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JOINT OPTIMAL INVENTORY, DYNAMIC PRICING AND ADVERTISEMENT POLICIES FOR NON-INSTANTANEOUS DETERIORATING ITEMS

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Abstract. In this paper, a novel model for dynamic pricing and inventory control of non-instantaneously deteriorating items is proposed. To reflect the dynamic nature of the problem, the selling price is modeled as a time-dependent function of the initial selling price and the discount rate. To this end, the product is sold at the initial price value for a time period; then its price is exponentially discounted to boost customer demands. The demand rate is a function of dynamic price, advertisement and changes in price over time. The model seeks to maximize total profit of the system by determining the optimal replenishment cycle, initial price, discount rate, and frequency of advertisement. In order to characterize the optimal solution, some useful theoretical results are derived upon which an iterative solution algorithm is developed. To demonstrate validity of the proposed model and applicability of the developed algorithm, numerical results are provided that are accompanied by an efficient sensitivity analysis on the important parameters of the model.

Mathematics Subject Classification. 90-XX.

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1. Introduction

In competitive and dynamic environment of business, meeting customer's expectations is an essential task. According to the classic approach, price was purely considered as a way to earn revenue. However, these days it is considered not only as a way to earn profit but also as an effective agent in customer's satisfaction and subsequently raising demand.

In most of the mathematical formulations of inventory management problems it is implicitly assumed that products have an infinite lifetime. However, most of products lose their initial value over time and for some of them the velocity of this process is more than usual. These products are called deteriorating items. Due to the imposed costs of deterioration on the system, the appropriate inventory control of these items is of great importance.

Most of the time there is a negative relationship between demand and the price of a product which can be modeled in a diverse variety of ways [1]. Despite of the mentioned diversity, these demand models fall into two main categories named additive and multiplicative demand models.

Keywords. Inventory control, non-instantaneous deterioration, dynamic pricing, advertisement.

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Previously, inventory management and pricing policies were considered separately, and dealt with by operational and marketing departments of businesses, respectively. However, trade-off between pricing decisions and replenishment policies in inventory management problems is of great significance which can efficiently enhance the profit of the firms.

Inspired by significance of pricing and inventory control of deteriorating items, in this paper, a novel model for dynamic pricing and inventory control of non-instantaneously deteriorating items is developed. To tackle with the practical conditions of the inventory systems, not only the selling price is assumed to be time-dependent but also the effect of changes in price is incorporated into the demand model. Since the demand rate is a decreasing function of the selling price, the negative impact of price increases (or positive influence of price reductions) on demand is obvious. This relation is more highlighted for some of the perishable items such as fashion accessories or Petroleum products. As reported in [1], Luxury brands such as Louis Vuitton, Calvin Klein and Versace incorporate successive price reductions in mid-season and end of the season sales. Following such a manner, when the discount rate increases by 20 percent much more customers are absorbed in comparison with 10 percent. Similarly, in oil industry, when the price reductions are high the demand rate increases drastically. Iranol and SPD incorporate this same selling strategy to stimulate the sale of oil motor.

Apart from price, the demand rate is dependent on advertisement as a powerful marketing parameter. In order to characterize the optimal solution, some useful theoretical results are derived based on which an iterative and simple solution algorithm is developed.

The remainder of the paper is organized as follows: in Section 2, the literature of the problem is reviewed and the related research gaps are distinguished. The assumptions and notations of the model are presented in Section 3. In Section 4, the mathematical model of the inventory system is formulated. Section 5 provides the theoretical results and the solution algorithm which are applied to derive the optimal solution. Numerical results and sensitivity analysis are represented in Section 6. Finally Section 7, finishes the paper with conclusion and recommended future research directions.

2. Literature review

The literature body of the problem is categorized by considering three types of demand including price-dependent demand, price and time-dependent demand, and price and stock-level-dependent demand. Each category involves a number of noteworthy studies.

2.1. Price-dependent demand

You [3] proposed a model which investigated the optimal times of price reductions where backorders were allowed and the demand curve might vertically shift down by reducing price. Chen and Sapra [4] also considered a two-period lifetime for products with a finite planning horizon and a periodic review inventory system. In Ghasemy Yaghin *et al.* [5] a similar problem was studied in a bi-level supply chain c for a multi-product and multi-period system.

Cai *et al.* [6] represented one of the rare studies on dynamic pricing that modeled price as a function of time. Wang *et al.* [7] also considered price as a function of time and modeled a non-instantaneous deterioration pattern. Yang *et al.* [8] investigated the possible effects of a supply price increase on retail pricing of deteriorating items. Zhang *et al.* [9] considered the problem of pricing and inventory management of deteriorating items under the assumption that the deterioration rate can be reduced by means of effective preservation technology investment.

Feng *et al.* [10] proposed a dynamic optimization model to maximize total profit by setting a joint pricing and advertising policy under advertising capacity limit. Liu *et al.* [11] developed an inventory model for perishable foods with price and quality dependent demand. The purpose of the paper was to determine a joint pricing and preservation technology investment strategy that optimizes the profit of the system. Chang *et al.* [12] proposed a model to determine optimal pricing and replenishment policies where linked-to-order trade credits were offered to the retailer. Shah *et al.* [13] provided another study that analyzed the impact of advertisement on demand. They considered an inventory system with non-instantaneous deteriorating items in which demand rate was a function of selling price and the frequency of advertisement in each replenishment cycle. Modeling the inventory holding cost as a time-dependent function was another notable feature of their model.

2.2. Price and time-dependent demand

Tsao and Sheen [14] presented one of the very few researches on effect of advertising on demand. Demand was modeled as a linear function of price, exponential function of time, and quadratic function of advertising costs. Zhang *et al.* [15] proposed a different structure. They developed a dynamic advertising model in which goodwill affected by advertising effort had a positive impact on the reference price.

