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A VARIATIONAL MODEL FOR EQUILIBRIUM PROBLEMS IN A TRAFFIC NETWORK *

Giandomenico Mastroeni[1] and Massimo Pappalardo[2]

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Abstract. We propose a variational model for one of the most important problems in traffic networks, namely, the network equilibrium flow that is, traditionally in the context of operations research, characterized by minimum cost flow. This model has the peculiarity of being formulated by means of a suitable variational inequality (VI) and its solution is called "equilibrium". This model becomes a minimum cost model when the cost function is separable or, more general, when the Jacobian of the cost operator is symmetric; in such cases a functional representing the total network utility exists. In fact in these cases we can write the first order optimality conditions which turn out to be a VI. In the other situations (*i.e.*, when global utility functional does not exist), which occur much more often in the real problems, we can study the network by looking for equilibrium solutions instead of minimum points. The Lagrangean approach to the study of the VI allows us to introduce dual variables, associated to the constraints of the feasible set, which may receive interesting interpretations in terms of potentials associated to the arcs and the nodes of the network. This interpretation is an extension and generalization of the classic Bellman conditions. Finally, we deepen the analysis of the networks having capacity constraints.

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[1] Department of Mathematics, via Buonarroti 2, 56127 Pisa, Italy;
e-mail: mastroen@dm.unipi.it

[2] Department of Applied Mathematics, via Bonanno 25/b, 56126 Pisa, Italy;
e-mail: pappalardo@dma.unipi.it

1. Introduction

The high increase of the volume of traffic registered in the recent years asks for effective mathematical models for the analysis of the road circulation especially in the urban areas.

Traffic assignment problems, that have been widely studied in the context of transportation network analysis, are characterized by several aspects among which we mention the management and the design of the road network (streets to be made one-way only, semaphorical waiting times, etc.), the knowledge of the traffic demand between origin-destination nodes and the definition of the equilibrium flows.

In this paper we study a variational model for the formulation of the equilibrium in traffic problems that overcomes, in the sense illustrated in the abstract, the minimal-cost approach.

In our analysis, we refer to a road traffic network, where each user aims to minimize its cost of transfer from a certain origin to a given destination; however, the considered models can be extended to other applications as economic or electric networks.

Historically, optimal flows were defined as the extremizers of a suitable functional. The natural criticism to this definition is based on the fact that there are very few real situations in which it is possible to ensure the existence of such a functional. Unlike the optimization models, the variational ones do not require the existence of such a functional. Indeed, in the variational model we consider as operator the variation of the cost with respect to the flow and we write the variational inequality. In this way when this operator "variation of the flow" is symmetric we have a functional to minimize (and our operator is the Jacobian of this functional), if it is not symmetric as many times occur in real situations, we study the VI but we do not have any functional to minimize.

We begin with a proposition that will be used in the sequel.

Let $K := \{x \in \mathbb{R}^n : g(x) \leq 0,\ h(x) = 0\}$. In the simplest form, a VI consists in finding $y \in K$ such that

$$\langle M(y), x - y\rangle \geq 0, \quad \forall x \in K, \qquad \mathrm{VI}(M, K)$$

where $M : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $h : \mathbb{R}^n \longrightarrow \mathbb{R}^p$.

Similarly to constrained extremum problem, we can associate to VI suitable Lagrangian-type optimality conditions in order to obtain primal-dual formulations of VI as stated by the following well-known result:

Proposition 1.1. *Suppose that the following conditions hold:*

i) g is a convex and differentiable function and there exists $\tilde{x} \in \mathbb{R}^n$ such that $g(\tilde{x}) < 0$;

ii) h is affine.

Then $x^* \in \mathbb{R}^n$ *is a solution of* VI*(M, K) if and only if there exist* $\lambda^* \in \mathbb{R}^p$, $\mu^* \in \mathbb{R}^m$ *such that* (x^*, μ^*, λ^*) *is a solution of the system*

$$\begin{cases} M(x) \ + \ \mu\nabla g(x) \ - \lambda\nabla h(x) = \ 0 \\ \langle \mu, g(x) \rangle \ = \ 0 \\ \mu \geq 0, \ g(x) \leq 0, \ h(x) = 0. \end{cases} \tag{1}$$

In Section 2 we will consider the equilibrium flow problem, formulated by means of a VI where the operator M is the variation of the cost with respect to the flows on the arcs of the network and it depends on the flow passing through. In Section 3, we show that the multipliers $\lambda_i^*, \ i = 1, \dots, p$ and $\mu_j^*, \ j = 1, \dots, m$ of Proposition 1.1, can be interpreted in terms of potentials associated to the nodes of the network, when VI represents the equilibrium condition of a traffic network problem and we can obtain a generalization of the classic Bellman conditions. In Section 4 we analyse the equivalence between a variational model with capacity constraints and this model without capacities, but with a suitable penalized operator.

