RAIRO-Oper. Res. 52 (2018) 895–901
https://doi.org/10.1051/ro/2018010

RAIRO Operations Research
www.rairo-ro.org

ECONOMIC-ORDER-TYPE INVENTORY MODELS FOR NON-INSTANTANEOUS DETERIORATING ITEMS AND BACKLOGGING

Bashair Ahmad and Lakdere Benkherouf*

Abstract. This paper is concerned with finding the optimal economic order quantity for the basic (EOQ) inventory model with backlogging in the presence of non-instantaneous deterioration. It is shown that the optimal EOQ is a threshold policy. That is, (i) if the time at which deterioration begins in the non-instantaneous deterioration model is greater than or equal to the time at which backlogging begins in the basic (EOQ) model, then the optimal policy is determined by the parameters of the basic (EOQ) model, else (ii) the optimal policy corresponds to the unique critical point of the objective function for the model with non-instantaneous deterioration. An approach for determining this policy is proposed. This approach is simple and easy to implement. Moreover, it does not suffer from the shortcomings of existing approaches in the literature. A numerical example is presented for illustration.

Mathematics Subject Classification. 90B05

Received July 28, 2017. Accepted January 31, 2018.

1. Introduction

This paper is concerned with the search for the optimal economic order model (EOQ) for the basic economic order quantity with non-instantaneous deteriorating items and backlogging. For such model, the onset of deterioration does not begin at the time of delivery but is delayed for some fixed known time. These phenomena can be observed in some food products such as vegetables, meat, etc. A new approach for determining the optimal order policy is proposed. Moreover, the optimal policy is shown to be of a threshold form. The policy states that if the time of the onset of deterioration is greater than or equal to the time of the start of backlogging for the basic (EOQ), then the optimal inventory policy is given by the parameters of the basic (EOQ) model. Else, the optimal policy is determined by the critical point of the objective function for the model where deterioration is present.

In our presentation in this paper, we opted for simplicity rather generality avoiding models which include partial backlogging, permissible delay in payment, pricing, inflation and others as in [1–4, 6–9] and the references therein. Existing approaches for finding the optimal (EOQ), for non-instantaneous deteriorating models, are based on solving optimization problems on separate branches of the decision variables and then selecting the overall minimum of these problems. This overall minimum supplies the optimal inventory policy. The process of

Keywords and phrases: Inventory models, non-instantaneous deterioration, optimal policy.

Department of Statistics and Operations Research, College of Science, Kuwait University, P.O. Box 5969, Safat 13060, Kuwait.

* Corresponding author: lakdere.benkherouf@ku.edu.kw

finding the policy can, in some instances, be laborious and tedious and may not always give the optimal policy for some parameters of the model: see for example [8]. The proposed approach of this paper, as we shall see, will always be able to find the optimal inventory policy. It is simple and easy to understand and involves very little computations unlike existing methods in the literature. Additionally, it is hoped that it could be extended to the models cited above and others.

The next section contains the assumptions and notation used throughout the paper. The mathematical model is presented in Section 3. Section 4 contains the main results of the paper along with an approach to find the optimal inventory policy. The last section contains illustrative examples of the applicability of the proposed approach and some general remarks.

2. Assumptions and notation

The mathematical model of this paper is developed under the following assumptions and notation:

(1) A single product is held in stock over an infinite planning horizon.
(2) The planning horizon is made-up of identical cycle of length T, where $T > 0$.
(3) The inventory level is at its maximum level at the beginning of the cycle.
(4) During a cycle the inventory level is affected by demand and possibly deterioration.
(5) Items experience deterioration once they spent at least some known time $\gamma > 0$ in stock.
(6) Deteriorated items are not repaired while in stock.
(7) Shortages are fully backlogged.
(8) The replenishment rate is infinite.
(9) The demand rate is denoted by $d > 0$,
the length of time of nonnegative inventory is denoted by $\tau \geq 0$ ($\tau \leq T$),
the deterioration rate by $\theta \geq 0$,
the holding cost $h > 0$,
the backlogging cost $p > 0$,
the unit cost $c > 0$,
the set-up cost $k > 0$.