Tripathy and Pradhan [16] considered demand as a decreasing function of price and time and assumed three-parameter Weibull distribution for deterioration rate. Dye [17] modeled demand as a general decreasing function of time and price. Purchasing price and product deterioration rate were defined as general functions of time. Finite time horizon and partial time-dependent backlog are considered in the structure of the proposed model. Delayed payments provided by both supplier and retailer are incorporated in the model as well.

Avinadav *et al.* [18] studied the pricing and ordering policies for products with time and price-dependent demand where demand was a linear decreasing function of the time elapsed after the last inventory system review. Panda *et al.* [19] studied the pricing and inventory management problem for perishable products by considering non-instantaneous deterioration. The price of the product in each period was obtained by defining price as a function of initial price and discount parameter. Krommyda *et al.* [20] studied a same problem for a two-warehouse inventory system.

2.3. Price and stock-level-dependent demand

Teng and Chang [21] were the first researchers who considered constant deterioration rate for products with price and stock-level-dependent demand. Giri and Bardhan [22] studied a single-period inventory system by considering price-stock-level-dependent demand in a bi-level supply chain including one supplier and one seller. In Soni and Patel [23] non-instantaneous deterioration and delayed payment by supplier are considered while shortage was not allowed. Table 1 represents a brief review of the mentioned papers.

Despite of the rich literature of the problem, a number of research gaps are distinguished: in the area of dynamic pricing, there are few studies formulating price as a function of time by defining discount variable, while this study has formulated the selling price as a time-dependent function by defining discount fraction variable as well as initial price. To the best of our knowledge, there is no research work incorporating the effect of changes in price into the demand function. However, this consideration enhances the dynamic feature of the model and is embedded into the structure of our demand function. The impact of the price of the substitute products has been rarely taken into account in the related literature. This consideration has been incorporated into the proposed demand model as well. Finally, despite of the high significance of advertisement in stimulating demand, there are few related research works linking demand to the advertisement factor. In this study the advertisement factor is modeled as the frequency of advertisement in each cycle.

3. Notations and assumptions

In order to have a uniform set of notations, the following notations are used throughout the paper:

Notations

Parameters

c	Unit purchasing cost.
h	Unit inventory holding cost per unit time.
O	The ordering cost per order.
B	Cost of each advertisement.
t_d	The period of time with no deterioration.
p_s	Unit purchasing cost of substitute products.

TABLE 1. Brief review of mentioned papers.

Reference	Demand factors	Deterioration rate	Deterioration pattern	Solution methodology
You [3]	Price	Constant	Instantaneous	Genetic algorithm and analytical approach
Chen and Sapra [4]	Price	Lifetime	–	Analytical approach
Ghasemy Yaghin *et al.* [5]	Price, advertisement	Lifetime	–	Fuzzy goal programming
Cai *et al.* [6]	Price	Constant	Instantaneous	Analytical approach
Wang *et al.* [7]	Price	Constant	Non-instantaneous	Analytical approach
Yang *et al.* [8]	Price	Constant	Instantaneous	Analytical approach
Zhang *et al.* [9]	Price	Constant	Instantaneous	Analytical approach
Feng *et al.* [10]	Price, advertisement	Constant	Instantaneous	Analytical approach
Liu *et al.* [11]	Price, quality	Constant	Instantaneous	Analytical approach
Chang *et al.* [12]	Price	Constant	Non-instantaneous	Analytical approach
Shah *et al.* [13]				
	Price, advertisement	Weibull	Non-instantaneous	Analytical approach and a heuristic algorithm
Tsao and Sheen [14]	Price, time	Constant	Instantaneous	Heuristic algorithm
Zhang *et al.* [15]				
Tripathy and Pradhan [16]	Price, time	Weibull	Instantaneous	Analytical approach
Dye [17]	Price, time	Constant	Instantaneous	PSO
Avinadav *et al.* [18]	Price, time	Lifetime	–	Analytical approach
Panda *et al.* [19]	Price, time	Constant	Instantaneous	Analytical approach
Krommyda *et al.* [20]	Price, time	Constant	Non-instantaneous	Analytical approach
Teng and Chang [21]	Price, stock level	Constant	Non-instantaneous	Analytical approach
Giri and Bardhan [22]	Price, stock level	Constant	Instantaneous	Analytical approach
Soni and Patel [23]	Price, stock level	Constant	Non-instantaneous	Analytical approach
This paper	Dynamic price, advertisement, price changes, price of substitute products	Weibull	Non-instantaneous	Analytical approach and a heuristic algorithm

Variables

A	The frequency of advertisement in each cycle (decision variable).
$s(t)$	The unit dynamic price of product at any time t (decision variable).
T	The replenishment cycle of the product (decision variable).
$I_1(t)$	The inventory level at time $t(0 \leqslant t \leqslant t_d)$.
$I_2(t)$	The inventory level at time $t(t_d \leqslant t \leqslant T)$.
I_0	The maximum inventory level.
Q	The order quantity.
$\dot{s}$	Changes in price per unit time.
OC	The total ordering cost.
HC	The total inventory holding cost.
PC	The total purchasing cost.
AC	The total advertisement cost.
$TP(s(t), A, T)$	The total profit per unit time of the inventory system.

