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THE EXPECTED CUMULATIVE OPERATIONAL TIME FOR FINITE SEMI-MARKOV SYSTEMS AND ESTIMATION

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Abstract. In this paper we, firstly, present a recursive formula of the empirical estimator of the semi-Markov kernel. Then a non-parametric estimator of the expected cumulative operational time for semi-Markov systems is proposed. The asymptotic properties of this estimator, as the uniform strongly consistency and normality are given. As an illustration example, we give a numerical application.

Keywords. Expected cumulative operational time, Semi-Markov process, Non-parametric Estimation.

Mathematics Subject Classification. 60K20.

1. Introduction

Semi-Markov models are common modelling tools in the analysis of machines subject to stochastic failures (*cf.* Limnios and Oprishan [4]). In this paper we consider systems with finite number of states and random holding times in each state. This consideration relaxes the exponential assumption and provides a rich class of models applicable in reliability, maintenance studies and survival analysis (*cf.* Janssen and Limnios [16]). However, in practice, data analysis for semi-Markov processes can be quite difficult. There are many competing models that relax the exponential assumption in modelling complex multistate systems in both reliability and survival analysis (see i.e., Huzurbazar [2]). In this paper, a recursive formula

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to estimate the semi-Markov kernel is proposed. The cumulative operational time $Co(t), t > 0$ of semi-Markov systems is the total time spent by the process in the set of up states during the time interval $[0, t]$. This indicator is of great interest in maintenance studies because it can be used to minimize the expected cost of the maintenance process by choosing the appropriate actions. The distribution of the cumulative operational time has been the subject of many papers (Smith *et al.* [12], Kulkarni *et al.* [3]). A closed form expression for the cumulative distribution function of $Co(t)$ in the semi-Markov case under the additional assumption that the sequences of operational and unoperational periods are independent was given by Rubino and Sericola [15]. Csenski [13], described a method for computing the cumulative operational time for semi-Markov processes.

To our knowledge there is no work on the estimation of the expected cumulative operational time of a semi-Markov system. In this paper we are concerned by developing an algorithm to estimate the expected cumulative operational time $E[Co(t)], t > 0$ of the semi-Markov systems. It is seen that this estimator is uniformly strongly consistent and converges weakly to a zero mean normal random variable.

This article begins by describing the model. In Section 3 we present a recursive method to estimate the embedded Markov transition matrix and the distribution of the holding time which in turns allow us to give an algorithm to estimate the semi-Markov kernel. In Section 4, we give the expression of the expected cumulative operational time $E[Co(t)], t > 0$ of the semi-Markov systems. Then we propose an estimator of $E[Co(t)], t > 0$. This estimator is seen to be consistent and to converge to a normal random variable as the time of observation becomes large. Finally, Section 5, presents a numerical example of a three state semi-Markov process.

2. Preliminaries

Consider a Markov renewal process (MRP) $(J, S) = (J_n, S_n)_{n \geq 0}$ defined on a probability complete space, where J_n is a Markov chain with values in $E = \{1, ..., s\}$, the state space of the process and S_n the jump times process. The random variables J_0,J_1, ...,J_n,... are the consecutive states to be visited by the MRP and $X_1, X_2, ...$ defined by $X_0 = 0$ and $X_n = S_n - S_{n-1}$, for $n \geq 1$, are the sojourn times in these states taking values in $[0, \infty)$.
A MRP can be completely determined if we know its initial law and its transition probabilities defined respectively by :$P(J_0 = k) = p(k)$ and

$$P[J_{n+1} = k, X_{n+1} \leq x | J_0, J_1, ..., J_n, X_1, X_2, ..., X_n] = Q_{J_n k}(x) \qquad (a.s.)$$

for all $x \in [0, +\infty)$ and $1 \leq k \leq s$.
The probabilities $p_{ij} = Q_{ij}(\infty)$ $(= \lim_{t \to +\infty} Q_{ij}(t))$ are the transition probabilities of the embedded Markov chain J_n.
Let us, also consider the distribution function associated to sojourn time in state

i before going to state j defined by:

$$F_{ij}(.) = \begin{cases} p_{ij}^{-1} \times Q_{ij}(.) & \text{if } p_{ij} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