3. The mathematical model

For the basic economic order quantity with non-instantaneous deteriorating items, a typical cycle begins with a maximum inventory model. If $\gamma \geq \tau$, then the inventory level is only affected by demand during the whole cycle. On the other hand, if $0 < \gamma < \tau$, then the level of inventory is affected by demand only on the time interval $[0, \gamma]$, and $(\tau, T]$ while the effect of deterioration takes place on the time interval $(\gamma, \tau]$. It follows that the changes in the level of inventory over a cycle may be described by a set of differential equations as follows:

If $I(t)$ denotes the inventory level at time t, then the dynamic of inventory level is described by

$$
\begin{aligned}
I'(t) &= -d, && \text{for } 0 < t \leq \min\{\gamma, \tau\}, && (3.1)\\
I'(t) &= -d - \theta I(t), && \text{for } \min\{\gamma, \tau\} \leq t < \tau, && (3.2)\\
I'(t) &= -d, && \text{for } \tau \leq t < T. && (3.3)
\end{aligned}
$$

with $I(\tau) = 0$, and $I(t)$ is continuous on the interval $[0, T]$. Figure 1 depicts a typical behaviour of the stock level when $\tau \geq \gamma$.

Differential equations (3.1)–(3.3) possess unique solutions. Indeed, if we assume that $\tau \geq \gamma$, as in Figure 1. These solutions are respectively given by:

$$I(t) = -d(t - \gamma) + I(\gamma), \quad \text{for } 0 < t \leq \gamma, \tag{3.4}$$

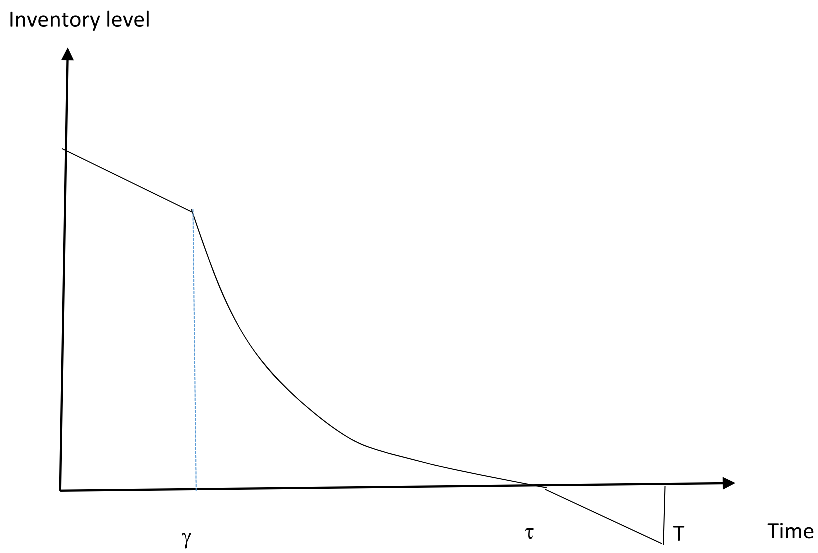

FIGURE 1. The inventory level for during the cycle.

$$I(t) = de^{-\theta t} \int_t^\tau e^{-\theta u} \mathrm{d}u, \quad \text{for } \gamma < t \leq \tau, \tag{3.5}$$

$$I(t) = -d(t-\tau). \tag{3.6}$$

It is also easy to see, from (3.5), that

$$I(\gamma) = de^{-\theta\gamma} \int_\gamma^\tau e^{-\theta u} \mathrm{d}u. \tag{3.7}$$

Note that we opted not to expand the integrals to keep the computations compact as our approach does not require their expanded value. We also think that this render the analysis handy for possible future generalization.

Next, computations for the amount of inventory on the interval $(0,\tau]$, the amount of deteriorated items on the interval $[\gamma,\tau)$, and the amount of shortages on $[\tau, T]$. are presented for completeness.