Assumptions

The following assumptions form our proposed dynamic pricing and inventory control model:

1. The planning horizon is infinite and shortage is not allowed.
2. The replenishment rate is infinite and the lead time is zero.
3. The inventory system involves single non-instantaneous deteriorating item.
4. The dynamic price of the product at any time t is formulated as: $p(t) = \begin{cases} s_0 & 0 \leqslant t \leqslant t_d \\ s_0 \exp(-\eta(t - t_d)) & t_d < t \leqslant T \end{cases}$ where s_0 is the initial price and $\eta \in \pi$ is variable of discount fraction for each unit time passing after the start of deterioration. In this paper $\pi = \{0.2, \ldots, 0.8, 0.9\}$.
5. The demand rate is a function of the selling price, frequency of advertisement and changes in price during time. The price of substitute products affects demand as well. Therefore, we set $D(s(t), A) = (D_0 - \mu s(t) + \sum_{s \in \Omega} \gamma_s p_s - \varepsilon \dot{s})(1 + A)^{\lambda}$ where D_0 is the potential demand where price is equal to zero, $\mu > 0$ is the price sensitivity factor, $\gamma_s > 0$ is the price sensitivity factor for substitute products, Ω is the set of substitute products, $\varepsilon > 0$ is the sensitivity factor of changes in price and $0 \leqslant \lambda < 1$ is the shape parameter of the advertisement. As in [24], this demand model is in form of additive-multiplicative demand models which has been vastly incorporated in recent literature.
6. The deterioration rate $\theta(t)$ at any time $t \geqslant 0$ follows Weibull distribution given by $\alpha\beta t^{\beta-1}$. Where $0 < \alpha \leqslant 1$ is the scale parameter and $\beta > 0$ is the shape parameter. Weibull distribution is the most widely used deterioration rate because of its power in projecting different decaying patterns. Figure 1 expresses this claim. As shown, when the shape parameter (β) is equal to unity the deterioration rate takes constant values. When $\beta > 1$ the deterioration rate is increasing and the initial rate is almost zero which is mostly applied in non-instantaneous deterioration case. On the other hand when $\beta < 1$ the initial decaying rate is extremely high and decreasing. As common in literature and for mathematical simplicity we have assumed $\beta = 2$.

4. The mathematical formulation

As mentioned in assumptions, the dynamic price of the product at any time t is formulated as:

$$s(t) = \begin{cases} s_0 & 0 \leqslant t \leqslant t_d \\ s_0 \exp(-\eta(t - t_d)) & t_d < t \leqslant T. \end{cases} \tag{4.1}$$

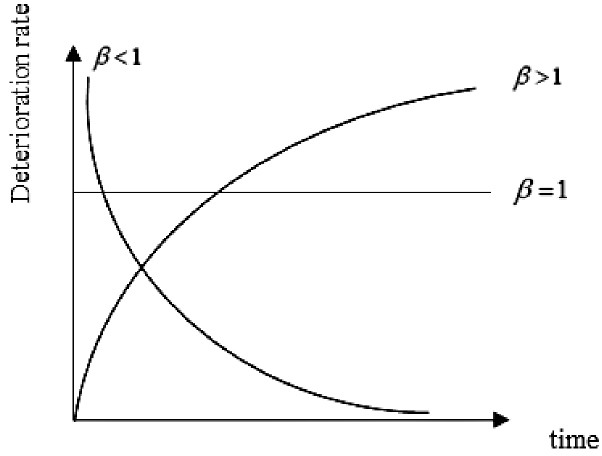

FIGURE 1. The Weibull deterioration rate.

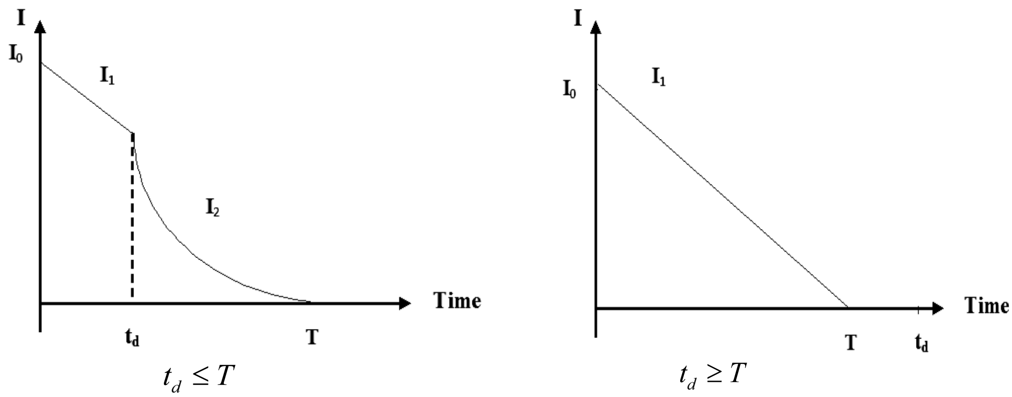

FIGURE 2. The inventory system for different cases.

Therefore, the changes in price per unit time are defined as:

$$\dot{s} = \begin{cases} 0 & 0 \leqslant t \leqslant t_d \\ -s_0\eta \, \exp(-\eta(t - t_d)) & t_d < t \leqslant T. \end{cases} \tag{4.2}$$

The inventory system evolves as follows: I_0 units of items arrive at the inventory system at the beginning of each cycle. Based on the values of t_d and T, two cases are possible ($t_d \leqslant T$ and $t_d \geqslant T$) which are shown in Figure 2. Each case is discussed in details as follows:

Case 1. $t_d \leqslant T$. In this case, during the interval $[0,\, t_d]$, the inventory system exhibits no deterioration and the inventory level decreases owing to demand only. Subsequently the inventory level declines due to demand and deterioration during time interval $[t_d,\, T]$. Based on this description during time interval $[0, T]$ the inventory status is represented by the following differential equations:

$$\frac{\mathrm{d}I_1(t)}{\mathrm{d}t} = -D(s(t), A) = -D(s_0, A) \quad 0 \leqslant t \leqslant t_d \tag{4.3}$$

$$\frac{\mathrm{d}I_2(t)}{\mathrm{d}t} = -D(s(t), A) - \theta(t)\, I_2(t) \quad t_d \leqslant t \leqslant T. \tag{4.4}$$

With boundary conditions $I_1(0) = I_0$ and $I_2(T) = 0$ solving equations (4.3) and (4.4) yields:

$$I_1(t) = -D(s_0, A)\ t + I_0 \tag{4.5}$$

$$I_2(t) = \mathrm{e}^{-\alpha t^2} \int_t^T D(s(u), A)\, \mathrm{e}^{\alpha u^2} \mathrm{d}u. \tag{4.6}$$

