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DUALITY OF VARIATIONAL PROBLEMS WITH A NEW APPROACH

S.K. Padhan*

Abstract. The present investigation introduces the third order duality in variational problems, as because, in certain situations, first and second order duality do not yield solutions but it succeeds in finding the desired results. The duality results for the pair of variational primal problems and their corresponding third order dual problems are demonstrated. Counterexamples are provided to justify the importance of the current research work. It is found that many reported results of the literature are particular cases of this paper.

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1. Introduction

Calculus of variation is an important tool for solving several real life problems like optimization of orbits, dynamics of rigid bodies, theory of variations, etc. To find the optimal value of a definite integral involving a certain function subject to fixed point boundary conditions is the main objective in the field of science and engineering. Based on the earlier work of Friedrichs [7], Courant and Hilbert [6] established the duality of an unconstrained variational problem. Afterwards, Hanson [10] observed some of the duality results of mathematical programming which had analogues in variational calculus. Mond and Hanson [16] formulated a constrained variational problem as a mathematical programming problem considering the above mentioned ideas in the classical calculus of variations. Again, they used the optimality conditions of Valentine [29] to obtain the Wolfe type dual variational problem under convexity. The first order duality for a class of variational problems with differential inequality constraints was discussed by Mond and Hanson [16]. Later, Bector *et al.* [2] studied the Mond-Weir type duality for the problems of Mond and Hanson [16] to weaken the convexity requirement using pseudoconvexity and quasiconvexity. After that, Chen [3] and Mond *et al.* [17] introduced the notion of invexity in variational problems. Nahak and Nanda [18] formulated Wolfe and Mond-Weir type duals for multiobjective variational problems and established different duality theorems under invexity assumptions. Later on, Nahak and Nanda [19] generalized their results [18] under pseudoinvexity assumptions. Further, Mishra and Mukherjee [15] generalized results under (F, ρ)-convexity introduced by Preda [26]. Based on it, Ahmad and Gulati [1] established the duality results in multiobjective variational problems. Nahak and Nanda [20, 21] studied the symmetric duality assuming pseudoinvexity in variational problems and established optimality conditions and duality results for the multiobjective variational problems under V-invexity assumptions. Since mathematical

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Department of Mathematics, Veer Surendra Sai University of Technology, Burla 768018, India.

* Corresponding author: skpadhan_math@vssut.ac.in

programming and classical calculus of variations have undergone independent developments, it felt that, the mutual adoption of ideas and techniques may prove fruitful for further development in this field. Targeting the above outputs, Husain and Jabeen [11] formulated a class of constrained variational problems with higher order derivatives and established a number of duality results for the Wolfe and Mond-Weir type duals under invexity and generalized invexity assumptions.

The central concept of optimization is the duality theory. Some recent developments of duality in vector optimization problems are [9, 28]. Mangasarian [14] introduced second and higher order duality for the nonlinear programming problems and proved suitable duality results. Following the procedure adopted by Mangasarian [14], Chen [5] formulated the second order duality for a class of constrained variational problems and obtained appropriate second order duality results (weak, strong and converse duality results) under invexity assumptions. Later, Gulati and Mehndiratta [8] studied the optimality conditions and the converse duality result for the second order multiobjective variational problems. A number of weakening situations in the second and higher order duality of the nonlinear variational problems had been discussed by many authors (see [4, 8, 12, 13, 22, 25]). Again, Sharma [27] established the higher order mixed type dual model for the variational control primal problem and proved weak, strong and strict converse duality results under higher order (F, G, ρ)-convexity assumptions. Recently, the concept of third order duality for the general nonlinear programming problems with a new approach was developed by Padhan and Nahak [24] and proved appropriate duality results. It was also shown that the third order duality of a nonlinear programming problem had solutions, whereas; the first and second order dual failed.

In this paper, especially the third order dual of the constrained variational primal problems have been studied and obtained duality relationships between the variational primal and its third order dual for the first time. Furthermore, the importance of the present study is exploited through numerical examples.

2. Notations and preliminaries

Through out this paper, let $I = [s_0, s_1]$ be a closed interval of the real line $\mathbb{R}$. Consider the function $f(t, s(t), \dot{s}(t))$, where $s : I \to \mathbb{R}^n$ and $\dot{s}$ denotes the derivative of s with respect to t. Here t is an independent variable. The symbol z^T stands for the transpose of a vector z. Let f_s and $f_{\dot{s}}$ denote the first partial derivatives of f with respect to s, and $\dot{s}$, respectively, that is, f with respect to s, and $\dot{s}$, respectively, that is,

$$f_s = \left(\frac{\partial f}{\partial s_1}, \frac{\partial f}{\partial s_2}, \dots, \frac{\partial f}{\partial s_n}\right)^T,$$

$$f_{\dot{s}} = \left(\frac{\partial f}{\partial \dot{s}_1}, \ \frac{\partial f}{\partial \dot{s}_2}, \dots, \frac{\partial f}{\partial \dot{s}_n}\right)^T.$$

The Hessian matrix of f with respect to $s(t)$ is denoted by f_{ss} and is defined as

$$f_{ss} = \begin{pmatrix} \frac{\partial^2 f}{\partial s_1 \partial s_1} & \frac{\partial^2 f}{\partial s_1 \partial s_2} & \cdot & \cdot & \cdot & \frac{\partial^2 f}{\partial s_1 \partial s_n} \\ \frac{\partial^2 f}{\partial s_2 \partial s_1} & \frac{\partial^2 f}{\partial s_2 \partial s_2} & \cdot & \cdot & \cdot & \frac{\partial^2 f}{\partial s_2 \partial s_n} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \frac{\partial^2 f}{\partial s_n \partial s_1} & \frac{\partial^2 f}{\partial s_n \partial s_2} & \cdot & \cdot & \cdot & \frac{\partial^2 f}{\partial s_n \partial s_n} \end{pmatrix}$$

and the $m \times n$ Jacobian matrix g_s with respect to s is written as

$$g_s = \begin{pmatrix} \frac{\partial g^1}{\partial s_1} & \frac{\partial g^1}{\partial s_2} & \cdot & \cdot & \cdot & \frac{\partial g^1}{\partial s_n} \\ \frac{\partial g^2}{\partial s_1} & \frac{\partial g^2}{\partial s_2} & \cdot & \cdot & \cdot & \frac{\partial g^2}{\partial s_n} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \frac{\partial g^m}{\partial s_1} & \frac{\partial g^m}{\partial s_2} & \cdot & \cdot & \cdot & \frac{\partial g^m}{\partial s_n} \end{pmatrix}.$$

Similarly, $f_{\dot{s}}$, $f_{\dot{s}s}$, $f_{s\dot{s}}$ and $g_{\dot{s}}$, are defined.

