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ON CO-BICLIQUES

Denis Cornaz[1]

Abstract. A co-biclique of a simple undirected graph $G = (V, E)$ is the edge-set of two disjoint complete subgraphs of G. (A co-biclique is the complement of a biclique.) A subset $F \subseteq E$ is an independent of G if there is a co-biclique B such that $F \subseteq B$, otherwise F is a dependent of G. This paper describes the minimal dependents of G. (A minimal dependent is a dependent C such that any proper subset of C is an independent.) It is showed that a minimum-cost dependent set of G can be determined in polynomial time for any nonnegative cost vector $x \in \mathbb{Q}^E_+$. Based on this, we obtain a branch-and-cut algorithm for the maximum co-biclique problem which is, given a weight vector $w \in \mathbb{Q}^E_+$, to find a co-biclique B of G maximizing $w(B) = \sum_{e \in B} w_e$.

Keywords. Co-bicyclique, signed graph, branch-and-cut.

Mathematics Subject Classification. 05C15, 90C09.

1. Introduction

Let $G = (V, E)$ be a simple undirected graph and w_e a nonnegative weight for each $e \in E$. We denote $E[U] = \{uv \in E : u, v \in U\}$ and $\overline{E}[U] = \{uv : u, v \in U, u \neq v, uv \notin E\}$. A subset of nodes $U \subseteq V$ is called a *clique* if $\overline{E}[U] = \emptyset$. A set $B \subseteq E$ is called a *co-biclique* if there are two disjoint cliques U_1 and U_2 such that $B = E[U_1] \cup E[U_2]$. Note that $\emptyset$ is a co-biclique. A co-biclique B is maximum if its weight $w(B) = \sum_{e \in B} w_e$ is maximum.

This paper adresses the *maximum co-biclique problem* which is to determine a maximum co-biclique of G. Note that finding a maximum cardinality clique in G can be reduced to finding a maximum cardinality co-biclique in $2G$, where $2G$

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[1] Équipe Combinatoire, UFR 921, Case 189, Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris Cedex 05 France; cornaz@math.jussieu.fr

consists in the graph G and a disjoint copy of G. This implies that the maximum co-biclique problem is NP-hard.

The structure of odd cycle is essential in the study of the bipartite subgraphs (see [8]). The collection of the odd cycles of G coincide with the minimal dependents of the independence system that is naturally formed by the bipartite subgraphs of G. (An *independence system* of a set E consists in a collection $\mathcal{I}$ of subsets of E such that $I \in \mathcal{I}$ and $I' \subseteq I$ implies $I' \in \mathcal{I}$.) Recently, it was showed that less natural independence systems could be associated to more complicated graph structures, with interesting polyhedral and algorithmic consequences. This approach is used for graph coloring in [4, 5]. The independence system associated to the (edge-set of) induced bipartite subgraphs have been defined and described in [3]. In [1, 2], the independence systems associated to the bicliques and to the complete multipartite subgraphs have been introduced and characterized. This paper studies the independence system associated to the co-bicliques. According to our knowledge, although bicliques have been studied a lot (see [7]), the maximum co-biclique problem has never been considered before. Knowing the importance of bicliques, we found natural to study co-bicliques.

Our approach is the following: We say that an edge set $F \subseteq E$ is *independent* if there is a co-biclique B such that $B \supseteq F$, otherwise F is dependent. In this way, solving the maximum co-biclique problem is equivalent to determine the maximum weight of an independent. Hence there is a 0-1 linear programming formulation of the maximum co-biclique problem in the natural variable space, namely

$$(P_I)\begin{cases} \max & \sum\limits_{e\in E} w_e x_e \\ \text{s.t.} & \\ & \qquad x_e \quad \in \{0,1\} \quad \text{for every } e \in E, \\ & \qquad x(C) \quad \le |C|-1 \quad \text{for every dependent set } C, \end{cases}$$

where $x(C) = \sum_{e\in C} x_e$. We are interested in solving (P_I) with a branch-and-cut algorithm. That method is efficient if the continuous relaxation (P) of (P_I) can be solved in polynomial time. The number of inequalities of (P) may be exponential (with respect to $n := |V|$) but we will show that indeed (P) can be solved in polynomial time.

