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OPTIMAL REPLENISHMENT AND CREDIT POLICY IN AN INVENTORY MODEL FOR DETERIORATING ITEMS UNDER TWO-LEVELS OF TRADE CREDIT POLICY WHEN DEMAND DEPENDS ON BOTH TIME AND CREDIT PERIOD INVOLVING DEFAULT RISK

Avik Mukherjee and Gour Chandra Mahata*

Abstract. In this paper, we examine an optimal dynamic decision-making problem for a retailer's inventory system of deteriorating items under two-level trade credit financing where the supplier, as well as the retailer, offers trade credit to the subsequent downstream member, the demand rate of which varies simultaneously with time and the length of credit period that is offered to the customers. The deterioration rate is non-decreasing over time. In addition, the risk of default increases with the credit period length. A generalized model is presented to determine the optimal trade credit and replenishment strategies that maximize the retailer's annual total profit. We then demonstrate that the retailer's optimal credit period and replenishment cycle time not only exist but also are unique. Thus, the search of the global optimal solution reduces to finding a local solution. Finally, we run several numerical examples to illustrate the problem and gain managerial insights.

Mathematics Subject Classification. 90B05.

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1. Introduction

The traditional economic order quantity (EOQ) model assumes that the retailer must pay immediately on receiving the items. However, in practice, the manufacturer (supplier or vendor) may offer the retailer a delay in payment (called trade credit) as a strategy to promote sales and increase market share. Usually, no interest is charged on the outstanding amount if the payment is settled within trade credit period. Therefore, the retailer can sell the goods and deposit the accumulated revenues in a bank to earn interest, and hence delay the payments up to the last moment of the permissible delay period. However, the manufacturer can charge a high interest, if the payment is not made by the retailer within the trade credit period on previously agreed terms and conditions. This brings some economic advantage to the retailers as they can earn interest from the revenue realized during the permissible delay period. Moreover, once a trade credit is offered, the amount of time the retailer's capital tied up in stock is reduced, and that leads to a reduction in holding cost. In fact, the

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Department of Mathematics (UG & PG), Dum Dum Motijheel College, 1, Motijheel Avenue, Kolkata 700074, West Bengal, India.

* Corresponding author: gourmahata@yahoo.co.in; gcmahata@yahoo.in

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retailers who have budget constraints or have limited number of financing opportunities rely on trade credit as a source of short-term fund. Wal-Mart, the largest retailer in the world, often uses trade credit as a larger source of capital than bank borrowing (Zhong and Zhou [60]). In India, the non-state-owned enterprises often obtain limited support from banks. Therefore, the trade credit policy is adopted as a very important short-term financing method. On the other hand, the policy of granting trade credit adds not only an additional cost but also an additional dimension of default risk (*i.e.*, the event in which the buyer will be unable to payoff its debt obligations) to the retailer (Teng *et al.* [52]).

Several authors discussing this topic have appeared in the literatures that investigate inventory problems under varying conditions. Some of the papers are discussed below. Goyal [9] first developed an EOQ model under the conditions of permissible delay in payments. Thereafter, most recent researchers in the field of inventory lot-sizing policies with trade credit financing have extended his basic model. Teng [47] derived an EOQ model under conditions of permissible delay in payments. Abad and Jaggi [1] presented a joint approach for setting unit price and the length of the credit period for a seller when end demand is price sensitive. Shinn and Hwang [45] determined the retailer's optimal price and order size simultaneously under the condition of order-size-dependent delay in payments. Huang and Chung [13] extended Goyal's [9] model to cash discount policy for early payment. Salameh *et al.* [35] extended this issue to the continuous review inventory model. Khanra *et al.* [18] derived an inventory model with time dependent demand and shortages under trade credit policy. Mahata and Mahata [30] presented an economic production quantity (EPQ) model of time varying quadratic non-decreasing demand pattern for a retailer where supplier's trade offer gives the retailer a credit period and price-discount on the purchase of merchandise. Recently, Shin *et al.* [44] observed the effects of human errors and trade-credit financing in a two-echelon supply chain model.

In fact, many products (*e.g.*, bakeries, fruits, meat, milk, vegetables, fashion-merchandises, and high-tech products) are perishable and deteriorate continuously due to several reasons, such as evaporation, spoilage, and obsolescence. Most of the above papers discussed trade-credit policy of inventory models with no deterioration. Aggarwal and Jaggi [2] extended Goyal's model to consider the deteriorating items. Later on, Jamal *et al.* [16] extended Aggarwal and Jaggi's [2] model to allow for shortages. Liao *et al.* [21] and Sarker *et al.* [41] presented an inventory model with deteriorating items under inflation and permissible delay in payments. Arcelus *et al.* [3] observed retailer's pricing, trade-credit policy, and inventory policies for deteriorating items. Mahata and Goswami [28] presented a fuzzy EPQ model for deteriorating items when delay in payment is permissible. Hsieh *et al.* [10] complemented the shortcomings of Jamal *et al.* [16] and showed that the optimal solution for each case not only exists but is unique under specific circumstances. Skouri *et al.* [46] developed the model for deteriorating items with ramp-type demand and permissible delay in payments. Sarkar *et al.* [40] discussed an inventory model with trade-credit policy and variable deterioration for fixed lifetime products. There are several papers related to deterioration such as Sana and Chaudhuri [36], Sarkar [37], Sarkar and Sarkar [39] investigated some inventory models with increasing demand and time varying deterioration.

The above works are based on one-level trade credit policy in which only the supplier or the manufacturer offers a trade credit to the retailer. However, from practical circumstances, whenever the manufacturer offers a trade credit to the retailer, the retailer may extend a similar offer to his customers. Keeping this scenario in mind, Huang [11] extended Goyal's model to develop an EOQ model in which the supplier offers the retailer a permissible delay period (supplier trade credit), and the retailer in turn provides a trade credit period to his customers (retailer trade credit). Huang [12] incorporated Huang's [11] model to investigate the two-level trade credit policy in the EPQ frame work. Mahata and Goswami [29] extended Huang's [11] model to an EOQ inventory model for deteriorating items in the fuzzy sense. Furthermore, Liao [20] extended Huang's model to an EPQ model for deteriorating items. The assumptions of Huang's model are modified by Teng and Goyal [50]. They stated that the retailer obtain its revenue from N to $N + T$, not from 0 to T. Subsequently, Teng [48] provided the optimal ordering policies for a retailer to deal with bad credit customers as well as good credit customers. Kreng and Tan [19] modified Huang's [11] model by developing optimal whole seller's replenishment decisions in the EOQ model under two levels of trade credit policy depending on the order quantity. Min *et al.* [32] developed an inventory model for deteriorating items under stock-dependent demand and two-level trade credit. Chung [5] addressed a simplified solution procedure for the optimal replenishment decision under two levels of trade-credit policy. Chen *et al.* [4] discussed the EPQ model with up-stream full trade credit and down-stream partial trade credit with a constant demand. Ouyang *et al.* [34] discuss the integrated inventory model with the order-size dependent trade credit and a constant demand. Mahata and De [27], Mahata [26] provided the optimal ordering policies for a retailer with trade credit and variable deterioration for fixed lifetime products. Recently, Shah *et al.* [43] developed an optimal replenishment policy for retailer under partial upstream prepayment and partial downstream overdue payment for quadratic demand.

The papers above discussed the EOQ or EPQ inventory models under trade credit financing based on the assumption that the demand rate is constant over time. However, in practice, the market demand is always changing rapidly and is affected by several factors such as price, time, inventory level, and delayed payment period, etc. Some researchers realize this phenomenon and extend their studies above to build the inventory models by assuming that the demand is variable. Chung and Liao [7] discussed the inventory replenishment problems with trade credit financing by considering a price-sensitive demand. Giri and Maiti [8] discussed the supply chain model with price and trade credit sensitive demand with trade credit by considering the fact that a retailer shares a fraction of the profit earned during the credit period. Mahata [24] discussed the inventory replenishment problems with up-stream full trade credit and down-stream partial trade credit financing by considering a price-sensitive demand. Sarkar [37], and Teng *et al.* [53] build the economic quantity model with trade credit financing for time-dependent demand. Min *et al.* [33] developed an inventory model under conditions of permissible delay in payments, assuming that the items are replenished with the demand rate of the items dependent on the current inventory level. The inventory model with the credit-linked demand are discussed by Jaggi *et al.* [14], Jaggi *et al.* [15]. Thangam and Uthayakumar [56] discussed trade credit financing for perishable items in a supply chain when demand depends on both selling price and credit period. Lou and Wang [22] studied optimal trade credit and order quantity by considering trade credit with a positive correlation of market sales, but are negatively correlated with credit risk. Teng *et al.* [55] discussed the optimal trade credit and lot size policies considering the demand and default risk sensitive to the credit period with learning curve production costs. Wang *et al.* [57] and Wu *et al.* [59] captured the relevant fact that the deterioration rate for a deteriorating product increases with time and reaches 100% by the time it reaches its expiration date, and then derived the optimal credit period and cycle time in a supply chain in which trade credit increases not only the sales revenue but also the default risk and the opportunity cost. Mahata [25] proposed an EOQ model for the retailer to obtain his/her optimal credit period and cycle time under two-level trade credit by considering demand dependence on delayed payment time with default risk for deteriorating items. Wu and Chan [58] analyzed the lot sizing policies for deteriorating items with expiration dates and partial trade credit to credit risk customer by considering demand dependence on trade credit and default risk related to credit period. However, all the above models make an implicit assumption that the demand rate is constant over an infinite planning horizon. This assumption is only valid during the maturity phase of a product life cycle. In the introduction and growth phase of a product life cycle, the firms face increasing demand with little competition. Mahata *et al.* [31] developed an EOQ model under two-level trade credit financing involving default risk by considering demand to a credit-sensitive and linear non-decreasing function of time. A linear trend demand implies a uniform change in the demand rate of the product per unit time. This is a fairly unrealistic phenomenon and it seldom occurs in the real market. One can usually observe in the electronic market that the sales of items increase rapidly in the introduction and growth phase of the life cycle because there are few competitors in market. However, the rates of demand and default risks in the above-mentioned studies are assumed to be specific functions of the credit period. Previous work on inventory control under two-levels of trade credit policy has been summarized in Table 1, which shows a brief comparison of published work and the present study.

When discussing a firm's operations under trade credit with default risk, it is necessary to consider the effect of market demand related to time varying non-decreasing function of time and the credit period. From a product life cycle perspective, it is only in the maturity stage that demand is near constant. During the growth stage of a product life cycle (especially for high-tech products), the demand function increases with time. While considering time-varying demands, inventory modellers usually take the demand to be either linearly dependent or exponentially dependent upon time. For the first case, the demand rate function is of the form $f(t) = a + bt, a > 0, b \geq 0$, which implies steady increase in demand, which may be rarely seen to occur in the real market. For the second

TABLE 1. The features of the this paper *versus* other ones in two-level credit period.