As shown in Figure 2, $I_1(t_d) = I_2(t_d)$. Then the maximum inventory level (I_0) is obtained as:

$$I_0 = \mathrm{e}^{-\alpha t_d^2} \int_{t_d}^T D(s(u), A)\, \mathrm{e}^{\alpha u^2} \mathrm{d}u + D(s(t_d), A)t_d. \tag{4.7}$$

Substituting (4.7) into (4.5) gives:

$$I_1(t) = \mathrm{e}^{-\alpha t_d^2} \int_{t_d}^T D(s(u), A)\, \mathrm{e}^{\alpha u^2} \mathrm{d}u - D(s(t), A)t + D(s(t_d), A)t_d. \tag{4.8}$$

The order quantity is equal to I_0 *i.e.*

$$Q = I_0 = \mathrm{e}^{-\alpha t_d^2} \int_{t_d}^T D(s(u), A)\, \mathrm{e}^{\alpha u^2} \mathrm{d}u + D(s(t_d), A)t_d. \tag{4.9}$$

The total profit of the inventory system involves the following components:

1. SR: the sales revenue

$$SR = \int_0^T s(t)\, D(s(t), A)\, \mathrm{d}t. \tag{4.10}$$

2. Oc: the ordering cost

$$OC = O. \tag{4.11}$$

3. HC: the inventory holding cost

$$\begin{aligned} HC &= h\left(\int_0^{t_d} I_1(t)\mathrm{d}t + \int_{t_d}^T I_2(t)\mathrm{d}t\right) \\ &= h\int_0^{t_d} \mathrm{e}^{-\alpha t_d^2}\left[\int_{t_d}^T D(s(u), A)\, \mathrm{e}^{\alpha u^2}\mathrm{d}u - D(s(t), A)t + D(s(t_d), A)t_d\right]\mathrm{d}t \\ &\quad + h\int_{t_d}^T \mathrm{e}^{-\alpha t^2}\left[\int_t^T D(s(u), A)\, \mathrm{e}^{\alpha u^2}\mathrm{d}u\right]\mathrm{d}t. \end{aligned} \tag{4.12}$$

4. PC: the purchasing cost

$$PC = cQ = c\left(\mathrm{e}^{-\alpha t_d^2}\int_{t_d}^{T} D(s(u),A)\,\mathrm{e}^{\alpha u^2}\mathrm{d}u + D(s(t_d),A)t_d\right). \tag{4.13}$$

5. AC: the advertisement cost

$$AC = B\cdot A. \tag{4.14}$$

Therefore, the total profit per unit time ($TP(s(t),\, A,\, T)$) is given by:

$$TP_1(s(t),A,T) = \frac{1}{T}\,(SR - OC - HC - PC - AC). \tag{4.15}$$

Case 2. $t_d \geqslant T$. In this case, the model turns into the traditional inventory model and the total profit of the system is given by:

$$TP_2(s(t),\, A,\, T) = \left(D_0 - \mu s_0 + \sum_{s\in\,\Omega}\gamma_s\, p_s\right)(1+A)^{\lambda}\left[s_0 - \frac{hT}{2} - c\right] - \frac{B\cdot A + O}{T}\cdot \tag{4.16}$$

Then, the total profit of the system is given by:

$$TP(s(t),A,T) = \begin{cases} TP_1(s(t),\, A,\, T) & t_d \leqslant T \\ TP_2(s(t),\, A,\, T) & t_d \geqslant T. \end{cases} \tag{4.17}$$

It should be noted that, the function $TP(s(t),\, A,\, T)$ is continuous at $T = t_d$. Next section provides theoretical results and solution methodology to determine the optimal solution.

5. Solution methodology

Due to high complexity of the formulated equations, the concavity of the total profit per unit time cannot be proved by using Hessian matrix. Therefore, the problem is solved applying the following search procedure a similar form of which has been used in Wu *et al.* [25] and Shah *et al.* [13] as well. We first prove that for given values of η, s_0 and T there exist a unique optimal value of A. Then for known values of η, s_0 and A, a unique optimal value of T is obtained and finally, for given η, A and T, a unique optimal value of s_0 is determined which maximizes the total profit per unit time. Since the discount fraction is defined as a discrete variable, the above mentioned procedure is applied for different values of η and finally the optimal solution is obtained by comparing the results.

First, for fixed η, s_0 and T, the second order derivative of $TP(s(t),\, A,\, T)$ is obtained as follows:

$$\frac{\partial^2 TP}{\partial A^2} = \frac{\lambda(\lambda-1)(1+A)^{\lambda-2}}{T}(Expression\, i) \qquad i = 1,2, \tag{5.1}$$

where:

$$\text{Expression } 1 = \text{Expression } a - \text{Expression } b - \text{Expression } c \tag{5.2}$$

$$\text{Expression } a = \int_0^{t_d} \left(D_0 + \sum_{s\in\Omega} \gamma_s\, p_s - \mu s_0 \right) p_0 \mathrm{d}t + \int_{t_d}^{T} \left(D_0 + \sum_{s\in\Omega} \gamma_s\, p_s + s_0(\varepsilon\eta - \mu) \left(\mathrm{e}^{-\eta(t-t_d)} \right) \right) s_0(\mathrm{e}^{-\eta(t-t_d)}) \mathrm{d}t \tag{5.3}$$

$$\text{Expression } b = h \left\{ \begin{array}{l} \int_0^{t_d} \left(\mathrm{e}^{-\alpha t_d^2} \right) \int_{t_d}^{T} \left(D_0 + \sum_{s\in\Omega} \gamma_s\, p_s + s_0(\varepsilon\eta - \mu)(\mathrm{e}^{-\eta(u-t_d)}) \right) \left(\mathrm{e}^{\alpha u^2} \right) \mathrm{d}u\, \mathrm{d}t \\ \int_{t_d}^{T} \left(\mathrm{e}^{-\alpha t^2} \right) \int_{t}^{T} \left(D_0 + \sum_{s\in\Omega} \gamma_s\, p_s + s_0(\varepsilon\eta - \mu)(\mathrm{e}^{-\eta(u-t_d)}) \right) (\mathrm{e}^{\alpha u^2}) \mathrm{d}u\, \mathrm{d}t \\ \int_0^{t_d} - \left(D_0 + \sum_{s\in\Omega} \gamma_s\, p_s - \mu s_0 \right) (t - t_d) \mathrm{d}t \end{array} \right\} \tag{5.4}$$

