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ON SEMIDEFINITE BOUNDS FOR MAXIMIZATION OF A NON-CONVEX QUADRATIC OBJECTIVE OVER THE ℓ_1 UNIT BALL

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Abstract. We consider the non-convex quadratic maximization problem subject to the ℓ_1 unit ball constraint. The nature of the l_1 norm structure makes this problem extremely hard to analyze, and as a consequence, the same difficulties are encountered when trying to build suitable approximations for this problem by some tractable convex counterpart formulations. We explore some properties of this problem, derive SDP-like relaxations and raise open questions.

Keywords. Non-convex quadratic optimization, L1-norm constraint, semidefinite programming relaxation, duality.

1. Introduction

The technique of semidefinite programming relaxation applied to combinatorial and related non-convex quadratic optimization problems became popular after the seminal papers of Goemans and Williamson [1] on the max-cut problem. Later on, Nesterov [12] generalized this result, and a vast literature in that direction for various types of non-convex quadratic problems can be found in the recent handbook [16], and references therein.

In this note we study a non-convex quadratic optimization problem that remained largely unexplored in the literature, and that appears quite challenging.

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More precisely, we study the problem which consists of maximizing a quadratic form over the ℓ_1 unit ball

$$[QPL1] \quad \max\{x^T Qx : \|x\|_1 \leq 1, x \in \mathbb{R}^n\}.$$

When the quadratic form is positive semidefinite the problem is trivial, and admits a closed form solution, unlike its ℓ_∞ counterpart which is as hard as an optimization problem can be! When the matrix is indefinite, this problem is particularly difficult as pointed out in the recent work of Nesterov [13]. As we shall see, the l_1-ball structure leads to challenging difficulties, and the main purpose of this note is to further clarify the difficulties involved, to report on our current findings, including the deceptive ones (for didactical purposes!), and to promote the study of this problem by the mathematical programming researchers. We derive several reformulations of the problem, including upper bounds formulated as SDP-like but nonconvex problems. We then show that a convex SDP-like relaxation proposed by Nesterov [13] without any justification on its source, can be derived by using a simple but quite unusual variational representation of the l_1 norm. It is also proved that the non-convex upper bound formulations are at least as good as the convex SDP relaxation. Finally, we establish by elementary arguments, that one of our bounds is exact, thus extending a recent result of [18] to a class of [QPL1] problems. The paper also includes an appendix on duality and lifting to make the exposition self-contained.

Since the SDP relaxation method is not the only approach to tackle optimization problems with quadratic objective and constraint functions, we mention here the work on global optimality conditions for non-convex optimization problems developed in a series of papers by Hiriart-Urruty, see *e.g.*, [2–4] and Jeyakumar *et al.* [6,7]. Hiriart-Urruty develops a general global optimality condition based on a generalized subdifferential concept, and specializes the condition to several problems of non-convex optimization, including maximization of a convex quadratic function subject to strictly convex quadratic inequalities, minimization of a quadratic function subject to a single quadratic inequality (trust-region problem), and subject to two quadratic inequalities (two-trust-region problem). Hiriart-Urruty also obtains conditions that are both necessary and sufficient in [2–4] for non-convex quadratic programs. Jeyakumar *et al.* use a generalized global subdifferential to give necessary and sufficient optimality conditions for minimization of a quadratic function subject to quadratic constraints. While global optimality conditions are not the subject of the present paper, a future application of the aforementioned results to problem [QPL1] may lead to further progress in the study of [QPL1].

We conclude this section by defining our notation which is fairly standard. We denote by S_n the space of $n \times n$ symmetric matrices equipped with the inner product $\langle X, Y \rangle := \mathrm{tr}(XY)$, $\forall X, Y \in S_n$, where tr denotes the trace operator. For $X \in S_n$, $X \succeq 0$ means X is positive definite; $d(X) \in \mathbb{R}^n$ is the diagonal of X; $\lambda_{\max}(X) \equiv \lambda_1(X) \geq \ldots \geq \lambda_n(X) \equiv \lambda_{\min}(X)$ are the eigenvalues of X. For $v \in \mathbb{R}^n$, we denote by $\mathrm{Diag}(v)$ the diagonal matrix with entries v_i. Finally, for any scalar function $f(t)$, and $x \in \mathbb{R}^n$, $f(x)$ denotes the vector with entries $f(x_i)$.