Now the $n \times n^2$ matrix $f_{sss} = f_s^T \otimes f_{ss}$ is defined as

$$f_{sss} = \begin{pmatrix} \frac{\partial^3 f}{\partial s_1 \partial s_1 \partial s_1} & \cdot\ \cdot\ \cdot & \frac{\partial^3 f}{\partial s_1 \partial s_1 \partial s_n} & \cdot\ \cdot\ \cdot & \frac{\partial^3 f}{\partial s_1 \partial s_n \partial s_1} & \cdot\ \cdot\ \cdot & \frac{\partial^3 f}{\partial s_1 \partial s_n \partial s_n} \\ \frac{\partial^3 f}{\partial s_2 \partial s_1 \partial s_1} & \cdot\ \cdot\ \cdot & \frac{\partial^3 f}{\partial s_2 \partial s_1 \partial s_n} & \cdot\ \cdot\ \cdot & \frac{\partial^3 f}{\partial s_2 \partial s_n \partial s_1} & \cdot\ \cdot\ \cdot & \frac{\partial^3 f}{\partial s_2 \partial s_n \partial s_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial^3 f}{\partial s_n \partial s_1 \partial s_1} & \cdot\ \cdot\ \cdot & \frac{\partial^3 f}{\partial s_n \partial s_1 \partial s_n} & \cdot\ \cdot\ \cdot & \frac{\partial^3 f}{\partial s_n \partial s_n \partial s_1} & \cdot\ \cdot\ \cdot & \frac{\partial^3 f}{\partial s_n \partial s_n \partial s_n} \end{pmatrix}.$$

Again, $f_{ss\dot{s}}$, $f_{s\dot{s}\dot{s}}$ and $f_{\dot{s}\dot{s}\dot{s}}$ can be defined accordingly.

Let $S(I, \mathbb{R}^n)$ be the set of all piecewise smooth functions s with norm $\|s\| = \|s\|_\infty + \|Ds\|_\infty$, where the differentiation operator D is given by

$$\bar{s} = Ds \Leftrightarrow s(t) = \bar{\alpha} + \int_{s_0}^{t} \bar{s}(x)\mathrm{d}x, \tag{2.1}$$

and $\bar{\alpha}$ is a given boundary value; thus $\frac{\mathrm{d}}{\mathrm{d}t} = D$ except at discontinuities. Consider the constrained variational primal problem

$$\text{(VP)} \quad \min \int_{s_0}^{s_1} f\left(t, s(t), \dot{s}(t)\right) \mathrm{d}t,$$

subject to

$$g\left(t, s(t), \dot{s}(t)\right) \leq 0, \quad t \in I, \tag{2.2}$$

$$s(s_0) = \gamma_1, \quad s(s_1) = \gamma_2; \quad \dot{s}(s_0) = \delta_1, \quad \dot{s}(s_1) = \delta_2, \tag{2.3}$$

where f and g are thrice continuously differentiable functions from $I \times \mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}$ and $\mathbb{R}^m$, respectively.

The first order dual of (VP) formulated by Mond and Hanson [16] is given by:

$$\text{(VFD)} \quad \max \int_{s_0}^{s_1} \left\{ f\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \bar{\alpha}(t)^T g\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) \right\} \mathrm{d}t$$

subject to

$$f_s\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)+g_s\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T\bar{\alpha}(t)-\frac{\mathrm{d}}{\mathrm{d}t}\Big[f_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)+g_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T\bar{\alpha}(t)\Big]=0,\quad t\in I, \tag{2.4}$$

$$\bar{s}(s_0)=\gamma_1,\quad \bar{s}(s_1)=\gamma_2;\quad \dot{\bar{s}}(s_0)=\delta_1,\quad \dot{\bar{s}}(s_1)=\delta_2, \tag{2.5}$$

$$\bar{\alpha}(t)\in\mathbb{R}^m_+,\quad t\in I. \tag{2.6}$$

Taking the approach of Mangasarian [14], Chen [5] formulated the following second order duality (VSD) for the class of constrained variational problems (VP).

$$\begin{aligned}
\text{(VSD)}\ \max\int_{s_0}^{s_1}&\Big\{f\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)+\bar{\alpha}(t)^Tg\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)\\
&-\frac{1}{2}\bar{\beta}(t)^T\Big[f_{ss}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)+\Big(g_s\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T\bar{\alpha}(t)\Big)_s\\
&-2\frac{\mathrm{d}}{\mathrm{d}t}\Big(f_{s\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)+\Big(g_s\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T\bar{\alpha}(t)\Big)_{\dot{s}}\Big)\\
&+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(f_{\dot{s}\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)+\Big(g_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T\bar{\alpha}(t)\Big)_{\dot{s}}\Big)\Big]\bar{\beta}(t)\Big\}\mathrm{d}t
\end{aligned}$$

subject to

$$\begin{aligned}
&f_s\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)+g_s\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T\bar{\alpha}(t)-\frac{\mathrm{d}}{\mathrm{d}t}\Big[f_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)\\
&+g_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T\bar{\alpha}(t)\Big]+\Big[f_{ss}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)+\Big(g_s\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T\bar{\alpha}(t)\Big)_s\\
&-2\frac{\mathrm{d}}{\mathrm{d}t}\Big(f_{s\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)+\Big(g_s\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T\bar{\alpha}(t)\Big)_{\dot{s}}\Big)\\
&+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(f_{\dot{s}\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)+\Big(g_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T\bar{\alpha}(t)\Big)_{\dot{s}}\Big)\Big]\bar{\beta}(t)=0,\quad t\in I,
\end{aligned} \tag{2.7}$$

$$\bar{s}(s_0)=\gamma_1,\quad \bar{s}(s_1)=\gamma_2;\quad \dot{\bar{s}}(s_0)=\delta_1,\quad \dot{\bar{s}}(s_1)=\delta_2, \tag{2.8}$$

$$\bar{\alpha}(t)\in\mathbb{R}^m_+,\quad \bar{\beta}(t)\in\mathbb{R}^n,\quad t\in I. \tag{2.9}$$

