(F)$ is compact. Only the case $\sigma \geq 0$ will really matter. Indeed, as we shall see below, the condition $\sigma < 0$ is equivalent to the fact that $U(F)$ has finite volume with respect to $\tau_{(X,D)}$.

Let $\mathscr{C}^{\mathrm{an}}_{F,(L,D)}(D)$ be the subcomplex of the analytic Clemens complex $\mathscr{C}^{\mathrm{an}}_F(D)$ consisting of all non-empty subsets $A \subset \bar{\mathscr{A}}$ such that $E_A(F) \neq \varnothing$ and $d_\alpha = \lambda_\alpha \sigma + 1$ for any $\alpha \in A$.

Let $b = \dim \mathscr{C}^{\mathrm{an}}_{F,(L,D)}(D)$. For any face A of maximal dimension b of $\mathscr{C}^{\mathrm{an}}_{F,(L,D)}(D)$, let $D_A = \bigcap_{\alpha \in A} D_\alpha$ be the corresponding stratum of X. The subset $D_A(F)$ carries a natural measure $\mathrm{d}\tau_{D_A}$ and we define

$$Z_A(s) = \int_{D_A(F)} \|\mathsf{f}_\Delta(x)\|^s \prod_{\alpha \notin A} \|\mathsf{f}_\alpha(x)\|^{s\lambda_\alpha - d_\alpha} \,\mathrm{d}\tau_A(x). \tag{4.3}$$

Proposition 4.6. *Let $\sigma, \mathscr{C}^{\mathrm{an}}_{F,(L,D)}(D)$ and b be defined as above. Then the integral defining $Z(s)$ converges for* $\mathrm{Re}(s) > \sigma$ *and defines a holomorphic function in that domain.*

Assume that $\sigma \neq -\infty$. Then there is a positive real number δ such that Z has a meromorphic continuation to a half-plane $\mathrm{Re}(s) > \sigma - \delta$, *with a pole of order $b = \dim \mathscr{C}^{\mathrm{an}}_{F,(L,D)}(D)$ at $s = \sigma$ with leading coefficient*

$$\lim_{s \to \sigma} (s - \sigma)^b Z(s) = \sum_{\substack{A \in \mathscr{C}^{\mathrm{an}}_{F,(L,D)}(D) \\ \dim A = b}} Z_A(\sigma) \prod_{\alpha \in A} \frac{1}{\lambda_\alpha}$$

and moderate growth in vertical strips.

When $F = \mathbf{R}$ or $\mathbf{C}$, Z has no other pole provided δ is chosen sufficiently small.

When F is ultrametric, there is a positive integer f such that $(1 - q^{f(\sigma - s)})^b Z(s)$ is holomorphic on the half-plane $\{\mathrm{Re}(s) > \sigma - \delta\}$, *again provided δ is sufficiently small.*

Proof. By definition,

$$Z(s) = \int_{X(F)} \|\mathsf{f}_\Delta\|^s \prod_\alpha \|\mathsf{f}_\alpha(x)\|^{s\lambda_\alpha - d_\alpha} \mathrm{d}\tau_X(x),$$

an integral of the type studied in Sec. 3. Precisely, using the notations introduced in Sec. 4.1, we have $Z(s) = \mathscr{I}(\mathbf{1}, (s + 1; s\lambda_\alpha - d_\alpha + 1))$, where the first parameter s refers to the divisor Δ, while for each $\alpha \in \mathscr{A}$, the parameter $s_\alpha = s\lambda_\alpha - d_\alpha$ corresponds to the divisor D_α. Similarly,

$$Z_A(s) = \mathscr{I}_A(\mathbf{1}, (s + 1; s\lambda_\alpha - d_\alpha + 1)).$$

By Lemma 4.1, this integral converges and defines a holomorphic function as long as $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(s)\lambda_\alpha > d_\alpha$. This shows the holomorphy of $Z(s)$ for $\mathrm{Re}(s) > \sigma$.

Assume that $\sigma \neq -\infty$. By Proposition 4.2, the function Z has a meromorphic continuation to the domain of $\mathbf{C}$ defined by the inequalities $\mathrm{Re}(s_\alpha) > -\frac{1}{2}$ and $\mathrm{Re}(s) > 0$, hence to some domain of the form $\mathrm{Re}(s) > \sigma - \delta$.

In the ultrametric case, the existence of a positive integer f such that $(1 - q^{(\sigma-s)f})^b Z(s)$ has no pole on such a half-plane also follows directly from Proposition 4.2 (one may take for f the l.c.m. of the f_α), as well as the growth in vertical strips.

It remains to prove the asserted behavior at $s = \sigma$. By Proposition 4.3,

$$\begin{aligned}\lim_{s\to\sigma}(s-\sigma)^b Z(s) &= \sum_{\substack{A\in\mathscr{A}_{L,D}^{\mathrm{an}}\\ \dim A=b}} \mathscr{I}_A(\mathbf{1}; (\sigma+1; \sigma\lambda_\alpha - d_\alpha + 1)_{\alpha\notin A}) \prod_{\alpha\in A}\frac{1}{\lambda_\alpha}\\ &= \sum_{\substack{A\in\mathscr{A}_{L,D}^{\mathrm{an}}\\ \dim A=b}} Z_A(\sigma)\prod_{\alpha\in A}\frac{1}{\lambda_\alpha}. \qquad \square\end{aligned}$$

Using the Tauberian Theorem A.1 recalled in the Appendix, we obtain the following estimate for the volume $V(B)$ in the archimedean case.

Theorem 4.7. *Assume that $F = \mathbf{R}$ or $\mathbf{C}$ and that $\sigma \geq 0$. This implies that $b \geq 1$.*

There exists a polynomial P and a positive real number δ such that

$$V(B) = B^\sigma P(\log B) + \mathrm{O}(B^{\sigma-\delta}).$$

Moreover, if $\sigma > 0$, then P has degree $b-1$ and leading coefficient

$$\mathrm{lcoeff}(P) = \frac{1}{\sigma(b-1)!}\sum_{\substack{A\subset\mathscr{C}_{F,(L,D)}^{\mathrm{an}}(D)\\ \dim A=b}} Z_A(\sigma)\prod_{\alpha\in A}\frac{1}{\lambda_\alpha};$$

otherwise, if $\sigma = 0$, then P has degree b and its leading coefficient satisfies

$$\mathrm{lcoeff}(P) = \frac{1}{b!}\sum_{\substack{A\subset\mathscr{C}_{F,(L,D)}^{\mathrm{an}}(D)\\ \dim A=b}} Z_A(\sigma)\prod_{\alpha\in A}\frac{1}{\lambda_\alpha}.$$

Considering integrals of the form

$$Z_\Phi(s) = \int_{U(F)} \Phi(x)\|\mathsf{f}_L(x)\|^s \,\mathrm{d}\tau_{(X,D)}(x),$$

or applying the abstract equidistribution theorem (Proposition 2.10), we deduce the following corollary ("equidistribution of height balls"):

Corollary 4.8. *Assume that $F = \mathbf{R}$ or $\mathbf{C}$ and that $\sigma > 0$. Then $b \geq 1$ and, when $B \to +\infty$, the family of measures*

$$V(B)^{-1}\mathbf{1}_{\{\|\mathsf{f}_L(x)\|\geq 1/B\}}\,\mathrm{d}\tau_{(X,D)}(x)$$

converges tightly to the unique probability measure which is proportional to

$$\sum_{\substack{A\subset \mathscr{A}^{\mathrm{an}}_{L,D} \\ \dim A=b}} \|\mathsf{f}_\Delta(x)\|^\sigma \prod_{\alpha\in A}\frac{1}{\lambda_\alpha}\prod_{\alpha\notin A}\|\mathsf{f}_\alpha(x)\|^{-1}\,\mathrm{d}\tau_{D_A}(x).$$

In the case of ultrametric local fields, our analysis leads to a good understanding of the Igusa zeta functions but the output of the corresponding Tauberian theorem, i.e. the discussion leading to Corollaries A.4–A.6 in the Appendix, is less convenient since one can only obtain an asymptotic expansions for $V(q^N)$, when N belongs to a fixed congruence class modulo some positive integer f. In fact, when all irreducible components of D are geometrically irreducible, one may take $f=1$ in Proposition 4.6. In that case, Corollary A.5 even leads to a precise asymptotic expansion in that case.

We leave the detailed statement to the interested reader and we content ourselves with the following corollary.

Corollary 4.9. *Assume that F is ultrametric and that $\sigma\geq 0$. Let $b^*=b$ if $\sigma=0$, and $b^*=b-1$ if $\sigma>0$. Then, when the integer N goes to infinity,*

$$0<\liminf\frac{V(q^N)}{q^{N\sigma}N^{b^*}}\leq\limsup\frac{V(q^N)}{q^{N\sigma}N^{b^*}}<\infty.$$

Proof. Changing metrics modifies the volume forms and the height functions by a factor which is lower- and upper-bounded; this does not affect the result of the corollary. Consequently, we may assume that all metrics are smooth and that the function $\|\mathsf{f}_L\|$ is $q^{\mathbf{Z}}$-valued. The Igusa zeta function $Z(s)$ is then $2i\pi/\log q$-periodic and has a meromorphic continuation of the form $\sum_A \Phi_A(s)\prod_{\alpha\in A}(1-q^{f_\alpha\lambda_\alpha(\sigma-s)})$, for some functions Φ_A which are holomorphic on an open half-plane containing the closed half-plane given by $\{\mathrm{Re}(s)\geq\sigma\}$. Let f be any positive integer such that f is an integral multiple of $f_\alpha\lambda_\alpha$, for any $\alpha\in\mathscr{A}$; we see that $Z(s)(1-q^{f(\sigma-s)})^b$ extends holomorphically to this open half-plane and it now suffices to apply Corollary A.4. □

4.3. *Adelic Igusa integrals*

4.3.1. *Geometric setup*

Let F be a number field, let $\mathbb{A}_F$ be the ring of adeles of F. More generally, if S is a finite set of places of F, let $\mathbf{A}_F^S$ be the restricted product of local fields F_v, for $v\notin S$.

We fix an algebraic closure $\bar F$ of F; for each place v, we also fix a decomposition group Γ_v at v in the Galois group $\Gamma_F=\mathrm{Gal}(\bar F/F)$.

Let $\bar X$ be a smooth projective variety over F, let $(D_\alpha)_{\alpha\in\mathscr{A}}$ be a family of irreducible divisors in $\bar X$ whose sum $\Delta=\sum_{\alpha\in\mathscr{A}}D_\alpha$ is geometrically a divisor with strict normal crossings. For $\alpha\in\mathscr{A}$, let f_{D_α} denote the canonical section of the line bundle $\mathscr{O}_{\bar X}(D_\alpha)$ and let F_α be the algebraic closure of F in the function field of D_α.

Let $\bar{\mathscr{A}}$ be the set of irreducible components of the divisor $\Delta_{\bar{F}}$; it carries an action of Γ_F, the set of orbits of which, $\bar{\mathscr{A}}/\Gamma_F$, identifies canonically with the set $\mathscr{A}$.

Endow all line bundles $\mathscr{O}_{\bar{X}}(D_\alpha)$, as well as the canonical line bundle $\omega_{\bar{X}}$, with adelic metrics.

Let $\mathscr{B}$ be any subset of $\mathscr{A}$, corresponding to a subset $\bar{\mathscr{B}}$ of $\bar{\mathscr{A}}$ which is stable under the action of Γ_F. Let $Z = \bigcup_{\alpha\in\mathscr{B}} D_\alpha$ be the union of the corresponding divisors, $D = \bigcup_{\alpha\notin\mathscr{B}} D_\alpha$ the union of the other divisors, and let us define $X = \bar{X}\setminus Z$ and $U = X\setminus D$.

The local spaces $X(F_v)$, $U(F_v)$, for any place v of F, and the adelic spaces $X(\mathbb{A}_F)$, $U(\mathbb{A}_F)$ are locally compact and carry Tamagawa measures $\tau_{X,v}$, $\tau_{U,v}$, τ_X and τ_U (see Definition 2.8) which are Radon measures, i.e. finite on compact subsets. The set of irreducible components $\mathscr{A}_v$ of the divisor Δ_{F_v} is in natural bijection with the set of Γ_v-orbits in $\bar{\mathscr{A}}$. For any such orbit α, we will denote by D_α the corresponding divisor on X_{F_v}.

Our aim here is to establish analytic properties of the adelic integral

$$\mathscr{I}(\Phi;(s_\alpha)) = \int_{U(\mathbb{A}_F)} \prod_{\alpha\in\mathscr{A}} \prod_v \|\mathsf{f}_{D_\alpha}\|_v^{s_\alpha-1}\Phi(x)\,\mathrm{d}\tau_U,$$

when Φ is the restriction to $U(\mathbb{A}_F)$ of a smooth function with compact support on $X(\mathbb{A}_F)$.

It will be convenient to view the map $\alpha\mapsto s_\alpha$ as a Γ_F-equivariant map from $\bar{\mathscr{A}}$ to $\mathbf{C}$. In other words, for each $\alpha\in\bar{\mathscr{A}}$, we let $s_\alpha = s_{[\alpha]}$, where $[\alpha]\in\mathscr{A}$ is the orbit of α under Γ_F. More generally, if β is any subset of such an orbit, we define $s_\beta = s_\alpha$, for any $\alpha\in\bar{\mathscr{A}}$ belonging to β (it does not depend on the choice of α).

4.3.2. *Convergence of local integrals*

Assume that $\Phi = \prod_v \Phi_v$ is a product of smooth functions and define, for any $v\in\mathrm{Val}(F)$,

$$\mathscr{I}_v(\Phi_v;(s_\alpha)) = \int_{U(F_v)} \prod_{\alpha\in\mathscr{A}} \|\mathsf{f}_{D_\alpha}\|_v^{s_\alpha-1}\Phi_v(x)\,\mathrm{L}_v(1,\mathrm{EP}(U))\mathrm{d}\tau_{X,v}.$$

When the local integrals $\mathscr{I}_v$ converge absolutely, as well as the infinite product $\prod_v \mathscr{I}_v(\Phi;(s_\alpha))$, then the integral $\mathscr{I}(\Phi;(s_\alpha))$ exists and one has an equality

$$\mathscr{I}(\Phi;(s_\alpha)) = \mathrm{L}_*(1,\mathrm{EP}(U))^{-1}\prod_v \mathscr{I}_v(\Phi_v;(s_\alpha)).$$

Let $\alpha\in\mathscr{A}$. Let us decompose the Γ_F-orbit α as a union of disjoint Γ_v-orbits $\alpha_1,\dots,\alpha_r$. Then $D_\alpha = \sum_{i=1}^r D_{\alpha_i}$ and $\mathsf{f}_{D_\alpha} = \prod_{i=1}^r \mathsf{f}_{D_{\alpha_i}}$. It follows that the integral $\mathscr{I}_v$ can be rewritten as

$$\mathscr{I}_v(\Phi_v;(s_\alpha)) = \int_{U(F_v)} \prod_{\alpha\in\mathscr{A}_v} \|\mathsf{f}_{D_\alpha}\|_v^{s_\alpha-1}\Phi_v(x)\,\mathrm{L}_v(1,\mathrm{EP}(U))\mathrm{d}\tau_{X,v}.$$

By Lemma 4.1, $\mathscr{I}_v(\Phi_v;(s_\alpha))$ converges absolutely when $\operatorname{Re}(s_\alpha)>0$ for each $\alpha\in\mathscr{A}$. If moreover Φ_v has compact support in $X(F_v)$, then the conditions $\operatorname{Re}(s_\alpha)>0$ for $\alpha\in\mathscr{B}$ are not necessary.