2. The trivial case and trivial bounds

Consider the problem of maximizing a quadratic form with $Q \in S_n$ on the l_1-unit ball $\mathbf{B}_1$ defined by

$$[QPL1] \quad v_* := \max\{x^T Qx : \|x\|_1 \leq 1\} \equiv \max\{x^T Qx : x \in \mathbf{B}_1\}.$$

Trivial case. $Q \succeq 0$ Interestingly with $Q \in S_n$ positive semidefinite the problem is trivial, unlike its ℓ_∞ ball constraint counterpart which is as hard as an optimization problem can be! The problem consists of maximizing a convex function over the l_1-unit ball, a compact convex set. It is well-known then, that the maximum must occur at some extreme point of $\mathbf{B}_1 = \{x : \|x\|_1 \leq 1\}$. Since the set of extreme points of $\mathbf{B}_1$ is simply given by $\{e_1, -e_1, \ldots, e_n, -e_n\}$, where e_i are the base vectors of $\mathbb{R}^n$, it follows immediately that $v_* := \max\{x^T Qx : \|x\|_1 \leq 1\} = \max_{1 \leq i \leq n} Q_{ii}$, with the max attained at $x^* = e_j$ for the index j corresponding to the maximum diagonal element Q_{jj}.

The trivial bounds. For any $x \in \mathbb{R}^n$, we have: $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$. Therefore one has,

$$\max\{x^T Qx : \|x\|_2 \leq n^{-1/2}\} \leq v_* \leq \max\{x^T Qx : \|x\|_2 \leq 1\},$$

namely $n^{-1}\lambda_{\max}(Q) \leq v_* \leq \lambda_{\max}(Q)$.

As usual, for any given nonconvex optimization problem, the main questions are then:

(a) Can we find better bounds that are computationally tractable?
(b) Can we determine the quality/tightness of such bounds?

As we shall see, trying to answer positively both questions for the [QPL1] problem leads to some conflicting situations.

3. Generating bounds: direct approach

Consider the following prototype problem:

$$(P) \quad \max\{x^T Qx : x^2 \in \mathcal{F}\},$$

where $\mathcal{F}$ is a closed and bounded, but non-convex, subset of $\mathbb{R}^n$.

When $\mathcal{F}$ is a convex set, problem (P) encompasses several interesting quadratic models, see *e.g.*, [17,18]. A standard way to build a relaxation of (P), *i.e.* to derive a bound, is *via* duality, (more precisely *via* bi-duality), see [15]), or *via* the lifting procedure of Lovasz-Schrijver, [11]. For completeness and the interested reader, we have included in an appendix a concise summary of the relevant results of both approaches.

The trouble with the [QLP1] problem, is that as formulated, it is *not* a quadratic representable problem, (see appendix). Indeed, the [QPL1] problem, can be written in the form of problem (P) as:

$$v_* := \max\{x^T Qx : x^2 \in \mathcal{F}\}$$

where $\mathcal{F} = \{y : \sum_{j=1}^n \sqrt{y_j} \leq 1, y \geq 0\}$. Then, the lifting procedure results in the *non-convex* bound obtained from problem (R):

$$v_R = \max_{X \in S_n} \left\{ \langle Q, X \rangle : \sum_{j=1}^n \sqrt{X_{jj}} \leq 1 \ X \succeq 0 \right\}.$$

On the other hand, for $\mathcal{F} = \left\{ y : \sum_{j=1}^n \sqrt{y_j} \leq 1, y \geq 0 \right\}$, its convex hull is simply,

$$\operatorname{conv} \mathcal{F} = \left\{ y : \sum_{j=1}^n y_j \leq 1, y \geq 0 \right\}.$$

Then, since $\mathcal{F} \subset \operatorname{conv} \mathcal{F}$, an additional relaxation yields the convex bound:

$$v_{\text{convex}} := v_c = \max\{\langle Q, X \rangle : \ \operatorname{tr}(X) \leq 1, \ X \succeq 0\}.$$