Again, Padhan and Nahak [23] studied the higher order duality (VHD) of the primal (VP) by introducing two different functions $h: I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\longrightarrow\mathbb{R}$ and $k: I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\longrightarrow\mathbb{R}^m$. That is,

$$\text{(VHD)}\ \max\int_{s_0}^{s_1}\Big[f\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)+\bar{\alpha}(t)^Tg\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)+h\left(t,\bar{s}(t),\dot{\bar{s}}(t),p\right)+\bar{\alpha}(t)^Tk\left(t,\bar{s}(t),\dot{\bar{s}}(t),p\right)\Big]\mathrm{d}t$$

subject to

$$\nabla_p h\left(t,\bar{s}(t),\dot{\bar{s}}(t),p\right)+\nabla_p\bar{\alpha}(t)^Tk\left(t,\bar{s}(t),\dot{\bar{s}}(t),p\right)=0,\quad t\in I, \tag{2.10}$$

$$\bar{s}(s_0) = \gamma_1, \quad \bar{s}(s_1) = \gamma_2; \quad \dot{\bar{s}}(s_0) = \delta_1, \quad \dot{\bar{s}}(s_1) = \delta_2, \tag{2.11}$$

$$\bar{\alpha}(t) \in \mathbb{R}^m_+, \quad t \in I, \tag{2.12}$$

where $\nabla_p h\left(t, \bar{s}(t), \dot{\bar{s}}(t), p\right)$ and $\nabla_p \bar{\alpha}(t)^T k\left(t, \bar{s}(t), \dot{\bar{s}}(t), p\right)$ are the gradient of h and $\bar{\alpha}k$, respectively, with respect to p. This is due to Mangasarian [14].

Motivated with the work of Padhan and Nahak [24], a new type of duality for the variational primal (VP) is introduced which is named variational third order dual (VTD) by taking the cubic approximations on f and g. The present study is quite different from Mangasarian [14] as it is not a particular case of any existing methods.

The third order duality (VTD) of (VP) is defined as

$$\begin{aligned}
\text{(VTD)}\ \max \int_{s_0}^{s_1} \Big\{ & f\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \bar{\alpha}(t)^T g\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) \\
& -\frac{1}{2}\bar{\beta}(t)^T \Big[f_{ss}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \Big)_s \\
& -2\frac{\mathrm{d}}{\mathrm{d}t}\Big(f_{s\dot{x}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \Big)_{\dot{s}} \Big) \\
& +\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(f_{\dot{s}\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(g_{\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \Big)_{\dot{s}} \Big) \Big] \bar{\beta}(t) \\
& -\frac{5}{6}\bar{\beta}(t) \Big[f_{sss}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(\Big(g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \Big)_s \Big)_s \\
& -3\frac{\mathrm{d}}{\mathrm{d}t}\Big(f_{ss\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(\Big(g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \Big)_s \Big)_{\dot{s}} \Big) \\
& +3\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(f_{s\dot{s}\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(\Big(g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \Big)_{\dot{s}} \Big)_{\dot{s}} \Big) \\
& -\frac{\mathrm{d}^3}{\mathrm{d}t^3}\Big(f_{\dot{s}\dot{s}\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(\Big(g_{\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \Big)_{\dot{s}} \Big)_{\dot{s}} \Big) \Big] \bar{\beta}^2(t) \Big\} \mathrm{d}t
\end{aligned}$$

subject to

$$\begin{aligned}
& f_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\Big[f_{\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) \\
& + g_{\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \Big] + \Big[f_{ss}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \Big)_s \\
& -2\frac{\mathrm{d}}{\mathrm{d}t}\Big(f_{s\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \Big)_{\dot{s}} \Big) \\
& +\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(f_{\dot{s}\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(g_{\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \Big)_{\dot{s}} \Big) \Big] \bar{\beta}(t) \\
& + \Big[f_{sss}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(\Big(g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \Big)_s \Big)_s \\
& -3\frac{\mathrm{d}}{\mathrm{d}t}\Big(f_{ss\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(\Big(g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \Big)_s \Big)_{\dot{s}} \Big) \\
& +3\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(f_{s\dot{s}\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(\Big(g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \Big)_{\dot{s}} \Big)_{\dot{s}} \Big) \\
& -\frac{\mathrm{d}^3}{\mathrm{d}t^3}\Big(f_{\dot{s}\dot{s}\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(\Big(g_{\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \Big)_{\dot{s}} \Big)_{\dot{s}} \Big) \Big] \bar{\beta}^2(t) = 0, \quad t \in I,
\end{aligned} \tag{2.13}$$

$$\bar{s}(s_0) = \gamma_1, \quad \bar{s}(s_1) = \gamma_2; \quad \dot{\bar{s}}(s_0) = \delta_1, \quad \dot{\bar{s}}(s_1) = \delta_2, \tag{2.14}$$

$$\bar{\alpha}(t) \in \mathbb{R}^m_+, \quad \bar{\beta}(t) \in \mathbb{R}^n, \quad t \in I, \tag{2.15}$$

where $\bar{\beta}^2(t) = \bar{\beta}(t) \otimes \bar{\beta}(t)$, a matrix of order $n^2 \times 1$.

Let

$$\begin{aligned} H\left(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)\right) = f_{ss}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\Big)_s \\ -2\frac{\mathrm{d}}{\mathrm{d}t}\Big[f_{s\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\Big)_{\dot{s}}\Big] \\ +\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big[f_{\dot{s}\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(g_{\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\Big)_{\dot{s}}\Big] \end{aligned}$$

and

$$\begin{aligned} I\left(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)\right) = f_{sss}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(\Big(g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\Big)_s\Big)_s \\ -3\frac{\mathrm{d}}{\mathrm{d}t}\Big[f_{ss\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(\Big(g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\Big)_s\Big)_{\dot{s}}\Big] \\ +3\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big[f_{s\dot{s}\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(\Big(g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\Big)_{\dot{s}}\Big)_{\dot{s}}\Big] \\ -\frac{\mathrm{d}^3}{\mathrm{d}t^3}\Big[f_{\dot{s}\dot{s}\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \Big(\Big(g_{\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\Big)_{\dot{s}}\Big)_{\dot{s}}\Big]. \end{aligned}$$

Then H is an $n \times n$ symmetric matrix and I an $n \times n^2$ matrix. Now the above dual (VTD) can be expressed as

$$\begin{aligned} \text{(VTD)} \quad \max \int_{s_0}^{s_1} \Big[f\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \bar{\alpha}(t)^T g\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) \\ -\frac{1}{2}\bar{\beta}(t)^T H\left(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)\right)\bar{\beta}(t) - \frac{5}{6}\bar{\beta}(t) I\left(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)\right)\bar{\beta}^2(t)\Big]\mathrm{d}t \end{aligned}$$

subject to

$$\begin{aligned} f_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\Big[f_{\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + g_{\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\Big] \\ +H\left(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)\right)\bar{\beta}(t) + I\left(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)\right)\bar{\beta}^2(t) = 0, \quad t \in I, \end{aligned} \tag{2.16}$$

$$\bar{s}(s_0) = \gamma_1, \quad \bar{s}(s_1) = \gamma_2; \quad \dot{\bar{s}}(s_0) = \delta_1, \quad \dot{\bar{s}}(s_1) = \delta_2, \tag{2.17}$$

$$\bar{\alpha}(t) \in \mathbb{R}^m_+, \quad \bar{\beta}(t) \in \mathbb{R}^n, \quad t \in I. \tag{2.18}$$

Remark 2.1.