References	Assumption						Variable		
	Trade credit		Deteriorating items	Expiration date	Default risk	Demand	T	N	P
	SR	RC							
Teng and Goyal [50]	*	*				C	*		
Liao [20]	*	*	*			C	*		
Thangam and Uthayakumar [56]	*	*	*			$D(P,N)$	*	*	*
Wu *et al.* [59]	*	*	*	*	*	$D(N)$	*	*	
Teng and Lou [51]	*	*			*	$D(N)$	*	*	
Mahata [23]	*	*	*			C	*		
Teng [48]	*	*			*	C	*		
Jaggi *et al.* [15]	*	*				$D(N)$	*	*	
Jaggi *et al.* [14]	*	*				$D(N)$	*	*	
Teng and Chang [49]	*	*				C	*		
Chung and Huang [6]	*	*				C	*		
Huang [11]	*	*				C	*		
Sarkar [37]		*	*	*		$D(t)$	*		
Mahata [25]	*	*	*		*	$D(N)$	*	*	
Mahata [26]	*	*	*	*		C	*		
Mahata and De [27]	*	*	*	*		C	*	*	
Mahata [24]	*	*	*	*		$D(P)$	*		*
Teng *et al.* [54]	*	*				$D(t)$	*		
Sarkar *et al.* [40]	*	*	*	*		C	*		
Mahata *et al.* [31]	*	*			*	$D(t,N)$	*	*	
This paper	*	*	*	*	*	$D(t,N)4$	*	*	

SR = supplier–retailer trade credit, RC = retailer–customer trade credit, C = constant, N = credit period, P = price, T = cycle time, $D(t)$ = time dependent demand, $D(N)$ = credit-sensitive demand, $D(P,N)$ = price and credit sensitive demand, $D(t,N)$ = time and credit-sensitive demand.

case, the demand rate function is of the form $f(t) = \alpha e^{\beta t}, \alpha > 0, \beta \geq 0$. In the real market situations, demand is unlikely to increase at a rate which is so high as exponential. Quadratic time-dependence of demand of the form $f(t) = a + bt + ct^2, a, b, c > 0$, seems to be a better representation of time-varying market demands. Here $a(> 0)$ stands for the initial demand rate, b is the rate with which the demand rate increases. The rate of change in the demand rate itself changes at a rate c. We have $\frac{\mathrm{d}f(t)}{\mathrm{d}t} = b + 2ct$ and $\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2} = 2c$. Now $\frac{\mathrm{d}f(t)}{\mathrm{d}t} = 0$ gives $t = -\frac{b}{2c}$. The rate of increase of $f(t)$ is an increasing function of time. This type of demand is known as accelerated growth in demand which is seen to occur in the case of the state-of-the-art aircrafts, computers, machines and their spare parts, etc. Moreover, the marginal influence of the credit period on sales is associated with the unrealized potential market demand. To obtain robust and generalized results, we extend the constant demand to a time varying and credit sensitive demand. Consequently, in this paper, we propose an EOQ model in a supplier–retailer–customer supply chain in which: (a) the supplier provides an up-stream trade credit and the retailer also offers a down-stream trade credit, (b) the retailer's down-stream trade credit to the buyer not only increases sales and revenue but also opportunity cost and default risk, (c) the demand rate of which varies simultaneously with time and the length of credit period that is offered to the customers, and (d) the product deteriorates at an increasing time-varying rate of deterioration, and there is no repair or replacement of deteriorated units during the inventory cycle. We model the retailer's inventory system under these conditions as a profit maximization problem. We then show that the retailer's optimal credit period and cycle time not only exist but also are unique. Furthermore, we run some numerical examples to illustrate the problem and provide managerial insights.

2. Notation and assumptions

In the study, we discuss the optimal inventory policy and the trade credit policy for deteriorating products under two-level trade credit financing with dynamic demand which varies simultaneously with time and the length of credit period that is offered to the customers involving the default risk. The items deteriorate at an increasing varying rate of deterioration. To build the mathematical models, the following notation and assumptions are adopted in this paper.

2.1. Notation

The following notations are used throughout this paper.

Parameters

A: ordering cost per order in dollars.
h: holding cost per unit per year in dollars excluding interest charge.
c: purchasing cost per unit in dollars.
p: selling price per unit in dollars, where $p > c$.
r: annual compound interest paid per dollar per year.
M: up-stream credit period in years by the supplier.
I_e: interest earned per dollar per year.
I_c: interest charged per dollar in stock per year by the supplier.

Decision variables

N: down-stream credit period in years by the retailer.
T: replenishment cycle time in years, where $T \geq 0$.

Functions

$F(N)$: the rate of default risk giving the credit period N.
$\lambda(t, N)$: demand rate at time t and credit period N.
$\theta(t)$: deterioration rate at time t, which is a non-decreasing function in t.
$I(t)$: inventory level at time t.
$\Pi\,(N, T)$: annual total profit, which is a function of N and T.

Optimal values

N^*: optimal down-stream credit period in years.
T^*: optimal replenishment cycle time in years.
Π^*: optimal annual total profit in dollars.

2.2. Assumptions

(i) The replenishment occurs instantaneously at an infinite rate.
(ii) In a supplier–retailer–customer supply chain system, the retailer buys deteriorating items from his/her supplier, and then sells them to his/her customers. We may assume without loss of generality (WLOG) that the supplier grants an up-stream credit period of M years to the retailer while the retailer in turn provides a down-stream credit period of N years to his/her customers.
(iii) During the replenishment period $[0, T]$, the customers arrived at a rate of $\lambda\,(t, N) = f\,(t)\,D(N)$, where $f\,(t)$ and $D(N)$ are non-negative, continuous twice differentiable functions, where $0 \leq t \leq T$. We consider that the demand would be time increasing, *i.e.*, we take $f'\,(t) > 0$ for all $t > 0$, as the demand rate in today's high tech products increases significantly during the growth stage. Because credit allows customers to enjoy the benefit of delayed payments, lengthening the credit period will stimulate sales; hence, we also assume that $D'\,(N) > 0$ for all $N > 0$.

(iv) Although sales can be stimulated by trade credit, longer credit period increase the probability of a customer default. For example, the default risk of a 30-year mortgage is higher than that of a 15-year mortgage. Therefore, we assume that $F(0) = 0, 0 < F(N) < 1$ and $F'(N) > 0$ for all $N > 0$.

(v) The items deteriorate at a varying rate of deterioration $\theta(t)$, where $\theta'(t) \geq 0$ and $0 < \theta(t) \ll 1$ and t represents the inventory holding time after receipt of the consignment. Here $\theta'(t)$ denotes the first derivative of $\theta(t)$ with respect to t. Note that $\theta'(t) \geq 0$ means that the deterioration rate is non-decreasing over time. Furthermore, there is no repair or replacement of deteriorated units during the planning horizon, and the items will be withdrawn from the warehouses immediately as they are deteriorated.

(vi) If $T \geq M$, then the retailer settles the account at time M and pays for the interest charges on items in stock with rate I_c over the interval $[M, T]$. If $T \leq M$, then the retailer settles the account at time M and there is no interest charge in stock during the whole cycle. On the other hand, if $M > N$, the retailer can accumulate revenue and earn interest during the period from N to M with rate I_e under the up-stream and down-stream trade credit conditions.

Given the above, it is possible to formulate a mathematical inventory EOQ model with trade credit financing.

3. Model formulation

Based on the assumptions above, the inventory system goes as follows: the retailer receives the order quantity Q at $t = 0$. Hence, the inventory starts with Q units at $t = 0$, and gradually reaches zero at $t = T$ due to the combined influence of the demand and deterioration. Therefore, inventory level $I(t)$ with respect to time is governed by the following differential equation:

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -\lambda(t, N) - \theta(t) I(t), \qquad 0 \leq t \leq T, \tag{3.1}$$

with the boundary condition $I(0) = Q, I(T) = 0$.

Substituting $\lambda(t, N) = f(t) D(N)$ into equation (3.1) and then solving the differential equation yields

$$I(t) = D(N) e^{-\int_0^t \theta(s)\mathrm{d}s} \int_t^T e^{\int_0^t \theta(s)\mathrm{d}s} f(u)\,\mathrm{d}u, \qquad 0 \leq t \leq T. \tag{3.2}$$

For notational convenience, let $g(t) = \int_0^t \theta(s)\,\mathrm{d}s$.

At time $t = 0$, the retailer's orders deteriorating items. Hence, the retailer's ordering cost per cycle time T is $OC = A$.

Utilizing the result of (3.2), the ordering quantity during the replenishment period $[0, T]$, denoted by Q, is

$$Q = I(0) = D(N) \int_0^T e^{g(t)} f(t)\,\mathrm{d}t. \tag{3.3}$$

Hence, the purchase cost (PC) during the replenishment period $[0, T]$ is then given by

$$\mathrm{PC} = cD(N) \int_0^T e^{g(t)} f(t)\,\mathrm{d}t. \tag{3.4}$$

Likewise, the holding cost (HC) is incurred by the retailer based on the instantaneous inventory level; this cost does not include interest paid to the supplier.

Hence, the holding cost during the replenishment period $[0, T]$ can be written as

$$\mathrm{HC} = h \int_0^T I(t)\,\mathrm{d}t = hD(N) \int_0^T e^{-g(t)} \int_t^T e^{g(u)} f(u)\,\mathrm{d}u\mathrm{d}t. \tag{3.5}$$

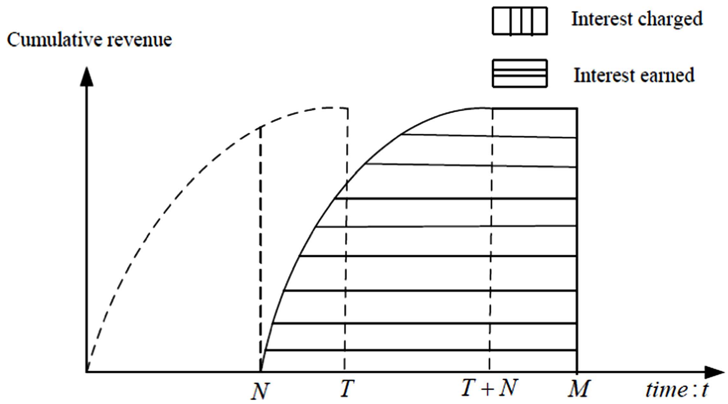

FIGURE 1. The retailer's interest earned and interest charged when $N \leq T+N \leq M$.

As the annual compound interest rate is r, a dollar received at time t is equivalent to e^{-rt} dollars received now. The retailer offers the buyer a credit period of N. Hence, the discounted sales revenue after the default risk during the replenishment period $[0, T]$ is

$$pe^{-rN} D\left(N\right) \left[1 - F\left(N\right)\right] \int_0^T f\left(t\right) \mathrm{d}t. \tag{3.6}$$

Regarding the exogenous variables, we have three potential cases: (i) $N \leq T+N \leq M$, (ii) $N \leq M \leq T+N$ and (iii) $M \leq N \leq T+N$.

Case I. $N \leq T+N \leq M$ (shown in Fig. 1).