This paper is organized as follows. In Section 2, we give some definitions and we characterize the independents. In Section 3, we give a complete description of the minimal dependents. In Section 4, we show that finding a minimum-cost dependent reduces to finding a minimum-cost odd cycle in an auxiliary signed graph $\hat{G}$ of G. We use this to show that (P) can be solved in polynomial time.

2. Preliminaries

First we collect some general terminology and facts on signed graphs (this can be found in [8], Vol. C, p. 1329). A *signed graph* is a triple (V, E, Σ), where (V, E) is an undirected graph and $\Sigma \subseteq E$. The subset of edges Σ is called a *signing*.

A *path* of (V, E, Σ) is a subset $P \subseteq E$ of the form $P = v_0v_1, v_1v_2, \dots, v_{k-1}v_k$ where each v_i is a distinct node of V. If all the v_i are distinct except $v_0 = v_k$, then P is called a *cycle*. We call a path, or a cycle *odd* (*even*, respectively) if it contains an odd (even, respectively) number of edges in Σ.
A *cut* of (V, E, Σ) is a set of edges of the form $\delta(U) = \{uv \in E : u \in U, v \in V \setminus U\}$ where $U \subseteq V$.

Lemma 2.1 ([8])**.** *Two signing Σ and Σ' give the same collection of odd cycles if and only if $\Sigma \Delta \Sigma'$ is a cut of (V, E).*

Let $G = (V, E)$ be a graph and $\overline{E} := \overline{E}[V]$. The signed graph associated to G is the signed graph $\tilde{G} = (V, E \cup \overline{E}, \overline{E})$. Denote $V(F)$ the set of nodes incident to an edge in $F \subseteq E$.

Definition 2.2. Let $F \subseteq E$ and $W = V(F)$. The rooted graph of F is the signed graph

$$\tilde{G}_F = (W, \overline{E}[W] \cup F, \overline{E}[W]).$$

The following lemma characterizes the independents (of G).

Lemma 2.3. *Let $F \subseteq E$. The following propositions are equivalent.*

(i) *$\tilde{G}_F$ has no odd cycle;*
(ii) *$\overline{E}[W]$ is a cut of $\tilde{G}_F$;*
(iii) *F is an independent.*

Proof. *(i)* $\Leftrightarrow$ *(ii)* : It follows by setting $V := W$, $E := \overline{E}[W] \cup F$, $\Sigma := \overline{E}[W]$ and $\Sigma' := \emptyset$ in Lemma 2.1.
(ii) $\Rightarrow$ *(iii)* : If $\overline{E}[W] = \delta(U)$ is a cut of $\tilde{G}_F$, then $\overline{E}[U] = \overline{E}[W \setminus U] = \emptyset$ and $F \subseteq E[U] \cup E[W \setminus U]$. Since $B = E[U] \cup E[W \setminus U]$ is a co-biclique, then F is an independent.
(iii) $\Rightarrow$ *(ii)* : If F is an independent, F is contained in a co-biclique $B = E[U] \cup E[W \setminus U]$. Hence $\overline{E}[W]$ is a cut $\delta(U)$ of $\tilde{G}_F$. □

A set F is a minimal dependent if F is a dependent and F' is an independent for every proper subset F' of F. Lemma 2.3 has the following corollary.