(1) If $\bar{\beta}(t) = 0$, then (VTD) becomes the first order duality defined by Mond and Hanson [16].
(2) If the third order derivatives are zero, then it reduces to second order dual of Chen [5] and Padhan and Nahak [22].
(3) If s is independent of t then the results of Padhan and Nahak [24] are particular cases of the present work.

(4) The third order duality can be extended for any positive integer order by taking suitable approximations on f and g.

Lemma 2.2. *[5] If (VP) attains a local (global) minimum at $\bar{s} \in S$, then there exist Lagrange multiplier $\tau \in \mathbb{R}$ and piecewise smooth $\lambda : I \to \mathbb{R}^m$ such that,*

$$\tau f_s(t, \bar{s}(t), \dot{\bar{s}}(t)) + g_s(t, \bar{s}(t), \dot{\bar{s}}(t))^T \lambda(t) = \frac{\mathrm{d}}{\mathrm{d}t}[\tau f_{\dot{s}}(t, \bar{s}(t), \dot{\bar{s}}(t)) + g_{\dot{s}}(t, \bar{s}(t), \dot{\bar{s}}(t))^T \lambda(t)], \quad t \in I \tag{2.19}$$

$$\lambda(t)^T g(t, \bar{s}(t), \dot{\bar{s}}(t)) = 0, \quad t \in I \tag{2.20}$$

$$(\tau, \lambda(t)^T) \geq 0, \quad t \in I. \tag{2.21}$$

Remark 2.3. Equations (2.19)–(2.21) give the Fritz-John necessary conditions for (VP), and they become Kuhn-Tucker conditions if $\tau = 1$.

3. Third order duality results

In this section, the weak, strong and converse duality relationships between the variational primal (VP) and the corresponding third order dual (VTD) are established. An alternative proof of converse duality theorem is also discussed.

Theorem 3.1 (Weak duality)**.** Let $s(t) \in S(I, \mathbb{R}^n)$, and $(\bar{s}(t), \bar{\alpha}(t), \bar{\beta}(t))$ be the feasible solutions of (VP) and (VTD), respectively. Suppose $\int_{s_0}^{s_1} f(t, ., .)\mathrm{d}t$ and $\int_{s_0}^{s_1} \bar{\alpha}(t)^T g(t, ., .)\mathrm{d}t$ are convex functions in s and $\dot{s}$. If there exist real valued functions $m\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)) > 0$, $M\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)) > 0$, $n\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)) > 0$ and $N\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)) > 0$ on $I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ with the following conditions:

$$\bar{\beta}(t) H\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t))\, \bar{\beta}(t) \geq m\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t))\, \|\bar{\beta}(t)\|^2, \quad t \in I, \tag{3.1}$$

$$\|H\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t))\,\| \leq M\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t))\,, \quad t \in I, \tag{3.2}$$

$$\bar{\beta}(t) I\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t))\, \bar{\beta}(t)^2 \geq n\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t))\, \|\bar{\beta}(t)\|^3, \quad t \in I, \tag{3.3}$$

$$\|I\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t))\,\| \leq N\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t))\,, \quad t \in I. \tag{3.4}$$

Then

$$\int_{s_0}^{s_1} f\,(t, s(t), \dot{s}(t))\,\mathrm{d}t \geq \int_{s_0}^{s_1} \Big[f\,(t, \bar{s}(t), \dot{\bar{s}}(t)) + \bar{\alpha}(t)^T g\,(t, \bar{s}(t), \dot{\bar{s}}(t)) - \frac{1}{2}\bar{\beta}(t)^T H\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t))\, \bar{\beta}(t)$$
$$- \frac{5}{6}\bar{\beta}(t)^T I\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t))\, \bar{\beta}(t)^2 \Big] \mathrm{d}t,$$

provided

$$\|\bar{\beta}(t)\| \geq \max\left\{ \frac{6}{5} \cdot \frac{N\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t))}{n\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t))}, 2\frac{M\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t))}{m\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t))} \right\} \|s(t) - \bar{s}(t)\|. \tag{3.5}$$

Proof. Now

$$\int_{s_0}^{s_1} f\,(t, s(t), \dot{s}(t))\,\mathrm{d}t - \int_{s_0}^{s_1} \Big[f\,(t, \bar{s}(t), \dot{\bar{s}}(t)) + \bar{\alpha}(t)^T g\,(t, \bar{s}(t), \dot{\bar{s}}(t))$$
$$- \frac{1}{2}\bar{\beta}(t)^T H\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t))\, \bar{\beta}(t) - \frac{5}{6}\bar{\beta}(t)^T I\,(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t))\, \bar{\beta}(t)^2 \Big] \mathrm{d}t$$