2. The variational arc-flow model

In this section we consider a VI model formulated in terms of the flows on the arcs. This kind of model is easier to handle in the applications than the one which considers the flows on the paths, since the real data are very often related to the arcs of the network, instead of the paths: an example is given by the capacities, which, in the real applications, are, in general, given on the arcs.

In this section we will also consider the following further assumptions and notations:

- f_i is the flow on the arc $A_i := (r, s)$ and $f := (f_1, \dots, f_n)^T$ is the vector of the flows on all arcs;
- we assume that each arc A_i is associated with an upper bound d_i on its capacity, $d := (d_1, \dots, d_n)$. More general capacity constraints of the form $g(f) \leq 0$ can be considered where g is a convex and continuously differentiable function. These constraints can be related to links, nodes, routes, O-D pairs or any combination of them;
- $c_i(f)$ is the cost-variation on the arc A_i as function of the flows, $\forall i = 1, \dots, n$ and $c(f) := (c_1(f), \dots, c_n(f))^T$; we assume that $c(f) \geq 0$;
- q_j is the balance at the node j, $j = 1, \dots, p$ and $q := (q_1, \dots, q_p)^T$;
- $\Gamma = (\gamma_{ij}) \in \mathbb{R}^p \times \mathbb{R}^n$ is the node-arc incidence matrix whose elements are

$$\gamma_{ij} = \begin{cases} -1, & \text{if } i \text{ is the initial node of the arc } A_j, \\ +1, & \text{if } i \text{ is the final node of the arc } A_j, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Definition 2.1. The feasible set of the arc-flow model is defined by:

$$K_f(d) := \{f \in \mathbb{R}^n \ : \ \Gamma f = q, \ 0 \le f \le d\};$$

or by more general side constraints:

$$K_f(g) := \{f \in \mathbb{R}^n \ : \ \Gamma f = q, \ f \ge 0, \ g(f) \le 0\}.$$

Consider now the following VI:

$$\text{find } f^* \in K_f \ \text{ s.t. } \ \langle c(f^*), f - f^* \rangle \ge 0, \quad \forall f \in K_f, \tag{3}$$

where

$$K_f := \{f \in \mathbb{R}^n \ : \ \Gamma f = q, \ f \ge 0\}.$$

Definition 2.2. A flow f is a variational pre-equilibrium flow for the capacitated model if and only if f solves (3); moreover a variational pre-equilibrium flow is called equilibrium flow if and only if $g(f) \le 0$ (or, in particular, $f \le d$).

The problem (3) collapses to the minimal-cost network-flow problem when the function $c(f)$ is independent of f, namely, $c(f) := (c_{ij}, (i,j) \in A)$.

Example 2.1. Easy examples shows that, also in the separable case, a solution of the variational model can substantially differ from that obtained using the minimum cost flow model, which is classic in the context of operations research. We solve (3) by means of the gap function approach solving the following constrained extremum problem:

$$\min_{f \in K_f} g(f) := \sup_{x \in K_f} \left\{ \langle c(f), f - x \rangle - \|x - f\|^2 \right\} \cdot$$

Let

$$\Gamma = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$q = (-2, 0, 0, 0, 0, 2)^T, \qquad d = (2, 1, 1, 1, 1, 1, 2, 2)^T.$$

The cost function is defined by $c(f) := Cf$ where C is the diagonal matrix with components on the diagonal given by the vector $D := (5.5, 1, 2, 3, 4, 50, 3.5, 1.5)$.

The solution of (3) and the potentials at the nodes are given by the vectors

$$f^* := \begin{pmatrix} 1.000 \\ 1.000 \\ 0.158 \\ 0.842 \\ 0.885 \\ 0.115 \\ 1.042 \\ 0.958 \end{pmatrix} \qquad \lambda^* := \begin{pmatrix} 9.464 \\ 3.964 \\ 7.189 \\ 3.649 \\ 1.437 \\ 0 \end{pmatrix}.$$

Suppose, now, that the cost function is a constant $c(f) := D$. In this case (3) collapses to the classic minimal-cost flow problem. The optimal solution and the potentials at the nodes are given by the vectors:

$$f^* := (1, 1, 0, 1, 1, 0, 1, 1)^T,$$

$$\lambda^* := \begin{pmatrix} 10.495 \\ 4.995 \\ 8.382 \\ 3.500 \\ 1.500 \\ 0 \end{pmatrix}.$$