$$\text{Expression } c = c \left[\left(\mathrm{e}^{-\alpha t_d^2} \right) \int_{t_d}^{T} \left(D_0 + \sum_{s\in\Omega} \gamma_s\, p_s + s_0(\varepsilon\eta - \mu) \left(\mathrm{e}^{-\eta(t-t_d)} \right) \left(\mathrm{e}^{\alpha t^2} \right) \right) \mathrm{d}t + (D_0 - \mu p_0) t_d \right] \tag{5.5}$$

$$\text{Expression } 2 = s_0 \left(D_0 + \sum_{s\in\Omega} \gamma_s\, p_s - \mu s_0 \right) T - \frac{h}{2} \left(D_0 + \gamma_s p_s - \mu s_0 \right) T^2 - c \left(D_0 + \gamma_s p_s - \mu s_0 \right) T. \tag{5.6}$$

Since $\lambda < 1$ it is obvious that $\frac{\partial^2 TP}{\partial A^2} < 0$, therefore $TP(s(t),\, A,\, T)$ is a concave function of A and the search to find the optimal frequency of advertisement is restricted to find a local optimum. Now we provide some useful theoretical results in order to find the optimal length of replenishment (T^*) and the optimal initial price (s_0^*) for two aforementioned possible cases.

For notational simplicity set:

$$\varDelta(A, s(t)) = O + B \cdot A - \frac{h}{2}((D_0 - \mu s_0)t_d + \sum_{s\in\Omega} \gamma_s\, p_s - \mu s_0)(1 + A)^{\lambda} t_d^2. \tag{5.7}$$

Case 1. $t_d \leqslant T$.

Lemma 5.1. *For fixed η and A and known s_0 we have*

(a) *If $\varDelta(A, s(t)) \geqslant 0$ then $TP_1(s(t),\, A,\, T)$ is concave and has a unique global optimum value at $T_1^* = T_1$ where $\frac{\partial TP_1(s(t),\, A,\, T)}{\partial T}|_{T=T_1} = 0$ (see Eq. (A.1) in Appendix A).*
(b) *If $\varDelta(A, s(t)) < 0$ then $TP_1(s(t),\, A,\, T)$ has a maximum value at $T_1^* = t_d$.*

Proof. See Appendix A. □

Therefore, For given s_0 and fixed η and A, T_1^* is given by:

$$T_1^* = \begin{cases} T_1 & \varDelta(A, s(t)) \geqslant 0 \\ t_d & \varDelta(A, s(t)) < 0. \end{cases} \tag{5.8}$$

Lemma 5.2. *For fixed A, η and given $T_1 \in [t_d, \infty)$ there exists a unique s_0^{1*} which maximizes $TP_1(s(t),\, A,\, T)$ where $\frac{\partial TP_1(s(t),\, A,\, T)}{\partial s_0}|_{s_0 = s_0^{1*}} = 0$ (see Eq. (B.2) in Appendix B).*

Proof. See Appendix B. □

Case 2. $t_d \geqslant T$.

Lemma 5.3. *For known* s_0 *and fixed* η *and* A*, we have*

(a) *If* $\Delta(A, s(t)) \leqslant 0$ *then* $TP_2(s(t),\ A,\ T)$ *is concave and has a unique global optimum value at* $T_2^* = T_2$ *where* $\frac{\partial TP_2(s(t),\ A,\ T)}{\partial T}|_{T=T_2} = 0$ *(see Eq.* (C.1) *in Appendix* C*).*
(b) *If* $\Delta(A, s(t)) > 0$ *then* $TP_2(s(t),\ A,\ T)$ *has a maximum value at* $T_2^* = t_d$.

Proof. See Appendix C. □

Therefore, for fixed and given η, s_0 and A, T_2^* is given by:

$$T_2^* = \begin{cases} T_2 & \Delta(A, s(t)) \leqslant 0 \\ t_d & \Delta(A, s(t)) > 0 \end{cases} \tag{5.9}$$

Lemma 5.4. *For fixed* A *and* η *and Known* $T_2 \in (0, t_d]$ *there exists a unique* s_0^{2*} *which maximizes* $\frac{\partial TP_2(s(t),\ A,\ T)}{\partial s_0}|_{s_0=s_0^{2*}} = 0$ *(see Eq.* (D.2) *in Appendix* D*).*

Proof. See Appendix D. □

Combining Lemmas 5.1 and 5.3, we have the following theorem

Theorem 5.5. *For given* A *and known* s_0 *we have:*

1. *If* $\Delta(A, s(t)) > 0$ *the optimal replenishment cycle is* $T^* = T_1$.
2. *If* $\Delta(A, s(t)) < 0$ *the optimal replenishment cycle is* $T^* = T_2$.
3. *If* $\Delta(A, s(t)) = 0$ *the optimal replenishment cycle is* $T^* = t_d$.

Proof. Regarding the fact that $TP_1(s(t),\ A,\ t_d) = TP_2(s(t),\ A,\ t_d)$, the proof follows from Lemmas 5.1 and 5.3. □

For fixed A and σ the unique optimal solution for (T, s_0) which maximizes $TP(s(t), A, T)$ exists. The optimal solution can be obtained through some iterative numerical procedure the convergence of which can be easily proved by adopting a similar graphical technique used in Hadley and Whitin [26]. The following algorithm which is similar to the one proposed by Wu *et al.* [25] and Shah *et al.* [13] is developed to identify global optimal solution for (A, s_0, η, T). As the algorithm is based upon proven lemmas, it ensures that the obtained solutions are optimal.