Corollary 2.4. *Let $F \subseteq E$. F is a minimal dependent if and only if*

(i) *$\tilde{G}_F$ has at least one odd cycle, and*
(ii) *for every odd cycle Q of $\tilde{G}_F$ and every edge $f \in F \setminus Q$, there is a node $v_f \in V(Q)$ such that f is the unique edge in F incident to v_f.*

Proof. *Necessity.* Let F be a minimal dependent. Then (i) follows from the fact that F is not an independent. If (ii) does not hold, then there is an odd cycle Q and an edge $f \in F \setminus Q$ such that $V(Q) \subseteq V(F \setminus \{f\})$. But then Q belongs to the rooted graph of $F \setminus \{f\}$, which is impossible since $F \setminus \{f\}$ is independent.
Sufficiency. By (i), F is a dependent. Assume that $F' = F \setminus \{f\}$ is a dependent for some $f \in F$. Then the rooted graph $\tilde{G}_{F'}$ of F' has an odd cycle Q. Since f is not an edge of $\tilde{G}_{F'}$, then $f \in F \setminus Q$. By (ii), there is a node $v_f \in V(Q)$ such that $v_f \notin V(F')$, a contradiction. □

3. Description of the minimal dependents

In what follows we introduce some definitions that are useful to give a complete description of the minimal dependents. Throughout the section we will use the following conventions: F will always represent an edge subset of G, W is the set of nodes of F, and $\tilde{G}_F$ will always represent the rooted graph of F. (Recall that $\tilde{G}_F$ is a signed graph.)

Definition 3.1. F induces an obstruction with an odd cycle Q of $\tilde{G}_F$ if for every edge f in $F \setminus Q$

(a) f is incident to exactly one node of Q, and
(b) f is adjacent to no edge in $F \setminus \{f\}$.

Definition 3.2. Let F be an edge set inducing an obstruction with the odd cycle $Q = v_0v_1, v_1v_2, \ldots, v_{k-1}v_k$ (where the indices are taken modulo k).
An edge $v_iv_{i+2} \in \overline{E}[W]$ is called short-chord if

(a) $v_{i+1}v_{i+2} \in F$ and $v_iv_{i+1}, v_{i+2}v_{i+3} \in \overline{E}[W]$, or
(b) $v_iv_{i+1} \in F$ and $v_{i-1}v_i, v_{i+1}v_{i+2} \in \overline{E}[W]$.

An edge $v_iv_{i+3} \in \overline{E}[W]$ is called a diagonal if

(c) $v_iv_{i+1}, v_{i+2}v_{i+3} \in F$, $v_{i-1}v_i, v_{i+1}v_{i+2}, v_{i+3}v_{i+4} \in \overline{E}[W]$.

An edge $v_iw \in \overline{E}[W]$ with $w \notin V(Q)$ is called a wing if

(d) $v_iv_{i+1}, wv_{i+2} \in F$ and $v_{i-1}v_i, v_{i+1}v_{i+2}, v_{i+2}v_{i+3} \in \overline{E}[W]$, or
(e) $v_iv_{i-1}, wv_{i-2} \in F$ and $v_{i-3}v_{i-2}, v_{i-2}v_{i-1}, v_iv_{i+1} \in \overline{E}[W]$.

Definition 3.3. We say that two wings v_iw and v_jw' overlap if $v_iv_j \in F$.

Figure 1 depicts the objects of the above definitions.

Theorem 3.4. *F is a minimal dependent if and only if F induces an obstruction with an odd cycle Q such that*

(i) *every edge in $\overline{E}[W] \setminus Q$ is either a short-chord, a diagonal, or a wing, and*
(ii) *no wings overlap.*

Proof. Necessity: Let F be a minimal dependent of G. By Corollary 2.4(i), $\tilde{G}_F$ contains an odd cycle. Let Q be an odd cycle of $\tilde{G}_F$ such that $|Q \cap F|$ is maximal.

Let P be a path of $\tilde{G}_F$ linking $v_i, v_j \in V(Q)$ such that $P \cap Q = \emptyset$ and $V(P) \cap V(Q) = \{v_i, v_j\}$. We let $P_1, P_2 \subseteq Q$ be the two distinct paths of Q linking v_i, v_j. So $P_1 \cap P_2 = \emptyset$ and $P_1 \cup P_2 = Q$. Note that $|P_1 \cap \overline{E}[W]|$ and $|P_2 \cap \overline{E}[W]|$ are of opposite parity. Hence we can assume without loss of generality that $Q_1 = P_1 \cup P$ is an odd cycle and $Q_2 = P_2 \cup P$ is an even cycle of $\tilde{G}_F$.