Clearly we have $v_* \leq v_R \leq v_c$, with the unfortunate situation that (R) is a non-convex problem. Moreover, it turns out that v_c is nothing else but the trivial bound! Indeed, a dual of v_c is:

$$\inf_{t \geq 0} \{t + \max_{X \succeq 0} \{\langle Q - tI, X \rangle\} = \inf_{t \geq 0} \{t : \ Q - tI \preceq 0\} = \inf_{t \in \mathbb{R}} \{t : Q - tI \preceq 0\} \equiv \lambda_{\max}(Q),$$

where we dropped the non-negativity restriction on t, since Q is indefinite.

We summarize this double deceptive situation in:

Proposition 1. *Let v_*, v_c, v_R denote respectively optimal value of [QPL1], its convex bound and its non-convex one. Then, $v_* \leq v_R \leq v_c = \lambda_{\max}(Q)$.*

This naturally gives rise to the following open problem:

Problem 1. Given the very special structure of v_R, can we find a suitable/efficient method to compute an approximate solution to this non-convex problem, and to evaluate the quality of the resulting approximation?

We provide a partial answer to this problem in Section 4.3.

4. Deriving bounds using other representations of the l_1 norm

4.1. A simple representation

The starting point is the following. Observe that for any $x \in \mathbb{R}^n$:

$$\|x\|_1 \leq 1 \iff \exists\, v_1, \ldots, v_n \geq 0 \ |x_i| \leq v_i,\ \forall i \in [1, n],\ \sum_{i=1}^{n} v_i \leq 1. \tag{1}$$

Indeed, if $\|x\|_1 \leq 1$, then with $v_i = |x_i|$, the RHS of (1) is satisfied. Conversely, summing the inequalities over $i = 1, \ldots, n$ yields $\|x\|_1 \leq \sum_i v_i \leq 1$.

Note that if $v_i = 0$ for some i then the corresponding $x_i = 0$. Thus in the sequel w.l.o.g., we will write $v \geq 0$, although it should be understood that $v > 0$.

Now,

$$|x_i| \leq v_i,\ v_i \geq 0 \iff x_i^2 \leq v_i^2, \forall i \in [1, n].$$

Define

$$V = \{(x, v) : x_i^2 \leq v_i^2,\ i = 1, \ldots, n, e^T v \leq 1, v \geq 0\}. \tag{2}$$

Thus,

$$x \in \mathbf{B}_1 \iff (x, v) \in V.$$

In view of this, the original problem [QPL1] can thus be written as

$$v_* = \max\{x^T Q x : (x, v) \in V\}.$$

Now, define

$$\mathcal{F} = \left\{ (y, z) \in \mathbb{R}_+^n \times \mathbb{R}^n :\ y \geq z,\ \sum_{j=1}^{n} \sqrt{y_j} \leq 1 \right\},$$

then using (2), [QPL1] can be written as

$$\max\{w^T Q' w :\ w^2 \in \mathcal{F}\}$$

where Q' is the block diagonal matrix in S^{2n}, with $0, Q$ in diagonal and 0 in off-diagonal, and $w = (v, x)$. In that case the lifting procedure yields the bound

$$\max \left\{ \langle Q, X \rangle :\ v \geq d(X), X \succeq 0,\ \sum_{j=1}^{n} \sqrt{v_j} \leq 1 \right\} \equiv \max\{\langle Q, X \rangle :\ (X, v) \in \mathcal{S}\}$$

where

$$\mathcal{S} = \left\{ v \geq d(X), X \succeq 0,\ \sum_{j=1}^{n} \sqrt{v_j} \leq 1 \right\}.$$

Thus, as in (1), since

$$\forall X \in S_n,\ \exists v \geq 0, v \geq d(X), X \succeq 0,\ \sum_{j=1}^{n} \sqrt{v_j} \leq 1 \iff \sum_{i=1}^{n} \sqrt{X_{ii}} \leq 1, X \succeq 0$$

it follows that the above bound is nothing else but the non-convex bound v_R:

$$\max\{\langle Q, X\rangle :\ \sum_{j=1}^{n} \sqrt{X_{jj}} \leq 1, X \succeq 0\}.$$

Applying one more relaxation by taking either the convex hull of $\mathcal{F}$ or the convex hull of $\mathcal{S}$ yields in both cases two convex bounds that look different, but which deceptively are equal to the trivial bound, *i.e.*,

$$\max\{\langle Q, X\rangle :\ v \geq d(X), X \succeq 0,\ e^T v \leq 1\} \equiv \\ \max\{\langle Q, X\rangle :\ \mathrm{tr}(X) \leq 1,\ X \succeq 0\} \equiv \lambda_{\max}(Q), \quad (3)$$

where the first equality follows by using the same argument of (1).