With $N \leq T+N \leq M$, the retailer receives the sales revenue at $T+N$ and is able to pay off the total purchase cost at time M. Therefore, there is no interest charged. During the period $[N, T+N]$, the retailer can obtain the interest earned on the sales revenues received and on the full sales revenue during the period $[T+N, M]$ as shown in Figure 1. Therefore, the annual interest earned is

$$\begin{aligned} &\frac{pI_e}{T}\left[\int_0^{T+N}\int_N^{t+N} \lambda\left(u-N, N\right) \mathrm{d}u\mathrm{d}t + \left(M-T-N\right)\int_0^T \lambda\left(u, N\right) \mathrm{d}u\right] \\ &= \frac{pI_e D(N)}{T}\left[\int_0^{T+N}\int_N^{t+N} f\left(u-N\right) \mathrm{d}u\mathrm{d}t + \left(M-T-N\right)\int_0^T f\left(u\right) \mathrm{d}u\right]. \end{aligned} \tag{3.7}$$

Combining the above results, the retailer's annual total profit can be expressed as follows: $\mathit{\Pi}_1\left(N, T\right) =$ discounted sales revenue after default risk − annual ordering cost − annual purchasing cost − annual holding cost + annual interest earned − annual interest charged.

$$\begin{aligned} \mathit{\Pi}_1\left(N, T\right) = &\frac{p}{T} e^{-rN} D\left(N\right)\left[1-F\left(N\right)\right]\int_0^T f\left(t\right)\mathrm{d}t - \frac{hD(N)}{T}\int_0^T e^{-g(t)}\int_t^T e^{g(u)} f(u)\mathrm{d}u\mathrm{d}t - \frac{A}{T} \\ &+ \frac{pI_e D(N)}{T}\left[\int_0^T\int_0^t f\left(z\right)\mathrm{d}z\mathrm{d}t + \left(M-T-N\right)\int_0^T f\left(u\right)\mathrm{d}u\right] - \frac{cD(N)}{T}\int_0^T e^{g(u)} f(u)\mathrm{d}u. \end{aligned} \tag{3.8}$$

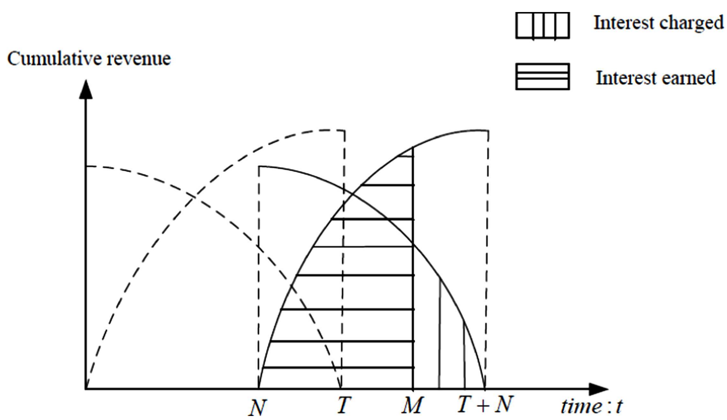

FIGURE 2. The retailer's interest earned and interest charged when $N \leq M \leq T+N$.

Case II. $N \leq M \leq T+N$ (shown in Fig. 2).

When $M \leq T+N$, the retailer cannot receive the last payment before the permissible delay time M. As a result, the retailer must finance all items sold after time $(M-N)$ at time M, and pay off the loan until $T+N$ at an interest rate I_c per dollar per year as shown in Figure 2. Therefore, we can have the interest charged in the following:

$$\frac{cI_c}{T}\int_{M-N}^{T} I(t)\,\mathrm{d}t = \frac{cI_c}{T} D(N) \int_{M-N}^{T} e^{-g(t)} \int_{t}^{T} e^{g(u)} f(u)\,\mathrm{d}u\mathrm{d}t. \tag{3.9}$$

On the other hand, the retailer starts selling product at time 0, and receiving the money at time N. Consequently, the retailer accumulates sales revenue in an account that earns I_e per dollar per year starting from N through M as shown in Figure 2. Therefore, we have the interest earned in the following:

$$\frac{pI_e}{T}\int_{N}^{M}\int_{N}^{t+N} \lambda(u-N,N)\,\mathrm{d}u\mathrm{d}t = \frac{pI_e}{T} D(N) \int_{0}^{M-N}\int_{0}^{t} f(u)\,\mathrm{d}u\mathrm{d}t. \tag{3.10}$$

Hence, similar to (3.8), we know that the retailer's annual total profit is

$$\begin{aligned} \Pi_2(N,T) = {} & \frac{p}{T} e^{-rN} D(N)\left[1-F(N)\right] \int_{0}^{T} f(t)\,\mathrm{d}t - \frac{hD(N)}{T}\int_{0}^{T} e^{-g(t)} \int_{t}^{T} e^{g(u)} f(u)\mathrm{d}u\mathrm{d}t - \frac{A}{T} \\ & + \frac{pI_e}{T} D(N) \int_{0}^{M-N}\int_{0}^{t} f(u)\,\mathrm{d}u\mathrm{d}t - \frac{cI_c}{T} D(N) \int_{M-N}^{T} e^{-g(t)} \int_{t}^{T} e^{g(u)} f(u)\,\mathrm{d}u\mathrm{d}t \\ & - \frac{cD(N)}{T}\int_{0}^{T} e^{g(u)} f(u)\mathrm{d}u. \end{aligned} \tag{3.11}$$

Case III. $M \leq N \leq T+N$ (shown in Fig. 3).

Since $M \leq N$, there is no interest earned for the retailer. In addition, the retailer must finance all the purchasing cost from $[M,N]$ and pay off the loan from $[N,T+N]$ as shown in Figure 3. Therefore, the interest charged per cycle is

$$\frac{cI_c}{T}\left[\int_{0}^{T} I(t)\,\mathrm{d}t + (N-M)\,Q\right]$$

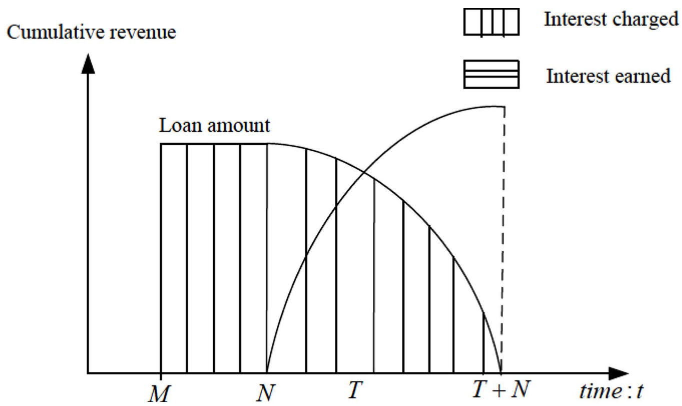

FIGURE 3. The retailer's interest earned and interest charged when $M \leq N \leq T + N$.

$$= \frac{cI_c}{T} D(N) \left[\int_0^T e^{-g(t)} \int_t^T e^{g(u)} f(u) \mathrm{d}u \mathrm{d}t + (N - M) \int_0^T e^{g(u)} f(u) \mathrm{d}u \right]. \tag{3.12}$$

Consequently, the retailer's annual total profit function per cycle can be expressed as

$$\begin{aligned} \Pi_3(N,T) = {} & \frac{p}{T} e^{-rN} D(N) [1 - F(N)] \int_0^T f(t) \, \mathrm{d}t - \frac{hD(N)}{T} \int_0^T e^{-g(t)} \int_t^T e^{g(u)} f(u) \mathrm{d}u \mathrm{d}t \\ & - \frac{cI_c}{T} D(N) \left[\int_0^T e^{-g(t)} \int_t^T e^{g(u)} f(u) \mathrm{d}u \mathrm{d}t + (N - M) \int_0^T e^{g(u)} f(u) \mathrm{d}u \right] \\ & - \frac{A}{T} - \frac{cD(N)}{T} \int_0^T e^{g(u)} f(u) \mathrm{d}u. \end{aligned} \tag{3.13}$$

Thus, we can combine them by

$$\Pi(N,T) = \begin{cases} \Pi_1(N,T), & N \leq T + N \leq M \\ \Pi_2(N,T), & N \leq M \leq T + N \\ \Pi_3(N,T), & M \leq N \leq T + N \end{cases}, \tag{3.14}$$

which is a function of two variables N and T.

Therefore, the retailer's objective is to determine the optimal credit period N^* and cycle time T^* such that the annual total profit $\Pi_i(N,T)$ for $i = 1, 2$, and 3 is maximized.

4. Solution methodology

In this section, we characterize the retailer's optimal credit period and cycle time in each case, and then obtain the conditions in which the optimal T^* is in either $T + N \leq M$ or $T + N \geq M$.

Case I: $N \leq T + N \leq M$.

Let us take

$$\Pi_1(N,T) = \frac{1}{T} \mathfrak{D}_1(N,T), \tag{4.1}$$

where

$$\mathfrak{D}_1(N,T) = pe^{-rN}D(N)[1-F(N)]\int_0^T f(t)\,\mathrm{d}t - hD(N)\int_0^T e^{-g(t)}\int_t^T e^{g(u)}f(u)\,\mathrm{d}u\mathrm{d}t$$
$$-A + pI_eD(N)\left[\int_0^T\int_0^t f(z)\,\mathrm{d}z\mathrm{d}t + (M-T-N)\int_0^T f(u)\,\mathrm{d}u\right]$$
$$-cD(N)\int_0^T e^{g(u)}f(u)\,\mathrm{d}u. \tag{4.2}$$

Now,

$$\frac{\partial \Pi_1(T|N)}{\partial T} = -\frac{1}{T^2}\mathfrak{D}_1(T|N) + \frac{1}{T}\mathfrak{D}_1'(T|N), \tag{4.3}$$

where,

$$\mathfrak{D}_1'(T|N) = pe^{-rN}D(N)[1-F(N)]f(T) + pI_eD(N)\left[\int_0^T f(z)\,\mathrm{d}z + (M-T-N)f(T)\right]$$
$$-cD(N)e^{g(T)}f(T) - hD(N)e^{g(T)}f(T)\int_0^T e^{-g(u)}\mathrm{d}u. \tag{4.4}$$

For fixed $N \geq 0$, the first order partial derivative of $\Pi_1(N,T)$ with respect to T is

$$\frac{\partial \Pi_1(T|N)}{\partial T} = -\frac{1}{T^2}\left[\mathfrak{D}_1(T|N) - T\mathfrak{D}_1'(T|N)\right] = 0.$$

Let, $\tilde{T}_1$ (may or may not exist) be the solution of

$$\left[\mathfrak{D}_1(T|N) - T\mathfrak{D}_1'(T|N)\right] = 0 \Rightarrow \frac{1}{\tilde{T}}\mathfrak{D}_1(\tilde{T}_1|N) = \mathfrak{D}_1'\left(\tilde{T}_1|N\right) = \Pi_1(N,\tilde{T}_1),$$
$$\frac{\partial^2 \Pi_1(T|N)}{\partial T^2} = \frac{2}{T^3}\mathfrak{D}_1(T|N) - \frac{2}{T^2}\mathfrak{D}_1'(T|N) + \frac{1}{T}\mathfrak{D}_1''(T|N). \tag{4.5}$$

If $T = \tilde{T}_1$ then

$$\frac{\partial^2 \Pi_1(\tilde{T}_1|N)}{\partial T^2} = \frac{1}{\tilde{T}}\mathfrak{D}_1''(\tilde{T}_1|N),$$

which implies

$$\frac{\partial^2 \pi_1(\tilde{T}_1|N)}{\partial T^2} < 0, \quad \text{if } \mathfrak{D}_1''(\tilde{T}_1|N) < 0. \tag{4.6}$$

Now,

$$\mathfrak{D}_1''(T|N) = -cD(N)e^{g(T)}\theta(T)f(T) - hD(N)f(T) - hD(N)e^{g(T)}\theta(T)f(T)\int_0^T e^{-g(u)}\mathrm{d}u + \Big[pe^{-rN}D(N)(1-F(N)) + pI_eD(N)(M-T-N) - cD(N)e^{g(T)} - hD(N)e^{g(T)}\int_0^T e^{-g(u)}\mathrm{d}u\Big] f'(T). \tag{4.7}$$

For simplicity let us define the discrimination term

$$\Delta = \mathfrak{D}_1(T|N) - T\mathfrak{D}_1'(T|N)|_{T=M-N}\,.$$

Theorem 4.1. *If* $pe^{-rN}(1-F(N)) + pI_e(M-T-N) - ce^{g(T)} - he^{g(T)}\int_0^T e^{-g(u)}\mathrm{d}u < 0$, *then* $\Pi_1(T|N)$ *is maximized at* $T = \tilde{T}_1$, *where*

(i) If $\Delta > 0$ *we can find* $\tilde{T}$ *such that* $0 \le \tilde{T}_1 < M-N$.
(ii) If $\Delta \le 0$ *we take* $\tilde{T}_1 = M-N$.