Claim 1. We claim that none of the following propositions can be true.
(1) $P = \{v_iv_j\}$ with $v_iv_j \in F$.
(2) $P = \{v_iw, wv_j\}$ with $v_iw, wv_j \in F$.
(3) $P = \{v_iw, ww', w'v_j\}$ with $v_iw, w'v_j \in F$.
Proof. If either (1), or (2), or (3) is true, then $V(Q_1) \subseteq V(F \setminus \{f\})$ for every

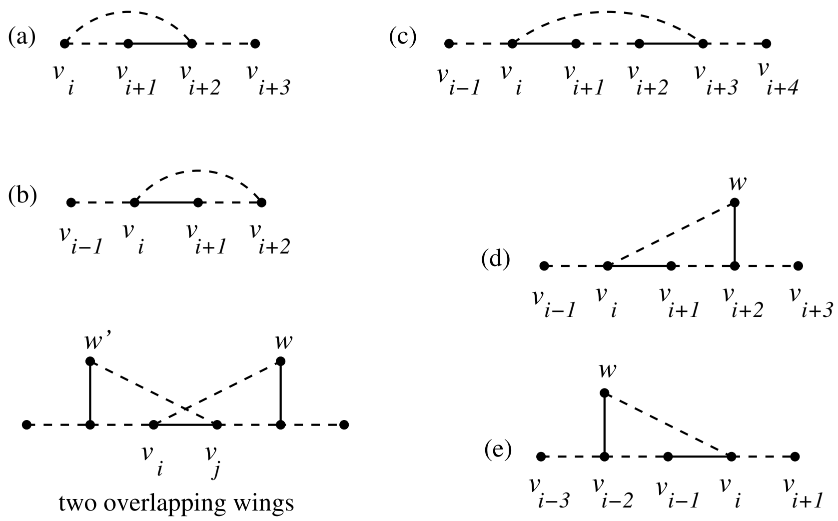

FIGURE 1. Short-chords (a,b), diagonal (c) and wings (d,e).

$f \in F \cap P_2$. If there is an edge $f \in F \cap P_2$, then, by Corollary 2.4(i), $F \setminus \{f\}$ is a dependent; this contradicts the minimality of F. So we can assume that $F \cap P_2 = \emptyset$. Therefore $|F \cap Q_1| > |F \cap Q|$; which contradicts the maximality of $|Q \cap F|$. *(End of the proof of Claim 1.)*

Claim 2. We claim that F induces an obstruction with Q. *Proof.* By Corollary 2.4(ii), every edge in F is incident to Q. If $f \in F$ is a chord of Q, we can assume that $P = \{f\}$; this is impossible by Claim 1(1). So Q has no chord in F, hence Definition 3.1(a) holds. Let f, f' be two adjacent edges such that $f \in F \setminus Q$ and $f' \in F$. By Corollary 2.4(ii), f' also belongs to $F \setminus Q$. Moreover the common node of f and f' is not in Q; thus we can assume that $P = \{f, f'\}$; this is impossible by Claim 1(2). Hence Definition 3.1(b) holds. *(End of the proof of Claim 2.)*

A node v of Q is said to be *exposed* if it is incident to no edge in $F \cap Q$.

Claim 3. We claim that the following propositions are true.

(1) P_2 has no internal edge in F.

(2) If Q_2 has no chord in F, then P_2 has no exposed node.

(3) Every node of P_2 is incident to (exactly) one edge in F.