In addition to the lifting strategy by Lovasz and Schrijver [11] that we used in the present paper, a recent paper by Lasserre [8] introduces a new lifting procedure which would be applicable to the formulation of QPL1 in terms of the set V as given above. This is important because, for the special case of 0-1 optimization, the Lasserre procedure is known to be tighter than Lovasz-Schrijver lifting procedure; see [10]. Therefore, it is possible that the more recent lifting procedure provides a means to make progress on this problem in the future. Since the procedure of Lasserre has been implemented [9] its application to the QPL1 problem of this paper can be an interesting line of further research.

4.2. Using a variational representation of l_1

We will work with the equivalent formulation of $[QPL1]$ written as:

$$\begin{array}{rl} \max & x^T Q x \\ \text{s.t.} & \|x\|_1^2 \leq 1. \end{array}$$

The next simple result gives a somewhat unusual variational representation of the l_1 norm in $\mathbb{R}^n$, which as we shall see turns out to be quite useful.

Lemma 1. *For any $x \in \mathbb{R}^n$ one has*

$$\|x\|_1^2 = \min\{x^T \operatorname{Diag}(v^{-1}) x : e^T v \leq 1, v > 0\},$$

where $\operatorname{Diag}(v^{-1})$ *stands for the positive definite diagonal matrix with diagonal elements* $\{v_j^{-1} :\ j = 1, \ldots, n.\}$.

Proof. This follows easily from the optimality conditions of the optimization problem defined in the right-hand side, which gives the unique minimizer $\bar{v} := |x| \|x\|_1^{-1}$, or directly just by using Cauchy-Schwarz inequality. □

Note that the constraint can include the boundary *i.e.*, $v \geq 0$; that does not change anything in our case. In the sequel we use the notation $\Delta := \{v \in \mathbb{R}^n : e^T v \leq 1, v > 0\}$.

Using Lemma 1, problem [QPL1] is equivalent to

$$\max_x \{x^T Q x : \min_{v \in \Delta} x^T \operatorname{Diag}(v^{-1}) x \leq 1, v \in \Delta\} = \max_{x,v} \{x^T Q x : x^T \operatorname{Diag}(v^{-1}) x \leq 1, v \in \Delta\}. \quad (4)$$

Using the lifting procedure we obtain the *non-convex* bound:

$$m_R := \max_{X \succeq 0} \{\langle Q, X \rangle : \min_{v \in \Delta} \langle X, \operatorname{Diag}(v^{-1}) \rangle \leq 1\} = \max_{X \succeq 0, v} \{\langle Q, X \rangle : \langle X, \operatorname{Diag}(v^{-1}) \rangle \leq 1, v \in \Delta\}.$$

This non-convex bound is in fact equal to the non-convex bound v_R. Indeed, taking $x_i := \sqrt{X_{ii}}$ in Lemma 1 gives

$$\left(\sum_{i=1}^n \sqrt{X_{ii}} \right)^2 = \min\{\langle X, \operatorname{Diag}(v^{-1}) \rangle : v \in \Delta\},$$

and hence it follows that $m_R = v_R$.

The variational representation of the l_1 norm given in Lemma 1 appears to produce equally deceptive results! The so-called Schur complement recalled below, is the cure to turn this situation around, and will allow us to explain how to derive a convex bound proposed by Nesterov [13], who did not give the source of its derivation, and indicated that its quality is unknown. We first recall two useful results from matrix analysis.