Proof. As $D(N) > 0$, we have from the condition that

$$pe^{-rN}D(N)(1-F(N)) + pI_eD(N)(M-T-N) - cD(N)e^{g(T)} - hD(N)e^{g(T)}\int_0^T e^{-g(u)}\mathrm{d}u < 0.$$

Thus from equation (4.7), we have $\mathfrak{D}_1''(\tilde{T}_1|N) < 0$:

(i) If $\Delta > 0$ then by intermediate value theorem we must find a solution $\tilde{T}_1$ of $\frac{\partial \Pi_1(T|N)}{\partial T} = 0$ within the range $0 \le \tilde{T}_1 < M-N$ and $\Pi_1(T,N)$ is maximized at $T = \tilde{T}_1$ as $f'(T) > 0$.
(ii) If $\Delta \le 0$ then $\frac{\partial \Pi_1(T|N)}{\partial T} > 0$ for $\tilde{T}_1 \in [0, M-N]$, *i.e.*, $\Pi_1(T,N)$ is an increasing function or satisfies the optimality condition for $\tilde{T}_1 = M-N$. Hence, we take $\tilde{T}_1 = M-N$ as the optimal solution.

□

Corollary 4.2. $pe^{-rN}(1-F(N)) + pI_e(M-N) - c < 0$ *holds for* $\forall N > \hat{N}$ *where* $\hat{N}$ *be the unique solution of* $pe^{-rN}(1-F(N)) + pI_e(M-N) - c = 0$ *if* $pe^{-rM}(1-F(M)) - c < 0$ *and consequently condition for Theorem 4.1 holds.*

Proof. As $\phi(N) = pe^{-rN}(1-F(N)) + pI_e(M-N) - c$ is a strictly decreasing function and $\phi(0) = (p + pI_eM - c) > 0$, we must find unique solution $\hat{N}$ of the equation $\phi(N) = pe^{-rN}(1-F(N)) + pI_e(M-N) - c = 0$ which exists as $\phi(M) = pe^{-rM}(1-F(M)) - c < 0$.

Now as

$$pe^{-rN}(1-F(N)) + pI_e(M-N) < c,$$

for all $N \in]\hat{N}, M]$,

$$pe^{-rN}(1-F(N)) + pI_e(M-N) < c < ce^{g(T)} + he^{g(T)}\int_0^T e^{-g(u)}\mathrm{d}u + pI_eT,$$

i.e., condition for Theorem 4.1 holds evidently in the aforesaid interval. □

Note: If $\mathfrak{D}_1''(T|N) \geq 0$ then $\pi_1(T|N)$ is a convex function of T. Thus the optimal solution attains at the boundary point, *i.e.*, either 0 or $+\infty$. In that case, we have to take $\tilde{T} = 0$ which implies an immediate replenishment is mandatory.

Now we have

$$\Pi_1(N,T) = \frac{1}{T}\left[pe^{-rN}D(N)(1-F(N))\int_0^T f(t)\,\mathrm{d}t - pI_eND(N)\int_0^T f(t)\,\mathrm{d}t - D(N)X - A\right], \tag{4.8}$$

where

$$X = c\int_0^T e^{g(u)}f(u)\,\mathrm{d}u + h\int_0^T e^{-g(t)}\int_t^T e^{g(u)}f(u)\,\mathrm{d}u\mathrm{d}t - pI_e\int_0^T\int_0^t f(z)\,\mathrm{d}z\mathrm{d}t$$
$$-p(M-T)I_e\int_0^T f(t)\,\mathrm{d}t. \tag{4.9}$$

Here, we have to consider $X > 0$ by suitable adjustments of I_e.

Using equation (4.8), we take the partial first derivative of $\Pi_1(N,T)$ with respect to N.

$$\frac{\partial \Pi_1(N|T)}{\partial N} = \frac{1}{T}\left[p\frac{\mathrm{d}}{\mathrm{d}N}\left\{e^{-rN}D(N)(1-F(N))\right\}\int_0^T f(t)\,\mathrm{d}t\right.$$
$$\left.-pI_e\left\{ND'(N)+D(N)\right\}\int_0^T f(t)\,\mathrm{d}t - D'(N)X\right], \tag{4.10}$$

$\frac{\partial \Pi_1(N|T)}{\partial N} = 0$ may hold if $\frac{\mathrm{d}}{\mathrm{d}N}\left\{e^{-rN}D(N)(1-F(N))\right\} > 0$, whereas if $\frac{\mathrm{d}}{\mathrm{d}N}\left\{e^{-rN}D(N)(1-F(N))\right\} < 0$, $\Pi_1(N|T)$ is a decreasing function and we must have $\tilde{N}_1 = 0$.

Let us consider that $\frac{\partial \Pi_1(N|T)}{\partial N} = 0$, for $N = \tilde{N}_1$ (may or may not exist).

Now

$$\frac{\partial^2 \Pi_1(N|T)}{\partial N^2} = \frac{1}{T}\left[p\frac{\mathrm{d}^2}{\mathrm{d}N^2}\left\{e^{-rN}D(N)(1-F(N))\right\}\int_0^T f(t)\,\mathrm{d}t - pI_e\{ND''(N)\right.$$
$$\left.+2D'(N)\}\int_0^T f(t)\,\mathrm{d}t - D''(N)X\right]. \tag{4.11}$$

Theorem 4.3. *If $e^{-rN}D(N)(1-F(N))$ is a strictly concave function with $D''(N) > 0$ then*

(i) If $\left.\frac{\partial \Pi_1(N|T)}{\partial N}\right|_{N=0} \leq 0$ then $\Pi_1(N,T)$ is maximized at $(0,\tilde{T}_1)$ subject to that $\left.\frac{\partial^2 \Pi_1(N|T)}{\partial T^2}\right|_{T=\tilde{T}_1} < 0$.

(ii) If the condition of Theorem 4.1 is valid for $N = \tilde{N}_1$ and $\left.\frac{\partial \Pi_1(N|T)}{\partial N}\right|_{N=0} > 0$ and $\left.\frac{\partial \Pi_1(N|T)}{\partial N}\right|_{N=M} < 0$ then $\exists$ a unique solution $\tilde{N}_1$ smaller than M such that $\Pi_1(N,T)$ is maximized at $(\tilde{N}_1,\tilde{T}_1)$.

(iii) If $\left.\frac{\partial \Pi_1(N|T)}{\partial N}\right|_{N=M} \geq 0$ then $\Pi_1(N,T)$ is maximized at $(M,\tilde{T}_1)$ subject to that $\left.\frac{\partial^2 \Pi_1(N|T)}{\partial T^2}\right|_{T=\tilde{T}_1} < 0$.

Proof. Since $e^{-rN}D(N)(1-F(N))$ is strictly concave thus

$$\frac{\mathrm{d}}{\mathrm{d}N}\left\{e^{-rN}D(N)(1-F(N))\right\} > 0 \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}N^2}\left\{e^{-rN}D(N)(1-F(N))\right\} < 0.$$

Then from (4.10) we have $\frac{\partial^2 \Pi_1(N|T)}{\partial N^2} < 0$. Hence, $\Pi_1(N,T)$ is a concave function on N.

(i) If $\left.\frac{\partial \Pi_1(N|T)}{\partial N}\right|_{N=0} \leq 0$ then $\frac{\partial \Pi_1(N|T)}{\partial N} < 0$ for $N \in [0, M]$, *i.e.*, $\Pi_1(N|T)$ is a strictly decreasing function or satisfies optimality condition at $N = 0$. Hence, $\Pi_1(N|T)$ is maximized for $N = 0$. Clearly $\Pi_1(N,T)$ is then maximized at $(0, \tilde{T}_1)$ if $\left.\frac{\partial^2 \Pi_1(T|N)}{\partial T^2}\right|_{T=\tilde{T}_1} < 0$.

(ii) If $\left.\frac{\partial \Pi_1(N|T)}{\partial N}\right|_{N=0} > 0$ and $\left.\frac{\partial \Pi_1(N|T)}{\partial N}\right|_{N=M} < 0$ then by intermediate value theorem we find a solution $\tilde{N}_1 \in [0, M]$ of $\frac{\partial \pi_1(N|T)}{\partial N} = 0$. As $\Pi_1(N|T)$ is a concave function thus $\exists$ a unique solution $\tilde{N}_1$ smaller than M such that $\Pi_1(N,T)$ is maximized at $N = \tilde{N}_1$. Now if the condition of Theorem 4.1 is valid for $N = \tilde{N}_1$ then $\Pi_1(N,T)$ is maximized at $(\tilde{N}_1, \tilde{T}_1)$ having the value $\max(\Pi_1(N,T)) = \mathfrak{D}_1'(\tilde{N}_1, \tilde{T}_1)$.

(iii) If $\left.\frac{\partial \Pi_1(N|T)}{\partial N}\right|_{N=M} \geq 0$ then $\frac{\partial \Pi_1(N|T)}{\partial N} > 0$ for $N \in [0, M]$, *i.e.*, $\Pi_1(N|T)$ is a strictly increasing function or satisfies optimality condition at $N = M$. Hence, $\Pi_1(N|T)$ is maximized for $N = M$. Clearly $\Pi_1(N,T)$ is then maximized at $(M, \tilde{T}_1)$ if $\left.\frac{\partial^2 \Pi_1(T|N)}{\partial T^2}\right|_{T=\tilde{T}_1} < 0$.