The decomposition $\alpha=\bigcup\alpha_i$ corresponds to the decomposition $F_\alpha\otimes_F F_v=\prod_{i=1}^r F_{\alpha_i}$. Observe that the local factor of Dedekind's zeta function ζ_{F_α} at v is given by the formula

$$\zeta_{F_\alpha,v}(s)=\prod_{i=1}^r(1-q_v^{-f_{\alpha,i}s})^{-1},$$

where for each i, $f_{\alpha,i}=[F_{\alpha,i}:F_v]$. Let $\zeta^*_{F_\alpha}(1)$ be the residue at $s=1$ of this zeta function.

4.3.3. *Convergence of an Euler product*

To study the convergence of the product, we may ignore a finite set of places.

Let S be a finite set of places containing the archimedean places so that, for all other places, all metrics are defined by good integral models $\overline{\mathscr{X}}$, $\mathscr{X}$, $\mathscr{U}$, …, over $\operatorname{Spec}\mathfrak{o}_{F,S}$. Assume moreover that for any $v\notin S$, Φ_v is the characteristic function of $\mathscr{X}(\mathfrak{o}_v)$.

By Denef's formula (Proposition 4.5), one has

$$\mathscr{I}_v(\Phi_v;(s_\alpha))=\sum_{A\subset\mathscr{A}_v\setminus\mathscr{B}_v}(q_v^{-1}\mu_v(\mathfrak{o}_v))^{\dim X}\,\#\,D_A^\circ(k_v)\prod_{\alpha\in A}\frac{q_v^{f_\alpha}-1}{q_v^{f_\alpha s_\alpha}-1}$$

for any place $v\notin S$. Combined with the estimate of Theorem 2.5, this relation implies that

$$\mathscr{I}_v(\Phi_v;(s_\alpha))\prod_{\alpha\in\mathscr{A}\setminus\mathscr{B}}\zeta_{F_\alpha,v}(s_\alpha)^{-1}=1+\mathrm{O}(q_v^{-1-\varepsilon})$$

provided $\operatorname{Re}(s_\alpha)>\frac12+\varepsilon$ for each $\alpha\notin\mathscr{B}$. (See Proposition 9.5 in [13] for a similar computation.) This asymptotic expansion will imply the desired convergence of the infinite product.

Proposition 4.10. *Assume that Φ is a smooth function with compact support on $X(\mathbb{A}_F)$. Then the integral $\mathscr{I}(\Phi;(s_\alpha))$ converges for $\operatorname{Re}(s_\alpha)>1$, for each $\alpha\notin\mathscr{B}$, and defines a holomorphic function in this domain. This function has a meromorphic continuation: there is a holomorphic function φ defined for $\operatorname{Re}(s_\alpha)>\frac12$ if $\alpha\notin\mathscr{B}$, such that*

$$\mathscr{I}(\Phi;(s_\alpha))=\varphi(s)\prod_{\alpha\notin\mathscr{B}}\zeta_{F_\alpha}(s_\alpha).$$

Moreover, if $s_\alpha=1$ for $\alpha\notin\mathscr{B}$, then

$$\varphi(s)=\prod_{\alpha\notin\mathscr{B}}\zeta^*_{F_\alpha}(1)^{-1}\int_{X(\mathbb{A}_F)}\Phi(x)\prod_{\beta\in\mathscr{B}}\prod_v\|\mathsf{f}_{D_\beta}\|_v^{s_\beta-1}\,\mathrm{d}\tau_X(x).$$

(Note that, in the last formula, the function under the integration sign has compact support on $X(\mathbb{A}_F)$, while $\mathrm{d}\tau_X$ is a Radon measure on that space.)

Proof. The first two parts of the theorem follow from the estimates that we have just derived and the absolute convergence for $\mathrm{Re}(s) > 1$ of the Euler product defining the Dedekind zeta function of a number field.

Let $s \in \mathbf{C}^{\mathscr{A}}$ be such that $s_\alpha = 1$ for $\alpha \notin \mathscr{B}$. One therefore has

$$\mathscr{I}_v(\Phi_v; s) = \mathrm{L}_v(1, \mathrm{EP}(U)) \int_{U(F_v)} \prod_{\beta\in\mathscr{B}} \|\mathsf{f}_{D_\beta}\|_v^{s_\beta - 1} \Phi_v(x)\, \mathrm{d}\tau_{X,v}(x).$$

Moreover, one has equalities of virtual representations of Γ_F (cf. Eq. (2.5)),

$$\mathrm{EP}(U) = \mathrm{EP}(\bar{X}) + \sum_{\alpha\in\mathscr{A}} \mathrm{Ind}_{\Gamma_{F_\alpha}}^{\Gamma_F} \mathbf{1},$$

$$\mathrm{EP}(X) = \mathrm{EP}(\bar{X}) + \sum_{\alpha\in\mathscr{B}} \mathrm{Ind}_{\Gamma_{F_\alpha}}^{\Gamma_F} \mathbf{1},$$

from which it follows that

$$\mathrm{EP}(X) = \mathrm{EP}(U) - \sum_{\alpha\notin\mathscr{B}} \mathrm{Ind}_{\Gamma_{F_\alpha}}^{\Gamma_F} \mathbf{1}.$$

In particular, for any finite place v of F,

$$\mathrm{L}_v(1, \mathrm{EP}(X)) = \mathrm{L}_v(1, \mathrm{EP}(U)) \prod_{\alpha\notin\mathscr{B}} \zeta_{F_\alpha, v}(1)^{-1}$$

and

$$\mathrm{L}^*(1, \mathrm{EP}(X)) = \mathrm{L}^*(1, \mathrm{EP}(U)) \prod_{\alpha\notin\mathscr{B}} \zeta^*_{F_\alpha}(1)^{-1}.$$

If S_∞ is the set of archimedean places of F, then (recall that $s_\alpha = 1$ for $\alpha \notin \mathscr{B}$)

$$\begin{aligned}
\varphi(s) &= \mathrm{L}^*(1, \mathrm{EP}(U))^{-1} \prod_{v\in S_\infty} \mathscr{I}_v(\Phi_v; s) \prod_{v\notin S_\infty} \left(\mathscr{I}_v(\Phi_v; s) \prod_{\alpha\notin\mathscr{B}} \zeta_{F_\alpha, v}(1)^{-1} \right) \\
&= \mathrm{L}^*(1, \mathrm{EP}(U))^{-1} \prod_v \int_{X(F_v)} \Phi_v(x) \prod_{\beta\in\mathscr{B}} \|\mathsf{f}_{D_\beta}\|_v^{s_\beta - 1} \mathrm{L}_v(1, \mathrm{EP}(X))\, \mathrm{d}\tau_{X,v}(x) \\
&= \mathrm{L}^*(1, \mathrm{EP}(U))^{-1} \mathrm{L}^*(1, \mathrm{EP}(X)) \int_{X(\mathbb{A}_F)} \Phi(x) \prod_{\beta\in\mathscr{B}} \prod_v \|\mathsf{f}_{D_\beta}\|_v^{s_\beta - 1}\, \mathrm{d}\tau_X(x) \\
&= \prod_{\alpha\notin\mathscr{B}} \zeta^*_{F_\alpha}(1)^{-1} \int_{X(\mathbb{A}_F)} \Phi(x) \prod_{\beta\in\mathscr{B}} \prod_v \|\mathsf{f}_{D_\beta}\|_v^{s_\beta - 1}\, \mathrm{d}\tau_X(x).
\end{aligned}$$ □

In fact, it is possible to establish a more general theorem. Let S be a finite set of places of F containing the archimedean places. For any place $v \in S$, let Φ_v be a smooth bounded function on $X(F_v)$; let also Φ^S be a smooth function with compact support on $X(\mathbb{A}_F^S)$; let Φ be the function $\Phi^S \prod_{v\in S} \Phi_v$ on $X(\mathbb{A}_F)$.

By the same arguments as above, the integral $\mathscr{I}(\Phi;(s_\alpha))$ converges provided $\operatorname{Re}(s_\alpha) > 1$ for each $\alpha \notin \mathscr{B}$ and $\operatorname{Re}(s_\beta) > 0$ for each $\beta \in \mathscr{B}$ and defines a holomorphic function in that domain.

Proposition 4.11. *Let $\Omega \subset \mathbf{C}^{\mathscr{A}}$ be the set of (s_α) such that $\operatorname{Re}(s_\alpha) > \frac{1}{2}$ for $\alpha \notin \mathscr{B}$ and $\operatorname{Re}(s_\beta) > -\frac{1}{2}$ for $\beta \in \mathscr{B}$. The function $\mathscr{I}(\Phi;(s_\alpha))$ admits a meromorphic continuation of the following form. For any place $v \in S$ and any face A of maximal dimension of $\mathscr{C}^{\mathrm{an}}_{F_v}(D)$, there is a holomorphic function φ_A on Ω such that*

$$\mathscr{I}(\Phi;(s_\alpha)) = \prod_{\alpha\notin\mathscr{B}} \zeta^S_{F_\alpha}(s_\alpha) \prod_{v\in S} \left(\sum_{A\in\mathscr{C}^{\mathrm{an},\max}_{F_v}(D)} \varphi_A(s) \prod_{\alpha\in A} \zeta_{F_\alpha,v}(s_\alpha) \right).$$

Moreover, the functions φ_A have moderate growth in vertical strips in the sense that for any compact subset K of $\mathbf{R}^{\mathscr{A}} \cap \Omega$, there are real numbers c and κ such that

$$|\varphi_A(s)| \leq c \prod_{\alpha\in\mathscr{A}} (1+|s_\alpha|)^\kappa,$$

for $s \in \mathbf{C}^{\mathscr{A}}$ such that $\operatorname{Re}(s) \in K$.

4.4. *Volume asymptotics over the adeles*

In this section we derive asymptotic estimates for volumes of height balls in adelic spaces, similar to those we established above for height balls over local fields.

Let F be a number field, X a smooth projective variety over F, purely of dimension n. Let D be an effective divisor in X and $\mathscr{A}$ its set of irreducible components; for $\alpha \in \mathscr{A}$, let D_α be the corresponding component and d_α its multiplicity in D. We have $D = \sum d_\alpha D_\alpha$. For $\alpha \in \mathscr{A}$, let f_α be the canonical section of $\mathscr{O}_X(D_\alpha)$; let f_D denote the canonical section of $\mathscr{O}_X(D)$. We have $\mathsf{f}_D = \prod \mathsf{f}_\alpha^{d_\alpha}$.

Let us endow these line bundles with adelic metrics, in such a way that the isomorphism $\mathscr{O}_X(D) \simeq \bigotimes \mathscr{O}_X(D_\alpha)^{d_\alpha}$ is an isometry.

Let $U = X \setminus D$; let us endow ω_X and $\omega_X(D)$ with adelic metrics in such a way that the isomorphism $\omega_X(D) \simeq \omega_X \otimes \mathscr{O}_X(D)$ is an isometry. By the constructions of Sec. 2, we obtain natural measures τ_X, τ_U and $\tau_{(X,D)}$ on $U(\mathbb{A}_F)$ or $X(\mathbb{A}_F)$, related by the equalities

$$\mathrm{d}\tau_{(X,D)}(x) = H_D(x)\mathrm{d}\tau_U(x),$$

where, for $x \in U(\mathbb{A}_F)$, $H_D(x) = \prod_x \|\mathsf{f}_D(x)\|_v^{-1}$.

Let L be an effective divisor in X whose support is equal to the support of D; for $\alpha \in \mathscr{A}$, let λ_α be the multiplicity of D_α in L, so that $L = \sum \lambda_\alpha D_\alpha$. We endow the line bundle $\mathscr{O}_X(L)$ with the natural adelic metric deduced from the metrics of the line bundles $\mathscr{O}_X(D_\alpha)$.

The canonical section of $\mathscr{O}_X(L)$ vanishes on L, hence on D. Let H_L denote the corresponding height function (denoted $H_{\mathscr{O}_X(L),\mathsf{f}_L}$ in Sec. 2.3) on the adelic space $U(\mathbb{A}_F)$; recall (Eq. (2.3)) that it is defined by the formula

$$H_L(\mathbf{x}) = \prod_{v\in\mathrm{Val}(F)} \|\mathsf{f}_L\|(x_v)^{-1}, \quad \text{for } \mathbf{x} = (x_v)_v.$$

By Lemma 2.1, H_L is bounded from below on $U(\mathbb{A}_F)$ and, for any real number B, the set of all $\mathbf{x} \in U(\mathbb{A}_F)$ such that $H_L(\mathbf{x}) \leq B$ is compact in $(X\backslash L)(\mathbb{A}_F)$. Let $V(B)$ denote its volume with respect to the measure $\tau_{(X,D)}$:

$$V(B) = \int_{\substack{U(\mathbb{A}_F)\\ \{H_L(\mathbf{x})\leq B\}}} \mathrm{d}\tau_{(X,D)}(\mathbf{x}); \tag{4.4}$$

it is a positive real number (for B large enough) and we want to understand its asymptotic behavior when $B \to \infty$. We are also interested in the asymptotic behavior of the probability measures

$$\frac{1}{V(B)} \mathbf{1}_{\{H_L(\mathbf{x})\leq B\}} \mathrm{d}\tau_{(X,D)}(\mathbf{x}).$$

As in the case of local fields, we introduce a geometric Igusa zeta function, namely

$$Z(s) = \int_{U(\mathbb{A}_F)} H_L(x)^{-s}\, \mathrm{d}\tau_{(X,D)}(x), \tag{4.5}$$

for any complex number s such that the integral converges absolutely. For any such s, we thus have

$$Z(s) = \int_{U(\mathbb{A}_F)} \prod_v \|\mathsf{f}_\alpha(x_v)\|^{s\lambda_\alpha - d\alpha}\, \mathrm{d}\tau_X(\mathbf{x}) = \mathscr{I}(\mathbf{1}; (s\lambda_\alpha - d_\alpha + 1))$$

with the notation of Sec. 4.3.

Let $\sigma = \max_{\alpha\in\mathscr{A}}(d_\alpha/\lambda_\alpha)$ and let $\mathscr{A}_{L,D}$ be the set of $\alpha \in \mathscr{A}$ such that $d_\alpha = \lambda_\alpha\sigma$.