Lemma 2. *(a) (Schur Complement) Let $A \succ 0$, and let $S = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ be a symmetric matrix. Then,*

$$S \succeq 0 \iff C - BA^{-1}B^T \succeq 0.$$

(b) Let $A, D \in S_n$, and assume $A \succ 0$, $D \succeq 0$. Then, $A - D \succeq 0 \iff \rho(DA^{-1}) \leq 1$, where for any $M \in S_n$, $\rho(M) := \max_{1 \leq i \leq n} |\lambda_i(M)|$.

Proof. (a) and (b), see *e.g.*, [5] (p. 472, and p. 471). □

Proposition 2. *A convex upper bound for [QPL1] is given by*

$$v_N := \max\{\langle Q, X \rangle : \operatorname{Diag}(v) - X \succeq 0, \ e^T v \leq 1, \ \ X \succeq 0\}.$$

Moreover, one has

$$v_* \leq v_R = m_R \leq v_N \leq \lambda_{\max}(Q).$$

Proof. Using (4), one has:

$$v_* = \max\{x^T Qx : \ x^T \operatorname{Diag}(v^{-1})x \leq 1, v \in \Delta\}.$$

Applying Lemma 2 with $A = \operatorname{Diag}(v), B = x$ and $C = 1$ one has:

$$x^T \operatorname{Diag}(v^{-1})x \leq 1 \iff \operatorname{Diag}(v) - xx^T \succeq 0.$$

Therefore, using the lifting procedure, the desired bound v_N follows. Now, since $\operatorname{Diag}(u) - X \succeq 0$ implies that $u \geq d(X)$ and also $u \geq 0$ by positive semidefiniteness of X, it follows *via* (3), that $v_N \leq \lambda_{\max}(Q)$.

Now, recall that

$$m_R := \max_{X,v}\{\langle Q, X\rangle : \ \langle X, \operatorname{Diag}(v^{-1})\rangle \leq 1, v \in \Delta\}.$$

Let (X, v) be a feasible solution of the latter. Then, $e^T v \leq 1, \ v > 0$, and since $X \operatorname{Diag}(v^{-1}) \succeq 0$, one has

$$\rho(X \operatorname{Diag}(v^{-1})) \leq \operatorname{tr}(X \operatorname{Diag}(v^{-1})) \leq 1.$$

Invoking Lemma 2b, this implies that $\operatorname{Diag}(v) - X \succeq 0$, and hence that the feasible set of the relaxation m_R is contained in the set of feasible solutions of the relaxation v_N, so that $m_R \leq v_N$. Since we already established that $v_R = m_R$, the proof is complete. □

4.3. Exact bounds

We now ask if it is possible to identify problems where any of the bounds derived in this note, can be exact, *i.e.*, can provide an optimal solution of [QPL1]. Interestingly, the nonconvex bound v_R allows for extending a result of Zhang (Th. 2 of [18]) to a class of problem [QPL1], and for which the bound v_R coincides with v_*. The proof is very simple and relies on elementary arguments, compare with [18], Theorem 2.

Proposition 3. *For any given $Q \in \mathcal{S}^n$ with $Q_{ij} \geq 0$ for all (i, j) such that $i \neq j$, one has $v_* = v_R$. Moreover, from any optimal solution X of (R) one obtains an optimal solution x of [QPL1] according to the formula*

$$x_j = \sqrt{X_{jj}}, \forall j = 1, \ldots, n.$$

Proof. From the development above, we immediately have that if X is an optimal solution of (R), then for the point $x_j = \sqrt{X_{jj}}, \ j = 1, \ldots, n$, one has:

$$\|x\|_1 = \sum_{j=1}^{n} \sqrt{X_{jj}} \leq 1,$$

i.e., a feasible point x for [QPL1] and $v_* \le v_R$. We seek conditions under which this inequality is an equality. Assume we have an optimal solution X of problem (R). Let us examine now the values of the respective objective functions in [QPL1] and (R) corresponding to x and X. As x is a feasible point for [QPL1] we have

$$v_* \ge \sum_{i=1}^{n} Q_{ii}x_i^2 + 2\sum_{i<j} Q_{ij}x_ix_j = \sum_{i=1}^{n} Q_{ii}X_{ii} + 2\sum_{i<j} Q_{ij}\sqrt{X_{ii}X_{jj}}.$$