Algorithm

Step 1: Set $j = 1$ and $\eta = 0.2$.
Step 2: Set $A^j = 0$.
Step 3: Set $k = 1$ and initialize the value of $s_0^{k,j} = c$.
Step 4: Calculate $\Delta(A^j, s^{k,j}(t))$.

4.1: If $\Delta(A^j, s^{k,j}(t)) > 0$, obtain the value of $T_1^{k,j}$ by solving $\frac{\partial TP_1(s(t),A,T)}{\partial T} = 0$. Substitute $T_1^{k,j}$ into equation (B.2) in order to calculate $s_{0(1)}^{k,j}$. Set $s_0^{k+1,j} = s_{0(1)}^{k,j}$ and $T^{k,j} = T_1^{k,j}$.

4.2: If $\Delta(A^j, s^{k,j}(t)) < 0$, obtain the value of $T_2^{k,j}$ by solving $\frac{\partial TP_2(s(t),A,T)}{\partial T} = 0$. Substitute $T_2^{k,j}$ into Equation (D.2) in order to calculate $s_{0(2)}^{k,j}$. Set $s_0^{k+1,j} = s_{0(2)}^{k,j}$ and $T^{k,j} = T_2^{k,j}$.

4.3: If $\Delta(A^j, s^{k,j}(t)) = 0$, set $T_1^{k,j} = T_2^{k,j} = t_d$. Substitute $T_2^{k,j}$ into equations (B.2) or (D.2) in order to calculate $s_0^{k,j}$. Set $s_0^{k+1,j} = s_0^{k,j}$ and $T^{k,j} = t_d$.

Step 5: If $|s_0^{k+1,j} - s_0^{k,j}| < Epsilon$ ($Epsilon$ is considered to be a very small value), then set $(s_0^{j*}, T^{j*}) = (s_0^{(k+1,j)^*}, T^{(k,j)*})$ and go to Step 6. Otherwise, $k = k + 1$ and go back to Step 4.
Step 6: Calculate $TP(s(t)^{j*}, A^j, T^{j*})$, then (s_0^{j*}, T^{j*}) is the optimal solution and $TP(s(t)^{j*}, A^j, T^{j*})$ is the maximum value of the objective function for fixed A^j and σ.
Step 7: Set $A^{'j} = A^j + 1$ and repeat Step 3 to Step 6 to obtain $TP(s(t)^{j*}, A^{'j}, T^{j*})$.
Step 8: If $TP(s(t)^{j*}, A^{'j}, T^{j*}) > TP(s(t)^{j*}, A^j, T^{j*})$, then $A^j = A^{'j}$ and go back to Step 7. Otherwise go to Step 9.
Step 9: Set $(s_0^{j*}, A^{j*}, T^{j*}) = (s_0^{j*}, A^j, T^{j*})$ which is the optimal solution for fixed σ.
Step 10: Set $j = j + 1$ and $\eta = \eta + 0.1$. If $\eta \leqslant 0.9$ go back to Step 2; otherwise go to Step 11.
Step 11: Set $TP(p(t)^{p*}, A^p, T^{p*}) = \max_j\{TP(p(t)^{j*}, A^j, T^{j*})\}$ and

$$(s_0^*, \eta^*, A^*, T^*) = (s_0^{p*}, \eta^{p*}, A^{p*}, T^{p*}).$$

Step 12: End.

6. Experimental results

In this section, the developed algorithm is applied to solve the following problem in order to show validity of the proposed model and applicability of the developed algorithm. The values of the parameters of the problem are defined in Table 2. Identical parameters of model are taken from Shah *et al.* [13] and adopted to our model. In order to show the effect of related deterioration and advertisement parameters, Table 3 provides the computational results for different values of α, t_d and λ.

Based on the computational results, the following managerial insights are obtained about deterioration and advertisement factors which are analogous to Shah *et al.* [13]:

1. For fixed α and t_d, increasing λ results in increasing optimal replenishment cycle (T^*), optimal order quantity (Q^*), optimal advertisement frequency (A^*), optimal initial price (s_0^*) and total profit per unit time of the inventory system (TP). In fact, changes in the shape of advertisement (λ) increase A^* and therefore, result in an increase in demand and the total profit of the system. The optimal discount fraction ($\eta*$) is not sensitive to changes in λ. This shows that if the retailer could sell the products in a market which is highly influenced by promotional activities, she will be able to earn more profits.
3. For fixed α and λ, it is observed when value of t_d increases, optimal replenishment cycle (T^*), optimal order quantity (Q^*) and total profit per unit time of the inventory system (TP) increase, whereas optimal initial price (s_0^*) decreases slightly and optimal advertisement frequency (A^*) and optimal discount fraction ($\eta*$) remain unchanged. As deterioration imposes excess costs on the inventory system and lowers the value of the total profit of the inventory system, the later the inventory starts to deteriorate (*i.e.* t_d has higher value) the higher profit is obtained. This also conveys a significant idea. Investing on advanced inventory holding technologies increases the functional costs of the firms. On the other hand, it can efficiently reduce the costs by delaying the time that the items start to deteriorate. Therefore, a balance between these two costs can raise the profits.
3. For fixed λ and t_d, with an increase in the value of α, optimal advertisement frequency (A^*) and optimal discount fraction ($\eta*$) remain unchanged, while increasing α results in a decrease in the value of optimal order quantity (Q^*) and total profit per unit time of the inventory system (TP). In comparison to the other variables, changes in the value of optimal initial price (s_0^*) is imperceptible. Similarly it can be concluded that, better inventory holding technologies can lower the deterioration rate and increase the firm's profit.

Table 2. The value of the parameters of the problem.

Parameter	D_0	γ_s	p_s	ε	μ	O	B	h	c
Value	4000	30	5	60	200	250	80	0.4	3

TABLE 3. Computational results for different values of α, t_d and λ.