Then, the integral defining $Z(s)$ converges for any complex number s such that $\mathrm{Re}(s) > \sigma$ (Proposition 4.10). Moreover, there exists a positive real number δ such that the function $s \mapsto Z(s)$ admits a meromorphic continuation to the half-plane given by $\mathrm{Re}(s) > \sigma - \delta$. Namely, by Proposition 4.10 again, there exists a holomorphic function φ defined for $\mathrm{Re}(s) > \sigma - \delta$ such that

$$Z(s) = \varphi(s) \prod_{\alpha\in\mathscr{A}_{L,D}} \zeta_{F_\alpha}(s\lambda_\alpha - d_\alpha + 1)$$

and

$$\varphi(1) = \prod_{\alpha\in\mathscr{A}_{L,D}} \zeta^*_{F_\alpha}(1) \int_{X(\mathbb{A}_F)} \prod_{\alpha\notin\mathscr{A}_{L,D}} H_{D_\alpha}(x)^{d_\alpha - \sigma\lambda_\alpha}\, \mathrm{d}\tau_X(x).$$

In particular, the function Z has a pole at $s=1$ of order $\#(\mathscr{A}_{L,D})$ and satisfies:

$$\lim_{s\to 1}(s-1)^{\#(\mathscr{A}_{L,D})}Z(z) = \varphi(1)\prod_{\alpha\in\mathscr{A}}\frac{\zeta^*_{F_\alpha}(1)}{\lambda_\alpha}$$
$$= \prod_{\alpha\in\mathscr{A}_{L,D}}\lambda_\alpha^{-1}\int_{X(\mathbb{A}_F)}\prod_{\alpha\notin\mathscr{A}_{L,D}} H_{D_\alpha}(x)^{d_\alpha-\sigma\lambda_\alpha}\,\mathrm{d}\tau_X(x).$$

Let E denote the $\mathbf{Q}$-divisor $\sigma L - D$. We have

$$E = \sigma L - D = \sum_{\alpha\in\mathscr{A}}(\sigma\lambda_\alpha - d_\alpha)D_\alpha = \sum_{\alpha\notin\mathscr{A}_{L,D}}(\sigma\lambda_\alpha - d_\alpha)D_\alpha.$$

Consequently,

$$\lim_{s\to\sigma}(s-\sigma)^{\#(\mathscr{A}_{L,D})}Z(\sigma) = \prod_{\alpha\in\mathscr{A}_{L,D}}\lambda_\alpha^{-1}\int_{X(\mathbb{A}_F)} H_E(x)^{-1}\,\mathrm{d}\tau_X(x)$$
$$= \int_{X(\mathbb{A}_F)}\mathrm{d}\tau_{(X,E)}(x). \tag{4.6}$$

We summarize the results obtained in the following proposition:

Proposition 4.12. *Let σ and $\mathscr{A}_{L,D}$ be defined as above. Then the integral defining $Z(s)$ converges for $\mathrm{Re}(s)>\sigma$ and defines a holomorphic function in that domain. Moreover, there is a positive real number δ such that Z has a continuation to the half-plane $\mathrm{Re}(s)>\sigma-\delta$ as a meromorphic function with moderate growth in vertical strips, whose only pole is at $s=\sigma$, with order $b=\#(\mathscr{A}_{L,D})$ and leading coefficient given by Eq. (4.6).*

Similarly to what we did in the local case, using the Tauberian Theorem A.1 and the abstract equidistribution theorem (Proposition 2.10), we obtain the following result.

Theorem 4.13. *There exists a monic polynomial P of degree b and a positive real number δ such that, when $B\to\infty$,*

$$V(B) = \frac{\prod_{\alpha\in\mathscr{A}_{L,D}}\lambda_\alpha^{-1}}{\sigma(b-1)!}B^\sigma P(\log B)\int_{X(\mathbb{A}_F)} H_E(x)^{-1}\,\mathrm{d}\tau_X(x) + \mathrm{O}(B^{\sigma-\delta}).$$

Moreover, we have the tight convergence of probability measures

$$\frac{1}{V(B)}\mathbf{1}_{\{H_L(x)\leq B\}}\mathrm{d}\tau_{(X,D)}(x) \to \frac{1}{\int_{X(\mathbb{A}_F)} H_E^{-1}\tau_X} H_E(x)^{-1}\,\mathrm{d}\tau_X(x).$$

Remark 4.14. Let S be a finite set of places of F. At least two variants of the preceding results may be useful in S-integral contexts.

Let $\mathbb{A}_F^S$ be the ring of adeles "outside S". For the first variant, we consider points of $U(\mathbb{A}_F^S)$ of bounded height, and their volume with respect to the Tamagawa measure $\tau^S_{(X,D)}$ in which the local factors in S. In that context, all infinite products of the set $\mathrm{Val}(F)$ of places of F are replaced by the products over the set $\mathrm{Val}(F)\backslash S$ of places which do not belong to S. The modifications to be made to the statement and proof of Theorem 4.13 are obvious and lead to an asymptotic expansion for the volumes of height balls in $U(\mathbb{A}_F^S)$.

A second variant is also possible which restricts to the subset Ω of $U(\mathbb{A}_F)$ consisting of points $(x_v)_{v\in\mathrm{Val}(F)}$ which are "integral" at each place $v \notin S$. We do not need to be specific about that condition; for all that matters in our analysis, we understand that this subset is relatively compact in $U(\mathbb{A}_F)$ and has non-empty interior. A scheme-theoretic definition would ask that we are given a projective and flat model of X over $\mathfrak{o}_F$; then a point $(x_v) \in U(\mathbb{A}_F)$ belongs to Ω if and only if, for any finite place $v \notin S$, the reduction mod v of x_v does not belong to $D(\mathbf{F}_v)$. An adelic definition would consider Ω to be defined by a condition of the form $\prod_{v\notin S} \|\mathsf{f}_L\|_v(x) \leq B_0$, for some real number B_0.

In that case, the analytic study of the adelic zeta function involved is straightforward from what has been done in Sec. 4.2. Namely, it decomposes as an infinite product over all places $v \in \mathrm{Val}(F)$ of v-adic zeta functions. The subproduct corresponding to places $v \notin S$ extends holomorphically to the whole complex plane, while each factor attached to a place $v \in S$ is the source of zeros and poles described by the v-adic analytic Clemens complex as in Proposition 4.6.

Although the procedure should be quite clear to the reader, in any specific example, a general statement would certainly be too obfuscating to be of any help. We would like the reader to observe than when all abscissae of convergence at places in S are equal to a common real number σ, the order of the pole at σ will be the sum of the orders b_v, and the leading coefficient the product of those computed in Proposition 4.6. Moreover, if S contains archimedean places, the order of the pole at σ will be *strictly* greater than the order of the other possible poles on the line $\mathrm{Re}(s) = \sigma$. In that case, the Tauberian Theorem A.7 gives a simple asymptotic formula for the volume of points of bounded height. With the notation of that Theorem, the q_j are powers of prime numbers and the non-Liouville property of the quotients $\log q_j / \log q_{j'}$ follows from Baker's theorem on linear forms in logarithms, [1], Theorem 3.1.

We expect that these asymptotic expansions for volumes can serve as a guide to understand the asymptotic number of S-integral solutions of polynomial equations, e.g. to a S-integral generalization of the circle method. For two positive results in that direction, we refer to the papers [14, 15].

5. Examples

5.1. *Clemens complexes of toric varieties*

Let T be an algebraic torus over a field F. Let M and N denote the groups of characters and of cocharacters of the torus $T_{\bar{F}}$, endowed with the action of the Galois group $\Gamma = \mathrm{Gal}(\bar{F}/F)$ and the natural duality pairing $\langle \cdot, \cdot \rangle$.

Let X be a smooth proper F-scheme which is an equivariant compactification of T. By the theory of toric varieties, X corresponds to a fan Σ in the space $N_{\mathbf{R}}$ which is invariant under the action of Γ. By definition, Σ is a set of convex polyhedral rational cones in $N_{\mathbf{R}}$ satisfying the following properties:

- for any cones σ, σ' in Σ, their intersection is a face of both σ and σ';
- all faces of each cone in Σ belong to Σ;
- for any $\gamma \in \Gamma_F$ and any cone $\sigma \in \Sigma$, $\gamma(\sigma) \in \Sigma$.

Assume for the moment that T is split, meaning that Γ acts trivially on M. To each cone σ of Σ corresponds a T-stable affine open subset $X_\sigma = \operatorname{Spec} F[\sigma^\vee \cap M]$ in X, where $\sigma^\vee$ is the cone in $M_{\mathbf{R}}$ dual to σ. The variety X is glued from these affine charts X_σ, along the natural open immersions $X_\sigma \to X_\tau$, for any two cones σ and τ in Σ such that $\tau \supset \sigma$. The zero-dimensional cone $\{0\}$ corresponds to T; in particular, each X_σ contains T and carries a compatible action of T.

Let t denote the dimension of T and let σ be a cone of maximal dimension of Σ. By the theory of toric varieties, the smoothness assumption on X implies that there exists a basis of N which generates σ as a cone. Consequently, the torus embedding (T, X_σ) is isomorphic to $(\mathbf{G}_m^t, \mathbf{A}^t)$. The irreducible components of $X_\sigma \subset T$ then correspond to the hyperplane coordinates in $\mathbf{A}^t$. In particular, $X_\sigma \backslash T$ is a divisor with strict normal crossings, all of whose components of $X_\sigma \backslash T$ meet at the origin. This shows that a set of components of $X \backslash T$ has a nontrivial intersection whenever the corresponding rays belong to a common cone in Σ. The converse holds since each point of X admits a T-invariant affine neighborhood, hence of the form X_σ.

In particular, what precedes holds over $\bar{F}$ and implies the following description of the geometric Clemens complex $\mathscr{C}(X, D)$ of (X, D). There is a bijection between the set of orbits of $T_{\bar{F}}$ in $X_{\bar{F}}$ and the set Σ. The irreducible components of $D_{\bar{F}}$ are in bijection with the set of one-dimensional cones (*rays*) of Σ and a set of components has a nontrivial intersection if and only if the corresponding rays belong to a common cone in Σ. The cone $\{0\}$ belongs to Σ and corresponds to the open orbit T. Consequently, the Clemens complex $\mathscr{C}(X, D)$ is equal to the set $\Sigma \backslash \{\{0\}\}$, partially ordered by inclusion, with the obvious action of the Galois group Γ_F.

If T is split, then Γ_F acts trivially on $\mathscr{C}(X, D)$, hence $\mathscr{C}_F(X, D) = \mathscr{C}(X, D)$. We have also observed in that case that the various strata of X even had F-rational points. In other words, the F-analytic Clemens complex $\mathscr{C}_F^{\mathrm{an}}(X, D)$ is also equal to $\mathscr{C}(X, D)$.

Let us now treat the general case by proving that $\mathscr{C}_F^{\mathrm{an}}(X, D) = \mathscr{C}_F(X, D)$. By definition, the F-rational Clemens complex $\mathscr{C}_F(X, D)$ is the subcomplex of $\mathscr{C}(X, D)$ consisting of Γ_F-invariant faces; in other words, it corresponds to Γ_F-invariant cones of positive dimension. We need to prove that for any such cone σ, the closure of the corresponding orbit $\mathscr{O}_\sigma$ in X, which is defined over F, has an F-rational point. For each ray r of σ, let n_r be the generator of $N \cap r$; the group Γ_F acts on the set of these n_r, and their sum n is fixed by Γ_F. It corresponds to a cocharacter $c_n \colon \mathbf{G}_m \to T$ whose limit at 0 is an F-rational point and this point belongs to $\overline{\mathscr{O}_\sigma}$.

In fact, it even belongs to $\mathscr{O}_\sigma$ because the point n belongs to the relative interior of the cone σ.

Lemma 5.1. *Let T_0 be the maximal F-split torus in T; the group N_0 of cocharacters of T_0 is equal to the subspace of N fixed by Γ_F. Let X_0 be the Zariski closure of T_0 in X. It is a smooth equivariant compactification of T_0 and its fan Σ_0 in $(N_0)_{\mathbf{R}}$ has as cones the intersections $\sigma \cap (N_0)_{\mathbf{R}}$, for all cones $\sigma \in \Sigma^{\Gamma_F}$ which are invariant under Γ_F.*

Proof. It suffices to prove the desired result in each affine chart of X. Consider a cone $\sigma \in \Sigma$ and let $X_\sigma = \operatorname{Spec} \bar{F}[M \cap \sigma^\vee]$ be the corresponding affine open subset of $X_{\bar{F}}$. A vector in $M_0 \cap \sigma$, being fixed under the action of Γ_F, belongs to all of the cones $\gamma(\sigma)$, for $\gamma \in \Gamma_F$, hence belongs to their intersection τ which, by the definition of a fan, is a cone in Σ, obviously Γ_F-invariant. As a consequence, the closure of T_0 in X_σ is contained in the toric open subvariety X_τ.

We can thus assume that σ is a Γ_F-invariant cone, and we may moreover assume that it is of maximal dimension. Since X is smooth, there is a basis $(e_1, \dots, e_d)$ of N and an integer $s \in \{1, \dots, d\}$ such that σ is generated by $S = \{e_1, \dots, e_s\}$. We can also assume that $e_{s+1}, \dots, e_r$ belong to M_0 and generate a complement to the (saturated) subgroup generated by $M_0 \cap \sigma$. Over $\bar{F}$, this identifies T with $\mathbf{G}_m^d$ and X_σ with $\mathbf{A}^s \times \mathbf{G}_m^{d-s}$. Moreover, Γ_F acts by permutations on S. This implies that $M_0 \cap \sigma$ is generated by the vectors $\sum_{i \in O} e_i$, where O runs over the set S/Γ_F of Γ_F-orbits in S. Consequently, $T_0 \cap X_\sigma$ is the set of elements $(x_1, \dots, x_d)$ in $\mathbf{G}_m^d$ such that x_i is constant on each orbit O, and $x_i = 1$ for $i > s$. Its closure is the set of such elements $(x_1, \dots, x_d) \in \mathbf{A}^r \times \mathbf{G}_m^{d-r}$ satisfying the same relations. This shows that the closure of T_0 in X_σ is an isomorphic to the toric embedding of $\mathbf{G}_m^{\dim M_0}$ in $\mathbf{A}^{\#(S/\Gamma_F)} \times \mathbf{G}_m^{\dim M_0 - \#(S/\Gamma_F)}$. □

Corollary 5.2. *By associating to a T_0-orbit in X_0 the corresponding T-orbit in X, we obtain a bijection between the Clemens complex of $(X_0, X_0 \backslash T_0)$ and the F-analytic Clemens complex of $(X, X \backslash T)$.*

5.2. *Clemens complexes of wonderful compactifications*

Let G be a connected reductive group over a field F. Let T_0 denote a maximal split torus of G and T a maximal torus of G containing T_0.

Let X be a smooth proper F-scheme which is a biequivariant compactification of G. We denote by Y and Y_0 the closures of the tori T and T_0 in X; these are toric varieties defined over F. We assume that over a separable closure $\bar{F}$ of F, the divisor $D = X \backslash G$ has strict normal crossings. Observe that this assumption is satisfied if X is a wonderful compactification of G.

Proposition 5.3. *With this notation, one has: $X(F) = G(F)Y(F)G(F)$ and $Y(F) = T(F)Y_0(F)$; in particular, $X(F) = G(F)Y_0(F)G(F)$.*

Proof. Let us first prove that $X(F) = G(F)Y(F)G(F)$. It suffices to prove that $X(F)$ is contained in the latter set. We use the arguments of [10], Lemma 6.1.4. Let $x_0 \in X(F)$; since X is assumed to be smooth, we can choose an arc $x \in X(F[\![t]\!])$ such that $x(0) = x_0$ and x is not contained in $X \backslash G$. We interpret x as a $F(\!(t)\!)$-point of G; by the Cartan decomposition ([11], Proposition 4.4.3. The fact that the group $G(F[\![t]\!])$ is good follows from the property that the $F(\!(t)\!)$-algebraic group $G_{F(\!(t)\!)}$ is split over an unramified extension, namely $\bar{F}(\!(t)\!)$, and descent properties of the building)

$$G(F(\!(t)\!)) = G(F[\![t]\!])T(F(\!(t)\!))G(F[\![t]\!]),$$

we may write $x = g_1 y g_2$, where $g_1, g_2 \in G(F[\![t]\!])$ and $y \in T(F(\!(t)\!))$. Writing $y = g_1^{-1} x g_2^{-1}$, we see that $y \in Y(F[\![t]\!])$. Specializing at $t = 0$, we now obtain $x_0 = g_1(0)y(0)g_2(0)$, as required.