The amount of inventory on the interval $(0,\tau]$ is equal, by (3.4) and (3.5), to

$$\begin{aligned} V(0,\tau) &:= \int_0^\gamma \left\{ -d(t-\gamma) + de^{-\theta\gamma} \int_\gamma^\tau e^{-\theta u} \mathrm{d}u \right\} \mathrm{d}t \\ &\quad + \int_\gamma^\tau \left(de^{-\theta t} \int_t^\tau e^{-\theta u} \mathrm{d}u \right) \mathrm{d}t \\ &= \frac{1}{2} d\gamma^2 + d\gamma \int_\gamma^\tau e^{\theta(t-\gamma)} \mathrm{d}t + \frac{d}{\theta} \int_x^y \{ e^{\theta(t-x)} - 1 \} \mathrm{d}t, \end{aligned} \tag{3.8}$$

the amount of shortages is

$$S(\tau, T) := \frac{1}{2} d(T-\tau)^2, \tag{3.9}$$

and the amount of deteriorated items is by (3.7)

$$W(\gamma,\tau) := I(\gamma) - d(\tau-\gamma) \tag{3.10}$$
$$= d\int_x^y \{e^{\theta(t-x)} - 1\}\mathrm{d}t. \tag{3.11}$$

4. The optimal inventory policy

We assume that the total cost over a cycle, $R(\tau,T)$, is made up set-up cost + deteriorating cost + holding cost + backlogging cost. It follows from the computations of the previous section that the deteriorating cost is equal to

$$cd\int_x^y \{e^{\theta(t-x)} - 1\}\mathrm{d}t,$$

the holding cost is

$$hd\left(\frac{1}{2}\gamma^2 + \gamma\int_\gamma^\tau e^{\theta(t-\gamma)}\mathrm{d}t + \frac{1}{\theta}\int_x^y \{e^{\theta(t-x)} - 1\}\mathrm{d}t\right).$$

Thus

$$R(\tau,T) = \begin{cases} R_\theta(\tau,T) & \text{if} \quad \tau > \gamma \\ R_0(\tau,T) & \text{if} \quad \tau \le \gamma, \end{cases} \tag{4.1}$$

where

$$R_\theta(\tau,T) := k + H(\gamma) + D_\theta(\gamma,\tau) + S(\tau,T) \tag{4.2}$$

with,

$$H(x) := \frac{1}{2}hdx^2, \tag{4.3}$$
$$D_\theta(x,y) := hdx\int_x^y e^{\theta(t-x)}\mathrm{d}t + \left(cd + \frac{hd}{\theta}\right)\int_x^y \{e^{\theta(t-x)} - 1\}\mathrm{d}t, \tag{4.4}$$
$$S(x,y) = \frac{1}{2}pd(y-x)^2, \tag{4.5}$$

for $y \ge x \ge 0$. Here, the function $R_0(\tau,T)$ coincides with $R_\theta(\tau,T)$ when $\theta = 0$. Also, it is easy to infer that $R_0(\gamma,T) = R_\theta(\gamma,T)$.

Note that the form of expression (4.2) is written in general with the view of using it for more general models if required.

The objective is to minimize

$$F(\tau,T) := \frac{R(\tau,T)}{T}, \tag{4.6}$$

on the set $\Omega := \{\tau,T) \in \mathbb{R}^2 : \ 0 < \tau \le T < \infty\}$.

Let $\partial_x(.,.)$ and $\partial_y(.,.)$ denote the partial derivatives of a bivariate function with respect to the first and the second component respectively.

We shall next examine the problem of determining the minimum of the function

$$F_\theta(\tau, T) = \frac{R_\theta(\tau, T)}{T}, \tag{4.7}$$

on the set $\Omega \cap \{(\tau, T), \tau \geq \gamma\}$, given that the parameters θ and γ are non-negative and known, and where $R_\theta(\tau, T)$ is defined in (4.2).

Write

$$\tau^* := \sqrt{\frac{2k}{hd}}\sqrt{\frac{p}{p+h}}. \tag{4.8}$$

The expression for τ^* in (4.8) corresponds to the time of the beginning of shortage period in the classical economic order quantity model with shortages: see [5].