α	t_d	λ	η	A	s_0	T	Q	TP
		0.03	0.9	2	6.7522	0.4926	2041.122	10 957.22
	0.02	0.04	0.9	5	6.8648	0.5203	2475.439	12 232.42
		0.05	0.9	11	7.0055	0.5556	3068.948	14 487.91
		0.03	0.9	2	5.9734	0.5109	2274.018	11 011.31
0.08	0.04	0.04	0.9	5	6.0501	05332	2705.243	12 942.44
		0.05	0.9	11	6.1501	0.5829	3301.836	15 107.45
		0.03	0.9	2	5.5348	0.5395	2577.558	11 729.42
	0.06	0.04	0.9	5	5.5908	0.5554	3051.005	13 320.61
		0.05	0.9	11	5.6643	0.6112	3714.032	15 993.74
		0.03	0.9	2	6.7051	0.4819	1976.374	9351.24
	0.02	0.04	0.9	5	6.8174	0.5091	2401.511	10 994.73
		0.05	0.9	11	6.9572	0.5437	2969.921	12 821.55
		0.03	0.9	2	5.9378	0.5033	2175.734	9992.21
0.1	0.04	0.04	0.9	5	6.0131	0.5149	2589.127	11 425.32
		0.05	0.9	11	6.1114	0.5538	3154.404	13 450.76
		0.03	0.9	2	5.5186	0.5158	2350.814	10 321.31
	0.06	0.04	0.9	5	5.5717	0.5439	2758.733	11 952.73
		0.05	0.9	11	5.6435	0.6089	3416.241	13 980.43
		0.03	0.9	2	6.6641	0.4731	1915.872	8251.11
	0.02	0.04	0.9	5	6.7759	0.4997	2331.834	9336.14
		0.05	0.9	11	6.9154	0.5337	2904.595	11 221.33
		0.03	0.9	2	5.9111	0.4987	2081.684	8865.74
0.12	0.04	0.04	0.9	5	5.9848	0.5098	2477.244	9931.01
		0.05	0.9	11	6.0816	0.5381	3011.036	11 784.99
		0.03	0.9	2	5.5145	0.5002	2219.547	9119.21
	0.06	0.04	0.9	5	5.5656	0.5378	2601.115	10 241.33
		0.05	0.9	11	5.6351	0.5522	3310.714	12 179.95

TABLE 4. Computational results for different values of μ and ε.

μ	ε	η	A	s_0	T	Q	TP
	4	0.5	11	7.6652	0.7102	3398.620	18 111.2545
8	6	0.5	16	8.7175	0.8003	4259.057	21 335.5333
	8	0.6	20	11.1641	0.8643	5224.991	29 714.1003
	4	0.6	6	6.0912	0.6191	2623.942	11 456.9989
10	6	0.7	8	7.1216	0.6715	3097.803	14 569.8085
	8	0.7	12	8.2852	0.7435	3838.153	18 107.8556
	4	0.8	4	5.2034	0.4527	2168.825	7606.0712
12	6	0.9	5	5.9843	0.5098	2477.244	9931.0119
	8	0.9	7	6.9047	0.5830	2958.286	12 688.6746

As observed, the optimal discount fraction ($\eta*$) is insensitive to the changes in the value of α, t_d and λ. Hence, it seems necessary to evaluate the effect of changes of other parameters in the value of the optimal discount fraction. Table 4 provides computational results for different value of μ and ε when $\alpha = 0.75$, $t_d = 0.04$ and $\lambda = 0.04$.

Based on the computational results, following managerial insights are obtained about the effect of ω and ε:

1. For fixed ε, by increasing price sensitivity factor (μ), optimal replenishment cycle (T^*), optimal order quantity (Q^*), optimal advertisement frequency (A^*), optimal initial price (s_0^*) and total profit per unit time of the inventory system (TP) decrease while the discount fraction (η^*) increases. When the price sensitivity

TABLE 5. Computational results for different value of p_s.

p_s	η	A	s_0	T	Q	TP
4	0.9	4	5.6461	0.4418	2226.803	8704.2281
5	0.9	5	5.9848	0.5098	2477.244	9931.0119
6	0.9	5	6.3218	0.5967	2727.537	11 064.4553
7	0.9	6	6.6576	0.6026	2978.181	12 305.4629

of the market increases, a fixed selling price establishes a lower demand rate. Therefore, by demand drops the order quantity decreases as well. The firms need to lower the initial selling price and increase the discount rate to compensate this negative effect. These all, influentially reduce the firm's profits.

2. For fixed μ, increasing sensitivity factor of changes in price (ε) results in an increase in optimal replenishment cycle (T^*), optimal order quantity (Q^*), optimal advertisement frequency (A^*) and total profit per unit time of the inventory system (TP). By increasingε, a fixed price reduction results in a higher demand rate. In deterioration free period, the price is fixed. Then the firm can earn more profits by increasing the initial selling price and increasing the discount rate slightly that can offset the initial price increase.

As mentioned before, the price of substitute product influences demand as well. Regarding the computational results provided in Table 5 for $\alpha = 0.75$, $t_d = 0.04$ and $\lambda = 0.04$, this influence is analyzed.

As it is observed, by increasing price of substitute product (p_s), optimal replenishment cycle (T^*), optimal order quantity (Q^*), optimal advertisement frequency (A^*), optimal initial price (s_0^*) and total profit per unit time of the inventory system (TP) gets larger due to the direct effect of p_s on demand rate.

7. Conclusion

This paper provided an integrated model for dynamic pricing and inventory control of non-instantaneous deteriorating items. The selling price was defined as a time-dependent function of the initial price and discount rate which is one of the novel features of the proposed model. The product was sold at an initial price value until the deterioration started; then its price was exponentially discounted to boost customer demands and compensate the negative impact of the deterioration. Apart from the selling price, the demand rate was a function of advertisement and changes in price over time. To the best of our knowledge, there is no research work incorporating the effect of changes in price into the demand function. Moreover, the impact of advertisement on stimulating sales was embedded into the demand model. An iterative algorithm was developed based on derived theoretical results. We illustrated through the experimental results the way the optimal solution was obtained by this simple algorithm. Computational results indicated that implementing better inventory holding technologies can efficiently enhance the profit of the system by lowering the negative effect of the deterioration.