$$
\begin{aligned}
\geq\ & \int_{s_0}^{s_1} \left[(s(t)-\bar{s}(t))\, f_s\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right) + (\dot{s}(t)-\dot{\bar{s}}(t))\, f_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)\right] \mathrm{d}t \\
& + \int_{s_0}^{s_1} \left[(s(t)-\bar{s}(t)) \left(g_s\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\right) + (\dot{s}(t)-\dot{\bar{s}}(t)) \left(g_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\right) \right] \mathrm{d}t \\
& + \frac{1}{2}\int_{s_0}^{s_1} \bar{\beta}^T(t) H\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right) \bar{\beta}(t)\mathrm{d}t + \frac{5}{6}\int_{s_0}^{s_1} \bar{\beta}^T(t) I\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right) \bar{\beta}^2(t)\mathrm{d}t \\
& \left(\text{by convexity of } \int_{s_0}^{s_1} f(t,.,.)\mathrm{d}t,\ \int_{s_0}^{s_1} \bar{\alpha}(t)^T g(t,.,.)\mathrm{d}t,\ (2.2) \text{ and } (2.18)\right) \\
=\ & \int_{s_0}^{s_1} (s(t)-\bar{s}(t)) \left[f_s\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right) + g_s\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\right] \mathrm{d}t \\
& + \int_{s_0}^{s_1} (\dot{s}(t)-\dot{\bar{s}}(t)) \left[f_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right) + g_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\right] \mathrm{d}t \\
& + \frac{1}{2}\int_{s_0}^{s_1} \bar{\beta}^T(t) H\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right) \bar{\beta}(t)\mathrm{d}t + \frac{5}{6}\int_{s_0}^{s_1} \bar{\beta}^T(t) I\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right) \bar{\beta}^2(t)\mathrm{d}t \\
=\ & \int_{s_0}^{s_1} (s(t)-\bar{s}(t)) \Big[\frac{\mathrm{d}}{\mathrm{d}t}\left\{f_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right) + g_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\right\} \\
& - H\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right)\bar{\beta}(t)\Big]\mathrm{d}t - \int_{s_0}^{s_1} (s(t)-\bar{s}(t))\, I\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right)\bar{\beta}^2(t)\mathrm{d}t \\
& + \int_{s_0}^{s_1} (\dot{s}(t)-\dot{\bar{s}}(t)) \left[f_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right) + g_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\right] \mathrm{d}t \\
& + \frac{1}{2}\int_{s_0}^{s_1} \bar{\beta}^T(t) H\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right) \bar{\beta}(t)\mathrm{d}t + \frac{5}{6}\int_{s_0}^{s_1} \bar{\beta}^T(t) I\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right) \bar{\beta}^2(t)\mathrm{d}t \\
& (\text{by } (2.16)) \\
=\ & -\int_{s_0}^{s_1} (\dot{s}(t)-\dot{\bar{s}}(t)) \left[f_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right) + g_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\right] \mathrm{d}t \\
& - \int_{s_0}^{s_1} (s(t)-\bar{s}(t)) \left[H\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right)\bar{\beta}(t) + I\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right)\bar{\beta}^2(t)\right] \mathrm{d}t \\
& + \int_{s_0}^{s_1} (\dot{s}(t)-\dot{\bar{s}}(t)) \left[f_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right) + g_{\dot{s}}\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\right] \mathrm{d}t \\
& + \frac{1}{2}\int_{s_0}^{s_1} \bar{\beta}^T(t) H\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right) \bar{\beta}(t)\mathrm{d}t + \frac{5}{6}\int_{s_0}^{s_1} \bar{\beta}^T(t) I\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right) \bar{\beta}^2(t)\mathrm{d}t \\
& (\text{by integrating by parts, } (2.3) \text{ and } (2.17)) \\
\geq\ & \frac{1}{2}\int_{s_0}^{s_1} m\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right) ||\bar{\beta}(t)||^2\mathrm{d}t + \frac{5}{6}\int_{s_0}^{s_1} n\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right) ||\bar{\beta}(t)||^3\mathrm{d}t \\
& - \int_{s_0}^{s_1} ||s(t)-\bar{s}(t)|| M\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right) ||\bar{\beta}(t)||\mathrm{d}t \\
& - \int_{s_0}^{s_1} ||s(t)-\bar{s}(t)|| M\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right) ||\bar{\beta}(t)||^2\mathrm{d}t \\
& (\text{by} (3.1), (3.2), (3.3) \text{ and } (3.4))
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{s_0}^{s_1}\Big[m\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right)||\bar{\beta}(t)||\Big(||\bar{\beta}(t)|| - 2\frac{M\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right)}{m\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right)}||s(t)-\bar{s}(t)||\Big)\Big]\mathrm{d}t \\
&\quad + \frac{5}{6}\int_{s_0}^{s_1}\Big[n\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right)||\bar{\beta}(t)||^2\Big(||\bar{\beta}(t)|| - \frac{6}{5}\cdot\frac{N\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right)}{n\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right)}||s(t)-\bar{s}(t)||\Big)\Big]\mathrm{d}t \\
&\geq 0. \\
&\quad \Big(\text{by } (3.5)\Big)
\end{aligned}
$$

Hence the result. □

Theorem 3.2 (Strong duality)**.** Let $u(t) \in S(I,\mathbb{R}^n)$ be a local (global) optimal solution of (VP), then there exists a piecewise smooth function $\alpha : I \longrightarrow \mathbb{R}^m$ such that $(u(t),\alpha(t),\beta(t)=0)$ is a feasible solution of (VTD) and the two objective values are equal. Again, if the Weak Duality Theorem 3.1 holds for every feasible solution $\big(\bar{s}(t),\bar{\alpha}(t),\bar{\beta}(t)\big)$ of (VTD), then $(u(t),\alpha(t),\beta(t)=0)$ is an optimal solution of (VTD).

Proof. Since $u(t)$ is a local optimal solution of (VP), by Lemma 2.2, there exists a piecewise smooth $\alpha : I \longrightarrow \mathbb{R}^m$ such that $(u(t),\alpha(t))$ satisfies

$$
f_s(t,u(t),\dot{u}(t)) + g_s(t,u(t),\dot{u}(t))^T\alpha(t) = \frac{\mathrm{d}}{\mathrm{d}t}[f_{\dot{s}}(t,u(t),\dot{u}(t)) + g_{\dot{s}}(t,u(t),\dot{u}(t))^T\alpha(t)], \quad t\in I \tag{3.6}
$$

$$
\alpha(t)^T g(t,u(t),\dot{u}(t)) = 0, \quad t\in I \tag{3.7}
$$

$$
\alpha(t) \geq 0, \quad t\in I. \tag{3.8}
$$

Hence, $(u(t),\alpha(t),\beta(t)=0)$ satisfies the constraints of (MTD). Now for every feasible solution $\big(\bar{s}(t),\bar{\alpha}(t),\bar{\beta}(t)\big)$ of (VTD), we have

$$
\begin{aligned}
&\int_{s_0}^{s_1}\Big[f\left(t,u(t),\dot{u}(t)\right) + \alpha(t)^T g\left(t,u(t),\dot{u}(t)\right) - \frac{1}{2}\beta(t)^T H\left(t,u(t),\dot{u}(t),\alpha(t)\right)\beta(t) \\
&\quad -\frac{5}{6}\beta(t)^T I\left(t,u(t),\dot{u}(t),\alpha(t)\right)\beta^2(t)\Big]\mathrm{d}t = \int_{s_0}^{s_1} f\left(t,u(t),\dot{u}(t)\right)\mathrm{d}t \\
&\Big(\text{by } \alpha(t)^T g(t,u(t),\dot{u}(t)) = 0 \text{ and } \beta(t)=0\Big) \\
&\geq \int_{s_0}^{s_1}\Big[f\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right) + \bar{\alpha}(t)^T g\left(t,\bar{s}(t),\dot{\bar{s}}(t)\right) \\
&\quad -\frac{1}{2}\bar{\beta}(t)^T H\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right)\bar{\beta}(t) - \frac{5}{6}\bar{\beta}(t) I\left(t,\bar{s}(t),\dot{\bar{s}}(t),\bar{\alpha}(t)\right)\bar{\beta}^2(t)\Big]\mathrm{d}t
\end{aligned}
$$