□

Case II: $N \leq M \leq T + N$

Let us take

$$\Pi_2(N,T) = \frac{1}{T}\mathfrak{D}_2(N,T), \tag{4.12}$$

where

$$\begin{aligned}\mathfrak{D}_2(N,T) = {} & pe^{-rN} D(N)[1 - F(N)] \int_0^T f(t)\,\mathrm{d}t - hD(N) \int_0^T e^{-g(t)} \int_t^T e^{g(u)} f(u)\,\mathrm{d}u\mathrm{d}t \\ & + pI_e D(N) \int_0^{M-N} \int_0^t f(u)\,\mathrm{d}u\mathrm{d}t - cI_c D(N) \int_{M-N}^T e^{-g(t)} \int_t^T e^{g(u)} f(u)\,\mathrm{d}u\mathrm{d}t \\ & - cD(N) \int_0^T e^{g(u)} f(u)\,\mathrm{d}u - A. \end{aligned} \tag{4.13}$$

Now we take the first order partial derivative of $\Pi_2(N,T)$ with respect to T to find the optimal cycle time,

$$\frac{\partial \Pi_2(T|N)}{\partial T} = -\frac{1}{T^2}\mathfrak{D}_2(T|N) + \frac{1}{T}\mathfrak{D}_2'(T|N) = -\frac{1}{T^2}\left[\mathfrak{D}_2(T|N) - T\mathfrak{D}_2'(T|N)\right], \tag{4.14}$$

where

$$\begin{aligned}\mathfrak{D}_2'(T|N) = {} & pe^{-rN} D(N)[1 - F(N)] f(T) - hD(N) e^{g(T)} f(T) \int_0^T e^{-g(u)}\mathrm{d}u \\ & - cI_c D(N) e^{g(T)} f(T) \int_{M-N}^T e^{-g(u)}\mathrm{d}u - cD(N) e^{g(T)} f(T). \end{aligned} \tag{4.15}$$

Let, $\forall N \geq 0$, $\tilde{T}_2$ (may or may not exist) be the solution of $[\mathfrak{D}_2(T|N) - T\mathfrak{D}_2'(T|N)] = 0$, which implies $\frac{1}{\tilde{T}_2}\mathfrak{D}_2(\tilde{T}_2|N) = \mathfrak{D}_2'\left(\tilde{T}_2|N\right) = \Pi_2(\tilde{T}_2|N)$.

Now, by taking the second order partial derivative of $\Pi_2\,(N,T)$ with respect to T, we have

$$\frac{\partial^2 \Pi_2(T|N)}{\partial T^2} = \frac{2}{T^3}\mathfrak{D}_2\,(T|N) - \frac{2}{T^2}\mathfrak{D}_2'\,(T|N) + \frac{1}{T}\mathfrak{D}_2''(T|N). \tag{4.16}$$

If $T = \tilde{T}_2$ then $\frac{\partial^2 \Pi_2(\tilde{T}_2|N)}{\partial T^2} = \frac{1}{\tilde{T}_2}\mathfrak{D}_2''(\tilde{T}_2|N)$, which implies $\frac{\partial^2 \Pi_2(\tilde{T}_2|N)}{\partial T^2} < 0$ if $\mathfrak{D}_2''(\tilde{T}_2|N) < 0$, where

$$\begin{aligned}\mathfrak{D}_2''(T|N) = & -hD(N)e^{g(T)}\theta(T)f(T)\int_0^T e^{-g(u)}\mathrm{d}u - hD(N)f(T) - cD(N)e^{g(T)}\theta(T)f(T)\\ & -cI_cD(N)e^{g(T)}\theta(T)f(T)\int_{M-N}^T e^{-g(u)}\mathrm{d}u - cI_cD(N)f(T)\\ & + \Bigg[pe^{-rN}D\,(N)\,\{1-F\,(N)\} - hD\,(N)\,e^{g(T)}\int_0^T e^{-g(u)}\mathrm{d}u - cD\,(N)\,e^{g(T)}\\ & -cI_cD\,(N)\,e^{g(T)}\int_{M-N}^T e^{-g(u)}\mathrm{d}u\Bigg] f'(T).\end{aligned} \tag{4.17}$$

It can be easily verified that if $pe^{-rN}\,\{1-F\,(N)\} - c < 0$ then $\lim_{T\to+\infty} \frac{\partial \Pi_2(T|N)}{\partial T} = -\infty$.

Theorem 4.4. *If*

$$pe^{-rN}\,\{1-F\,(N)\} - he^{g(T)}\int_0^T e^{-g(u)}\mathrm{d}u - ce^{g(T)} - cI_ce^{g(T)}\int_{M-N}^T e^{-g(u)}\mathrm{d}u < 0,$$

then $\Pi_2(T|N)$ is maximized for $T = \tilde{T}_2$ where

(i) $M - N < \tilde{T}_2 < +\infty$ if $\Delta < 0$.
(ii) $\tilde{T}_2 = M - N$ if $\Delta \geq 0$.

Proof. As $D(N) > 0$, we have

$$pe^{-rN}D\,(N)\,\{1-F\,(N)\} - hD\,(N)\,e^{g(T)}\int_0^T e^{-g(u)}\mathrm{d}u - cD\,(N)\,e^{g(T)} - cI_cD\,(N)\,e^{g(T)}\int_{M-N}^T e^{-g(u)}\mathrm{d}u < 0.$$

Thus from equation (4.17), $\mathfrak{D}_2''(T|N) < 0$, *i.e.*, $\Pi_2(T|N)$ is a concave function on T.

It is easy to verify that

$$\Delta = \mathfrak{D}_1\,(N,T) - T\mathfrak{D}_1'(N,T)|_{T=M-N} = \mathfrak{D}_2\,(N,T) - T\mathfrak{D}_2'(N,T)|_{T=M-N}.$$

(i) Now if $\Delta < 0$ then $\left.\frac{\partial \Pi_2(T|N)}{\partial T}\right|_{T=M-N} > 0$. Thus, by intermediate value theorem, we have a solution $\tilde{T}_2$ of $\frac{\partial \Pi_2(T|N)}{\partial T} = 0$ such that $M - N < \tilde{T}_2 < +\infty$.
(ii) If $\Delta \geq 0$ then $\left.\frac{\partial \Pi_2(T|N)}{\partial T}\right|_{T=M-N} < 0$, *i.e.*, $\Pi_2(T|N)$ is a strictly decreasing function of $T \in [M-N, +\infty]$ or satisfies optimality condition at $\tilde{T}_2 = M - N$. Then $\Pi_2(N,T)$ is maximum for $\tilde{T}_2 = M - N$.

□

Corollary 4.5. *$pe^{-rN}(1-F(N))-c<0$ holds for $\forall N>\hat{N}$ where $\hat{N}$ is the unique solution of $pe^{-rN}(1-F(N))-c=0$ if $pe^{-rM}(1-F(M))-c<0$ and consequently condition for Theorem 4.4 holds for all $N\in\left]\hat{N},M\right]$.*

Proof. As $pe^{-rN}(1-F(N))-c$ is a strictly decreasing function, by similar arguments as Corollary 4.2, we must find unique solution $\hat{N}$ of the equation $pe^{-rN}(1-F(N))-c=0$ which exists as $pe^{-rM}(1-F(M))-c<0$.

As $pe^{-rN}\{1-F(N)\}<c<ce^{g(T)}+cI_ce^{g(T)}\int_{M-N}^{T}e^{-g(u)}\mathrm{d}u+he^{g(T)}\int_0^T e^{-g(u)}\mathrm{d}u$, therefore condition of Theorem 4.4 evidently holds in the said interval.

$$\begin{aligned}\Pi_2(N,T)=\frac{1}{T}\Bigg[&pe^{-rN}D(N)[1-F(N)]\int_0^T f(t)\,\mathrm{d}t-D(N)Y-A\\&-cI_cD(N)\int_{M-N}^{T}e^{-g(t)}\int_t^T e^{g(u)}f(u)\,\mathrm{d}u\mathrm{d}t+pI_eD(N)\int_0^{M-N}\int_0^t f(u)\,\mathrm{d}u\mathrm{d}t\Bigg],\end{aligned}\tag{4.18}$$

where

$$Y=h\int_0^T e^{-g(t)}\int_t^T e^{g(u)}f(u)\,\mathrm{d}u\mathrm{d}t+c\int_0^T e^{g(u)}f(u)\,\mathrm{d}u.\tag{4.19}$$

We now take the partial derivative of $\Pi_2(N,T)$ with respect to N.

$$\begin{aligned}\frac{\partial\Pi_2(N|T)}{\partial N}=\frac{1}{T}\Bigg[&p\frac{\mathrm{d}}{\mathrm{d}N}\left\{e^{-rN}D(N)(1-F(N))\right\}\int_0^T f(t)\,\mathrm{d}t-D'(N)Y-pI_eD(N)\int_0^{M-N}f(u)\,\mathrm{d}u\\&-cI_cD'(N)\int_{M-N}^{T}e^{-g(t)}\int_t^T e^{g(u)}f(u)\,\mathrm{d}u\mathrm{d}t-cI_cD(N)e^{-g(M-N)}\int_{M-N}^{T}e^{g(u)}f(u)\,\mathrm{d}u\\&+pI_eD'(N)\int_0^{M-N}\int_0^t f(u)\,\mathrm{d}u\mathrm{d}t\Bigg].\end{aligned}\tag{4.20}$$

Let us consider that $\frac{\partial\Pi_2(N|T)}{\partial N}=0$, for $N=\tilde{N}_2$ (may or may not exist).

Now, by taking the second order partial derivative with respect to N, we have

$$\begin{aligned}\frac{\partial^2\Pi_2(N|T)}{\partial N^2}=\frac{1}{T}\Bigg[&p\frac{\mathrm{d}^2}{\mathrm{d}N^2}\left\{e^{-rN}D(N)(1-F(N))\right\}\int_0^T f(t)\,\mathrm{d}t-D''(N)Y+pI_eD(N)f(M-N)\\&-cI_cD''(N)\int_{M-N}^{T}e^{-g(t)}\int_t^T e^{g(u)}f(u)\,\mathrm{d}u\mathrm{d}t-2cI_cD'(N)e^{-g(M-N)}\int_{M-N}^{T}e^{g(u)}f(u)\,\mathrm{d}u\\&+pI_eD''(N)\int_0^{M-N}\int_0^t f(u)\,\mathrm{d}u\mathrm{d}t-2pI_eD'(N)\int_0^{M-N}f(u)\,\mathrm{d}u-cI_cD(N)f(M-N)\\&-cI_cD(N)e^{-g(M-N)}\theta(M-N)\int_{M-N}^{T}e^{g(u)}f(u)\,\mathrm{d}u\Bigg].\end{aligned}\tag{4.21}$$

□

Theorem 4.6. *If $e^{-rN}D(N)(1-F(N))$ is a strictly concave function with $D''(N)>0, cI_c>pI_e$, $cI_c\int_M^T e^{-g(t)}\int_t^T e^{g(u)}f(u)\,\mathrm{d}u\mathrm{d}t>pI_e\int_0^M\int_0^t f(u)\,\mathrm{d}u\mathrm{d}t$ then* *(i) If* $\left.\frac{\partial \Pi_2(N|T)}{\partial N}\right|_{N=0} \leq 0$ *then* $\Pi_2\,(N,T)$ *is maximized at* $(0,\tilde{T}_2)$ *subject to that* $\left.\frac{\partial^2 \Pi_2(N|T)}{\partial T^2}\right|_{T=\tilde{T}_2} < 0$.