On the other hand, we have in (R):$v_R = \sum_{i=1}^{n} Q_{ii}X_{ii} + 2\sum_{i<j} Q_{ij}X_{ij}$, and thus to have the inequality $v_* \ge v_R$ it is sufficient to verify that the following holds:

$$\sum_{i=1}^{n} Q_{ii}X_{ii} + 2\sum_{i<j} Q_{ij}\sqrt{X_{ii}X_{jj}} \ge \sum_{i=1}^{n} Q_{ii}X_{ii} + 2\sum_{i<j} Q_{ij}X_{ij},$$

which is the same as

$$\sum_{i<j} Q_{ij}(\sqrt{X_{ii}X_{jj}} - X_{ij}) \ge 0.$$

But, since we assumed that $Q_{ij} \ge 0 \ \forall\ i \ne j$, and since for $X \succeq 0$, one has $\sqrt{X_{ii}X_{jj}} - X_{ij} \ge 0$ for all i, j such that $i \ne j$, the above inequality trivially holds. □

Another open question was raised during our computational experimentation with the Nesterov relaxation, *i.e.* the bound v_N. We have observed that whenever Q has non-negative off-diagonal elements, v_N was exact, although we were not able to prove a result to that effect, *i.e.*, a result similar to Proposition 3 seems to hold for the Nesterov relaxation. We pose this as another open problem raised by our study.

Problem 2. Given Q with non-negative off-diagonal elements, is it true that $v_* = v_N$?

5. Appendix: duality and lifting

5.1. The dual approach

When building a relaxed problem, for short a relaxation, the main objective is to get a tractable relaxation and to have it as tight as possible. A well-known way of generating relaxed problems is *via* duality, or more precisely *via* bi-duality, which provides a built-in convexification process. Shor [15], was the first to realize the power of such a duality framework for generating useful bounds to non-convex quadratic optimization problems.

Let us first recall some useful well-known convex analysis results, for more details and notations see [14]. For a non-convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ there is a natural procedure to convexify it *via* the use of the convex hull of the epigraph of f. Let $\operatorname{conv} f$ denotes the convex hull of f, which is the greatest convex function majorized by f.

A key player in any duality framework is the conjugate of a given function.

Definition 1. For any function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(y) := \sup_x \{\langle x, y\rangle - f(x)\}$$

is called the conjugate to f. The bi-conjugate of f is defined by

$$f^{**}(x) := \sup_y \{\langle x, y\rangle - f^*(y)\}.$$

Whenever $\operatorname{conv} f$ is proper, one always has that both f^* and f^{**} are proper, lower semi-continuous (lsc), and convex, and the following relations hold:

$$f^{**} = \operatorname{cl}(\operatorname{conv} f) \quad \text{and} \quad f^{**} \le \operatorname{cl} f,$$

where $\operatorname{cl} f(x) := \liminf_{y \to x} f(x)$. In particular, if $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex, then the conjugate function f^* is proper, lsc, and convex if and only if f is proper. Moreover, $(\operatorname{cl} f)^* = f^*$ and $f^{**} = \operatorname{cl} f$.

For any set $C \subset \mathbb{R}^n$, we denote by δ_C its indicator function, namely $\delta_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise. From the above results, it follows that when $C \subset \mathbb{R}^n$ is a nonempty convex set, then:

$$(\delta_C)^{**} = \operatorname{cl} \delta_C = \delta_{\operatorname{cl} C}. \tag{5}$$

Consider now the prototype quadratic problem

$$(P) \quad \max\{x^T Q x : \ x^2 \in \mathcal{F}\},$$

where $\mathcal{F}$ is a closed and bounded, but non-convex, subset of $\mathbb{R}^n$. This problem is equivalent to

$$(P) \quad v_P := \max_{x,y}\{x^T Q x : \ x^2 = y, \ y \in \mathcal{F}\}.$$

Thus, a dual of (P) is:

$$(D) \quad \inf_{u \in \mathbb{R}^n} \sup_{x,y} \{x^T (Q - \operatorname{Diag}(u))x + u^T y - \delta_{\mathcal{F}}(y)\},$$

and since $\max_{x \in \mathbb{R}^n} x^T A x < \infty$ if and only if $A \preceq 0$, this reduces to

$$(D) \quad v_D := \inf\{\delta^*_{\mathcal{F}}(u) : \ Q - \operatorname{Diag}(u) \preceq 0\}.$$