So, $(u(t),\alpha(t),\beta(t)=0)$ is also an optimal solution of (VTD). Hence the result follows. □

Theorem 3.3 (Converse duality)**.** Let f and g be four times continuously differentiable functions. Suppose $(u(t),\alpha(t),\beta(t))$ is a local (global) solution of (VTD). Again, if the following assumptions are satisfied:

(i) $-\rho\big(H\beta(t) + I\beta^2(t)\big) + \big(H\eta(t) + I\eta^2(t)\big) = 0 \Rightarrow \eta(t) = \rho\beta(t), \quad t\in I,$

(ii) $\left[\frac{1}{2}\left(\gamma(t)^T H\gamma(t)\right) + \frac{1}{6}\left(\gamma(t)^T I\gamma^2(t)\right)\right]_s - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\left(\gamma(t)^T H\gamma(t)\right) + \frac{1}{6}\left(\gamma(t)^T I\gamma^2(t)\right)\right]_{\dot{s}}$
$-I\gamma^2(t) = 0 \Rightarrow \gamma(t) = 0, \ \forall\ \gamma(t)\in S(I,\mathbb{R}^n), \quad t\in I,$

then $u(t)$ is a feasible solution of (VP), and the two objective functions are same. Further, if the conditions of Theorem 3.1 are included, then $u(t)$ is an optimal solution of (VP).

Proof. For simple representations denote $H(t,u(t),\dot{u}(t),\alpha(t))$, $I(t,u(t),\dot{u}(t),\alpha(t))$, $f(t,u(t),\dot{u}(t))$ and $g(t,u(t),\dot{u}(t))$ by H, I, f and g, respectively. As $(u(t),\alpha(t),\beta(t))$ is a local (global) solution of (VTD), by Lemma 2.2, $\exists$ Lagrange multiplier $\rho\in\mathbb{R}$, and piecewise smooth functions $\eta\longrightarrow\mathbb{R}^n$ and $\theta\longrightarrow\mathbb{R}^m$ such that the following Fritz-John conditions hold at $(u(t),\alpha(t),\beta(t))$:

$$\begin{aligned}\rho\Big\{&f_s+g_s^T\alpha(t)-\frac{1}{2}\left[\beta(t)^TH\beta(t)\right]_s-\frac{5}{6}\left[\beta(t)^TI\beta^2(t)\right]_s\\&-\frac{\mathrm{d}}{\mathrm{d}t}\left[f_{\dot{s}}+g_{\dot{s}}^T\alpha(t)-\frac{1}{2}\left(\beta(t)^TH\beta(t)\right)_{\dot{s}}-\frac{5}{6}\left(\beta(t)^TI\beta^2(t)\right)_{\dot{s}}\right]\Big\}\\&+\Big\{f_{ss}+\left(g_s^T\alpha(t)\right)_s-\frac{\mathrm{d}}{\mathrm{d}t}\left[f_{\dot{s}s}+\left(g_{\dot{s}}^T\alpha(t)\right)_s\right]+\left(H\beta(t)\right)_s+\left(I\beta^2(t)\right)_s\\&-\frac{\mathrm{d}}{\mathrm{d}t}\left[f_{s\dot{s}}+\left(g_s^T\alpha(t)\right)_{\dot{s}}-\frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{s}\dot{s}}+\left(g_{\dot{s}}^T\alpha(t)\right)_{\dot{s}}\right)\right.\\&\left.+\left(H\beta(t)\right)_{\dot{s}}+\left(I\beta^2(t)\right)_{\dot{s}}\right]\Big\}\eta(t)=0,\quad t\in I,\end{aligned}\tag{3.9}$$

$$-\rho\big(H\beta(t)+I\beta^2(t)\big)+\big(H\eta(t)+I\eta^2(t)\big)=0,\quad t\in I,\tag{3.10}$$

$$\begin{aligned}&\rho\left[g_j-\frac{1}{2}\beta(t)^Tg_{jss}\beta(t)-\frac{5}{6}\beta(t)^Tg_{jsss}\beta^2(t)\right]\\&\quad+\left\{g_{js}-\frac{\mathrm{d}}{\mathrm{d}t}g_{j\dot{s}}+\left[g_{jss}-2\frac{\mathrm{d}}{\mathrm{d}t}g_{js\dot{s}}+\frac{\mathrm{d}^2}{\mathrm{d}t^2}g_{j\dot{s}\dot{s}}\right]\beta(t)\right\}\eta(t)\\&\quad+\left\{\left[g_{jsss}-3\frac{\mathrm{d}}{\mathrm{d}t}g_{jss\dot{s}}+3\frac{\mathrm{d}^2}{\mathrm{d}t^2}g_{js\dot{s}\dot{s}}-\frac{\mathrm{d}^3}{\mathrm{d}t^3}g_{j\dot{s}\dot{s}\dot{s}}\right]\beta^2(t)\right\}\eta(t)\\&\quad+\theta_j(t)=0,\quad t\in I,\ j=1,2,\dots,m\end{aligned}\tag{3.11}$$

$$\theta(t)^T\alpha(t)=0,\quad t\in I,\tag{3.12}$$

$$f_s+g_s^T\alpha(t)-\frac{\mathrm{d}}{\mathrm{d}t}\Big[f_{\dot{s}}+g_{\dot{s}}^T\alpha(t)\Big]+H\beta(t)+I\beta^2(t)=0,\quad t\in I,\tag{3.13}$$

$$(\rho,\theta(t))\geq0,\quad(\rho,\eta(t),\theta(t))\neq0,\quad t\in I.\tag{3.14}$$