(ii) If the condition of Theorem 4.3 is valid for $N=\tilde{N}$ *and* $\left.\frac{\partial \Pi_2(N|T)}{\partial N}\right|_{N=0} > 0$ *and* $\left.\frac{\partial \Pi_2(N|T)}{\partial N}\right|_{N=M} < 0$ *then* $\exists$ *a unique solution* $\tilde{N}$ *smaller than* M *such that* $\Pi_2\,(N,T)$ *is maximized at* $(\tilde{N}_2,\tilde{T}_2)$.

(iii) If $\left.\frac{\partial \Pi_2(N|T)}{\partial N}\right|_{N=M} \geq 0$ *then* $\Pi_2\,(N,T)$ *is maximized at* $(M,\tilde{T}_2)$ *subject to that* $\left.\frac{\partial^2 \Pi_2(N|T)}{\partial T^2}\right|_{T=\tilde{T}_2} < 0$.

Proof. Since $e^{-rN}D(N)(1-F(N))$ is strictly concave thus $\frac{\mathrm{d}}{\mathrm{d}N}\left\{e^{-rN}D(N)(1-F(N))\right\} > 0$ and $\frac{\mathrm{d}^2}{\mathrm{d}N^2}\left\{e^{-rN}D(N)(1-F(N))\right\} < 0$.

Now it is evident that $cI_c\int_0^T e^{-g(t)}\int_t^T e^{g(u)} f(u)\,\mathrm{d}u\mathrm{d}t > 0$.

Thus, if $cI_c\int_M^T e^{-g(t)}\int_t^T e^{g(u)} f(u)\,\mathrm{d}u\mathrm{d}t > pI_e\int_0^M\int_0^t f(u)\,\mathrm{d}u\mathrm{d}t$, then $cI_c\int_{M-N}^T e^{-g(t)}\int_t^T e^{g(u)} f(u)\,\mathrm{d}u\mathrm{d}t > pI_e\int_0^{M-N}\int_0^t f(u)\,\mathrm{d}u\mathrm{d}t$, for all $N\in[0,M]$. So, from equation (4.21) we can conclude that $\frac{\partial^2 \Pi_2(N|T)}{\partial N^2} < 0$, *i.e.*, $\Pi_2\,(N,T)$ is a concave function on N.

(i) If $\left.\frac{\partial \Pi_2(N|T)}{\partial N}\right|_{N=0} \leq 0$ then $\frac{\partial \Pi_2(N|T)}{\partial N} < 0$ for $N\in[0,M]$, *i.e.*, $\Pi_2\,(N|T)$ is a strictly decreasing function or satisfies optimality condition at $N=0$. Hence, $\Pi_2\,(N|T)$ is maximized for $N=0$. Clearly $\Pi_2\,(N,T)$ is then maximized at $(0,\tilde{T}_2)$ if $\left.\frac{\partial^2 \Pi_2(T|N)}{\partial T^2}\right|_{T=\tilde{T}_2} < 0$.

(ii) If $\left.\frac{\partial \Pi_2(N|T)}{\partial N}\right|_{N=0} > 0$ and $\left.\frac{\partial \Pi_2(N|T)}{\partial N}\right|_{N=M} < 0$ then by intermediate value theorem we find a solution $\tilde{N}_2\in[0,M]$ of $\frac{\partial \Pi_2(N|T)}{\partial N} = 0$. As $\Pi_2\,(N|T)$ is a concave function thus $\exists$ a unique solution $\tilde{N}_2$ smaller than M such that $\Pi_2\,(N|T)$ is maximized at $N=\tilde{N}_2$. Now if the condition of Theorem 4.4 is valid for $N=\tilde{N}_2$ then $\Pi_2\,(N,T)$ is maximized at $(\tilde{N}_2,\tilde{T}_2)$ having the value $\max\left(\Pi_2\,(N,T)\right) = \mathfrak{D}_2'(\tilde{N}_2,\tilde{T}_2)$.

(iii) If $\left.\frac{\partial \Pi_2(N|T)}{\partial N}\right|_{N=M} \geq 0$ then $\frac{\partial \Pi_2(N|T)}{\partial N} > 0$ for $N\in[0,M]$, *i.e.*, $\Pi_2\,(N|T)$ is a strictly increasing function or satisfies optimality condition at $N=M$. Hence, $\Pi_2\,(N|T)$ is maximized for $N=M$. Clearly $\Pi_2\,(N,T)$ is then maximized at $(M,\tilde{T}_2)$ if $\left.\frac{\partial^2 \Pi_2(T|N)}{\partial T^2}\right|_{T=\tilde{T}} < 0$.

□

Case III: $M \leq N \leq T+N$.

Let us take

$$\Pi_3\,(N,T) = \frac{1}{T}\mathfrak{D}_3(N,T), \tag{4.22}$$

where

$$\begin{aligned}\mathfrak{D}_3\,(N,T) = {} & pe^{-rN}D\,(N)\,[1-F\,(N)]\int_0^T f\,(t)\,\mathrm{d}t - hD\,(N)\int_0^T e^{-g(t)}\int_t^T e^{g(u)} f\,(u)\,\mathrm{d}u\mathrm{d}t\\ & -cI_cD(N)\left\{\int_0^T e^{-g(t)}\int_t^T e^{g(u)} f\,(u)\,\mathrm{d}u\mathrm{d}t + (N-M)\int_0^T e^{g(u)} f\,(u)\,\mathrm{d}u\right\}\\ & -cD\,(N)\int_0^T e^{g(u)} f\,(u)\,\mathrm{d}u - A.\end{aligned} \tag{4.23}$$

By taking the first order partial derivative of $\Pi_3\,(N,T)$ with respect to T, we have

$$\frac{\partial \Pi_3(T|N)}{\partial T} = -\frac{1}{T^2}\mathfrak{D}_3\,(T|N) + \frac{1}{T}\mathfrak{D}_3'\,(T|N) = -\frac{1}{T^2}\left[\mathfrak{D}_3\,(T|N) - T\mathfrak{D}_3'(T|N)\right], \tag{4.24}$$

where

$$\mathfrak{D}_3'(T|N) = pe^{-rN}D(N)[1-F(N)]f(T) - hD(N)e^{g(T)}f(T)\int_0^T e^{-g(u)}\mathrm{d}u - cD(N)e^{g(T)}f(T)$$
$$-cI_cD(N)\left[e^{g(T)}f(T)\int_0^T e^{-g(u)}\mathrm{d}u + (N-M)e^{g(T)}f(T)\right]. \tag{4.25}$$

For extreme solutions $\forall N \geq 0$, $\frac{\partial \Pi_3(T|N)}{\partial T} = -\frac{1}{T^2}\left[\mathfrak{D}_3(T|N) - T\mathfrak{D}_3'(T|N)\right] = 0$.

Let $\tilde{T}_3$ (may or may not exist) be the solution of

$$\left[\mathfrak{D}_3(T|N) - T\mathfrak{D}_3'(T|N)\right] = 0 \Rightarrow \frac{1}{\tilde{T}}\mathfrak{D}_3(\tilde{T}|N) = \mathfrak{D}_3'\left(\tilde{T}_3|N\right) = \Pi_3(\tilde{T}_3|N).$$

Now, we take the second order partial derivative to determine the concavity of $\Pi_3(N,T)$ with respect to T.

$$\frac{\partial^2 \Pi_3(T|N)}{\partial T^2} = \frac{2}{T^3}\mathfrak{D}_3(T|N) - \frac{2}{T^2}\mathfrak{D}_3'(T|N) + \frac{1}{T}\mathfrak{D}_3''(T|N). \tag{4.26}$$

If $T = \tilde{T}_3$ then $\frac{\partial^2 \Pi_3(\tilde{T}_3|N)}{\partial T^2} = \frac{1}{\tilde{T}}\mathfrak{D}_3''(\tilde{T}_3|N)$, which implies $\frac{\partial^2 \Pi_3(\tilde{T}_3|N)}{\partial T^2} < 0$ if $\mathfrak{D}_3''(\tilde{T}_3|N) < 0$.

Now,

$$\mathfrak{D}_3''(T|N) = -hD(N)e^{g(T)}\theta(T)f(T)\int_0^T e^{-g(u)}\mathrm{d}u - hD(N)f(T)) - cI_cD(N)f(T)$$
$$-cI_cD(N)e^{g(T)}\theta(T)f(T)\left[\int_0^T e^{-g(u)}\mathrm{d}u + (N-M)\right] - cD(N)e^{g(T)}\theta(T)f(T)$$
$$+\left[pe^{-rN}D(N)\{1-F(N)\} - hD(N)e^{g(T)}\int_0^T e^{-g(u)}\mathrm{d}u\right.$$
$$\left.-cI_cD(N)e^{g(T)}\left\{\int_0^T e^{-g(u)}\mathrm{d}u + (N-M)\right\} - cD(N)e^{g(T)}\right]f'(T). \tag{4.27}$$

Let us define the discrimination term

$$\Delta = \mathfrak{D}_3(T|N) - T\mathfrak{D}_3'(T|N)\big|_{T=M-N}.$$

It can be easily verified that if $pe^{-rN}\{1-F(N)\} - cI_c(N-M) - c < 0$ then

$$\lim_{T\to+\infty}\frac{\partial \Pi_2(T|N)}{\partial T} = -\infty.$$

Theorem 4.7. *If $pe^{-rN}\{1-F(N)\} - he^{g(T)}\int_0^T e^{-g(u)}\mathrm{d}u - cI_ce^{g(T)}\left\{\int_0^T e^{-g(u)}\mathrm{d}u + (N-M)\right\} - ce^{g(T)} < 0$, then $\Pi_3(N,T)$ is maximized for $T = \tilde{T}$ where*

(i) $M - N < \tilde{T}_3 < +\infty$ if $\Delta < 0$.
(ii) $\tilde{T}_3 = M - N$ if $\Delta \geq 0$.

Proof. We have

$$pe^{-rN}D(N)\{1-F(N)\} - hD(N)e^{g(T)}\int_0^T e^{-g(u)}\mathrm{d}u - cI_cD(N)e^{g(T)}\left\{\int_0^T e^{-g(u)}\mathrm{d}u + (N-M)\right\}$$
$$-cD(N)e^{g(T)} < 0.$$

Thus, from equation (4.27) we have $\mathfrak{D}_3''(T|N) < 0$, *i.e.*, $\Pi_3(T|N)$ is a concave function of T.

(i) Now if $\Delta < 0$ then $\left.\frac{\partial \Pi_3(T|N)}{\partial T}\right|_{T=M-N} > 0$. Thus, by intermediate value theorem, we have a solution $\tilde{T}_3$ of $\frac{\partial \Pi_3(T|N)}{\partial T} = 0$ such that $M-N < \tilde{T}_3 < +\infty$.

(ii) If $\Delta \geq 0$ then $\left.\frac{\partial \Pi_3(T|N)}{\partial T}\right|_{T=M-N} < 0$, *i.e.*, $\Pi_3(N,T)$ is a strictly decreasing function of $T \in [M-N,+\infty]$ and hence satisfies optimality condition at $\tilde{T}_3 = M-N$. Then $\Pi_3(N,T)$ is maximum for $\tilde{T}_3 = M-N$.