So, we have:

$$P[J_n = j/J_0, J_1, ...J_{n-1} = i] = p_{ij} \quad \text{for all} \quad n > 0.$$

$$P[X_n \le x/J_0, ...J_{n-1} = i, J_n = j] = F_{ij}(x) \text{ for all } n \ge 0 \text{ and } x \ge 0.$$

$$P[X_{n_1} \le x_1, X_{n_2} \le x_2, ...X_{n_k} \le x_k/J_n, n \ge 0] = \prod_{i=1}^{k} F_{J_{n_i-1}J_{n_i}}(x_i) \qquad (a.s.)$$

for $0 \le n_1 \le n_2 \le ... \le n_k$ and $x_i \ge 0$ for $i = 1, ..., k$.

The Markov renewal matrix, $\psi(t)$ is defined by

$$\psi(t) = E[N(t)] = \sum_{l=0}^{\infty} Q^{(l)}(t),$$

where $N(t)$ is the counting process of transitions of the process up to time t and $Q_{ij}^{(1)}(t) = Q_{ij}(t)$ and for $l > 1$, $Q^{(l)}(t)$ is the lth convolution of $Q(t)$ in the Stieltjes sense and

$$Q_{ij}^{(0)}(.) = \begin{cases} 1_{\{i=j\}}(t) & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let us recall that since the state space, E, is finite, $\psi(t)$ is element wise finite for every $t \ge 0$, *i.e.* the MRP is normal see [8]

e semi-Markov transition matrix function of the semi-Markov process, $(Z_t)_{t\ge 0}$, is defined by:

$$P_{ij}(t) = P[Z_t = j|Z_0 = i] = P[J_{N_t} = j|J_0 = i] \qquad i,\ j \in E,$$

It is known, *cf.* [11], that

$$P_{ij}(t) = 1_{\{i=j\}}(1 - \sum_{k=1}^{s} Q_{ik}(t)) + \sum_{k\in E} \int_0^t P_{kj}(t-s) Q_{ik}(ds).$$

By solving the above Markov renewal equation, *cf.* [4], it is seen that, in matrix notation,

$$P(t) = (I - Q(t))^{(-1)} * (I - diag(Q(t)\mathbf{1})), \tag{1}$$

where $diag(\cdot)$ is a diagonal matrix of ith entry $\sum_{j=1}^{s} Q_{ij}(t)$ and $\mathbf{1} = (1, 1, ..., 1)^t$.

The semi-Markov transition matrix is a very useful matrix function in studying the semi-markov processes asymptotic properties and in their applications see [9].

3. Non-parametric estimation of the semi-Markov kernel

In the following, we will perform the non-parametric estimator given by [7], by giving a recursive formula which allows us to make estimation procedures easy in computation.

Let us also define the process $(N(t), t \in \mathbb{R}^+)$, by $N(t) := \sup\{n \geq 0 : S_n \leq t\}$, which counts the number of jumps of Z in the time interval $(0, t]$. Let $N_i(t, t')$ be the number of visits of Z to state $i \in E$ between time t and time t' and let $N_{ij}(t, t')$ be the number of jumps from state i to state j between time t and time t'. In the rest, we will denote $N_i(t) = N_i(0, t)$ and $N_{ij}(t) = N_{ij}(0, t)$. that is,

$$N_i(t) := \sum_{k=1}^{N(t)} \mathbf{1}_{\{J_k = i\}} = \sum_{k=1}^{\infty} \mathbf{1}_{\{J_k = i, S_k \leq t\}},$$
$$N_{ij}(t) := \sum_{k=1}^{N(t)} \mathbf{1}_{\{J_{k-1} = i, J_k = j\}} = \sum_{k=1}^{\infty} \mathbf{1}_{\{J_{k-1} = i, J_k = j, S_k \leq t\}}.$$

Let us suppose that we have one observation in the time interval $[0, T_1]$, $(Z(t), 0 \leq t \leq T_1)$, that is $\{J_0, X_1, \cdots, J_{N(T)}, X_{N(T_1)}, U_{T_1}\}$, where $U_{T_1} := T_1 - S_{N(T_1)}$.
The main problem here is that, when we have observed the history of the process until time T_1, we can get the maximum likelihood estimator of the semi-Markov kernel from [7], but what one can do to actualize the estimator if the data from T_1 to T_2, $T_1 < T_2$, is available? Of course, one can consider the history of the process on $[0, T_2]$, and then do as in [7], but this increases the complexity of calculation. In the sequel of this section we will present a recursive formula to answer this problem.