Let us now prove the second equality, namely $Y(F) = T(F)Y_0(F)$. Again, it suffices to prove that $Y(F)$ is contained in $T(F)Y_0(F)$. Let $x_0 \in Y(F)$; as above, there is an arc $x \in X(F[\![t]\!]) \cap G(F(\!(t)\!))$ such that $x(0) = x_0$. Using the Cartan decomposition, we may replace x by an arc of the form $y = g_1^{-1} x g_2$, with $g_1, g_2 \in G(F[\![t]\!])$ satisfying $y \in X(F[\![t]\!]) \cap T(F(\!(t)\!))$. Moreover, we may also assume that $g_1(0) = g_2(0)$ is the neutral element of $G(F)$. In particular, $y \in Y(F[\![t]\!])$ and $y(0) = x_0$.

Let S denote the anisotropic torus T/T_0 and $\pi \colon T \to S$ the quotient map. One has $\pi(y) \in S(F(\!(t)\!))$. Since S is anisotropic, one has $S(F[\![t]\!]) = S(F(\!(t)\!))$ (Lemma 5.4 below), hence $\pi(y) \in S(F[\![t]\!])$. Looking at the exact sequence of tori

$$1 \to T_0 \to T \to S \to 1$$

and applying Hilbert's Theorem 90 over the discrete valuation ring $F[\![t]\!]$, it follows that there exists $z \in T(F[\![t]\!])$ such that $\pi(z) = \pi(y)$, hence $z^{-1}y \in T_0(F(\!(t)\!))$. Moreover, $z^{-1}y \in Y(F[\![t]\!])$. Specializing t to 0, we see that $z(0)^{-1}y(0) \in Y_0(F)$, that is $y(0) = x_0 \in T(F)Y_0(F)$, as claimed.

The last equality follows immediately. ☐

The following lemma is well known; we include a proof for the convenience of the reader.

Lemma 5.4. *Let S be an anisotropic torus over a field F. Then, $S(F(\!(t)\!)) = S(F[\![t]\!])$.*

Proof. Let F' be a finite extension of F which splits S; then, for any F-algebra E, $S(E)$ is the set of morphisms from $\mathfrak{X}^*(S)$ to $E \otimes_F F'$ which commute with the actions of $\Gamma_{F'/F}$ on both sides. It follows that a point P in $S(F(\!(t)\!))$ corresponds to a Γ_F-equivariant group morphisms φ from $\mathfrak{X}^*(S)$ to $F'(\!(t)\!)^*$.

By composing φ with the order map $F'(\!(t)\!)^* \to \mathbf{Z}$, we obtain a $\Gamma_{F'/F}$-invariant morphism $\operatorname{ord}\circ\varphi \colon \mathfrak{X}^*(S) \to \mathbf{Z}$, which is necessarily 0 since S is anisotropic. Consequently, $\varphi(c) = 0$ for any $c \in \mathfrak{X}^*(S)$, which means that $\varphi(c) \in F'[\![t]\!]^*$. In other words, φ is a $\Gamma_{F'/F}$-equivariant morphism from $\mathfrak{X}^*(S)$ to $F'[\![t]\!]^*$, and the corresponding point P belongs to $S(F'[\![t]\!])$. $\square$

Our goal now is to describe the various Clemens complexes attached to the pair (X, D). Let $W = \mathrm{N}_G(T)/T$ and $W_0 = \mathrm{N}_G(T_0)/\mathrm{Z}_G(T_0)$ be the Weyl groups of G relative to the tori T and T_0.

Since the Weyl group W acts on G via the conjugation by an element of G, this extends to an action on X. This action induces the trivial action on $\mathscr{C}(X, D)$. Indeed, the group G being connected, each irreducible component of $D_{\bar{F}}$ is preserved by the actions of $G_{\bar{F}}$.

Proposition 5.5. *Over $\bar{F}$, the open strata of the stratification of $X_{\bar{F}}$ deduced from $D_{\bar{F}}$ are exactly the orbits of $(G \times G)_{\bar{F}}$ in $X_{\bar{F}}$. Moreover, associating to an orbit its closure defines a Γ_F-equivariant bijection from $G(\bar{F})\backslash X(\bar{F})/G(\bar{F})$ to $\mathscr{C}(X, D)$.*

Proof. We may assume that the field F is separably closed. We then have $T = T_0$.

By Proposition 5.3, the map

$$Y(F)/T(F) \to G(F)\backslash X(F)/G(F)$$

which associates to the $T(F)$-orbit of a point x in $Y(F)$ the orbit $G(F)xG(F)$ in $X(F)$ is surjective. By the theory of toric varieties, $T(F)$ has only finitely many orbits in $Y(F)$. Consequently, $G \times G^{\mathrm{opp}}$ has only finitely many orbits in X.

Let Z be an element of $\mathscr{C}(X, D)$, viewed as an irreducible closed subvariety of D. It is smooth, and possesses a $G \times G^{\mathrm{opp}}$-action. The group $G \times G^{\mathrm{opp}}$ acts on Z with only finitely many orbits; necessarily, one of these orbits, say Z_0, is open in Z. Since Z is irreducible, it follows that $Z = \overline{Z_0}$, and Z is the closure of a $G \times G^{\mathrm{opp}}$-orbit. By Theorem 2.1 of [33], there exists an open affine subset X_0 of X such that Z_0 is the unique closed orbit of $G \times G^{\mathrm{opp}}$ in GX_0G.

Let us show that $Z\backslash Z_0$ is contained in $X\backslash GX_0G$ or, equivalently, that $Z \cap GX_0G = Z_0$. Indeed, let O be any orbit in Z distinct from Z_0; we need to prove that $O \cap GX_0G = \varnothing$. Let us denote by $\bar{O}$ the closure of O in X; since Z_0 is open in Z, $\bar{O}$ has empty interior in Z, hence is an irreducible subset of X whose dimension satisfies $\dim \bar{O} < \dim Z$. However, any action of an algebraic group on an algebraic variety has a closed orbit; in particular $\bar{O} \cap GX_0G$ contains a closed orbit, which implies that $\bar{O} \cap GX_0G \supset Z_0$, and this contradicts the assumption $\dim \bar{O} < \dim Z_0$.

Since X_0 is open in X, it contains at least one point of G, hence GX_0G contains G. Consequently, $X\backslash GX_0G$ is contained in $X\backslash G$. Since GX_0G is an $G \times G^{\mathrm{opp}}$-invariant open subset of X, each irreducible component of $Z\backslash Z_0$ is contained in some irreducible component of $X\backslash GX_0G$ which does not meet Z_0.

Consequently, the orbit Z_0 is an open stratum of the stratification defined by the boundary divisor D.

Conversely, let O be an orbit of $G \times G^{\mathrm{opp}}$ in X and let Z be the stratum of minimal dimension in $\mathscr{C}(X, D)$ which contains O. By the preceding argument, Z is the closure of an orbit Z_0 whose complement $Z\backslash Z_0$ is a union of lower dimensional strata. By the minimality assumption on Z, $Z_0 = O$, which proves that O is an open stratum.

The rest of the proposition follows at once. □

The group of characters M_0 of T_0 is the group of coinvariants of Γ in M (it is a quotient of M), and its group of cocharacters N_0 is the group of invariants of Γ in N (a subgroup of N). Let $t_0 = \dim T_0$. Let Σ_0 denote the fan of N_0 induced by Σ; it is simplicial albeit not obviously smooth in general. This fan defines a toric variety X_0 which in fact is isomorphic to the Zariski closure of T_0 in X.

5.3. *Volume estimates for compactifications of semi-simple groups*

5.3.1. *Introduction*

Let G be a semisimple algebraic group of adjoint type over a field F of characteristic 0. Let $\iota\colon G \to \mathrm{GL}(V)$ be a faithful algebraic representation of G in a finite dimensional F-vector space V. The natural map

$$\mathrm{GL}(V) \to \mathrm{End}(V)\backslash\{0\} \to \mathbf{P}\mathrm{End}(V)$$

induces a map $\bar\iota\colon G \to \mathbf{P}\mathrm{End}(V)$. Let X_ι be the Zariski closure of its image; it is a bi-equivariant compactification of G. Let ∂X_ι be the complement to G in X_ι.

When ι is irreducible with regular highest weight, X_ι is the wonderful compactification defined by De Concini and Procesi in [19], Sec. 3.4. In that case, X_ι is smooth and ∂X_ι is a divisor with strict normal crossings.

When F is a local field, we endow the vector space $\mathrm{End}(V)$ with a norm $\|\cdot\|$. When F is a number field, let us choose, for any place v of F, a v-adic norm on $\mathrm{End}(V) \otimes F_v$, so that there is an $\mathfrak{o}_F$-lattice in $\mathrm{End}(V)$ inducing these norms for almost all finite places. As it was explained in Secs. 2.1.6 and 2.2.4, such choices give rise to a metric on the line bundle $\mathscr{O}(1)$ on the projective space $\mathbf{P}\mathrm{End}(V)$ (resp. an adelic metric) when F is a number field.

We claim that the line bundle $\mathscr{O}(1)$ on X_ι has a nonzero global section s_ι which is invariant under G. Such a section s_ι is unique up to multiplication by a scalar, since the quotient of two of them is a G-invariant rational function on X_ι. (See also [19], 1.7, p. 9, proposition.)

To prove the existence, let us first observe that the line in $\mathrm{End}(V)$ generated by id_V is G-invariant. By semi-simplicity of G, there exists a linear form ℓ_ι on $\mathrm{End}(V)$ which is invariant under G and maps id_V to 1. Let s_ι be the restriction to G of the global section of $\mathscr{O}(1)$ over $\mathbf{P}\mathrm{End}(V)$ defined by ℓ_ι. Now, we have

$$\|s_\iota\|(\bar\iota(g)) = \frac{|\ell_\iota(g)|}{\|\iota(g)\|} = \frac{|\ell_\iota^g(e)|}{\|\iota(g))\|} = \|\iota(g)\|^{-1},$$

for any $g \in G(F)$ when F is a local field, and similar equalities at all places of F in the number field case.

Consequently, for $F = \mathbf{R}$ or $\mathbf{C}$, the results of Sec. 4.2 imply, as a particular case, an asymptotic formula for the volume of sets of $g \in G(F)$ with $\|\iota(g)\| \leq B$, when $B \to \infty$, as well as similar (but weaker) estimates when F is a p-adic field. Similarly, when F is a number field, the volume estimates established in Sec. 4.4 imply estimates for the volume of adelic sets in $G(\mathbb{A}_F)$ consisting of adelic points of bounded height.

In the case of local fields, it requires further computations for making these estimates explicit, in terms of the representation ι. In particular, we will need to describe the analytic Clemens complex of ∂X_ι.

5.3.2. *The wonderful compactification of De Concini and Procesi*

For simplicity, we assume for the moment that G is split. We fix a maximal torus T of G which is split, as well as a Borel subgroup B of G containing T. We identify the groups of characters $\mathfrak{X}^*(T)$ and $\mathfrak{X}^*(B)$, as well as the groups of cocharacters $\mathfrak{X}_*(T)$ and $\mathfrak{X}_*(B)$; we also let $\mathfrak{a} = \mathfrak{X}_*(T) \otimes_{\mathbf{Z}} \mathbf{R}$ and $\mathfrak{a}^* = \mathfrak{X}^*(T) \otimes_{\mathbf{Z}} \mathbf{R}$.

Let Φ (resp. Φ^+), be the set of roots of G (resp. of positive roots in $\mathfrak{X}^*(T)$); let β denote the sum of all positive roots. Let $\Delta \subset \Phi^+$ be the set of simple roots; they form a basis of the real cone in $\mathfrak{a}^*$ generated by positive roots, hence we may write $\beta = \sum_{\alpha \in \Delta} m_\alpha \alpha$ for some positive integers m_α.

Let $\iota\colon G \to \mathrm{GL}(V)$ be a representation as above; we assume here that ι is irreducible and that its highest weight λ is regular, i.e. can be written as $\lambda = \sum_{\alpha \in \Delta} d_\alpha \alpha$, for some positive integers d_α. In that case, as recalled above, X_ι is the "wonderful compactification of G" defined by De Concini and Procesi. The variety X_ι does not depend on the actual choice of ι, but the projective embedding does.

The irreducible components of $X_\iota \backslash G$ are naturally indexed by the set Δ; we will write D_α for the divisor corresponding to a simple root α. Let $D = \sum D_\alpha$. The Clemens complex $\mathscr{C}(\partial X_\iota)$ is simplicial, and coincides with the F-analytic Clemens complex $\mathscr{C}^{\mathrm{an}}_F(\partial X_\iota)$ since G is assumed to be split.

The line bundles $\mathscr{O}(D_\alpha)$ form a basis of $\mathrm{Pic}(X_\iota)$, as well as generators of the cone of effective divisors (which is therefore simplicial). The restriction of $\mathscr{O}_{\mathbf{P}}(1)$ to X corresponds precisely to λ, and there is a canonical isomorphism

$$\mathscr{O}_{\mathbf{P}\mathrm{End}(V)}(1)|_{X_\iota} \simeq \mathscr{O}\left(\sum_{\alpha \in \Delta} d_\alpha D_\alpha\right).$$

According to [19], the anticanonical line bundle of X_ι is given by

$$K_{X_\iota}^{-1} \simeq \mathscr{O}\left(\sum_{\alpha \in \Delta} (m_\alpha + 1) D_\alpha\right).$$

Let $\mathscr{C} \subset \mathfrak{a}^*$ be the convex hull of the characters of T appearing in the representation ι. This is also the convex hull of the images of the highest weight λ under the action of the Weyl group. This is a convex and compact polytope which contains 0 in its interior by [36], Lemma 2.1.

Lemma 5.6. *Let $\sigma = \max_{\alpha\in\Delta}(m_\alpha/d_\alpha)$, let t be the number of elements $\alpha \in \Delta$ where equality holds. Then σ is the smallest positive real number such that $\beta/\sigma \in \mathscr{C}$ and t is the maximal codimension of a face of $\mathscr{C}$ containing β/σ.*

Proof. For any simple root α, let $\alpha^\vee$ be the corresponding coroot, so that the simple reflexion s_α associated to α is given by $s_\alpha(x) = x - 2\langle\alpha^\vee, x\rangle\alpha$, for $x \in \mathfrak{a}^*$. We have $\langle\alpha^\vee, \alpha\rangle = 1$ while $\langle\alpha^\vee, \alpha'\rangle = 0$ for any other simple root α'.