Proof. By Corollary 2.4(ii), every edge in F is incident to a node in Q_1. Thus (1) is true. Suppose that (2) is not true. Let v be an exposed node of P_2. There is an edge f in $F \setminus P_2$ incident to v and to a node in Q_1. By Claim 2, Q has no chord, hence f is incident to a node in P. This is impossible since Q_2 has no chord. Suppose now that (3) is false. Let v be a node of P_2 incident to two edges f_1, f_2 in F. By Claim 2, $f_1, f_2 \in Q$. If $f_1 \in P_1$ and $f_2 \in P_2$, then $P_2 = \{f_2\}$ since f_2 must be incident to a node in $V(Q_1 \setminus f_1)$; this is impossible. So $f_1, f_2 \in P_2$. Since f_1 and f_2 are incident to Q_1, then $P_2 = \{f_1, f_2\}$. Since Q_2 is even in $\tilde{G}_F$, then $|P \cap \overline{E}[W]|$ is even. Hence, because of the maximality of $|Q \cap F|$ and the minimality of F, there are only two cases: either $P = \{v_iv_j\}$ with $v_iv_j \in F$, or $P = \{v_iw, wv_j\}$ with $v_iw, wv_j \in F$. This is impossible by Claim 1. *(End of the proof of Claim 3.)*

Now we can prove that necessity is true. Denote $e_i = v_iv_{i+1}$ for $i = 0, 1, \ldots, k-1$. Let e be an edge in $\overline{E}[W] \setminus Q$. Suppose that $P = \{e\}$. Since Q has no chord, then Q_2 has no chord. As P has exactly one edge in $\overline{E}[W]$, P_2 has an odd number of edges in $\overline{E}[W]$. By Claim 3, P_2 contains exactly one edge in $\overline{E}[W]$. Note that since $\tilde{G}_F$ has no multiple edge, P_2 contains at least one edge in F. Suppose that P_2 contains exactly one edge f in F, by Claim 3, f is either e_i or e_{j-1}. First we assume that $f = e_j$. Then e is a short-chord (see Fig. 1a). If $f = e_i$, then e is a short-chord (see Fig. 1b). Now suppose that P_2 contains more than one edge in F. Claim 3 implies that the edges in $F \cap P_2$ are e_i and e_{j-1}. Finally, as P_2 has no exposed node, e is a diagonal (see Fig. 1c).

Assume now that $e = ww'$ with $w, w' \in W \setminus V(Q)$. This is impossible by Claim 1(3). We can assume now that $e = v_iw$ with $v_i \in V(Q)$ and $w \in W \setminus V(Q)$. Note that since F induces an obstruction with Q, there is an edge in F, say $f = v_jw$ (with $v_j \in V(Q)$), which is the unique edge in F incident to w and the unique edge in F incident to v_j. Thus e_{j-1} and e_j are in $\overline{E}[W]$. Let $P = \{e, f\}$. The path P contains one edge in $\overline{E}[W]$, therefore P_2 contains an odd number of edges in $\overline{E}[W]$. If P_2 contains no edge in F, then the odd cycle Q_1 has more edges in F than Q has, contradiction. We can assume that Q_2 has no chord. Assume first that $i < j$. By Claim 3, e_i is the unique edge of $F \cap P_2$. Also, e_{i-1} is in $\overline{E}[W]$. Moreover v_j is the unique exposed node of P_2. Thus $j = i + 2$ and the edge e is a wing (see Fig. 1d). The case $j < i$ is similar: $j = i - 2$ and e is a wing (see Fig. 1e). Finally the only possible neighbours of w besides v_i are v_{i-2} and v_{i+2}; if w is adjacent to these three nodes, w is incident to two wings.

Assume now that there exist two nodes $w, w' \in W \setminus V(Q)$, a wing $e = v_iw$ and a wing $e' = v_{i+1}w'$ which overlap. The path $P' = \{v_{i-1}w', e', e_i, e, wv_{i+2}\}$ has three edges in F and the path $P'' = \{e_{i-1}, \ldots, e_{i+1}\}$ has only one edge in F. The cycle obtained by replacing P'' by P' in the sequence describing P'' is an odd cycle in $\tilde{G}_F$ and has a larger number of edges in F than Q, which contradicts the maximality of $|F \cap Q|$.