Proposition 1. *The non-parametric estimator of the transition matrix of the embedded Markov chain and the non-parametric estimator of the distribution of the sojourn time can be actualized in the following manner:*

$$\hat{p}_{ij}(T+1) = A_i(T+1).\hat{p}_{ij}(T) + W_{ij}(T+1),$$
$$\hat{F}_{ij}(t, T+1) = C_{ij}(T+1).\hat{F}_{ij}(t, T) + D_{ij}(t, T+1),$$

where,

$$A_i(T+1) = \frac{N_i(T)}{N_i(T+1)} \quad \text{and} \quad W_{ij}(T+1) = \frac{N_{ij}(T, T+1)}{N_i(T+1)}$$

and

$$C_{ij}(T+1) = \frac{N_{ij}(T)}{N_{ij}(T+1)} \quad \text{and} \quad D_{ij}(t, T+1) = \frac{\sum_{l=N_{ij}(T)+1}^{N_{ij}(T+1)} 1_{\{X_{ijl} \leq t\}}}{N_{ij}(T+1)},$$

where X_{ijl} is the l^{th} sojourn time in state i before going to state j, $N_{ij}(T+1) = N_{ij}(T) + N_{ij}(T, T+1)$ and $N_i(T+1) = N_i(T) + \sum_{j=1}^{s} N_{ij}(T, T+1)$.

Proof. From [7], the non-parametric estimator of the transition matrix of the embedded Markov chain and the non-parametric estimator of the distribution of the sojourn time are given respectively by:

$$\hat{p}_{ij}(T+1) = \frac{N_{ij}(T+1)}{N_i(T+1)}$$
$$\hat{F}_{ij}(t,T+1) = \frac{1}{N_i(T+1)} \sum_{l=1}^{N_{ij}(T+1)} 1_{\{X_{ijl} \le t\}}.$$

For the first part of the proposition, see that :

$$\begin{aligned}
\hat{p}_{ij}(T+1) &= \frac{N_{ij}(T+1)}{N_i(T+1)} \\
&= \frac{N_{ij}(T) + N_{ij}(T,T+1)}{N_i(T) + N_i(T,T+1)} \\
&= \frac{N_i(T)}{N_i(T+1)}.\hat{p}_{ij}(T) + \frac{N_{ij}(T,T+1)}{N_i(T+1)}, \qquad (2)\\
&= A_i(T+1).\hat{p}_{ij}(T) + W_{ij}(T+1),
\end{aligned}$$

the Equation (2) was obtained by dividing the numerator and the denominator by $N_i^a(T)$. For the second part, we have that:

$$\begin{aligned}
\hat{F}_{ij}(t,T+1) &= \frac{1}{N_{ij}(T+1)} \sum_{l=1}^{N_{ij}(T+1)} 1_{\{X_{ijl} \le t\}}, \\
&= \frac{N_{ij}(T)}{N_{ij}(T+1)} \hat{F}_{ij}(t,T+1) + \frac{1}{N_i(T+1)} \sum_{l=N_{ij}(T)+1}^{N_{ij}(T+1)} 1_{\{X_{ijl} \le t\}}, \\
&= C_{ij}(T+1).\hat{F}_{ij}(t,T) + D_{ij}(t,T+1), \qquad (3)
\end{aligned}$$

which is the desired result. □

Remark 1.

(1) From (2) and (3) we get a recursive formula to estimate the semi-Markov kernel
$$\hat{Q}_{ij}(t,T) = \hat{p}_{ij}(T)\hat{F}_{ij}(t,T).$$

(2) Since the process that we consider is recurrently positive, we have that $\hat{p}_{ij}(T) \longrightarrow \nu_j$, the equilibrium measure of the embedded Markov chain, and
$$\max_{i,j \in U} \sup_{t \in \mathbb{R}^+} |\hat{F}_{ij}(t,T) - F_{ij}(t)| \to 0 \qquad \text{a.s.},$$
the estimation algorithm is convergent and we must stop the algorithm when the needed degree of precision is reached.