By construction the dual problem (D) is convex and weak duality implies that for any pair of primal-dual feasible points for problems (P)-(D), one has $v_P \le v_D$, so that v_D yields an upper bound for (P). Furthermore, since Slater's condition is here trivially satisfied for (D), we can generate another bound equal to v_D by taking the dual of (D), namely the bi-dual of (P) is:

$$(DD) \quad \sup_{X \succeq 0} \inf_{u \in \mathbb{R}^n} \{\langle Q, X\rangle + \delta^*_{\mathcal{F}}(u) - \langle Q, \mathrm{Diag}(u)\rangle\},$$

which reduces using (5) to

$$\sup\{\langle Q, X\rangle - \delta^{**}_{\mathcal{F}}(d(X)), X \succeq 0\} = \sup\{\langle Q, X\rangle : \ d(X) \in \mathrm{cl}\,\mathrm{conv}\,\mathcal{F}, \ X \succeq 0\}.$$

Since we assumed that $\mathcal{F}$ is bounded and closed, then

$$\mathrm{cl}\,\mathrm{conv}\,\mathcal{F} = \mathrm{conv}\,\mathrm{cl}\,\mathcal{F} = \mathrm{conv}\,\mathcal{F},$$

so that the bi-dual reads $(DD) \quad \sup\{\langle Q, X\rangle : \ d(X) \in \mathrm{conv}\,\mathcal{F}, \ X \succeq 0\}$, and we have $v_P \leq v_D = v_{DD}$.

5.2. The lifting approach

For quadratic problems, there is another systematic way for building relaxed problems and corresponding bounds, often called lifting, which is due to Lovasz and Schrijver [11]. The idea is based on enlarging the feasible set of the original problem to get an upper bound. For quadratic problems, this process is essentially *automatic*, *via* a rewriting of such problems with new variables. This is due to the simple well-known fact that:

$$x^T Q x = \langle Q, xx^T\rangle.$$

Indeed, using this in (P), the original problem in $\mathbb{R}^n$ can be equivalently rewritten in $(X, x) \in S^n \times \mathbb{R}^n$ as:

$$\max_{X,x}\{\langle Q, X\rangle : \ X = xx^T, d(X) \in \mathcal{F}\}.$$

Replacing the non-convex constraint $X = xx^T$, by its convex counter part $X - xx^T \succeq 0$, and noting that here x does not play any role in the max problem, we end-up with the relaxed problem:

$$(R) \quad \max\{\langle Q, X\rangle : \ d(X) \in \mathcal{F}, \ X \succeq 0\},$$

and we have $v_P \leq v_R$, where v_R denotes the optimal value of (R).

The interesting point to notice here is that since $\mathcal{F}$ was assumed to be non-convex, the resulting lifting procedure *does not* automatically produce a *convex relaxation*, as opposed to the dual approach. Since one obviously has

$$\{(X, x) : \ d(X) \in \mathcal{F}, \ X \succeq 0\} \subset \{(X, x) : \ d(X) \in \mathrm{conv}\,\mathcal{F}, \ X \succeq 0\},$$

then we have $v_R \leq v_{DD}$, namely we end up with the evil situation where the *non-convex* relaxation provides a "better" bound than its convex counterpart. Evidently, using one more relaxation for v_R, namely using $\mathcal{F} \subset \operatorname{conv} \mathcal{F}$, we end-up with the problem that we call v_{RR} and the two procedures coincide, *i.e.*, one has $v_* \leq v_{RR} = v_{DD}$.

The above discussion is to emphasize that when we are not dealing with *truly quadratically representable* optimization problems, the lifting procedure does not provide *automatically* a tractable convex bound. This is precisely the situation we have encountered with the [QLP1], which *is not*, a quadratic representable problem in the sense described above, but which has been shown to be quadratic transformable, thanks to the particular variational representation of the l_1-norm given in Lemma 1, and the Schur complement.

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