Within the Weyl chamber of λ, the polytope $\mathscr{C}$ is bounded by the affine hyperplanes orthogonal to the simple roots α and passing through λ. Consequently, a point x in this chamber belongs to $\mathscr{C}$ if and only if $\langle\alpha^\vee, x\rangle \leq \langle\alpha^\vee, \lambda\rangle$. Moreover, such a point x belongs to the boundary of $\mathscr{C}$ if and only if equality is achieved for some simple root α; then, the number of such α is the maximal codimension of a face of $\mathscr{C}$ containing x. For $x = \beta/\sigma = \sum_{\alpha\in\Delta}(m_\alpha/\sigma)\alpha$, we find

$$\langle\alpha^\vee, x - \lambda\rangle = \frac{m_\alpha}{\sigma} - \lambda_\alpha,$$

hence the lemma. □

Consequently, our geometric estimates (Theorem 4.7) imply the following result of Maucourant [36] under the assumption that G is split and V has a unique highest weight.

Corollary 5.7. *Define $\sigma = \max_{\alpha\in\Delta}(m_\alpha/d_\alpha)$ and let t be the number of $\alpha \in \Delta$ where the equality holds.*

Assume that $F = \mathbf{R}$ or $\mathbf{C}$. When $B \to \infty$, the volume $V(B)$ of all $g \in G(F)$ such that $\|\iota(g)\| \leq B$ satisfies an asymptotic formula of the form:

$$V(B) \sim cB^\sigma(\log B)^{t-1}.$$

According to our theorem, the positive constant c can be written as a product of the combinatorial factor

$$\frac{1}{\sigma(t-1)!}\prod_{\alpha\in A}\frac{1}{\lambda_\alpha}$$

and an explicit integral on the stratum of $X_\iota(F)$ defined by A, with respect to its normalized residue measure τ_{D_A}.

In the p-adic case, Corollary 4.9 similarly implies the following result:

Corollary 5.8. *Keep the same notation, assuming that F is a p-adic field. Then, when $B \to \infty$,*

$$0 < \liminf \frac{V(B)}{B^\sigma(\log B)^{t-1}} \leq \limsup \frac{V(B)}{B^\sigma(\log B)^{t-1}} < +\infty.$$

5.3.3. *The case of a general representations* (*split group*)

We now explain how to treat non-irreducible representations, still assuming that the group G is split.

By Proposition 6.2.5 of [10], there is a diagram of equivariant compactifications

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\pi"'] \arrow[dr] & \\ X_\iota & & X_w \end{tikzcd}$$

where X_w is the wonderful compactification previously studied and $\tilde{X}$ is smooth. Since the boundary divisor of a smooth toric variety is a divisor with strict normal crossings, it then follows from [10], Proposition 6.2.3, that the boundary $\partial\tilde{X}$ of $\tilde{X}$ is a divisor with strict normal crossings in $\tilde{X}$. Moreover, the proof of this proposition and the local description of toroidal G-embeddings in *loc. cit.*, Sec. 6.2, show that $\tilde{X}$ is obtained from X_w by a sequence of T-equivariant blow-ups. Therefore, the boundary $\partial\tilde{X}$ consists of the strict transforms $\tilde{D}_\alpha$ of the divisors D_α, indexed by the simple roots, and of the exceptional divisors E_i (for i in some finite index set I). These divisors form a basis of the effective cone in the Picard group. The anticanonical line bundle of $\tilde{X}$ decomposes as a sum

$$\sum_{\alpha\in\mathscr{A}}(m_\alpha+1)D_\alpha+\sum_{i\in I}E_i.$$

Let $\tilde{L}$ be the line bundle $\pi^*\mathscr{O}(1)$ on $\tilde{X}$; let us write it as $\tilde{L}=\sum\tilde{\lambda}_\alpha D_\alpha+\sum\tilde{\lambda}_i E_i$ in the above basis. Since we can compute volumes on $\tilde{X}$, our estimates imply that $V(B)$ has an asymptotic expansion of the form given in Corollary 5.7, σ being given by

$$\sigma=\max\left(\max_{\alpha\in\mathscr{A}}\frac{m_\alpha}{\tilde{\lambda}_\alpha},\max_{i\in I}\frac{0}{\tilde{\lambda}_i}\right)=\max_{\alpha\in\mathscr{A}}\frac{m_\alpha}{\tilde{\lambda}_\alpha},$$

and t is again the number of indices $\alpha\in\mathscr{A}$ where equality holds.

Observe that the definition of σ precisely means that the line bundle $\sigma\pi^*L-(K_{\tilde{X}}+\partial\tilde{X})$ belongs to the boundary of the effective cone of $\tilde{X}$; the integer t is then the codimension of the face of minimal dimension containing that line bundle. These properties can be checked on restriction to the toric variety $\tilde{Y}$ given by the closure of T in $\tilde{X}$; indeed the restriction map $\operatorname{Pic}(\tilde{X})\to\operatorname{Pic}(\tilde{Y})$ is an isomorphism onto the part of $\operatorname{Pic}(\tilde{Y})$ invariant under the Weyl group, and similarly for the effective cone. To determine σ, it now suffices to test for the positivity of the piecewise linear (PL) function on the vector space $\mathfrak{a}^*$ associated to this divisor.

The above formula for the anticanonical line bundle of $\tilde{X}$ implies that the PL function for $K_{\tilde{X}}+\partial\tilde{X}$ is that of $K_{X_w}+\partial X_w$.

On the other hand, the theory of heights on toric varieties relates the PL function corresponding to a divisor to the normalized local height function (see [4]). The formula is as follows. Let D be an effective T-invariant divisor, the corresponding

T-linearized line bundle has a canonical T-invariant nonzero global section s_D. For $t \in T(F)$, let $\ell(t)$ be the linear form on $\mathfrak{a}$ defined by $\chi \mapsto \log|\chi(t)|$; dually, one has $\ell(t)(\chi) = \langle \chi, \log(t) \rangle$. Then, the norm of s_D with respect to the canonical normalization is given by

$$\log \|s_D(t)\|^{-1} = \varphi_D(\ell(t)).$$

If the local height function is not normalized, the previous equality only holds up to the addition of a bounded term.

In our case, let Φ be the set of weights of T in the representation ι. Then, for $t \in T(F)$,

$$\|\iota(t)\| \approx \max_{\chi \in \Phi} |\chi(t)|,$$

hence

$$\log \|s_\iota(t)\|^{-1} = \log \|\iota(t)\| = \max_{\chi \in \Phi} \log|\chi(t)| + \mathrm{O}(1).$$

In other words, the line bundle $\pi^* L$ corresponds to the PL function $\varphi_\iota = \max_{\chi \in \Phi} \langle \chi, \cdot \rangle$ on $\mathfrak{a}^*$.

As explained above, σ is the least positive real number such that the PL function $s\varphi_\iota$ is greater than the PL function φ_β associated to $K_{X_w}^{-1}(-\partial X)$. Since the Weyl group acts trivially on the Picard group of $\tilde{X}$, these PL functions are invariant under the action of the Weyl group and it is sufficient to test the inequality on the positive Weyl chamber C in $\mathfrak{a}$.

Let (ϖ_α) denote the basis of $\mathfrak{a}^*$ dual to the basis (α) — up to the usual identification of $\mathfrak{a}$ with its dual given by the Killing form of G, this is the basis of fundamental weights. One has $C = \sum_\alpha \mathbf{R}_+ \varpi_\alpha$.

Recall also that β is the sum of the fundamental weights of G, hence belongs to the positive Weyl chamber in $\mathfrak{a}^*$. It follows that $w\beta \geq \beta$ for any element w in the Weyl group W (Bourbaki, LIE VI, §1, Proposition 18, p. 158). It follows that $\langle w\beta, \varpi_\alpha \rangle \leq \langle \beta, \varpi_\alpha \rangle$ for any $w \in W$. Consequently,

$$\varphi_\beta(y) = \max \langle w\beta, y \rangle = \langle \beta, y \rangle$$

for any $y \in C$. Moreover, if $y = \sum y_\alpha \varpi_\alpha \in \mathfrak{a}$, then

$$\langle \beta, y \rangle = \left\langle \sum m_\alpha \alpha, \sum y_\alpha \varpi_\alpha \right\rangle = \sum m_\alpha y_\alpha.$$

Let Λ be the set of dominant weights of ι with respect to C. For any $\lambda \in \Lambda$, let us write $\lambda = \sum \lambda_\alpha \alpha$, for some non-negative $\lambda_\alpha \in \mathbf{Z}$. Consequently, for any $y = \sum y_\alpha \varpi_\alpha \in C$, with $y_\alpha \geq 0$ for all α, one has

$$\varphi_\iota(y) = \max_{\chi \in \Phi} \langle \chi, y \rangle = \max_{\lambda \in \Lambda} \langle \lambda, y \rangle = \max_{\lambda \in \Lambda} \left(\sum_\alpha \lambda_\alpha y_\alpha \right).$$

The condition that $s\varphi_\iota(y) \geq \varphi_\beta$ on C therefore means that $s\max_{\lambda\in\Lambda}\lambda_\alpha \geq m_\alpha$ for any $\alpha\in\Phi$. In other words, $\sigma = \max(m_\alpha/\tilde\lambda_\alpha)$, where we have set $\tilde\lambda_\alpha = \max_{\lambda\in\Lambda}\lambda_\alpha$. Moreover, t is the number of simple roots α such that $\tilde\lambda_\alpha\sigma = m_\alpha$.

As in [36], let $\mathscr{C}$ be the convex hull of Φ in $\mathfrak{a}^*$; it is a compact polytope whose dual $\mathscr{C}^*$ is defined by the inequality $\varphi_\iota(\cdot)\leq 1$ in $\mathfrak{a}^*$. Let s be a positive real number. By definition, β/s belongs to $\mathscr{C}$ if and only if $\langle\beta/s, y\rangle \leq 1$ for any $y\in\mathfrak{a}^*$ such that $\varphi_\iota(\mathfrak{a})\leq 1$. This is precisely equivalent to the fact that $s\varphi_\iota - \langle\beta,\cdot\rangle$ is non-negative on $\mathfrak{a}^*$.

Let $y\in\mathfrak{a}^*$ and let $w\in W$ be such that $wy\in C$. Since φ_ι is invariant under W,

$$s\varphi_\iota(y) - \langle\beta, y\rangle = s\varphi_\iota(wy) - \langle w^{-1}\beta, wy\rangle \leq s\varphi_\iota(wy) - \langle\beta, wy\rangle,$$

with equality if and only if $w = e$. We have thus shown that $\beta/s\in\mathscr{C}$ if and only if the PL function $s\varphi_\iota - \varphi_\beta$ is non-negative. In other words, σ is the least positive real number such that β/σ belongs to $\mathscr{C}$, as claimed by Maucourant in [36].

Let $\mathscr{F}$ be the face of $\mathscr{C}$ containing β/σ. It is also explained in [36] that the dual face of $\mathscr{F}$ is contained in the positive Weyl chamber C (Lemma 2.3) and is given by

$$\mathscr{F}^* = \{y\in\mathscr{C}^*; \langle\beta, y\rangle = \sigma\}.$$

If again we decompose y as $\sum y_\alpha\varpi_\alpha$, it follows that $\mathscr{F}^*$ identifies as the set of non-negative (y_α) such that $\varphi_\iota(y)\leq 1$ and $\langle\beta, y\rangle = \sigma$. The first condition gives us $\sum\tilde\lambda_\alpha y_\alpha \leq 1$ and the second is equivalent to $\sum m_\alpha y_\alpha = \sigma$. This implies $\sum y_\alpha = 1$ and $y_\alpha = 0$ if $\sigma\neq m_\alpha/\tilde\lambda_\alpha$. Consequently, $\operatorname{codim}\mathscr{F} = \dim\mathscr{F}^* = t$, showing the agreement of our general theorem with the result obtained by Maucourant in [36], under the assumption that G is split.

5.3.4. *The general case*

We now treat the general case of a possibly non-split group. Let T be a maximal split torus in G, so that its Lie algebra $\mathfrak{a}_{\mathbf{R}}$ is a Cartan subalgebra of $\operatorname{Lie}(G)$. We have already explained how the F-analytic Clemens complex of X is related to the toric variety Y given by the closure of T in X. As a consequence, all positivity conditions and dimensions of faces which intervene in our geometric result rely only on the divisors which are "detected" by Y, hence are expressed in terms of PL functions in $\mathfrak{a}^*$. The previous analysis now applies verbatim and allow us again to recover Maucourant's theorem.

5.3.5. *Adelic volumes*

Let us now assume that F is a number field. Theorem 4.13 describes the analytic behavior of the volume — with respect to the Haar measure — of adelic points in $G(\mathbb{A}_F)$ of height $\leq B$, when $B\to\infty$. This allows to recover Theorem 4.13 in [27]. In fact, that theorem itself is proved as a corollary of the analytic behavior of the associated Mellin transform which had been previously shown in [45], Theorem 7.1. Similarly, this analytic behavior is a particular case of our Proposition 4.12.

5.4. *Relation with the output of the circle method*

Let X be a non-singular, geometrically irreducible, closed subvariety of an affine space V of dimension n over a number field F. When $F=\mathbf{Q}$, $W=\mathbf{A}^n$ and X is defined as the proper intersection of r hypersurfaces defined. Let n be a positive integer and let $f_1,\dots,f_r\in\mathbf{Z}[X_1,\dots,X_n]$ be polynomials with integer coefficients.

The circle method eventually furnishes an estimate, when the real number B grows to ∞, of the number $N(X,B)$ of solutions $x\in\mathbf{Z}^n$ of the system X given by $f_1(x)=\dots=f_r(x)=0$ whose "height" $\|x\|$ is bounded by B. A set of conditions under which the circle method applies is given by [6]; it suffices that the hypersurfaces $\{f_i=0\}$ meet properly and define a non-singular codimension r subvariety of $\mathbf{A}^n$, and that n is very large in comparison to the degrees of the f_i.

Let us assume that $X=V(f_1,\dots,f_r)$ is smooth and has codimension r. By the circle method, an approximation for $N(B)$ is given by a product of "local densities": for any prime number p, let

$$\mu_p(X)=\lim_{k\to\infty}\frac{\#\,X(\mathbf{Z}/p^k\mathbf{Z})}{p^{k\dim X}};$$

for the infinite prime, let

$$\mu_\infty(X,B)=\lim_{\varepsilon\to 0}\varepsilon^{-r}\operatorname{vol}\{x\in\mathbf{R}^n;\|x\|\leq B,\ |f_i(x)|<\varepsilon/2\},$$

where vol refers to the Euclidean volume in $\mathbf{R}^n$. In some cases, one can indeed prove that $N(X,B)\sim\mu_\infty(X,B)\prod_{p<\infty}\mu_p(X)$ when $B\to\infty$.

As already observed by [7] (Sec. 1.8) we first want to recall that the right-hand side $V(X,B)$ of this asymptotic expansion is really a volume. Under the transversality assumption we have made on the hypersurfaces defined by the f_i, there exists a differential form $\tilde\omega$ on a neighborhood of X in $\mathbf{A}^n$ such that

$$\tilde\omega\wedge\mathrm{d}f_1\wedge\dots\wedge\mathrm{d}f_r=\mathrm{d}x_1\wedge\dots\wedge\mathrm{d}x_n.$$

The restriction of $\tilde\omega$ to X is a well-defined gauge form ω on X and

$$\mu_\infty(X,B)=\int_{\substack{X(\mathbf{R})\\ \|x\|\leq B}}\mathrm{d}|\omega|_\infty(x),\quad \mu_p(X)=\int_{X(\mathbf{Z}_p)}|\omega|_p\quad (p\text{ prime}).$$

Let us write $[t:x]$ for the homogeneous coordinates of a point in $\mathbf{P}^n(K)$, so that $t\in K$ and $x=(x_1,\dots,x_n)\in K^n$. One then defines a metric on $\mathscr{O}(1)$ by the formula

$$\|s_F\|([t:x])=\frac{|F(t,x)|}{\max(|t|,\|x\|)^{\deg F}}$$

for any point $[t:x] \in \mathbf{P}^n(\mathbf{R})$ and any homogeneous polynomial F in $\mathbf{R}[t,x]$, s_F being the section of $\mathscr{O}(\deg F)$ attached to F. Let f_0 be the section corresponding to $F=t$; for $B \geq 1$, the condition $\|x\| \leq B$ can thus be translated to $\|\mathsf{f}_0\|([1:x]) \geq 1/B$.