(3) In Markov case, a similar result was obtained in Boudi [1].

4. Non-parametric estimation of the expected cumulative operational time

In reliability studies, the state space E is usually partitioned into two subsets. The first one, say U, consists in the up states and the second one, say D, consists in the down states. The entrance into a state might correspond to the occurrence of a critical event such component failure due to some cause or repair achivement. We assume here that the system is repairable and thus the process alternates between U and D.

The cumulative operational time is defined by

$$Co(t) = \int_0^t 1_{\{Z_u \in U\}} \mathrm{d}u. \tag{4}$$

It is the total spent by the semi-Markov process Z in the set of up states U during the time interval [0, t] (see for example [13], who described a method for computing the cumulative operational time for semi-Markov processes). It is easy to see that when the embedded Markov Chain, $((J_n)_{n\geq 0})$ is irreducible, recurrent, and aperiodic with equilibrium measure ν, then

$$\lim_{t\to+\infty} \frac{Co(t)}{t} = \frac{\sum_{i\in U} \nu_i m_i}{\sum_{k=1}^{s} \nu_k . m_k},$$

where m_i is the mean sojourn time in state i. A mathematical proof of this result can be found in [4] or in Ross [14]. However, one can reason heuristically that since ν_i is the proportion of transitions that are into state i, since m_i is the mean sojourn time in state i, the average system availability, if stationary, should be

$$\frac{\displaystyle\sum_{i\in U} \nu_i m_i}{\displaystyle\sum_{k=1}^{s} \nu_k . m_k}.$$

The problem of estimation of the stationary distribution of ergodic semi-Markov processes, was considered by [5].

The quantity that we want to study is the expected cumulative operational time of a semi-Markov system which is given by

$$\begin{aligned} E[Co(t)] &= E[\int_0^t 1_{\{Z_u \in U\}} \mathrm{d}u] \\ &= \int_0^t P(Z_u \in U) \mathrm{d}u \\ &= \sum_{j \in U} \int_0^t P(Z_u = j) \mathrm{d}u \\ &= \sum_{i=1}^{s} \sum_{j \in U} \int_0^t \alpha_i P_{ij}(u) \mathrm{d}u, \end{aligned} \tag{5}$$

where $P_{ij}(u)$ is given in (1) and α is the initial probability of the process.

The expected cumulative operational time is an important indicator in the maintenance studies since it allows us to derive the average system availability given by

$$\bar{A}(t) = \frac{1}{t} \sum_{j \in U} \int_0^t P_j(u) \mathrm{d}u.$$

Theorem 1. *The expected cumulative operational time (5) can be written as:*

$$E[Co(t)] = \sum_{i=1}^{s} \sum_{j \in U} \int_0^t \alpha_i M_j(t-x) \mathrm{d}\psi_{ij}(x),$$

where $M_j(x) = \int_0^x \bar{H}_j(u) \mathrm{d}u$,

Proof. Of course, this quantity exists because the process which we consider is finite positive ($\psi_{ij}(t) < \infty$ *cf.* [6]) and $M_j(t) < t$.

From (5), we have that

$$\begin{aligned} E[Co(t)] &= \sum_{i=1}^{s} \sum_{j \in U} \int_0^t \alpha_i P_{ij}(u) \mathrm{d}u, \\ &= \sum_{i=1}^{s} \sum_{j \in U} \int_0^t \alpha_i \int_0^u \bar{H}_j(u-x) \mathrm{d}\psi_{ij}(x) \mathrm{d}u, \\ &= \sum_{i=1}^{s} \sum_{j \in U} \int_0^t \alpha_i (\int_0^{t-x} \bar{H}_j(u) \mathrm{d}u) \mathrm{d}\psi_{ij}(x), \\ &= \sum_{i=1}^{s} \sum_{j \in U} \int_0^t \alpha_i M_j(t-x) \mathrm{d}\psi_{ij}(x), . \end{aligned} \tag{6}$$

which is the desired result.