Remark 2.1. The arc-flow model can be related to the path-flow model. Some further notations are:

- m is the total number of the considered paths and $F := (F_1, \dots, F_m)^T$ is the vector of the relative flows;
- we will suppose that, the nodes of the couple W_j are connected by the (oriented) paths, R_i, $i \in P_j \subseteq \{1, \dots, m\}$, $\forall j = 1, \dots, \ell$;
- ρ_j is the traffic demand for W_j, $j = 1, \dots, \ell$, $\rho := (\rho_1, \dots, \rho_\ell)^T$;
- $\Phi = (\phi_{ij}) \in \mathbb{R}^\ell \times \mathbb{R}^m$ is the couplets-paths incidence matrix whose elements are

$$\phi_{ij} = \begin{cases} 1, & \text{if } W_i \text{ is connected by the path } R_j, \\ 0, & \text{otherwise;} \end{cases} \tag{4}$$

- let $\Delta = \{\delta_{is}\}$ be the Kronecker matrix, where

$$\delta_{is} = \begin{cases} 1, & \text{if } A_i \in R_s \quad i = 1, \dots n \\ 0, & \text{if } A_i \notin R_s \quad s = 1, \dots m \end{cases}$$

and assume that the cost $C_s(F)$ can be expressed as the sum of the costs on the arcs of R_s:

$$C_s(F) = \sum_{i=1}^{n} \delta_{is} c_i(f).$$

In order to obtain an arc-flow formulation of the path-flow model it is necessary to define the feasible set of the *path-arc-flow* model

$$K_{f,F} := \{f \in \mathbb{R}^n : f = \Delta F, \Phi F = \rho,\ F \geq 0\}\cdot$$

Proposition 2.1. *$VI(c, K_f)$ is equivalent to $VI(C, K_{f,F})$.*

Proof. By the Kronecker matrix the flows on the arcs can be expressed in terms of the flows on the paths $f_i = \sum_{s=1}^{m} \delta_{is} F_s$. Therefore $f = \Delta F$ and $C(F) = \Delta^T c(f)$. The previous relations lead to the following equalities:

$$\langle C(H), F - H\rangle = c^T(f^*)\Delta(F - H) = \langle c(f^*), f - f^*\rangle,$$

where we have put $f^* := \Delta H$. The transformation of K_f in $K_{f,F}$ is obtained using standard arguments (see, for example, [8]). □

An advantage of the path-arc-flow model lies, for example, in the possibility of adding capacity constraints on the arcs in the feasible set $K_{f,F}$, even though the traffic demand is related to the couples O-D. We remark that in order to adopt the arc-flow model in the standard form (3), it is necessary that the traffic demand is related only to the nodes of the network.

3. Potentials and dual variables

We can apply Proposition 1.1 in order to obtain a primal-dual formulation of the $VI(c, K_f(d))$.

Proposition 3.1. *f^* is a solution of the* $\mathrm{VI}(c, K_f(d))$ *if and only if there exists $(\lambda^*, \mu^*) \in \mathbb{R}^{p\times n}$ such that (f^*, λ^*, μ^*) is a solution of the system*

$$\begin{cases} c(f)\ +\ \lambda\Gamma\ +\ \mu\ \geq\ 0 \\ \langle c(f) + \lambda\Gamma + \mu, f\rangle\ =\ 0 \\ \langle f - d, \mu\rangle = 0 \\ 0 \leq f \leq d,\ \Gamma f = q,\ \mu \geq 0. \end{cases} \tag{5}$$

We remark that, in order to follow the notation used in the theory of potentials, with no loss of generality, we have changed in (5) the sign of the multiplier λ (w.r.t. the statement of Prop. 1.1). Now we analyse the system (5).