First assume that the Zariski closure of X in $\mathbf{P}^n$ is smooth; let D be the divisor $\mathbf{V}(\mathsf{f}_0)$ in X. Letting $d = \sum \deg(f_i)$, the divisor of ω is equal to $d-n-1$. In that case, the measure $|\omega|$ coincides with the measure $\tau_{(\bar{X},(n+1-d)D}$ on $X(\mathbf{R})$, so that $\mu_\infty(X,B)$ is an integral of the form studied in this paper, namely

$$\mu_\infty(X,B) = \tau_{(X,(n+1-d)D)}(\|\mathsf{f}_0\| \geq 1/B).$$

Assume $n > d$. Then, when B converges to infinity, Theorem 4.7 implies that $\mu_\infty(X,B) \approx B^{n-d}(\log B)^{b-1}$, where b is the dimension of the Clemens complex of the divisor $D(\mathbf{R})$, that is the maximal number of components of $D(\mathbf{R})$ that have a common intersection point. (When $n=d$, a similar result holds except that $b-1$ has to be replaced by b.)

The paper [22] showed that this asymptotic does not hold for the specific example of a generic SL(2)-orbit of degree d binary forms. For example, when $d=3$, this amounts to counting points on a hypersurface of degree 4 in $\mathbf{P}^4$ and the expected exponent $0 = 4-4$ is replaced by $2/3$. This discrepancy is explained by [29]: observing that $(\bar{X}, D)$ is not smooth in that case, they computed a log-desingularization of $(\bar{X}, D)$ on which the behavior of the integral $\mu_\infty(X,B)$ can be predicted.

In the general case, let us thus consider a projective smooth compactification $\bar{Y}$ of X together with a projective, generically finite, morphism $\pi\colon \bar{Y} \to \bar{X}$ such that, over $\bar{\mathbf{Q}}$, the complementary divisor E to X in $\bar{Y}$ has strict normal crossings. Let E_α be the irreducible components of E. It is customary to assume that π induces an isomorphism over X, but this is not necessary in the following analysis; it is in fact sufficient to assume that the degree of π is constant on the complement to a null set.

Let us set $\eta = \pi^*\omega$; then, the divisor of η has the form $\operatorname{div}(\eta) = -\sum \rho_\alpha E_\alpha = -E'$, so that

$$\mu_\infty(X,B) = \int_{\substack{\bar{Y}(\mathbf{R}) \\ \|\pi^*\mathsf{f}_0\| \geq 1/B}} |\eta| = \tau_{(\bar{Y},E')}(\{\|\pi^*\mathsf{f}_0\| \geq 1/B\}).$$

Write also $\pi^*D = \sum \lambda_\alpha E_\alpha$. According to Theorem 4.7, $\mu_\infty(X,B) \approx B^a(\log B)^{b-1}$ where now, $a = \max(\rho_\alpha - 1)/\lambda_\alpha$, the minimum being restricted to those α such that $E_\alpha(\mathbf{R}) \neq \varnothing$; the integer b is the dimension of the subcomplex of the analytic Clemens complex consisting of those E_α achieving this minimum. (When $a=0$, $b-1$ is replaced by b.)

Let us give the specific example of an SL(2)-orbit in the affine space of binary forms of degree d, as treated in [29]. That paper constructs a pair $(\bar{Y}, E)$ with an action of $\mathfrak{S}_n$ such that $E = E_1 + E_2$ has two irreducible components, the divisors E_1 and E_2 (denoted $A[n-1]$ and $A[n]$ in that paper) forming a basis of the $\mathfrak{S}_n$-invariant part of $\operatorname{Pic}(\bar{Y})$. Moreover, $K_{\bar{Y}}^{-1} = E_1 + 2E_2$. They compute the inverse image of

$\mathscr{O}(1)$ and obtain $\frac{n-2}{2}E_1 + \frac{n}{2}E_2$ (*loc cit.*, Lemma 3.3). Consequently, $(\rho_1 - 1)/\lambda_1 = 0$ and $(\rho_2 - 1)/\lambda_2 = 2/n$. Finally, $a = 2/n$ and $b = 1$, implying that $\mu_\infty(X, B) \approx B^{2/n}$.

Let us return to the general case, and assume that $K_{\bar{X}}$ is $\mathbf{Q}$-Cartier. Then, the *log-discrepancies* (ε_α) are defined by the formula

$$K_{\bar{Y}}(E) = \pi^*(\bar{K}_{\bar{X}}(D)) + \sum \varepsilon_\alpha E_\alpha,$$

so that $1 - \rho_\alpha = (d - n)\lambda_\alpha + \varepsilon_\alpha$ for any α. Then,

$$a = n - d + \min_\alpha \frac{-\varepsilon_\alpha}{\lambda_\alpha},$$

where, again, the minimum is restricted to those α such that $E_\alpha(\mathbf{R}) \neq \varnothing$. When $(\bar{X}, D)$ has log-canonical singularities, $\varepsilon_\alpha \geq 0$ for each α and one obtains again $a \leq n - d$ since the log-discrepancy corresponding to the strict transform of the components of D is 0.

However, in the case treated by [29], $\varepsilon_1 = 0$ and $\varepsilon_2 = -1$, leading to the opposite inequality $a = \frac{2}{n} > 0$.

5.5. *Matrices with given characteristic polynomial*

5.5.1. Let V_P be the $\mathbf{Z}$-scheme of matrices in the affine 4-space whose characteristic polynomial is $X^2 + 1$, and let B_T be the Euclidean ball of radius T in $\mathbf{R}^4$. Shah gives in [44] the following asymptotic expansion:

$$\#(V_P(\mathbf{Z}) \cap B_T) \sim T\zeta_K^*(1)\frac{\pi^{1/2}}{\Gamma(3/2)}\frac{\pi}{\Gamma(2/2)\zeta(2)} = T\zeta_K^*(1)\frac{2\pi}{\zeta(2)} = C_{X^2+1}T,$$

where $\zeta^*_{\mathbf{Q}(i)}$ is the leading term at 1 of Dedekind's zeta function relative to the number field $\mathbf{Q}(i)$.

A matrix in V_P has the form $\left(\begin{smallmatrix} x & z \\ y & -x \end{smallmatrix}\right)$ with $x^2 + yz + 1 = 0$. Let X denote the subvariety of $\mathbf{P}^3$ defined by $X^2 + YZ + T^2 = 0$. It is smooth of dimension 2 over $\mathbf{Z}[1/2]$. The scheme V_P is exactly the open subset $U \subset X$ defined by $T \neq 0$. The divisor at infinity $D = X \backslash U$ is defined by $T = 0$; it is smooth as well (still over $\mathbf{Z}[1/2]$).

5.5.2. Let p be an odd prime number. One has $\#U(\mathbf{F}_p) = p^2 + (\frac{-1}{p})p$. Indeed, $z \neq 0$ gives $p(p-1)$ points. If $z = 0$, y may be arbitrary and x has to be $\pm\sqrt{-1}$, hence $2p$ points if -1 is a square and 0 else. Finally,

$$\operatorname{vol} U(\mathbf{Z}_p) = p^{-2}\#U(\mathbf{F}_p) = 1 + \left(\frac{-1}{p}\right)\frac{1}{p}.$$

For $p = 2$, we split $U(\mathbf{Z}_2)$ into two parts:

- y odd, so that $z = -(x^2+1)/y$. This has measure $1/2$.
- $y = 2y'$ even, which implies that y' and $x = 2x'+1$ odd, so that
$$z = -(x^2+1)/y = -(1+2x'+2(x')^2)/y'.$$
This has measure $1/4$. Indeed, it parametrizes as
$$\begin{cases} x = 1+2u, \\ y = \dfrac{2}{1+2v}, \\ z = (1+2u+2u^2)(1+2v) = 1+2u+2v+2u^2(1+2v)+4uv, \end{cases}$$
with $(u,v) \in (\mathbf{Z}_2)^2$. The differential of this map is given by the matrix
$$\begin{pmatrix} 2 & 0 \\ 0 & -4(1+2v)^{-2} \\ 2+4u(1+3v) & 2+(4u^2+4u) \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & -2(1+2v)^{-2} \\ 1+2u(1+3v) & 1+2u(1+u) \end{pmatrix}$$
and is twice a 2×3 matrix with coefficients in $\mathbf{Z}_2$, of which one of the 2×2 is invertible. Therefore, the measure of its image is $|2|_2^2 = 1/4$.

Finally, $\operatorname{vol} U(\mathbf{Z}_2) = 1/2 + 1/4 = 3/4$.

5.5.3. The virtual character $\mathrm{EP}(U)$ is $-[\chi]$, for $\chi = (-1/\cdot)$ the quadratic character corresponding to $\mathbf{Q}(i)$. We will prove this in general below, however, this can be seen directly as follows.

First, U is a hypersurface of the affine 3-space, so it has no non-constant invertible function. Let us now study its Picard group. Over $\mathbf{Q}(i)$, X is the hypersurface of $\mathbf{P}^3$ defined by the equation $(x+it)(-x+it) = yz$, so is isomorphic to $\mathbf{P}^1\times\mathbf{P}^1$, the two factors being interchanged by the complex conjugation. Consequently, $\operatorname{Pic}(X_{\bar F})$, being generated by these two lines, is the Galois module $\mathbf{1}\oplus[\chi]$. The result then follows from the fact that $\operatorname{Pic}(X_{\bar F})$ maps surjectively to $\operatorname{Pic}(U_{\bar F})$, its kernel being the submodule generated by the class of the hyperplane of equation $t=0$ in $\mathbf{P}^3$.

The L-function of $\mathrm{EP}(U)$ has local factors $\mathrm{L}_p(\mathrm{EP}(U), s) = 1 - \left(\frac{-1}{p}\right)p^{-s}$. The product $\mathrm{L}(\mathrm{EP}(U), s)$ of all L_p for $p>2$ is exactly $\mathrm{L}(\chi,s)^{-1}$. At $s=1$, it has neither a pole, nor a zero.

5.5.4. One has
$$\operatorname{vol} U(\mathbf{Z}_2)\prod_{p>2}\operatorname{vol} U(\mathbf{Z}_p)\mathrm{L}_p(1) = \frac{3}{4}\prod_{p>2}(1-p^{-2}) = \frac{6}{\pi^2} = \zeta(2)^{-1}.$$
As $\mathrm{L}(\chi,s)\zeta(s) = \zeta_{\mathbf{Q}(i)}(s)$, one has $\mathrm{L}(\chi,1) = \zeta^*_{\mathbf{Q}(i)}(1)$ and the normalized volume of $U(\mathbb{A}_f)$ is equal to
$$\zeta^*_{\mathbf{Q}(i)}(1)\frac{1}{\zeta(2)}.$$

5.5.5. Now we have to compute the volume of the divisor at infinity. Due to the way matrices are counted in [44] (one wants $2x^2+y^2+z^2 \leq T^2$) the natural metric on the tautological bundle of $\mathbf{P}^3$ is given by the formula

$$\|s_P\|(x:y:z:t) = \frac{P(x,y,z,t)}{\max(2|x|^2+|y|^2+|z|^2,|t|^2)^{\deg P/2}},$$

where P is a homogeneous polynomial in four variables and s_P the corresponding section of $\mathscr{O}(\deg P)$.

The divisor D has equations $s_1 = X^2+YZ+T^2=0$ and $s_2 = T = 0$ in $\mathbf{P}^3$. We will compute its volume using affine coordinates (y,z,t) for $\mathbf{P}^3$ and x for D. Affine equations of D corresponding to the two previous sections are $1+yz+t^2=0$ and $t=0$. One has

$$\mathrm{d}(1+yz+t^2)\wedge \mathrm{d}z\wedge \mathrm{d}t = z\mathrm{d}y\wedge \mathrm{d}z\wedge \mathrm{d}t,$$

and by definition,

$$\|\mathrm{d}y\wedge \mathrm{d}z\wedge \mathrm{d}t\| = \max(2+y^2+z^2,t^2)^2.$$

Consequently, on D, one obtains the following equalities

$$\lim \frac{\|s_1\|}{|1+yz+t^2|} = \frac{1}{\max(2+y^2+z^2,t^2)} = \frac{1}{2+y^2+z^2}$$

(since $t=0$ on D) and

$$\lim \frac{\|s_2\|}{|t|} = \frac{1}{\max(2+y^2+z^2,t^2)^{1/2}} = \frac{1}{(2+y^2+z^2)^{1/2}}.$$

Finally,

$$\begin{aligned}\|\mathrm{d}z\| &= \|z\mathrm{d}y\wedge \mathrm{d}z\wedge \mathrm{d}t\| \lim \frac{\|s_1\|}{|1+yz+t^2|} \lim \frac{\|s_2\|}{|t|} \\ &= |z|\max(2+y^2+z^2,t^2)^{1/2} = |z|(2+(1/z)^2+z^2)^{1/2} \\ &= (1+2z^2+z^4)^{1/2} = 1+z^2,\end{aligned}$$

so that the canonical measure on $D(\mathbf{R})$ is given by $|\mathrm{d}z|/(1+z^2)$. Hence, $\operatorname{vol} D(\mathbf{R}) = \pi$.

5.5.6. Finally, we obtain that Shah's constant C_{X^2+1} satisfies

$$C_{X^2+1} = 2\operatorname{vol} D(\mathbf{R})\operatorname{vol} U(\mathbb{A}_f),$$

compatibly with Theorem 4.7.

Appendix A. Tauberian Theorems

Let (X, μ) be a measured space and f a positive measurable function on X. Define

$$Z(s) = \int_X f(x)^{-s}\, d\mu(x) \quad \text{and} \quad V(B) = \mu(\{f(x) \leq B\}).$$

One has

$$Z(s) = \int_0^{+\infty} B^{-s}\, dV(B).$$

Theorem A.1. *Let a be a real number; we assume that $Z(s)$ converges for $\mathrm{Re}(s) > a$ and extends to a meromorphic function A.1 in the neighborhood of the closed half-plane $\mathrm{Re}(s) \geq a - \delta$, for some positive real number δ.*

If $a < 0$, then $V(B)$ has the limit $Z(0)$ when $B \to \infty$.