In the sequel, we are concerned with the non-parametric estimation of the expected cumulative operational time of a semi-Markov system.

From $\hat{P}(t)$, we propose to estimate $E[Co(t)]$ by:

$$E[\hat{Co(t)}] = \sum_{i=1}^{s} \sum_{j \in U} \alpha_i (\hat{M}_j * \hat{\psi}_{ij})(t) \tag{7}$$

where $\hat{M}_j(t,T) = \frac{1}{N_j(T)} \sum_{l=1}^{N_j(T)} (X_{jl} \wedge t)$.

In the sequel we will prove that the proposed estimator is unifomly strongly consistent and converges in law to a normal random variable. □

Theorem 2. *The estimator of the expected operational time, $E[\hat{Co(t)}]$ is uniformly strongly consistent in the sense that*

$$\max_{i,j} \sup_{t \in [0,L]} |E[\hat{Co(t,T)}] - E[Co(t)]| \longrightarrow 0 \quad a.s., \quad as \quad T \to \infty.$$

Proof. From (7), we have

$$\begin{aligned}\max_{i,j} \sup_{t \in [0,L]} |E[\hat{Co(t,T)}] - E[Co(t)]| &\leq \max_{i,j} \sup_{t \in [0,L]} \sum_{i=1}^{s} \sum_{j \in U} \alpha_i |(\hat{M}_j * \hat{\psi}_{ij})(t) - (\mathcal{M}_j * \psi_{ij})(t)| \\ &\leq \max_{i,j} \sup_{t \in [0,L]} \sum_{i=1}^{s} \sum_{j \in U} \alpha_i |(\hat{M}_j - M_j)(t)|.\psi_{ij}(t) \\ &\quad + |\hat{M}_j(t)|.|\hat{\psi}_{ij}(t) - \psi_{ij}(t)|.\end{aligned}$$

From [6], we have that,

$$\max_{j} \sup_{t \in \mathbb{R}^+} |\hat{H}_j(t,T) - H_j(t)| \longrightarrow 0 \qquad a.s. \qquad \text{when} \qquad T \longrightarrow \infty,$$

which implies that

$$\max_{j} \sup_{t \in [0,L]} \sum_{i=1}^{s} \sum_{j \in U} \alpha_i |(\hat{M}_j - M_j)(t)| \longrightarrow 0 \qquad a.s. \qquad \text{when} \qquad T \longrightarrow \infty.$$

For any $L \in \mathbb{R}^+$, we have

$$\max_{i,j} \sup_{t \in [0,L]} |\hat{\psi}_{ij}(t,T) - \psi_{ij}(t)| \longrightarrow 0 \qquad a.s. \qquad \text{when} \qquad T \longrightarrow \infty,$$

and the fact that $\psi_{ij}(t) < \infty$ and $\hat{M}_j(t) < t$, we get the desired result. □

Theorem 3. *For any fixed t, $t \in [0, \infty)$, $T^{1/2}(E[\hat{Co}(t,T)] - E[Co(t)])$ converges in law to a zero mean normal random variable with variance*

$$\begin{aligned}
\sigma^2(t) \quad = \quad & \sum_{i=1}^{s}\sum_{j=1}^{s} \mu_{ii} \cdot \{(\phi_{ij})^2 * Q_{ij} - (\phi_{ij} * Q_{ij})^2 \\
& + 1_{\{j \in U\}} \int_0^\infty \left[\int_0^\infty (x \wedge (t-u) - M_i) \mathrm{d}A_i(u)\right]^2 \mathrm{d}Q_{ij}(x) \\
& - \left[1_{\{j \in U\}} \int_0^\infty \int_0^\infty (x \wedge (t-u) - M_i) \mathrm{d}A_i(u) \mathrm{d}Q_{ij}(x)\right]^2 \\
& + 2.1_{\{j \in U\}} \int_0^\infty \phi_{ij}.(t-x) \int_0^\infty (x \wedge (t-u) - M_i) \mathrm{d}A_i(u) \mathrm{d}Q_{ij}(x) \\
& - 2.1_{\{j \in U\}} (\phi_{ij} * Q_{ij})(t).(A_i * ((x \wedge . - M_i))(t)\}
\end{aligned}$$

where $A_i(t) = \sum_{k=1}^{s} \alpha_k \psi_{ki}(t)$ *and* $\phi_{kl}(t) = \sum_{i=1}^{s} \sum_{j \in U} \alpha_i (\psi_{ik} * \psi_{lj} * M_j)(t)$.