The case without capacity constraints

Suppose, at first, that there are no capacity constraints on the arcs so that $d_{ij} = +\infty$, $\forall (i,j) \in A$. Then (5) becomes

$$\begin{cases} c(f)\ +\ \lambda\Gamma\ \geq\ 0 \\ \langle c(f) + \lambda\Gamma, f\rangle\ =\ 0 \\ \Gamma f = q,\ f \geq 0. \end{cases} \tag{6}$$

The question that now arises is to establish whether or not it is possible to find an equivalent formulation of (6) in terms of an equilibrium principle. It is easy to see that f^* fulfills (6) if and only if there exists $\lambda^* \in \mathbb{R}^p$ such that, $\forall (i,j) \in A$:

$$f^*_{ij} > 0 \Longrightarrow c_{ij}(f^*) = \lambda^*_i - \lambda^*_j \tag{7}$$

$$f^*_{ij} = 0 \Longrightarrow c_{ij}(f^*) \geq \lambda^*_i - \lambda^*_j. \tag{8}$$

We observe that the dual variables corresponding to the flow conservation constraints can be interpreted in terms of potentials at the nodes of the network. Actually, from (7) we deduce that

$$f^*_{ij} > 0 \Longrightarrow \lambda^*_i - \lambda^*_j \geq 0, \tag{9}$$

that is, a necessary condition for the arc (i,j) to have a positive flow is that the difference of potential between the nodes i and j is positive. *Vice versa*, from (9) we deduce that

$$\lambda^*_i - \lambda^*_j < 0 \Longrightarrow f^*_{ij} = 0, \tag{10}$$

that is, the negativity of the difference of potentials between nodes i and j is a sufficient condition in order to have $f^*_{ij} = 0$.

The case of capacity constraints

A straightforward extension of the relations (7) and (8) can be obtained in the presence of capacity constraints applying directly Proposition 3.1. We can state the following equilibrium principle, which is of immediate proof.

Theorem 3.1. *f^* is a solution of the* VI *(3) if and only if there exist $\lambda^* \in \mathbb{R}^p$ and $\mu^* \in \mathbb{R}^n_+$ such that, $\forall (i,j) \in A$:*

$$0 < f^*_{ij} < d_{ij} \Longrightarrow c_{ij}(f^*) = \lambda^*_i - \lambda^*_j, \quad \mu^*_{ij} = 0, \tag{11}$$

$$f^*_{ij} = 0 \Longrightarrow c_{ij}(f^*) \geq \lambda^*_i - \lambda^*_j, \quad \mu^*_{ij} = 0, \tag{12}$$

$$f^*_{ij} = d_{ij} \Longrightarrow c_{ij}(f^*) = \lambda^*_i - \lambda^*_j - \mu^*_{ij}. \tag{13}$$

We observe that the relations (9) and (10) are still valid and that (11)–(13) collapse to (7) and (8) when $d_{ij} = +\infty$, $\forall (i,j) \in A$.

Example 3.1. Consider the problem introduced in the Example 2.1. Note that the existence of a positive flow between the nodes i and j implies that $\lambda_i > \lambda_j$, according to (9).

We also remark that the multipliers μ_{ij} that appear in (13) can be interpreted as an additional cost to be added to $c_{ij}(f^*)$ in order to achieve the equivalence with the difference of potentials $\lambda^*_i - \lambda^*_j$. This aspect of the analysis will be developed in the next section.

4. About the equivalence between a model with capacities and one without capacities

In this section we will deepen the analysis of the model with capacity constraints. When the feasible set contains capacity constraints on the arcs, a research direction is given by the reformulation of the model with capacities by means of a model without capacities, but with a different operator (see, for example, [3]). We will show that a variational pre-equilibrium flow is a variational equilibrium flow if we make a perturbation of the cost-operator and we will show that such a perturbation can be defined through the potentials (dual variables) in a way generalizing the results existing in literature.

In other words, under suitable assumptions, we will show that f^* is a solution of the $\mathrm{VI}(c, K_f(g))$ if and only if it is a solution of the following:

$$\langle c(f^*) - \Psi'_f(\alpha; f^*, f^*), f - f^* \rangle \geq 0, \quad \forall f \in K_f, \qquad \mathrm{VI}_\alpha(c, K_f)$$

where $\Psi : \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a function depending on the parameter $\alpha \in \mathbb{R}^k$ and Ψ'_f denotes the gradient of Ψ, w.r.t. its third component.

Let us begin with a first result.