Sufficiency. Let $f \in F$ and let $\tilde{G}_{F'}$ be the signed rooted graph of $F' = F \setminus \{f\}$. Assume now that $F \setminus \{f\}$ is not a independent of G; by Lemma 2.3, there is an odd cycle D of $\tilde{G}_{F'}$. Note that D is also an odd cycle of $\tilde{G}_F$. If f is an edge of Q, Q cannot be a subgraph of $\tilde{G}_{F'}$. In the other case f links a node in $W \setminus V(Q)$ to an exposed node v of Q, v is not a node of $\tilde{G}_{F'}$ and again Q is not a subgraph of $\tilde{G}_{F'}$.

Assume that D contains a diagonal $e = v_iv_{i+3}$; f cannot be e_i or e_{i+2} since v_i and v_{i+3} have not been deleted from $\tilde{G}_F$. If we replace in D the subsequence $\ldots, e, \ldots$ by $\ldots, e_i, e_{i+1}, e_{i+2}, \ldots$ (which is not a subsequence of D since D is odd in $\tilde{G}_{F'}$) we obtain a new cycle which does not contain e and which is odd in $\tilde{G}_{F'}$. Reiterating this process, we can eliminate all the diagonals, and similarly all the short-chords. If D contains a node w in $W \setminus V(Q)$, D contains one or two wings incident to w. If D contains a subsequence $\dots, e, f', \dots$ where e is a wing, we replace in D that subsequence by $\dots, e_i, e_{i+1}, \dots$. If D contains a subsequence $\dots, e, e', \dots$ where e and e' are wings, we replace in D that subsequence by $\dots, e_i, \dots, e_{i+3}, \dots$. Again this new cycle is odd in $\tilde{G}_{F'}$ and we can eliminate similarly all the wings. Finally D contains edges of the cycle Q only, a contradiction. □

4. Solving (P)

Let

$$(P)\begin{cases} \max & \sum_{e\in E} w_e x_e \\ \text{s.t.} & \\ & 0 \le x_e \le 1 \quad \text{for every } e \in E, \\ & x(C) \le |C|-1 \quad \text{for every dependent set } C. \end{cases}$$

We state now the main result of the paper. We give a proof based on [9] which is simpler than our original proof.

Theorem 4.1. (P) *can be solved in polynomial time.*

Proof. By [6], the problem reduces to the following separation problem: given $x \in \mathbb{R}^E$, decide if x satisfies the constraints of (P), and if not, find a violated inequality. We can check in polynomial time if $0 \le x_e \le 1$ for every $e \in E$. Note that $x(C) \le |C| - 1$ is equivalent to $w(C) \ge 1$ with $w_e = 1 - x_e$ for every $e \in E$. Hence our separation problem reduce to the following problem: Does there exist a dependent with cost strictly smaller than 1?

In the following we describe a polynomial algorithm which answers this question. We reduce the problem to finding a minimum-cost odd cycle in an auxiliary signed graph $\hat{G}$ of G. For any depend of G, there is an odd cycle of $\hat{G}$ with the same cost, and *vice-versa*.

Let G be a graph with a nonnegative cost $c(e)$ for each $e \in E$. For every node $v \in V$, we define

$$c(v) = \min_{uv\in E} c(uv),$$

and we choose an edge $uv \in E$ such that $c(uv) = c(v)$; denote uv by $f(v)$.
Let $\hat{G}$ be the signed graph constructed from G as follows (this is illustrated with an example depicted in Fig. 2):

Let be the signed graph $\tilde{G} = (V, E \cup \overline{E}, \overline{E})$ associated with G. Note that each node has degree $n-1$ in $\tilde{G}$. For every edge $e = uv$ of $\tilde{G}$ we make a copy $\hat{e} = u_e v_e$ of e in $\hat{G}$, in this way, all the edges of $\hat{G}$ are disjoint. We will call e the mate of $u_e v_e$. We will use the following notation: $\hat{E}$ is the set of copies of edges in E and Σ is the set of copies of edges in $\overline{E}$. Note that a node v of $\tilde{G}$ has $n-1$ copies $v_e, v_f, \dots$ in $\hat{G}$. For every node v of $\tilde{G}$ we create the $\binom{n-1}{2}$ possible transition edges $v_e v_f$ in $\hat{G}$ between the different copies of v. The node v will be called the node