Now we have,

$$\Pi_3(N,T) = \frac{1}{T}\left[\begin{array}{c} pe^{-rN}D(N)[1-F(N)]\int_0^T f(t)\,\mathrm{d}t - cI_cD(N)(N-M)\int_0^T e^{g(u)}f(u)\,\mathrm{d}u \\ -D(N)Z - A \end{array}\right], \tag{4.28}$$

where

$$Z = (h+cI_c)\int_0^T e^{-g(t)}\int_t^T e^{g(u)}f(u)\,\mathrm{d}u\mathrm{d}t + c\int_0^T e^{g(u)}f(u)\,\mathrm{d}u. \tag{4.29}$$

By taking the first order partial derivative of $\Pi_3(T,N)$ with respect to N, we have

$$\begin{aligned}\frac{\partial \Pi_3(N|T)}{\partial N} = \frac{1}{T}\Bigg[&p\frac{\mathrm{d}}{\mathrm{d}N}\left\{e^{-rN}D(N)(1-F(N))\right\}\int_0^T f(t)\,\mathrm{d}t - D'(N)Z \\ &-cI_c\{D'(N)(N-M)+D(N)\}\int_0^T e^{g(u)}f(u)\,\mathrm{d}u\Bigg],\end{aligned} \tag{4.30}$$

$\frac{\partial \Pi_3(N|T)}{\partial N} = 0$ may hold if $\frac{\mathrm{d}}{\mathrm{d}N}\left\{e^{-rN}D(N)(1-F(N))\right\} > 0$ whereas if $\frac{\mathrm{d}}{\mathrm{d}N}\left\{e^{-rN}D(N)(1-F(N))\right\} < 0$, $\Pi_3(N|T)$ is a decreasing function and we must have $\tilde{N}_3 = M$.

Let us consider that $\frac{\partial \Pi_3(N|T)}{\partial N} = 0$, for $N = \tilde{N}_3$ (may or may not exist).

Now,

$$\begin{aligned}\frac{\partial^2 \Pi_3(N|T)}{\partial N^2} = \frac{1}{T}\Bigg[&p\frac{\mathrm{d}^2}{\mathrm{d}N^2}\left\{e^{-rN}D(N)(1-F(N))\right\}\int_0^T f(t)\,\mathrm{d}t - D''(N)X \\ &-cI_c\{D''(N)(N-M)+2D'(N)\}\int_0^T e^{g(u)}f(u)\,\mathrm{d}u\Bigg].\end{aligned} \tag{4.31}$$

If $e^{-rN}D(N)(1-F(N))$ is a strictly concave function then $\lim_{N\to+\infty}\frac{\partial \Pi_3(N|T)}{\partial N} = -\infty$.

□

Theorem 4.8. *If* $e^{-rN}D(N)(1-F(N))$ *is a strictly concave function and* $D''(N) > 0$ *then* *(i) If the condition of Theorem 4.7 is valid and* $\left.\frac{\partial \Pi_3(N|T)}{\partial N}\right|_{N=M} > 0$*,* $\exists$ *a unique* $\tilde{N}_3 > M$ *such that* $\Pi_3(N,T)$ *is maximized at* $(\tilde{N}_3, \tilde{T}_3)$*.*

(ii) If $\left.\frac{\partial \Pi_3(N|T)}{\partial N}\right|_{N=M} \leq 0$*,* $\Pi_3(N,T)$ *is maximized at* $(M, \tilde{T}_3)$ *subject to that* $\left.\frac{\partial^2 \Pi_1(T|N)}{\partial T^2}\right|_{T=\tilde{T}_3} < 0$*.*

Proof. Since $e^{-rN}D(N)(1-F(N))$ is strictly concave thus

$$\frac{\mathrm{d}}{\mathrm{d}N}\left\{e^{-rN}D(N)(1-F(N))\right\} > 0 \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}N^2}\left\{e^{-rN}D(N)(1-F(N))\right\} < 0.$$

So, from equation (4.31) we get $\frac{\partial^2 \Pi_3(N|T)}{\partial N^2} < 0$, *i.e.*, $\Pi_3(N|T)$ is a concave function.

(i) If $\left.\frac{\partial \Pi_3(N|T)}{\partial N}\right|_{N=M} > 0$, then by intermediate value theorem we find a solution $\tilde{N}_3 \in [M, +\infty]$ of $\frac{\partial \Pi_3(N|T)}{\partial N} = 0$. As $\Pi_3(N,T)$ is a concave function thus there exists a unique solution $\tilde{N}_3$ greater than M such that $\Pi_3(N,T)$ is maximized at $N = \tilde{N}_3$. Now if the condition of Theorem 4.7 is valid for $N = \tilde{N}_3$ then $\Pi_3(N,T)$ is maximized at $(\tilde{N}_3, \tilde{T}_3)$ having the value $\max(\Pi_2(N,T)) = \mathfrak{D}_2'(\tilde{N}_3, \tilde{T}_3)$.

(ii) If $\left.\frac{\partial \Pi_3(N|T)}{\partial N}\right|_{N=M} \leq 0$, then $\Pi_3(N,T)$ is strictly decreasing on $[M, +\infty]$ or having extremum at $N = M$. Thus, $\Pi_3(N,T)$ is maximized at $(M, \tilde{T}_3)$ subject to that $\left.\frac{\partial^2 \Pi_1(N|T)}{\partial T^2}\right|_{T=\tilde{T}_3} < 0$.

□

Now, it is time to present some numerical examples in the next section.

5. Numerical example

In this section, in order to show the applicability of the presented model and also the solution procedure, three numerical examples are presented. In addition, these examples provide the materials for sensitivity analysis as well as extracting some managerial insights, which will be discussed in the next section.

In our proposed model, we assume that

(i) The deterioration rate $\theta(t)$ is non-decreasing and $\theta(m) = 1$.

(ii) The demand rate is $\lambda(t,N) = f(t)\,D(N)$, where $f(t)$ and $D(N)$ are both increasing in t $(0 \leq t \leq T)$ and N $(N > 0)$, respectively.

(iii) The rate of default risk $F(N)$ given the credit period N offered by the retailer is positive, and increasing in N.

(iv) The up-stream credit period is M and the down-stream credit period is N.

In numerous previous inventory models, $\theta(t)$, $f(t)$, $D(N)$, and $F(N)$ are assumed in a specific form. Consequently, they are indeed special cases of the proposed model here.

Recently, many researchers have adopted the deterioration rate as $\theta(t) = 1/(1+m-t)$ with $0 \leq t \leq T \leq m$ to incorporate the fact that the deterioration rate is 100% near to its expiration date such as Sarkar [37], Mahata [24], Wu *et al.* [59], Sarkar *et al.* [40], Sarkar [38], and Sett *et al.* [42]. Hence, we use the newly adopted deterioration rate to run the numerical examples. Next, the rate of default risk $F(N) = 1 - e^{-bN}, b > 0$ defined in the models of Teng and Lou [51], Lou and Wang [22], and Mahata *et al.* [31] is used to demonstrate the impacts of default risk on the optimal credit period and cycle time decisions. In addition, we have also adopted $f(t) = a + bt + ct^2, a, b, c > 0$ used in Khanra *et al.* [17] and Mahata and Mahata [30] and $D(N) = e^{dN}, 0 < d < 1$ used in Teng and Lou [51], Wu *et al.* [59], and Mahata [25] to incorporate the fact that the demand rate $\lambda(t,N) = f(t)\,D(N)$ varies simultaneously with time and the length of credit period that is offered to the customers.

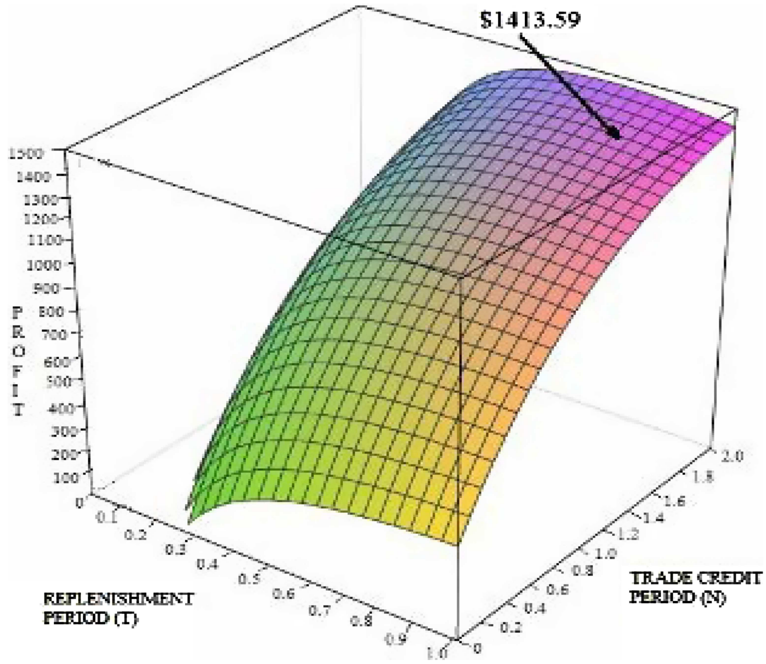

FIGURE 4. Optimal profit graph for Example 5.1.

Example 5.1. Let us assume that $\theta(t) = \frac{1}{1+m-t}$, $\lambda(t, N) = f(t) D(N)$, where $f(t) = 1000 + 100t + 20t^2$ and $D(N) = e^{0.075N}$, $F(N) = 1 - e^{-bN}, b = 0.03$, $r = 0.04$, $M = 0.8$ year, $p = \$20$/unit, $c = \$5$/unit, $A = \$200$/order, $h = \$3$/ unit/year, $I_c = 0.18$, $I_e = 0.1$, $m = 1$ year. By using software LINGO 16.0 x64, we have the maximum solution to $\Pi_i(N, T)$ for $i = 1$, 2, and 3 as follow:

$$N_1^* = 0.3748366 \text{ years},\ T_1^* = 0.4251634 \text{ years},\ \Pi_1^* = \$823.03.$$
$$N_2^* = 0.8 \text{ years},\ T_2^* = 1.121782 \text{ years},\ \Pi_2^* = \$1253.07.$$
$$N_3^* = 1.658679 \text{ years},\ T_3^* = 0.9433776 \text{ years},\ \Pi_3^* = \$1413.59.$$

Consequently the retailer's optimum solution is

$$N^* = 1.658679 \text{ years},\ T^* = 0.9433776 \text{ years},\ \Pi^* = \$1413.59.$$

For this type of demand pattern, the average profit function is highly non-linear. So, it is impossible to find closed type formula for N and T. But Figure 4 shows the concavity of the annual profit function in both N and T. Hence, the better optimal solution is a global maximum.