Furthermore, assume that:

(1) *Z has a pole of order b at $s = a$ and no other pole in the half-plane $\mathrm{Re}(s) \geq a - \delta$.*
(2) *Z has moderate growth in vertical strips*, i.e. *there exists a positive real number κ such that for any $\tau \in \mathbf{R}$,*

$$|Z(a - \delta + i\tau)| \ll (1 + |\tau|)^{\kappa}.$$

Then, there exist a monic polynomial P, a real number Θ and a positive real number ε such that, when $B \to \infty$,

$$V(B) = \begin{cases} \Theta B^a P(\log B) + \mathrm{O}(B^{a-\varepsilon}) & \text{if } a \geq 0; \\ Z(0) + \Theta B^a P(\log B) + \mathrm{O}(B^{a-\varepsilon}) & \text{if } a < 0. \end{cases}$$

Moreover, if $a \neq 0$, then

$$\deg(P) = b - 1 \quad \textit{and} \quad \Theta a (b-1)! = \lim_{s \to a} (s-a)^b Z(s)$$

while

$$\deg(P) = b \quad \textit{and} \quad \Theta b! = \lim_{s \to a} (s-a)^b Z(s)$$

if $a = 0$.

For any integer $k \geq 0$, let us define

$$V_k(B) = \frac{1}{k!} \int_{f(x) \leq B} \left(\log \frac{B}{f(x)} \right)^k d\mu(x) = \frac{1}{k!} \int_X \left(\log^+ \frac{B}{f(x)} \right)^k d\mu(x),$$

where $\log^+(u) = \max(0, \log u)$ for any positive real number u.

Lemma A.2. *Let k be an integer such that $k > \kappa$. Then there exist polynomials P and Q with real coefficients, a real number $\delta > 0$ such that*

$$V_k(B) = B^a P(\log B) + Q(\log B) + \mathrm{O}(B^{a-\delta}).$$

Moreover, the polynomials P and Q satisfy

$$P(T) = \begin{cases} \dfrac{1}{a^{k+1}(b-1)!}\Theta T^{b-1} + \cdots & \text{if } a \neq 0, \\ 0 & \text{if } a = 0; \end{cases} \tag{A.1}$$

$$Q(T) = \begin{cases} 0 & \text{if } a > 0; \\ \dfrac{1}{(b+k)!}\Theta T^{b+k} + \cdots & \text{if } a = 0; \\ \dfrac{1}{k!} Z(0) T^k + \dots & \text{if } a < 0. \end{cases} \tag{A.2}$$

Proof. We begin with the classical integral

$$\int_{\sigma+i\mathbf{R}} \lambda^s \frac{\mathrm{d}s}{s^{k+1}} = \frac{2i\pi}{k!} (\log^+(\lambda))^k,$$

where σ and λ are positive real numbers. For $\sigma > \max(0, a)$, this implies that

$$\begin{aligned} V_k(B) &= \frac{1}{k!} \int_X \log^+(B/f(x))^k \, \mathrm{d}\mu(x) \\ &= \int_X \frac{1}{2i\pi} \int_{\sigma+i\mathbf{R}} (B/f(x))^s \frac{\mathrm{d}s}{s^{k+1}} \, \mathrm{d}\mu(x) \\ &= \frac{1}{2i\pi} \int_{\sigma+i\mathbf{R}} B^s Z(s) \, \frac{\mathrm{d}s}{s^{k+1}}, \end{aligned}$$

where the written integrals converge absolutely. We now move the contour of integration to the left of the pole $s = a$, the estimates for $Z(s)$ in vertical strips allowing us to apply the residue theorem. Only $s = a$ and $s = 0$ may give a pole within the two vertical lines $\mathrm{Re}(s) = \sigma$ and $\mathrm{Re}(s) = a - \delta$ and we obtain

$$V_k(B) = \sum_{\substack{u \in \{0,a\} \\ a-\delta < u < \sigma}} \mathrm{Res}_{s=u} \left(\frac{B^s Z(s)}{s^{k+1}} \right) + \frac{1}{2i\pi} \int_{a-\delta+i\mathbf{R}} B^s Z(s) \, \frac{\mathrm{d}s}{s^{k+1}}.$$

If $a > 0$ and $a - \delta > 0$, or if $a < 0$, one checks that there exists a polynomial P of degree $b - 1$ and of leading coefficient $\Theta/a^{k+1}(b-1)!$ such that

$$\mathrm{Res}_{s=a} \left(\frac{B^s Z(s)}{s^{k+1}} \right) = B^a P(\log B).$$

In the case $a > 0$, moreover we set $Q = 0$.

If $a = 0$, let us set $P = 0$. There exists a polynomial Q of degree $b+k$ and of leading coefficient $\Theta/(b+k)!$ such that

$$\mathrm{Res}_{s=0}\left(\frac{B^s Z(s)}{s^{k+1}}\right) = Q(\log B).$$

Finally, if $a < 0$, one verifies that there exists a polynomial Q of degree k and of leading coefficient $Z(0)/k!$ such that

$$\mathrm{Res}_{s=0}\left(\frac{B^s Z(s)}{s^{k+1}}\right) = Q(\log B).$$

This implies the lemma. ◻

Lemma A.3. *Assume that there are polynomials P and Q, and a positive real number δ, such that*

$$V_k(B) = B^a P(\log B) + Q(\log B) + \mathrm{O}(B^{a-\delta}),$$

where we moreover assume that $P = 0$ if $a = 0$ and $Q = 0$ if $a > 0$. Then for any positive real number δ' such that $\delta' < \delta/2$, one has the asymptotic expansion

$$V_{k-1}(B) = B^a(aP(\log B) + P'(\log B)) + Q'(\log B) + \mathrm{O}(B^{a-\delta'}).$$

Proof. For any $u \in (-1, 1)$, one has

$$\begin{aligned}
V_k(B(1+u)) - V_k(B) &= B^a(P(\log(B(1+u))) - P(\log B)) \\
&\quad + Q(\log(B(1+u))) - Q(\log B) + \mathrm{O}(B^{a-\delta}) \\
&= B^a P(\log B)((1+u)^a - 1) \\
&\quad + B^a(1+u)^a(P(\log B + \log(1+u)) - P(\log B)) \\
&\quad + (Q(\log B + \log(1+u)) - Q(\log B)) + \mathrm{O}(B^{a-\delta}) \\
&= B^a P(\log B)(au + \mathrm{O}(u^2)) + B^a P'(\log B)u \\
&\quad + B^a \mathrm{O}(\log B)^{\deg P - 2} u^2 \\
&\quad + Q'(\log B)u + \mathrm{O}((\log B)^{\deg Q - 2} u^2) + \mathrm{O}(B^{a-\delta}).
\end{aligned}$$

Let ε be a positive real number; if $u = \pm B^{-\varepsilon}$, we obtain

$$\begin{aligned}
\frac{V_k(B(1+u)) - V_k(B)}{\log(1+u)} &= B^a(aP(\log B) + P'(\log B)) + \tilde{\mathrm{O}}(B^{a-\varepsilon}) \\
&\quad + Q'(\log B) + \tilde{\mathrm{O}}(B^{-\varepsilon}) + \mathrm{O}\big(B^{a-\delta+\varepsilon}\big),
\end{aligned}$$

where the $\tilde{\mathrm{O}}$ notation indicates unspecified powers of $\log B$. If $\delta' < \delta/2$, we may choose ε such that $\delta' < \varepsilon < \delta/2$ and then,

$$\begin{aligned}\frac{V_k(B(1+u)) - V_k(B)}{\log(1+u)} &= B^a(aP(\log B) + P'(\log B)) \\ &\quad + Q'(\log B) + \mathrm{O}(B^{a-\delta'}).\end{aligned} \tag{A.3}$$

On the other hand, for any real number u such that $0 < u < 1$, and any positive real number A, one has

$$\frac{\log(A(1-u))^k - \log(A)^k}{\log(1-u)} \le k\log(A)^{k-1} \le \frac{\log(A(1+u))^k - \log(A)^k}{\log(1+u)},$$

which implies the inequality

$$\frac{V_k(B(1-u)) - V_k(B)}{\log(1-u)} \le V_{k-1}(B) \le \frac{V_k(B(1+u)) - V_k(B)}{\log(1+u)}. \tag{A.4}$$

Substituting in the estimates of Eq. (A.3), we deduce the asymptotic expansion

$$V_{k-1}(B) = B^a(aP(\log B) + P'(\log B)) + Q'(\log B) + \mathrm{O}(B^{a-\delta'}), \tag{A.5}$$

as claimed. □

Proof. (Proof of Theorem A.1) We can now prove our Tauberian theorem. Let k be any integer such that $k > \kappa$. Lemma A.2 implies an asymptotic expansion for $V_k(B)$; let P, Q, δ be as in this lemma. Applying successively k times Lemma A.3, we obtain the existence of an asymptotic expansion for $V(B)$ of the form

$$V(B) = B^a D_a^k P(\log B) + D_0^k Q(\log B) + \mathrm{O}(B^{a-\delta'}),$$

for some positive real number δ' (any positive real number δ' such that $\delta' < \delta/2^k$ is suitable), where we have denoted by D_a and D_0 the differential operators $P \mapsto aP + P'$ and $P \mapsto P'$.

For $a \neq 0$, the operator D_a does not change the degree but multiplies the leading coefficient by a. Consequently,

$$D_a^k P(T) = \begin{cases} \dfrac{1}{a(b-1)!}\Theta^{b-1} + \cdots & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$

Similarly, the operator D decreases the degree by 1 and multiplies the leading coefficient by the degree. It follows that

$$D_0^k Q(T) = \begin{cases} Z(0) & \text{if } a < 0; \\ \dfrac{1}{b!}\Theta T^b + \cdots & \text{if } a = 0; \\ 0 & \text{if } a > 0. \end{cases}$$

Theorem A.1 now follows easily. □

In ultrametric contexts, the function f usually takes values of the form q^n, with $n \in \mathbf{Z}$, where q is a real number such that $q > 1$. In that case, the function Z is $(2i\pi)/\log q$-periodic, its poles form arithmetic progressions of common difference $2i\pi/\log q$, and Theorem A.1 does not apply.

Let q be a real number with $q > 1$ and assume that $f(x) \in q^{\mathbf{Z}}$ for any $x \in X$.

Let a be a non-negative real number; let us assume that $Z(s)$ converges for $\mathrm{Re}(s) > a$ and extends to a meromorphic function in the neighborhood of the closed half-plane $\mathrm{Re}(s) \geq a - \delta$, for some positive real number δ.

Suppose furthermore that the poles of Z belong to finitely many arithmetic progressions of the form $a_j + \frac{1}{\log q} 2i\pi\mathbf{Z}$, where $a_1, \dots, a_t \in \mathbf{C}$ are complex numbers of real part a, not two of them being congruent mod $2i\pi/\log q$; let $b_j = \mathrm{ord}_{s=a_j} Z(s)$ and let $c_j = \lim_{s\to a_j} (s - a_j)^{b_j} Z(s)$.

Let us make the change of variable $u = q^{-s}$ and set $Z(s) = \Phi(u)$. The function Φ is defined by

$$\Phi(u) = \sum_{n\in\mathbf{Z}} Z_n u^n, \quad \text{where } Z_n = \mu(\{f(x) = q^n\});$$

it is defined for $0 < |u| < q^{-a}$ and is holomorphic in that domain; moreover, it extends to a meromorphic function on the domain $0 < |u| < q^{\delta-a}$, with poles at q^{-a_j} (for $1 \leq j \leq t$) such that

$$\lim_{u\to q^{-a_j}} (1 - q^{a_j} u)^{b_j} \Phi(u) = \lim_{s\to a_j} (1 - q^{a_j - s})^{b_j} Z(s) = (\log q)^{b_j} c_j.$$

Consequently, there are polynomials p_j of degree b_j and leading coefficient $(\log q)^{b_j} c_j$ such that

$$Z(u) - \sum_{j=1}^{t} \frac{p_j(u)}{(1 - q^{a_j} u)^{b_j}}$$

is holomorphic in the domain defined by $0 < |u| < q^{\delta-a}$. Therefore, the Cauchy formula implies an asymptotic expansion of the form

$$Z_n = \sum_{j=1}^{t} P_j(n) q^{a_j n} + \mathrm{O}(q^{(a-\delta')n}),$$

where δ' is any positive real number such that $0 < \delta' < \delta$ and, for $1 \leq j \leq t$, P_j is a polynomial of degree $b_j - 1$ and of leading coefficient $c_j (\log q)^{b_j}/(b_j - 1)!$.

Since the Laurent series defining Φ has non-negative coefficients, the inequality $|Z(u)| \leq Z(|u|)$ holds for any $u \in \mathbf{C}$ such that $0 < |u| < q^{-a}$. In particular, we see, as is well known, that Φ has a pole on its circle of convergence. This means that, unless $t = 0$, we can assume that $a_1 = a$ and that $b_j \leq b_1$ for all j.

We are not interested in the sequence (Z_n) itself, but rather on the sums $V(q^n) = \sum_{m\leq n} Z_n$, when $n \to \infty$. (Their convergence follows from the fact that $\Phi(u)$ converges for arbitrary small nonzero complex numbers u.)

We begin by observing that for any integer j, there exists a polynomial Q_j such that

$$P_j(m) = Q_j(m) - q^{-a_j} Q_j(m-1)$$

for any $m \in \mathbf{Z}$. Moreover, if $q^{a_j} \neq 1$, then there is only one polynomial satisfying these relations, its degree satisfies $\deg(Q_j) = \deg(P_j)$ while its leading coefficient is equal to $\operatorname{lcoeff}(P_j)/(1-q^{-a_j})$; however, if $q^{a_j} = 1$, then $\deg(Q_j) = \deg(P_j) + 1$ and $\operatorname{lcoeff}(Q_j) = \operatorname{lcoeff}(P_j)/(\deg(P_j)+1)$.

Let us now separate the discussion according to the value of a.

Case $a < 0$. Then, $V(q^n)$ has the limit $\Phi(1)$ when $n \to +\infty$, and

$$V(q^n) = \Phi(1) - \sum_{m>n} Z_n = \Phi(1) - \sum_{j=1}^{t} \sum_{m>n} q^{a_j m} P_j(m) + \mathrm{O}(q^{(a-\delta')n}),$$

provided that $\delta' < \delta$ is chosen so that $a < a + \delta' < 0$. Since

$$\sum_{m>n} q^{a_j m} P_j(m) = \sum_{m>n} (q^{a_j m} Q_j(m) - q^{a_j(m-1)} Q_j(m-1)) = -q^{a_j n} Q_j(n),$$

we obtain that

$$V(q^n) = \Phi(1) + \sum_{j=1}^{t} q^{a_j n} Q_j(n) + \mathrm{O}(q^{(a-\delta')n}). \tag{A.6}$$

Case $a > 0$. In this case, we have

$$V(q^n) = \sum_{j=1}^{t} \sum_{m \leq n} (q^{a_j m} Q_j(m) - q^{a_j(m-1)} Q_j(m-1)) + \mathrm{O}(q^{a-\delta' n}),$$

so that

$$V(q^n) = \sum_{j=1}^{t} q^{a_j n} Q_j(n) + \mathrm{O}(q^{a-\delta'} n). \tag{A.7}$$

Moreover, $\deg(Q_j) = b_j$ for all j.