Proof. From (7), we see that

$$\begin{aligned}
T^{1/2}(E[\hat{Co}(t,T)] - E[Co(t)]) &= \sum_{i=1}^{s} \sum_{j \in U} \alpha_i T^{1/2} (\hat{M}_j * \hat{\psi}_{ij})(t) - (M_j * \psi_{ij})(t) \\
&= \sum_{i=1}^{s} \sum_{j \in U} \alpha_i T^{1/2} [(\hat{M}_j - M_j) * (\hat{\psi}_{ij} - \psi_{ij})(t) \\
&\quad + (\hat{M}_j - M_j) * \psi_{ij})(t) + M_j * (\hat{\psi}_{ij} - \psi_{ij})(t)] \quad (8)
\end{aligned}$$

From [6], the first term of (8) converges to zero. Then $T^{1/2}(E[\hat{Co}(t,T)] - E[Co(t)])$ has the same limit as

$$\sum_{i=1}^{s} \sum_{j \in U} \alpha_i T^{1/2} \left[(\hat{M}_j - M_j) * \psi_{ij})(t) + M_j * (\hat{\psi}_{ij} - \psi_{ij})(t)\right].$$

From [6], has the same limit as

$$\begin{aligned}
\sum_{i=1}^{s} \sum_{j \in U} \alpha_i T^{1/2} \Bigg[& \frac{1}{N_j(T)} \sum_{r=1}^{N_j(T)} (X_{jl} \wedge t - M_j) * \psi_{ij})(t) + \\
& \left(\sum_{k=1}^{s} \sum_{l=1}^{s} M_j * \psi_{ik} * \psi_{lj}\right) * (\hat{Q}_{kl} - Q_{kl})(t) \Bigg],
\end{aligned}$$

which is equal to

$$\sum_{i=1}^{s}\sum_{j\in U}\alpha_i T^{1/2}\left[\frac{1}{N_j(T)}\sum_{r=1}^{N_j(T)}(X_{jl}\wedge t-M_j)*\psi_{ij})(t)\right.$$
$$\left.+\left(\sum_{k=1}^{s}\sum_{l=1}^{s}M_j*\psi_{ik}*\psi_{lj}\right)*\left(\frac{1}{N_k(T)}\sum_{r=1}^{N_k(T)}(1_{\{J_{r+1}=l,X_r\le .\}}-Q_{kl})(t)\right)1_{\{J_r=k\}}\right],$$

which can be written as

$$\sum_{k=1}^{s}\sum_{l=1}^{s}\frac{T^{1/2}}{N_k(T)}\sum_{r=1}^{N_k(T)}[1_{\{J_r=k,k\in U,J_{r+1}=l\}}(X_r\wedge t-M_k)*A_k)(t)$$
$$+(B_{kl}*(1_{\{J_{r+1}=l,X_r\le .\}}-Q_{kl})(t))1_{\{J_r=k\}}],$$

where $A_k(t)=\sum_{i=1}^{s}\alpha_i\psi_{ik}(t)$ and $\phi_{kl}(t)=\sum_{i=1}^{s}\sum_{j\in U}\alpha_i(\psi_{ik}*\psi_{lj}*M_j)(t)$. Since $\frac{T}{N_k(T)}\to\mu_{kk}$, we can consider the function

$$f(J_r,J_{r+1},X_n)=\mu_{kk}A_k*(X_n\wedge t-M_k)1_{\{J_r=k,k\in U,J_{r+1}=l\}}$$
$$+\mu_{kk}\phi_{kl}*(1_{\{J_{r+1}=l,X_r\le .\}}-Q_{kl})(t)1_{\{J_r=k\}}.$$

By the central limit theorem of semi-Markov process (see [10]) or as in [6] for this function we get the desired result. □