Theorem 4.1. *The function F_θ is differentiable on the set $\Omega \cap \{(\tau, T),\ \tau \geq \gamma\}$. Moreover, it has a unique minimum*
(i) given by $\tau = \gamma$, $T_\gamma = \sqrt{\dfrac{2k}{pd} + \dfrac{p+h}{p}\gamma^2}$ for $\gamma \geq \tau^*$, else,
(ii) corresponding to the stationary point of F_θ.

Proof. The differentiability of F_θ is clear. We shall initially show part (ii). To do that we prove that the gradient $\nabla F_\theta(\tau, T) = 0$ has a unique solution. Computations, using (4.2)–(4.7), show that $(\partial_x F_\theta)(\tau, T) = 0$, or

$$(\partial_y D_\theta)(\gamma, \tau) + (\partial_x S)(\tau, T) = 0. \tag{4.9}$$

This leads after some simple algebra to

$$T = \tau + \frac{h\gamma e^{\theta(\tau-\gamma)} + \left(c + \frac{h}{\theta}\right)\left\{e^{\theta(\tau-\gamma)} - 1\right\}}{p}. \tag{4.10}$$

The right-hand side of (4.10) is strictly increasing in τ for $\tau > \gamma$. Also, T is uniquely defined a function of τ with $T'(\tau) > 1$. Write $T(\tau) := T$, so

$$T(\gamma) = \left(\frac{h+p}{p}\right)\gamma. \tag{4.11}$$

The expression $(\partial F_\theta)_y(\tau, T(\tau)) = 0$, gives

$$T(\tau)(\partial_y S)(\tau, T(\tau)) - R_\theta(\tau, T(\tau)) = 0. \tag{4.12}$$

Define a function G_θ as the left hand-side of (4.12)

$$G_\theta(\tau) := T(\tau)(\partial_y S)(\tau, T(\tau)) - R_\theta(\tau, T(\tau)). \tag{4.13}$$

It follows, that

$$\begin{aligned}G'_\theta(\tau) &= T'(\tau)(\partial_y S)(\tau,T(\tau)) + T(\tau)\left\{(\partial_x\partial_y S)(\tau,T(\tau)) + T'(\tau)(\partial_y^2 S)(\tau,T(\tau))\right\}\\ &\quad - (\partial_x R_\theta)(\tau,T(\tau)) - T'(\tau)(\partial_y R_\theta)(\tau,T(\tau))\\ &= T(\tau)\left\{(\partial_x\partial_y S)(\tau,T(\tau)) + T'(\tau)(\partial_y^2 S)(\tau,T(\tau))\right\}.\end{aligned}$$

The last equality is justified since $(\partial_y S)(\tau,T(\tau)) = (\partial_y R_\theta)(\tau,T(\tau))$, and $(\partial_x R_\theta)(\tau,T(\tau)) = 0$ by (4.9). It then follows from the definition of S in (4.5) that

$$G'_\theta(\tau) = pdt(\tau)\{T'(\tau)-1\}.$$

Hence G_θ is a strictly increasing function in τ since $T'(\tau) > 1$. Also, $G_\theta(\gamma) = \frac{1}{2}(\frac{p+h}{p})dh\gamma^2 - k$. Thus, $G_\theta(\gamma) < 0$ if and only if $\gamma < \sqrt{\frac{2k}{hd}}\sqrt{\frac{p}{p+h}} = \tau^*$. If $\gamma < \tau^*$, it can be shown that $G_\theta(\tau)$ goes to ∞, as $\tau \to \infty$. Hence, $G_\theta(\tau) = 0$, has a unique root on the interval (γ,∞). This completes the proof of Part (ii). Part (i) is now immediate. This finishes the proof of the theorem. □

Write

$$T^* := \sqrt{\frac{2k}{hd}}\sqrt{\frac{p+h}{p}}. \tag{4.14}$$

The value of T^* corresponds to the length of the period in the classical economic order quantity model with shortages: see [5].