Case $a = 0$. Then $V(q^n)$ also satisfies the asymptotic expansion (A.7). However, $\deg Q_1 = b_1 + 1$ and $\deg Q_j \leq b_j \leq b_1$ for $j \neq 1$, and we obtain

$$\lim_{n\to\infty} V(q^n) n^{-b_1-1} \operatorname{lcoeff}(Q_1) = \frac{1}{b_1+1} \operatorname{lcoeff}(P_1) = c_1 \frac{(\log q)^{b_1}}{b_1+1}. \tag{A.8}$$

Corollary A.4. *Let us retain the previous hypotheses, assuming moreover that $a > 0$. Then, we have the following weak asymptotic behavior:*

$$0 < \liminf V(q^n) q^{-an} n^{-b_1} \leq \limsup V(q^n) q^{-an} n^{-b_1} < \infty.$$

Corollary A.5. *Let us assume that for some positive integers b and d, the function $Z(s)(1-q^{(a-s)d})^b$ has a holomorphic expansion in some neighborhood of the half-plane $\{\mathrm{Re}(s) \geq a - \delta\}$. Then, where n is restricted to belong to any arithmetic progression mod. d, the sequence $(V(q^n) q^{-na} \log(q^n)^{-b})$ has a limit.*

Proof. The assumptions allow us to set $t = d$ and $a_j = a + 2i\pi(j-1)/d\log(q)$, for $1 \leq j \leq d$, and $b_j = b$. Let us fix $m \in \mathbf{N}$ and let us write $n = m + kd$, where $k \in \mathbf{N}$ goes to infinity. For $n \in \mathbf{N}$, we may write

$$V(q^n)q^{-na}\log(q^n)^{-b}\sum_{j=1}^{d} q^{(a_j-a)n}\frac{Q_j(n)}{(n\log q)^b} + \mathrm{O}(q^{-\delta' n}).$$

The asserted convergences follow from the fact the observation that for any j, $q^{(a_j-a)n}$ is a dth root of unity. More precisely, we find

$$\lim_{\substack{n\to\infty\\ n\equiv n_0 \pmod d}} V(q^n)q^{-na}\log(q^n)^{-b} = \frac{1}{(b-1)!}\sum_{j=1}^{d}\exp(2i\pi(j-1)/d)\frac{c_j}{1-q^{-a_j}},$$

where

$$c_j = \lim_{s\to a_j}(s-a_j)^b Z(s). \qquad \square$$

The following corollary has been inspired by the recent paper [17] by Cluckers, Comte and Loeser.

Corollary A.6. *Let $V^*(q^n)$ be the Cesáro mean of $V(q^n)$, namely*

$$V^*(q^n) = \frac{1}{n+1}\sum_{m=0}^{n} V(q^m).$$

If $a > 0$, then V^* satisfies

$$\lim_{n\to\infty} V^*(q^n)q^{-an}n^{-b_1} = c_1\frac{(\log q)^{b_1}}{1-q^{-a}}.$$

If $a < 0$, then

$$\lim_{n\to\infty}(V^*(q^n) - Z(0))q^{-an}n^{-b_1} = c_1\frac{(\log q)^{b_1}}{1-q^{-a}}.$$

Proof. This follows from the main result and the fact that for any complex number $z \in \mathbf{C}$ such that $|z| = 1$ but $z \neq 1$, the sequence (z^n) Cesáro-converges to 0. $\square$

We conclude this Appendix by a Tauberian result which is useful in S-integral contexts.

Theorem A.7. *Let a be a real number; we assume that $Z(s)$ converges to a holomorphic function for $\mathrm{Re}(s) > a$. Let us furthermore assume that it has a meromorphic continuation of the following form: there exists a positive integer $b \geq 1$ and a finite family $(q_j, b_j)_{j\in J}$ where q_j is a real number such that $q_j > 1$ and b_j is an integer satisfying $1 \leq b_j \leq b-1$ such that, setting $b_0 = b - \sum_{j\in J} b_j$, the function Z_0 defined by*

$$Z_0(s) = Z(s)\left(\frac{s-a}{s-a+1}\right)^{b_0}\prod_{j\in J}(1-q_j^{s-a})^{b_j}$$

extends to a holomorphic function with moderate growth in vertical strips, i.e. *there exists a positive real number κ such that for any $\tau \in \mathbf{R}$,*

$$|Z(a-\delta+\mathrm{i}\tau)| \ll (1+|\tau|)^{\kappa}.$$

Assume also that for any two $j, j' \in J, \log q_j / \log q_{j'}$ is not a Liouville number.

Then, there exist a monic polynomial P, a real number Θ and a positive real number ε such that, when $B \to \infty$,

$$V(B) = \begin{cases} \Theta B^a P(\log B) + \mathrm{O}(B^a (\log B)^{\max(b_j)}) & \text{if } a \geq 0; \\ Z(0) + \Theta B^a P(\log B) + \mathrm{O}(B^a (\log B)^{\max(b_j)}) & \text{if } a < 0. \end{cases}$$

Moreover, if $a \neq 0$, then

$$\deg(P) = b-1 \quad \text{and} \quad \Theta a(b-1)! = \lim_{s\to a} (s-a)^b Z(s)$$

while

$$\deg(P) = b \quad \text{and} \quad \Theta b! = \lim_{s\to a} (s-a)^b Z(s)$$

if $a = 0$.

Proof. For any integer $k \geq 0$, let us define

$$V_k(B) = \frac{1}{k!} \int_{f(x)\leq B} \left(\log \frac{B}{f(x)}\right)^k d\mu(x) = \frac{1}{k!} \int_X \left(\log^+ \frac{B}{f(x)}\right)^k d\mu(x),$$

where $\log^+(u) = \max(0, \log u)$ for any positive real number u.

As in Theorem A.1, we begin by proving an asymptotic expansion for $V_k(B)$, where k is an integer satisfying $k > \kappa$. As above, we have

$$V_k(B) = \frac{1}{2i\pi} \int_{\sigma+i\mathbf{R}} B^s Z(s) \frac{\mathrm{d}s}{s^{k+1}},$$

for $\sigma > a$, and we will move the line of integration to the left of $s = a$, the novelty being the presence of infinitely many poles on the line $\mathrm{Re}(s) = a$, namely at any complex number of the form $\alpha_{j,m} = a + 2\mathrm{i}m\pi/\log q_j$, for some $j \in J$ and some integer $m \in \mathbf{Z}$.

Let F_k be the holomorphic function given by $F_k(s) = B^s Z(s)/s^{k+1}$. Let μ be any common irrationality measure for the real numbers $\log q_j / \log q_{j'}$, namely a real number such that for any two integers m and m' such that $m \log q_j + m' \log q_{j'} \neq 0$, we have

$$|m \log q_j + m' \log q_{j'}| \geq \max(|m|, |m'|)^{-\mu}.$$

A straightforward computation based on Leibniz and Cauchy formulae shows the existence of a positive real number c such that for any $j \in J$ and any nonzero $m \in \mathbf{Z}$,

$$|\mathrm{Res}_{s=\alpha j,m} F_k(s)| \leq c(1 + |\mathrm{Im}\, \alpha_{j,m}|)^{\kappa-k-1-\mu\beta} B^a (\log B)^{\max(b_j)},$$

where $\beta = \sum_{j\in J} b_j$. Moreover, as in the proof of Lemma A.2, there exists a polynomial P of degree $\deg(P) = b - 1$ with leading coefficient $\Theta/a^{k+1}(b-1)!$ such that

$$\operatorname{Res}_{s=a} F_k(s) = B^a P(\log B).$$

We choose k such that $k > \kappa + \mu\beta$. Since $b > \max(b_j)$, these upper bounds imply that the sum of residues of F_k at all poles of real part a is dominated by the residue at $a = 0$ and satisfies the following estimate

$$\sum_{\operatorname{Re}(s)=a} \operatorname{Res} F_k(s) = B^a P(\log B).$$

If $a > 0$, we set $Q = 0$; if $a \le 0$, there exist a polynomial Q of degree $b + k$ and leading coefficient $\Theta/(b+k)!$ such that

$$\operatorname{Res}_{s=0} F_k(s) = Q(\log B).$$

We may then continue as in the proof of Lemma A.2 and conclude that

$$V_k(B) = B^a P(\log B) + Q(\log B) + \mathrm{O}(B^a(\log B)^\beta).$$

An application of Lemma A.3 similar to that of Theorem A.1 then implies Theorem A.7. □

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References

1. A. Baker, *Transcendental Number Theory* (Cambridge Univ. Press, 1975).
2. V. V. Batyrev, Birational Calabi–Yau n-folds have equal Betti numbers, in *New Trends in Algebraic Geometry* (Warwick, 1996) (Cambridge Univ. Press, 1999), pp. 1–11.
3. V. V. Batyrev and Yu. I. Manin, Sur le nombre de points rationnels de hauteur bornée des variétés algébriques, *Math. Ann.* **286** (1990) 27–43.
4. V. V. Batyrev and Yu. Tschinkel, Rational points on bounded height on compactifications of anisotropic tori, *Internat. Math. Res. Notices* **12** (1995) 591–635.
5. ——— , Height zeta functions of toric varieties, *J. Math. Sci.* **82** (1996) 3220–3239.
6. B. J. Birch, Forms in many variables, *Proc. London Math. Soc.* **265A** (1962) 245–263.
7. M. Borovoi and Z. Rudnick, Hardy–Littlewood varieties and semisimple groups, *Invent. Math.* **119** (1995) 37–66.

8. R. Brauer, On Artin's L-series with general group characters, *Ann. Math.* **48** (1947) 502–514.
9. R. de la Bretèche, Compter des points d'une variété torique, *J. Number Th.* **87** (2001) 315–331.
10. M. Brion and S. Kumar, *Frobenius Splitting Methods in Geometry and Representation Theory*, Progress in Mathematics, Vol. 231 (Birkhäuser, 2005).
11. F. Bruhat and J. Tits, Groupes réductifs sur un corps local, I. Données radicielles valuées, *Publ. Math. Inst. Hautes Études Sci.* (1972) 5–251.
12. A. Chambert-Loir and Yu. Tschinkel, Points of bounded height on equivariant compactifications of vector groups, II, *J. Number Th.* **85** (2000) 172–188.
13. ——— , On the distribution of points of bounded height on equivariant compactifications of vector groups, *Invent. Math.* **148** (2002) 421–452.
14. ——— , Integral points of bounded height on partial equivariant compactifications of vector groups, arXiv:0912.4751.
15. ——— , Integral points of bounded height on toric varieties, arXiv:1006.3345.
16. H. Clemens, Picard–Lefschetz theorem for families of nonsingular algebraic varieties acquiring ordinary singularities, *Trans. Amer. Math. Soc.* **136** (1969) 93–108.
17. R. Cluckers, G. Comte and F. Loeser, Local metric properties and regular stratifications of p-adic definable sets, arXiv:0910.0799.
18. J.-L. Colliot-Thélène and J.-J. Sansuc, La descente sur les variétés rationnelles. II, *Duke Math. J.* **54** (1987) 375–492.
19. C. De Concini and C. Procesi, Complete symmetric varieties, in *Invariant Theory* (Montecatini, 1982), Lecture Notes in Math., Vol. 996 (Springer, 1983), pp. 1–44.
20. P. Deligne, La conjecture de Weil, I, *Publ. Math. Inst. Hautes Études Sci.* **43** (1974) 273–307.
21. J. Denef, On the degree of Igusa's local zeta function, *Amer. J. Math.* **109** (1987) 991–1008.
22. W. Duke, Z. Rudnick and P. Sarnak, Density of integer points on affine homogeneous varieties, *Duke Math. J.* **71** (1993) 143–179.
23. A. Eskin and C. McMullen, Mixing, counting and equidistribution on Lie groups, *Duke Math. J.* **71** (1993) 181–209.
24. A. Eskin, S. Mozes and N. Shah, Unipotent flows and counting lattice points on homogeneous varieties, *Ann. Math.* **143** (1996) 253–299.
25. J. Franke, Yu. I. Manin and Yu. Tschinkel, Rational points of bounded height on Fano varieties, *Invent. Math.* **95** (1989) 421–435.
26. G. L. Gordon, On a simplicial complex associated to the monodromy, *Trans. Amer. Math. Soc.* **261** (1980) 93–101.
27. A. Gorodnik, F. Maucourant and H. Oh, Manin's and Peyre's conjectures on rational points and adelic mixing, *Ann. Sci. École Norm. Sup.* **41** (2008) 385–437.
28. A. Gorodnik, H. Oh and N. Shah, Integral points on symmetric varieties and Satake compatifications, *Amer. J. Math.* **131** (2009) 1–57.
29. B. Hassett and Yu. Tschinkel, Integral points and effective cone of moduli spaces of stable maps, *Duke Math. J.* **120** (2003) 577–599.
30. M. Hindry and J. H. Silverman, *Diophantine Geometry. An Introduction*, Graduate Texts in Mathematics, Vol. 201 (Springer, 2000).
31. J.-I. Igusa, Complex powers and asymptotic expansions. I. Functions of certain types, *J. Reine Angew. Math.* **268/269** (1974), 110–130, Collection of articles dedicated to Helmut Hasse on his 75th birthday, II.
32. ——— , Complex powers and asymptotic expansions. II. Asymptotic expansions, *J. Reine Angew. Math.* **278/279** (1975) 307–321.

33. F. Knop, The Luna-Vust theory of spherical embeddings, in *Proc. of the Hyderabad Conference on Algebraic Groups* (Hyderabad, 1989) (Madras) (Manoj Prakashan, 1991), pp. 225–249.
34. S. Lang, *Fundamentals of Diophantine Geometry* (Springer, 1983).
35. S. Lang and A. Weil, Number of points of varieties in finite fields, *Amer. J. Math.* **76** (1954) 819–827.
36. F. Maucourant, Homogeneous asymptotic limits of haar measures of semisimple linear groups and their lattices, *Duke Math. J.* **136** (2007) 357–399.
37. J. S. Milne, *Étale Cohomology*, Math. Notes, No. 33 (Princeton Univ. Press, 1980).
38. E. Peyre, Hauteurs et mesures de Tamagawa sur les variétés de Fano, *Duke Math. J.* **79** (1995) 101–218.
39. ——— (éd.), *Nombre et Répartition des Points de Hauteur Bornée*, Astérisque, No. 251, 1998.
40. ——— , Terme principal de la fonction zêta des hauteurs et torseurs universels, in *Nombre et Répartition des Points de Hauteur Bornée* [39], pp. 259–298.
41. P. Salberger, Tamagawa measures on universal torsors and points of bounded height on Fano varieties, in *Nombre et Répartition des Points de Hauteur Bornée* [39], pp. 91–258.
42. J.-P. Serre, Quelques applications du théorème de densité de Chebotarev, *Inst. Hautes Études Sci. Publ. Math.* (1981), 323–401.
43. ——— , *Lectures on the Mordell–Weil Theorem*, 3rd edn., Aspects of Mathematics (Friedr. Vieweg & Sohn, 1997) [translated from the French and edited by Martin Brown from notes by Michel Waldschmidt, With a foreword by Brown and Serre].
44. N. A. Shah, Counting integral matrices with a given characteristic polynomial, *Sankhyā Ser. A* **62** (2000) 386–412, Ergodic theory and harmonic analysis (Mumbai, 1999).
45. J. Shalika, R. Takloo-Bighash and Yu. Tschinkel, Rational points on compactifications of semi-simple groups, *J. Amer. Math. Soc.* **20** (2007) 1135–1186 (electronic).
46. A. Skorobogatov, *Torsors and Rational Points*, Cambridge Tracts in Mathematics, Vol. 144 (Cambridge Univ. Press, 2001).
47. A. Weil, *Adeles and Algebraic Groups*, Progr. Math., No. 23 (Birkhäuser, 1982).
48. ——— , *Basic Number Theory* (Springer, 1995), reprint of the 2nd (1973) edition.