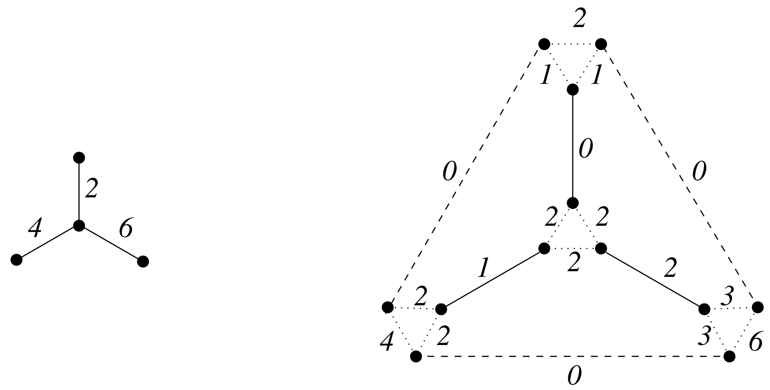

FIGURE 2. On the left the graph G, on the right the signed graph $\hat{G}$.

associated with the transition edge $v_e v_f$. Note that a transition edge is adjacent to exactly two edges in $\hat{E} \cup \Sigma$. The set of the transition edges will be denoted by T. The signing of $\hat{G}$ will be Σ, so $\hat{G} = (\hat{V}, T \cup \hat{E} \cup \Sigma, \Sigma)$. Now we define the costs in $\hat{G}$. The cost of an edge $\hat{e} \in \hat{G}$ is denoted by $d(\hat{e})$:

* $d(\hat{e}) = 0$ for each $\hat{e} \in \Sigma$;
* $d(\hat{e}) = c(v)/2$ for each $\hat{e} \in T$ associated with a node $v \in V$ adjacent to an edge in Σ and an edge in $\hat{E}$;
* $d(\hat{e}) = c(v)$ for each $\hat{e} \in T$ associated with a node $v \in V$ adjacent to two edges of the same type (two edges in Σ or two edges in $\hat{E}$);
* $d(\hat{e}) = c(e) - \frac{c(u)+c(v)}{2}$ for each $\hat{e} = u_e v_e \in \hat{E}$ where $e = uv \in E$.

Note that the cost $d(\hat{e})$ is nonnegative for each edge $\hat{e}$ of $\hat{G}$. The problem of finding a minimum-cost odd cycle in a signed graph can be solved in polynomial time for every nonnegative edge cost function (see [8]). Let $\hat{Q}$ be an odd cycle of the signed graph $\hat{G}$ minimizing its cost $d(\hat{Q}) = \sum_{\hat{e} \in \hat{Q}} d(\hat{e})$. We will show now that the cost of $\hat{Q}$ is equal to the minimum cost of a dependent set of G.

Remark that $\hat{Q}$ has at least one edge in Σ. We can assume that $\hat{Q}$ does not contain two consecutive edges that are transition edges in T. Thus $\hat{Q}$ can be decomposed into paths of the two following forms:

(P_1) $P_1 = \{t_1, \hat{e}_1, t_2, \hat{e}_2, \ldots, t_k, \hat{e}_k, t_{k+1}\}$, where $\hat{e}_i \in \hat{E}$, $t_i \in T$, and t_1 (t_{k+1}) is adjacent to an edge in $\Sigma \cap \hat{Q}$.

(P_2) $P_2 = \{\hat{e}_1, t_1, \hat{e}_2, t_2, \ldots, t_{k-1}, \hat{e}_k\}$ where $\hat{e}_i \in \Sigma$ and $t_i \in T$.