By assumption (i) and (3.10), we have

$$\eta(t)=\rho\beta(t),\quad t\in I.\tag{3.15}$$

Now, if $\rho=0$, then $\eta(t)=0$ and (3.11) yields $\theta(t)=0$, which contradicts to (3.14). Hence

$$\rho>0.\tag{3.16}$$

Using (3.15) and (3.16) in (3.9), we have

$$
\begin{aligned}
& f_s + g_s^T \alpha(t) - \frac{1}{2}\left[\beta(t)^T H \beta(t)\right]_s - \frac{5}{6}\left[\beta(t)^T I \beta^2(t)\right]_s \\
& -\frac{\mathrm{d}}{\mathrm{d}t}\left[f_{\dot{s}} + g_{\dot{s}}^T \alpha(t) - \frac{1}{2}\left(\beta(t)^T H \beta(t)\right)_{\dot{s}} - \frac{5}{6}\left(\beta(t)^T I \beta^2(t)\right)_{\dot{s}}\right] \\
& +\Big\{ f_{ss} + \left(g_s^T \alpha(t)\right)_s - \frac{\mathrm{d}}{\mathrm{d}t}\left[f_{\dot{s}s} + \left(g_{\dot{s}}^T \alpha(t)\right)_s\right] + \left(H\beta(t)\right)_s + \left(I\beta^2(t)\right)_s \\
& -\frac{\mathrm{d}}{\mathrm{d}t}\left[f_{s\dot{s}} + \left(g_s^T \alpha(t)\right)_{\dot{s}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{s}\dot{s}} + \left(g_{\dot{s}}^T \alpha(t)\right)_{\dot{s}}\right) \right.\\
& \left. + \left(H\beta(t)\right)_{\dot{s}} + \left(I\beta^2(t)\right)_{\dot{s}}\right]\Big\}\beta(t) = 0, \quad t \in I.
\end{aligned} \tag{3.17}
$$

By the definition of H and(2.16), (3.17) gives

$$
\left[\frac{1}{2}\left(\beta(t)^T H \beta(t)\right) + \frac{1}{6}\left(\beta(t)^T I \beta^2(t)\right)\right]_s - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\left(\beta(t)^T H \beta(t)\right) + \frac{1}{6}\left(\beta(t)^T I \beta^2(t)\right)\right]_{\dot{s}} - I\beta^2(t) \tag{3.18}
$$

From assumption (ii), we get

$$
\beta(t) = 0, \quad t \in I. \tag{3.19}
$$

By (3.15), we have

$$
\eta(t) = 0, \quad t \in I. \tag{3.20}
$$

using (3.19) and (3.20) in (3.11), we obtain

$$
\rho g_j + \theta_j(t) = 0, \quad t \in I, \quad j = 1, 2, ..., m. \tag{3.21}
$$

As $\rho > 0$ and $\theta(t) \geq 0$, we obtain

$$
g\left(t, u(t), \dot{u}(t)\right) \leq 0, \quad t \in I. \tag{3.22}
$$

So, $u(t)$ is a feasible solution of (VP).

Again $\rho > 0$, (3.12) and (3.21) yields,

$$
\alpha(t)^T g\left(t, u(t), \dot{u}(t)\right) = 0, \quad t \in I. \tag{3.23}
$$

From (3.19) and (3.23), it is easily concluded that the objective value of (VP) and (VTD) is equal. Further, if the conditions of Theorem 3.1 are satisfied, then $u(t)$ is an optimal solution of (VP). □

Theorem 3.4 (Converse duality)**.** Let $(\bar{s}(t), \bar{\alpha}(t), \bar{\beta}(t))$ be an optimal feasible solution of (VTD). Suppose $\int_{s_0}^{s_1} f(t, ., .)\mathrm{d}t$ and $\int_{s_0}^{s_1} \bar{\alpha}(t)^T g(t, ., .)\mathrm{d}t$ are convex functions in s and $\dot{s}$. Assume that

$$
\int_{s_0}^{s_1} \bar{\alpha}(t)^T g\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) - \left(s(t) - \bar{s}(t)\right)\left[H\left(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)\right)\bar{\beta}(t) + I\left(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)\right)\bar{\beta}^2(t)\right]\mathrm{d}t \geq 0. \tag{3.24}
$$

Then $\bar{s}(t)$ is an optimal solution of (VP).

Proof. Suppose $\bar{s}(t)$ is not an optimal solution of (VP). Then there exists a feasible solution $s(t)$ of the primal (VP) such that

$$\int_{s_0}^{s_1} f\left(t, s(t), \dot{s}(t)\right) \mathrm{d}t < \int_{s_0}^{s_1} f\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) \mathrm{d}t. \tag{3.25}$$

Now

$$\begin{aligned}
&\int_{s_0}^{s_1} f\left(t, s(t), \dot{s}(t)\right) \mathrm{d}t - \int_{s_0}^{s_1} f\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) \mathrm{d}t \\
\geq &\int_{s_0}^{s_1} \left[\left(s(t) - \bar{s}(t)\right) f_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + \left(\dot{s}(t) - \dot{\bar{s}}(t)\right) f_{\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)\right] \mathrm{d}t \\
&\left(\text{by convexity of } \int_{s_0}^{s_1} f(t, ., .)\mathrm{d}t\right) \\
= &\int_{s_0}^{s_1} \left(s(t) - \bar{s}(t)\right) \Big[- g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \\
&+ \frac{\mathrm{d}}{\mathrm{d}t}\Big\{ f_{\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) + g_{\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t)\Big\} \\
&- H\left(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)\right) \bar{\beta}(t) - I\left(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)\right) \bar{\beta}^2(t)\Big] \mathrm{d}t \\
&(\text{by } (2.16)) \\
= &- \int_{s_0}^{s_1} \Big[\left(s(t) - \bar{s}(t)\right) \Big(g_s\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \Big) \\
&+ \left(\dot{s}(t) - \dot{\bar{s}}(t)\right) \Big(g_{\dot{s}}\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right)^T \bar{\alpha}(t) \Big) \Big] \mathrm{d}t \\
&- \int_{s_0}^{s_1} \left(s(t) - \bar{s}(t)\right) \Big[H\left(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)\right) \bar{\beta}(t) \\
&+ I\left(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)\right) \bar{\beta}^2(t) \Big] \mathrm{d}t \\
&(\text{by integrating by parts}) \\
\geq &\int_{s_0}^{s_1} \bar{\alpha}(t)^T g\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) \mathrm{d}t - \int_{s_0}^{s_1} \bar{\alpha}(t)^T g\left(t, s(t), \dot{s}(t)\right) \mathrm{d}t \\
&- \int_{s_0}^{s_1} \left(s(t) - \bar{s}(t)\right) \Big[H\left(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)\right) \bar{\beta}(t) + I\left(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)\right) \bar{\beta}^2(t) \Big] \mathrm{d}t \\
&\left(\text{by convexity of } \int_{s_0}^{s_1} \bar{\alpha}(t)^T g(t, ., .)\mathrm{d}t\right) \\
\geq &\int_{s_0}^{s_1} \bar{\alpha}(t)^T g\left(t, \bar{s}(t), \dot{\bar{s}}(t)\right) - \left(s(t) - \bar{s}(t)\right) \Big[H\left(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)\right) \bar{\beta}(t) \\
&+ I\left(t, \bar{s}(t), \dot{\bar{s}}(t), \bar{\alpha}(t)\right) \bar{\beta}^2(t) \Big] \mathrm{d}t \\
&(\text{by } (2.2) \text{ and } (2.18)) \\
\geq &\ 0. \\
&(\text{by } (3.24))
\end{aligned}$$