Proposition 4.1. *Assume that* $\Psi : \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *fulfill the following assumptions,* $\forall \alpha \in \mathbb{R}^k$*:*

(i) $\Psi(\alpha; f, f) = 0, \quad \forall f \in K_f$*;*

(ii) $\Psi(\alpha; f, \cdot)$ *is a differentiable concave function on* $K_f, \quad \forall f \in K_f$*.*

Then f^* *is a solution of* $\mathrm{VI}_\alpha(c, K_f(g))$ *if and only if it is a solution of the* VI

$$\langle c(f^*), f - f^* \rangle - \Psi(\alpha; f^*, f) \geq 0, \quad \forall f \in K_f. \tag{14}$$

Proof. Taking into account assumption *(i)*, f^* is a solution of (14) if and only if f^* is an optimal solution of the problem

$$\min_{f \in K_f} \left[\langle c(f^*), f - f^* \rangle - \Psi(\alpha; f^*, f) \right]. \tag{15}$$

By *(ii)*, we have that (15) is a convex problem, so that $\mathrm{VI}_\alpha(c, K_f(g))$ is a necessary and sufficient optimality condition for (15), which completes the proof. □

We now have:

Theorem 4.1. *Suppose that* $\Psi(\alpha; f^*, f) := \sum_{i=1}^{v} \alpha_i (g_i(f^*) - g_i(f))$*, where* $g : \mathbb{R}^n \longrightarrow \mathbb{R}^v$ *is convex,* $\alpha_i \geq 0,\ i = 1, \ldots, v$ *and* $\exists f \in \mathbb{R}^n$ *such that* $g(f) < 0$*. Then* f^* *is a solution of* $\mathrm{VI}_\alpha(c, K_f(g))$ *if and only if there exists* $\alpha \in \mathbb{R}^v_+$ *such that* f^* *is a solution of* $\mathrm{VI}_\alpha(c, K_f)$ *with* $\sum_{i=1}^{v} \alpha_i g_i(f^*) = 0$*.*

Proof. First of all, we observe that Ψ fulfill the assumptions *(i)*,*(ii)* of the Proposition 4.1. Assume that f^* be a solution of $\mathrm{VI}_\alpha(c, K_f)$ for a suitable $\alpha \in \mathbb{R}^v_+$ with $\sum_{i=1}^{v} \alpha_i g_i(f^*) = 0$.

By Proposition 4.1 we have that

$$\langle c(f^*), f - f^* \rangle \;\geq\; \Psi(\alpha; f^*, f), \quad \forall f \in K_f(g),$$

taking into account that $K_f(g) \subseteq K_f$. Since, in our hypotheses,

$$\Psi(\alpha; f^*, f) := -\sum_{i=1}^{v} \alpha_i g_i(f),$$

we obtain that $\Psi(\alpha; f^*, f) \geq 0$, $\forall f \in K_f(g)$, so that f^* is a solution of $\mathrm{VI}(c, K_f(g))$.

Let f^* be a solution of $\mathrm{VI}(c, K_f(g))$.

By Proposition 1.1, it is known that f^* is a solution of $\mathrm{VI}(c, K_f(g))$ if and only if there exists $(\mu^*, \lambda^*, s^*) \in \mathbb{R}^{v \times p \times n}$ such that $(f^*, \mu^*, \lambda^*, s^*)$ is a solution of the system

$$\begin{cases} c(f) \;-\; \lambda\Gamma \;+\; \mu\nabla g(f) \;- s \;=\; 0 \\ \langle \mu, g(f) \rangle \;=\; \langle s, f \rangle = 0 \\ g(f) \leq 0, \;\; \Gamma f = q, \;\; f \geq 0 \\ \mu \geq 0, \;\; s \geq 0. \end{cases} \tag{16}$$

Consider now $\mathrm{VI}_\alpha(c, K_f)$. If we put $\alpha := \mu^*$, then $\mathrm{VI}_{\mu^*}(c, K_f)$ becomes

$$\langle c(f^*) + \mu^* \nabla g(f^*), f - f^* \rangle \geq 0, \quad \forall f \in K_f.$$

Still by Proposition 1.1, we have that f^* is a solution of $\mathrm{VI}_{\mu^*}(c, K_f)$ if and only if there exists $(\lambda^0, s^0) \in \mathbb{R}^{p \times n}$ such that (f^*, λ^0, s^0) is a solution of the system

$$\begin{cases} c(f) + \mu^* \nabla g(f) - \lambda\Gamma - s = 0 \\ \langle s, f \rangle = 0 \\ \Gamma f = q, \;\; f \geq 0, \;\; s \geq 0. \end{cases} \tag{17}$$

Therefore, if $(f^*, \mu^*, \lambda^*, s^*)$ is a solution of (16) then (f^*, λ^*, s^*) is also a solution of (17) and f^* solves $\mathrm{VI}_{\mu^*}(c, K_f)$. $\square$

Remark 4.1. By the proof of Theorem 4.1, it follows that the parameter α, which ensures the equivalence between the capacitated VI and the incapacitated VI, can be chosen as the multiplier μ^*, associated to the constraint $g(f) \leq 0$.

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