The cost of a path P_1 is

$$\begin{aligned}
d(P_1) &= d(t_1) + d(\hat{e}_1) + d(t_2) + \cdots + d(t_k) + d(\hat{e}_k) + d(t_{k+1}) \\
&= \frac{c(u_1)}{2} + c(e_1) - \frac{c(u_1) + c(u_2)}{2} + c(u_2) + \ldots \\
&\quad \ldots + c(u_k) + c(e_k) - \frac{c(u_k) + c(u_{k+1})}{2} + \frac{c(u_{k+1})}{2} \\
&= c(e_1) + c(e_2) + \ldots + c(e_k).
\end{aligned}$$

Thus the cost of P_1 is equal to the sum of costs $c(e)$ of mates $e \in E$ of edges $\hat{e} \in \hat{E} \cap P_1$. The cost of a path P_2 is

$$\begin{aligned}
d(P_2) &= d(\hat{e}_1) + d(t_2) + d(\hat{e}_2) + d(t_3) + \cdots + d(t_k) + d(\hat{e}_k) \\
&= 0 + c(u_2) + 0 + c(u_3) + \ldots + c(u_k) + 0.
\end{aligned}$$

The cost of P_2 is equal to the sum of the costs $c(u)$ of nodes u associated with transition edges in $T \cap P_2$. Let $F \subseteq E$ be the union of mates of the edges in $\hat{E} \cap Q$ and edges $f(u) \in E$ where u is the node associated with a transition edge of a path P_2. We have $d(\hat{Q}) \geq c(F)$. Besides, by Lemma 2.3, F is a dependent set. Now let F be a minimum-cost dependent set. By Theorem 3.4, F induces an obstruction with an odd cycle Q in $\tilde{G}$. Let $W = V(F)$. In the graph $\hat{G}$ there is a cycle $\hat{Q}$ such that the edges in $\Sigma \cap \hat{Q}$ (resp. $\hat{E} \cap \hat{Q}$) are the mates of edges in $\overline{E}[W] \cap Q$ (resp. $F \cap Q$), and the edges in $T \cap \hat{Q}$ are the transition edges associated with exposed nodes of Q. Clearly $\hat{Q}$ has an odd number of edges in Σ. Since Q has no chord in F, we have $c(F) \geq d(\hat{Q})$.

□

Conclusion

This paper establishes a link between the only apparently distant notions of co-bicliques and odd cycles. More precisely, the link concerns the subsets of co-bicliques only, but this is appropriate to the resolution of the maximum co-biclique problem. The odd cycles in signed graphs are used to handle naturally the complicated minimal forbidden structures for (subsets of) co-bicliques.

A theorem by Guenin gives a full characterization of those signed graphs for which the odd-cycle constraints define an integral polytope (see [8]). A remaining question is whether a characterization of the graphs for which the dependent-set constraints describe the co-biclique polytope can be deduced from Guenin's theorem?

References

[1] D. Cornaz, A linear programming formulation for the maximum complete multipartite subgraph problem. *Math. Program. B* **105** (2006) 329–344.

[2] D. Cornaz and J. Fonlupt, Chromatic characterization of biclique cover. *Discrete Math.* **306** (2006) 495–507.

[3] D. Cornaz and A.R. Mahjoub, The maximum induced bipartite subgraph problem with edge weights. *SIAM J. on Discrete Math.* to appear.

[4] D. Cornaz, *On forests, stable sets and polyhedra associated with clique partitions.* Manuscript available on Optimization Online.

[5] V. Jost, *Ordonnancement chromatique : polyèdres, complexité et classification.* Thèse de l'Université Joseph Fourier, Grenoble (2006).

[6] M. Grötschel, L. Lovàsz and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica* **1** (1981) 169–197.

[7] D. Monson, N.J. Pullman and R. Rees, A survey of clique and biclique coverings and factorizations of (0,1)-matrices. *Bull. I.C.A.* **14** (1995) 17–86.

[8] A. Schrijver, *Combinatorial Optimization*. Springer-Verlag, Berlin Heidelberg (2003).

[9] A. Sebő, private communication.