Hence the result. □

4. Importance of third order duality

Two numerical examples are discussed to justify the importance of the present work. The first example shows that the first order variational dual (VFD) and second order variational dual (VSD) have no solution, whereas the third order variational dual (VTD) has solutions. And from the second numerical example (Example 4.2), it can be easily seen that the third order dual provides better lower bound as compare to first and second order dual.

Example 4.1. Let $f,\ g : I \times [0,1) \times [0,1) \longrightarrow \mathbb{R}$ be defined by

$$f\left(t, s(t), \dot{s}(t)\right) = s^3(t) + s^2(t),$$
$$g\left(t, s(t), \dot{s}(t)\right) = \frac{1}{3}s^3(t) + s^2(t) - s(t),$$

where $t \in I$ and $s : I \longrightarrow [0,1)$. Clearly f and g are thrice continuously differentiable convex functions.

Now consider the following variational primal problem

$$\text{(VP)} \quad \min \int_{s_0}^{s_1} \left[s^3(t) + s^2(t)\right] \mathrm{d}t,$$

subject to

$$\frac{1}{3}s^3(t) + s^2(t) - s(t) \leq 0, \quad t \in I, \tag{4.1}$$
$$s(s_0) = \gamma_1, \quad s(s_1) = \gamma_2. \tag{4.2}$$

Let us check the feasibility of (VFD), (VSD) and (VTD).

For (VFD): From (2.4), we have

$$\bar{\alpha}(t) = -\frac{3\bar{s}^2(t) + 2\bar{s}(t)}{\bar{s}^2(t) + 2\bar{s}(t) - 1} < 0, \forall\ \bar{s}(t) \geq \frac{1}{2}.$$

Which contradicts to (2.6). Hence, $\bar{s}(t)$ is not the feasible solutions of (VFD).

For (VSD): $\bar{\beta}(t) = -1$ and (2.7) yields

$$\bar{\alpha}(t) = \frac{-3\bar{s}^2(t) + 4\bar{s}(t) + 2}{\bar{s}^2(t) - 3} < 0, \forall\ \bar{s}(t).$$

Which contradicts to (2.9). Hence, $\bar{s}(t)$ is not the feasible solutions of (VSD).

For (VTD): $\bar{\beta}(t) = -1$ and (2.13) gives

$$\bar{\alpha}(t) = \frac{-3\bar{s}^2(t) + 4\bar{s}(t) - 4}{\bar{s}^3(t) - 1} > 0, \forall\ \bar{s}(t).$$

Hence, $\bar{s}(t)$ are feasible solutions of (VSD). From the above analysis, it is observed that for $\bar{s}(t) \geq \frac{1}{2}$, the first as well as the second order variational dual has no solution, whereas the third order variational dual has solutions for all $\bar{s}(t)$.

Example 4.2. Let $f,\ g : [0,1] \times [0,1] \times [0,1] \longrightarrow \mathbb{R}$ be defined by

$$f\left(t, s(t), \dot{s}(t)\right) = s^3(t) + s(t),$$

$$g\left(t, s(t), \dot{s}(t)\right) = -s^3(t) + 3s^2(t) - 2s(t),$$

where $t \in [0,1]$ and $s : [0,1] \longrightarrow [0,1]$. Clearly f and g are thrice continuously differentiable convex functions. Now consider the following variational primal problem

$$\text{(VP)} \quad \min \int_0^1 \left[s^3(t) + s(t)\right] \mathrm{d}t,$$

subject to

$$-s^3(t) + 3s^2(t) - 2s(t) \leq 0, \quad t \in [0,1], \tag{4.3}$$

$$s(0) = \gamma_1, \quad s(1) = \gamma_2. \tag{4.4}$$

Since $s(t) \geq 0$ by the definition of s, the optimal solution of (VP) is attained at 0 and the minimal value is 0. The value of (VP) at $s(t) = 1$ and $I = [0,1]$ is 2, which is thus an upper bound to the true optimal value. Now compare a lower bound to the minimal value given by this approximation $\bar{s}(t) = 1$ and $I = [0,1]$ in (VP) with a lower bound in (VSD) and (VTD).

For (VFD): It is clearly seen that $\bar{s}(t) = 1$ is not a feasible solution of (VFD).

For (VSD): It is very clear that $\left(\bar{s}(t) = 1, \bar{\beta}(t) = -1, \bar{\alpha}(t) = 2\right)$ is a feasible solution of (VSD) and the value of the objective function is -1.

For (VTD): It can be easily observed that $\left(\bar{s}(t) = 1, \bar{\beta}(t) = -1, \bar{\alpha}(t) = \frac{4}{3}\right)$ is a feasible solution of (VTD) and the value of the objective function is $-\frac{8}{3}$. Fortunately the optimal value of the variational primal problem (VP) is known and it is 0. Hence, it can be easily concluded that (VTD) gives better bound than (VSD).

5. Conclusion

The third order duality of variational problems is formulated with a novel approach. Weak, strong and converse duality theorems are proved between the variational primal (VP) and its third order dual (VTD). A suitable numerical example (Example 4.2) is given to show that the third order duality gives better solutions compared to the results given by the first and second order dual. Further, it is observed that there are some problems in which, first as well as second order dual have no solution where as the third order dual has a solution (see Example 4.1). Our earlier work [24] is a particular case of the present results, when s, in the objective as well as constraint functions, is independent of t. Hence it generalizes the results in the references [5], [22] and [24].

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