5. Numerical application

Let us consider a three state semi-Markov system as illustrated in Figure 1, where $F_{12}(x)=1-\exp(-\lambda_1 x)$, $F_{21}(x)=1-\exp[-(\frac{x}{\alpha_1})^{\beta_1}]$, $F_{23}(x)=1-\exp[-(\frac{x}{\alpha_2})^{\beta_2}]$, and $F_{31}(x)=1-\exp(-\lambda_2 x)$, for $x\ge 0$. $\lambda_1=0.1$, $\lambda_2=0.2$, $\alpha_1=0.3$, $\beta_1=2$, $\alpha_2=0.1$ and $\beta_2=2$.

The transition probability matrix of the embedded Markov chain (J_n) is:

$$P=\begin{pmatrix}0&1&0\\p&0&1-p\\1&0&0\end{pmatrix},$$

where p is given by

$$p=\int_0^\infty[1-F_{23}(x)]\mathrm{d}F_{21}(x).$$

In Figure 2, we present the estimation of the cumulative operational time for the three state semi-Markov system given in Figure 1. As an appliccation to maintenance studies, we give in Figure 3 the estimation of the average system availability for this system.

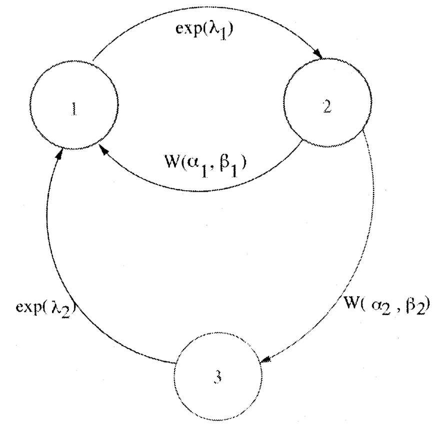

FIGURE 1. A three state semi-Markov system.

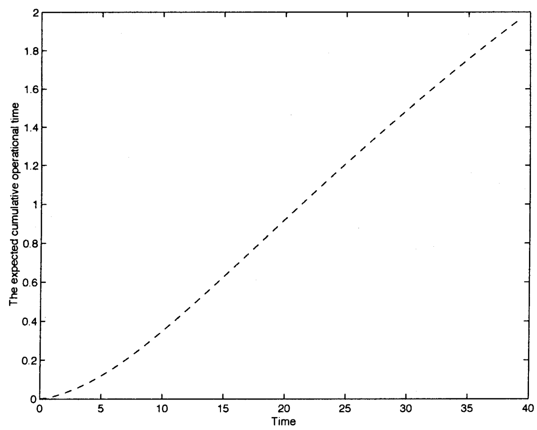

FIGURE 2. The estimation of the cumulative operational time for the three state semi-Markov system.

6. CONCLUDING REMARKS

The main impediment to the semi-Markov use in practice is computational complexity. In this paper, we have developed a recursive formula of the empirical estimator of the semi-Markov kernel. By making use of this estimator, we have proposed an estimator for the expected cumulative operational time of semi-Markov systems and we have proved its asymptotic properties. This indicator is of great interest in maintenance studies. Finally, it should be mentioned that there is a need for applications of this quantity to minimize the expected cost of the maintenance process by choosing the appropriate actions.

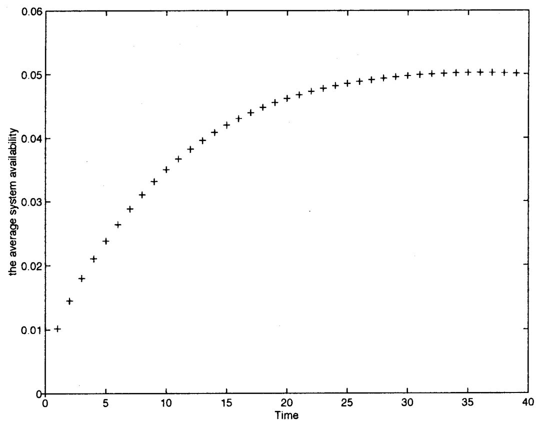

FIGURE 3. The estimation of the average system availability.

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