Theorem 4.2. *The function F is differentiable on the set $\Omega = \{\tau, T) : \ 0 < \tau \le T < \infty\}$. Moreover, it has a unique minimum*

(a) given by (τ^*, T^*) if $\gamma \ge \tau^*$, and
(b) if $\gamma < \tau^*$, the minimum is given by the stationary point of F_θ defined in (4.7).

Proof. It is clear that F is differentiable on $\Omega\backslash\{(\tau,T), \tau=\gamma,\ T>\tau\}$. It remains to check differentiability on the line segment $\tau=\gamma$. Direct algebra leads to $(\partial R_\theta)_x(\gamma,T) = hd\gamma - pd(T-\tau)$, $(\partial R_0)_x(\gamma,T) = hd\gamma - pd(T-\tau)$, or $(\partial R_\theta)_x(\gamma,T) = (\partial R_0)_x(\gamma,T)$. This settles the differentiability property. Part (a) is justified since $R_\theta(\tau,T) > R_0(\tau,T)$ for $\gamma \ge \tau$, and in particular for $\gamma \ge \tau^*$. Part (b) follows from Theorem 4.1. This completes the proof of the theorem. □

Remark 4.3.

(i) Theorems 4.2 provides a simple way of identifying the form of the optimal policy without the need to go through the optimization of the respective costs functions. This as far as we know appears to be new.
(ii) Note that although the optimal values of τ and T may depend on θ, the form of the optimal policy is independent of θ. Also, the case $\gamma=\tau$ occurs only when γ coincides with the optimal value τ.
(iii) Note that Theorem 4.2 and the proof of Theorem 4.1 contains a methodology for finding the optimal values of the optimal inventory policy. Indeed, (τ^*,T^*) is optimal if $\gamma \ge \tau^*$, where τ^* and T^* are given by (4.8)and (4.14) respectively. If $\gamma < \tau^*$, then the optimal value of τ can be as the root of the function $G_\theta(\tau) = 0$ defined in (4.13). This root can be easily found numerically, using any univariate search techniques or any off the shelves software, since the function G_θ is well behaved.

5. Numerical examples and conclusion

This section contains two simple examples to illustrate the applicability of Theorem 4.2 as well as some general remarks. The smoothness of the objective function in the model means that the optimal solution can be easily found.

For the examples, the values of k, h, c, p and d are fixed, so that $k = 250$, $h = 0.5$, $c = 1.5$, $p = 2.5$, and $d = 600$, $\theta = 0.08$, giving $\tau^* = 1.17851$, $T^* = 1.41421$, and $F(\tau^*, T^*) = 353.553$. The optimal policies are found for two values of γ.

(i) Set $\gamma = 1.4$. Clearly $\gamma > \tau$, giving that the optimal parameter values for (τ, T) corresponds to τ^* and T^*.
(ii) Set $\gamma = 0.8$. The value of $\gamma < \tau^*$. It follows by Theorem 4.2 that the optimal parameter values for (τ, T) corresponds to critical point of F_θ and these are found to be $\tau = 1.09660$, $T = 1.33488$, and $F(\tau, T) = 357.419$.

The above examples demonstrated the easiness of the applicability of the approach proposed in this paper.

To sum up, this paper proposed a simple procedure for finding the optimal inventory policy for the basic economic order quantity for non-instantaneous deteriorating items. It was shown than the optimal policy is of a threshold type. That is, there exists a critical value τ^*, which corresponds to the time of the onset of backlogging for the basic EOQ model with no deterioration. If this time is less than or equal to the time of the start of deterioration, then the optimal inventory policy coincides with that of the basic EOQ model. Else, the optimal policy corresponds to the critical point of the objective function for the inventory model with non-instantaneous deteriorating items. This critical point can be inferred from finding a unique root of some simple univariate function. Additionally, it was shown that the procedure is easy to implement and straightforward to explain. The applicability of the procedure was demonstrated by two examples.

Finally, it is worth pointing that threshold policy proposed in this paper appears to be new. Its simplicity makes it a prime candidate to be used by inventory management practitioners as it requires very little knowledge of optimization techniques. Also, possible extension of this policy to more general models is worth investigating.

Acknowledgements. The authors would like to thank an anonymous referee for useful comments on an earlier version of the paper.

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