Example 5.2. Using the same data as those in Example 5.1 except $A = \$90$/order, $c = \$13$/unit, $I_c = 0.25$, we obtained the following results:

$$N_1^* = 0.3545723 \text{ years},\ T_1^* = 0.3649281 \text{ years},\ \Pi_1^* = \$381.01.$$
$$N_2^* = 0.3358878 \text{ years},\ T_2^* = 0.4641122 \text{ years},\ \Pi_2^* = \$333.79.$$
$$N_3^* = 0.8 \text{ years},\ T_3^* = 0.5059718 \text{ years},\ \Pi_3^* = \$321.62.$$

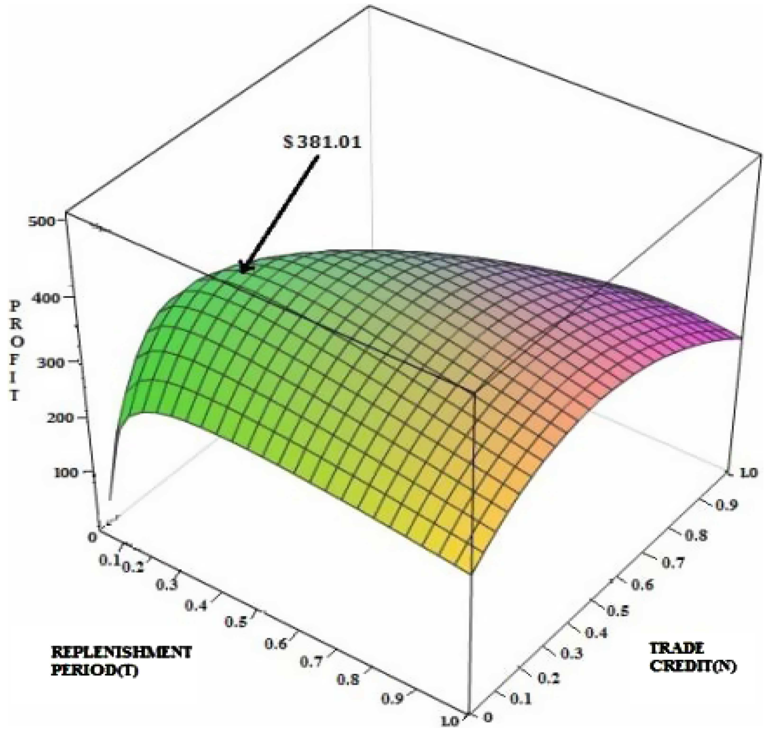

FIGURE 5. Optimal profit graph for Example 5.2.

So, the optimal solution for the retailer is

$$N^* = 0.3545723 \text{ years}, \; T^* = 0.3649281 \text{ years}, \; \Pi^* = \$381.01.$$

Figure 5 reveals that $\Pi(N,T)$ is a strictly pseudo-concave function in both N and T. Hence, the better optimal solution is a global maximum.

Example 5.3. Using the same data as those in Example 5.1 except $A = \$90/\text{order}$, $c = \$12/\text{unit}$, $I_c = 0.15$, we obtain the following results:

$$N_1^* = 0.4054832 \text{ years}, \; T_1^* = 0.3945168 \text{ years}, \; \Pi_1^* = \$469.81.$$
$$N_2^* = 0.8 \text{ years}, \; T_2^* = 0.5961728 \text{ years}, \; \Pi_2^* = \$488.82.$$
$$N_3^* = 0.8 \text{ years}, \; T_3^* = 0.5961728 \text{ years}, \; \Pi_3^* = \$488.82.$$

So the optimal solution for the retailer will be

$$N^* = 0.8 \text{ years}, \; T^* = 0.5961728 \text{ years}, \; \Pi^* = \$488.82.$$

Figure 6 reveals that $\Pi(N,T)$ is a strictly pseudo-concave function in both N and T. Hence the better optimal solution is a global maximum.

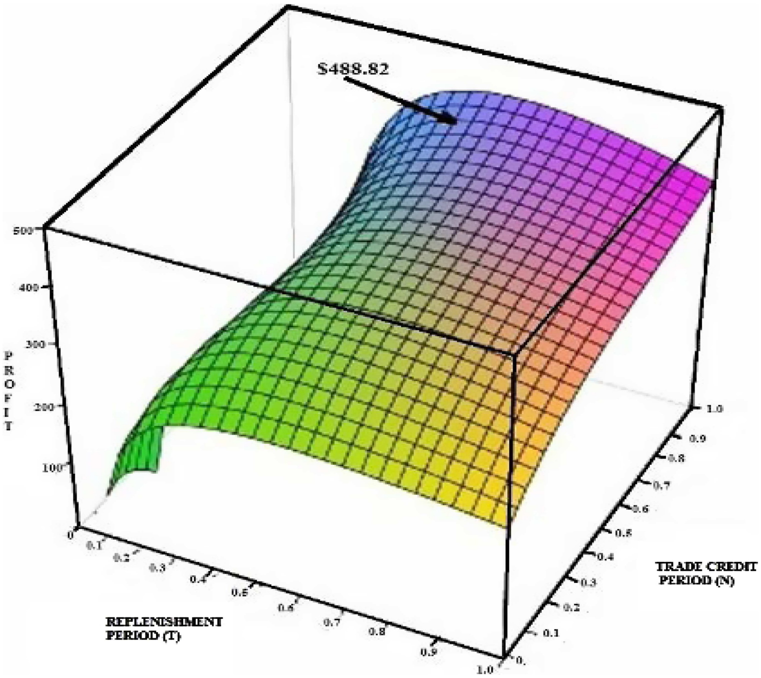

FIGURE 6. Optimal profit graph for Example 5.3.

5.1. Managerial insights

Sensitivity analysis has been performed in order to determine the robustness of the model presented above. Using the same data as those in Example 5.1, we study the sensitivity analysis on the optimal solutions with respect to the parameters in appropriate units. The computational results are shown in Table 2.

The sensitivity analysis reveals that,

(a) As the ordering cost A increases, the replenishment period T^* increases whereas the trade credit period N^* decreases along with total profit $\Pi^*(N,T)$. Thus, the retailer's objective is to reduce the ordering cost, anyhow to make a more profit. In case of a higher ordering cost the retailer should order for a longer replenishment period to reduce the order frequency.

(b) When the selling price p increases both the replenishment period T^* and the trade credit period N^* increases so total profit $\Pi^*(N,T)$ as well. Hence, in-line with the selling price the retailer should offer a longer trade credit period and reduce the order frequency to make more profit.

(c) Whereas the holding cost h and the purchasing cost c show a very different character over here. As both the holding and purchasing cost increase the retailer has to shorten the credit period and increase the order frequency to meet the optimal profit. The retailer should take measure to reduce the purchasing and holding cost by transportation cost reduction, online purchasing, reduced chain purchasing to reduce these costs, and earn more profit.

(d) The value of m reveals the obvious character of deterioration factor $\theta(t)$, *i.e.*, as deterioration factor decreases profit increases with increasing order cycle and credit period. That is the retailer should take necessary measure to reduce the deterioration of item to make sufficient profit.

(e) If the trade credit period of the supplier M increases the retailer should lengthen both the credit period and replenishment period to make more profit.

(f) If the value of b increases, the value of N^* and the total profit $\Pi^*(N,T)$ decreases whereas the order cycle T^* increases. When the default risk of the customer is higher, the retailer should offer a shorter delay

TABLE 2. Sensitivity analysis on parameters.

Parameter	Values	N^*	T^*	$\Pi^*(N,T)$
A	100	1.720941	0.6981161	1535.080
	150	1.686789	0.8333156	1469.845
	200	1.658679	0.9433776	1413.590
p	15	1.287579	0.8950283	710.1683
	20	1.658679	0.9433776	1413.590
	25	1.918966	1.032760	2156.027
c	5	1.658679	0.9433776	1413.590
	7	1.276843	0.8843640	1031.467
	10	0.8247592	0.8443106	564.8598
h	3	1.658679	0.9433776	1413.590
	5	1.634498	0.7235645	1276.525
	7	1.608521	0.6100265	1168.804
m	1	1.646732	0.8765509	1384.492
	1.5	1.658679	0.9433776	1413.590
	2	1.662810	0.9681793	1423.550
M	0.6	1.639831	0.9400124	1383.562
	0.8	1.658679	0.9433776	1413.590
	1.0	1.677467	0.9468381	1443.753
b	0.027	1.692726	0.9407193	1428.262
	0.030	1.658679	0.9433776	1413.590
	0.033	1.626103	0.9459927	1399.347

payment time to his customer and stretch the order cycle. The retailer can take some measure to reduce the default risk of the customer by adopting the partial trade credit policies with different delay payment schemes such as down payment scheme, annual installment scheme, EMI scheme, etc.

6. Conclusions

The results in this paper not only provide a valuable reference for decision-makers in planning and controlling the inventory but also provide a useful model for many organizations that use the decision rule to improve their sales profit. Taking care of upstream and downstream trade credits simultaneously for deteriorating items and considering demands to a time varying and credit sensitive involving default risk has received relatively little attention from researchers. Most of the existing inventory models under trade credit financing are assumed that the demand rate remains constant. However, in practice the market demand is always changing rapidly and is affected by several factors such as price, time, inventory level, and delayed payment period. In today's high-tech products, demand rate increases significantly during the growth stage. Moreover, the marginal influence of the credit period on sales is associated with the unrealized potential market demand. To obtain robust and generalized results, we extend the constant demand to a time varying and credit sensitive demand. In this paper, we formulate a supplier–retailer–customer supply chain inventory model for deteriorating items such as volatile liquids, blood banks, fresh fruits, vegetables, pharmaceutical products, etc. under two levels of trade credit policy with default risk consideration. The items deteriorate at an increasing varying rate of deterioration. The supplier frequently offers the retailer a trade credit of M periods, and the retailer in turn provides a trade credit of N periods to her/his buyer to stimulate sales and reduce inventory. From the seller's perspective, granting trade credit increases sales and revenue but also increases default risk (*i.e.*, the percentage that the buyer will not be able to pay off her/his debt obligations). Here, the demand is assumed to be dynamic which varies simultaneously with time and the length of credit period that is offered to the customers. The aim of this paper is to maximize the retailer's annual total profit by making decisions regarding the credit period and lot size. The concavity of all derived objective functions (*i.e.*, annual total profit of the retailer in all cases) are proved and a closed form optimal solution obtained for determining the global optimal of the model. At the end, several numerical examples are presented and a sensitivity analysis is performed to show the applicability of the developed models and also to provide managerial insights.

In practice, the contributions of this paper and the approach we considered to solve the problem are significant because the retailer has to decide whether it is worthwhile to alter the regular ordering pattern to exploit other opportunities and assess their monetary impact to find the optimal ordering policy under realistic conditions linking marketing as well as operations management concerns. Finally, this paper brings attention into the trade credit that is of major importance in the operations of enterprises in many economics.

All models have their limitations. In practice, deterioration rate depends on product, time, place, weather, etc. Based on the best of our knowledge, due to the complexity of the problem, none has scientifically measured and quantified the deterioration rate of a product before. Likewise, none has quantified demand rate as a function of time and trade credit. Hence, how to quantity deterioration rate as well as demand rate is a major obstacle to implement the proposed model into a real-world application. Additionally, in traditional marketing and economic theory, price is a major factor on the demand rate. As a result, one could take pricing strategy into consideration in the future research. Furthermore, there are four distinct economic equilibrium solutions between the seller and the buyer: (i) non-cooperative Nash equilibrium solution without a dominating player, (ii) non-cooperative Stackelberg equilibrium solution with a dominating player, (iii) cooperative Pareto equilibrium solution through negotiation, and (iv) integrated optimal solution for an integrated organization. Therefore, for the future research one should derive and compare those four distinct economic equilibrium solutions among players in a supplier–retailer–customer supply chain. Finally, one might generalize our proposed model in the directions of recent research papers such as Teng *et al.* [55], Shin *et al.* [44], and Shah *et al.* [43].

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