The proposed model can be extended by considering shortages, trade credit and time value of money. Considering variable ordering and holding costs is another future research direction. Since variable holding cost is particularly common and practical in the storage of deteriorating items this extension seems really beneficial.

Appendix A. Proof of Lemma 5.1

The first order partial derivative of $TP_1(s(t), A, T)$ with respect to T is given by:

$$\frac{\partial TP_1}{\partial T} = \frac{(SR' - HC' - PC')T - (SR - OC - HC - PC - AC)}{T^2}, \tag{A.1}$$

where:

$$SR' = s_0(1+A)^{\lambda}(\mathrm{e}^{-\eta(T-t_d)})\left(D_0 + \sum_{s\in\Omega}\gamma_s\,p_s + s_0(\varepsilon\eta-\mu)(\mathrm{e}^{-\eta(T-t_d)})\right) \tag{A.2}$$

$$HC' = h(1+A)^{\lambda}\left[\begin{array}{l}\left(D_0 + \sum\limits_{s\in\Omega}\gamma_s\,p_s + s_0(\varepsilon\eta-\mu)(\mathrm{e}^{-\eta(T-t_d)})\right)(\mathrm{e}^{\alpha T^2})\int\limits_0^{t_d}(\mathrm{e}^{-\alpha t_d^2})\mathrm{d}t\\ +\left(D_0 + \sum\limits_{s\in\Omega}\gamma_s\,p_s + s_0(\varepsilon\eta-\mu)(\mathrm{e}^{-\eta(T-t_d)})\right)(\mathrm{e}^{\alpha T^2})\int\limits_{t_d}^{T}(\mathrm{e}^{-\alpha t^2})\mathrm{d}t\end{array}\right] \tag{A.3}$$

$$PC' = c(\mathrm{e}^{-\alpha t_d^2})\left(D_0 + \sum_{s\in\Omega}\gamma_s\,p_s + s_0(\varepsilon\eta-\mu)(\mathrm{e}^{-\eta(T-t_d)})\right)(\mathrm{e}^{\alpha T^2})(1+A)^{\lambda}. \tag{A.4}$$

Motivated by equation (A.1) the auxiliary function $R(T), T\in[t_d,\infty)$ is defined as:

$$R(T) = (SR' - HC' - PC')T - (SR - OC - HC - PC - AC). \tag{A.5}$$

The first order derivative of $R(T)$ with respect to $T\in[t_d,\infty)$ gives:

$$\frac{\mathrm{d}R(T)}{\mathrm{d}T} = (SR'' - HC'' - PC'')T, \tag{A.6}$$

where:

$$SR'' = s_0(1+A)^{\lambda}\left[\begin{array}{l}-\eta\mathrm{e}^{-\eta(T-t_d)}\left(D_0 + \sum\limits_{s\in\Omega}\gamma_s\,p_s + s_0(\varepsilon\eta-\mu)(\mathrm{e}^{-\eta(T-t_d)})\right)\\ -\eta\mathrm{e}^{-2\eta(T-t_d)}s_0(\varepsilon\eta-\mu)\end{array}\right] \tag{A.7}$$

$$HC'' = h(1+A)^{\lambda}\left\{\begin{array}{l}\left[\begin{array}{l}-\eta s_0(\varepsilon\eta-\mu)(\mathrm{e}^{-\eta(T-t_d)})(\mathrm{e}^{\alpha T^2})\\ +2\alpha T\mathrm{e}^{\alpha T^2}(D_0 + \sum\limits_{s\in\Omega}\gamma_s\,p_s + s_0(\varepsilon\eta-\mu)(\mathrm{e}^{-\eta(T-t_d)}))\end{array}\right]\\ \left[\int\limits_0^{t_d}(\mathrm{e}^{-\alpha t_d^2})\mathrm{d}t + \int\limits_{t_d}^{T}(\mathrm{e}^{-\alpha t^2})\mathrm{d}t\right]\\ +(D_0 + \sum\limits_{s\in\Omega}\gamma_s\,p_s + s_0(\varepsilon\eta-\mu)(\mathrm{e}^{-\eta(T-t_d)}))\end{array}\right\} \tag{A.8}$$

$$PC'' = c(\mathrm{e}^{-\alpha t_d^2})(1+A)^{\lambda}\left[\begin{array}{l}-\eta s_0(\varepsilon\eta-\mu)(\mathrm{e}^{-\eta(T-t_d)})(\mathrm{e}^{\alpha T^2})\\ +2\alpha T\mathrm{e}^{\alpha T^2}(D_0 + \sum\limits_{s\in\Omega}\gamma_s\,p_s + s_0(\varepsilon\eta-\mu)(\mathrm{e}^{-\eta(T-t_d)}))\end{array}\right]. \tag{A.9}$$

It is shown that $\frac{\mathrm{d}R(T)}{\mathrm{d}T} < 0$ and hence $R(T)$ is a strictly decreasing function with respect to $T\in[t_d,\infty)$ (simply if $\eta < \frac{\mu}{\varepsilon}$ which is logically true since demand should be more sensitive to price than changes in price (*i.e.* $\varepsilon < \mu$) and we have assumed $\eta < 1$), moreover

$$R(t_d) = O + B\cdot A - \frac{h}{2}\left(D_0 + \sum_{s\in\Omega}\gamma_s\,p_s - \mu s_0\right)(1+A)^{\lambda}t_d^2 = \Delta(A, s(t)) \tag{A.10}$$

and $\lim R(T) = -\infty_{T\to\infty}$.

Part (a). If $\Delta \geqslant 0$, applying the intermediate value theorem, there exist a unique value of T (say $T_1\in[t_d,\infty)$) where $R(T_1) = 0$ which means that T_1 is the unique solution of $\frac{\partial TP_1(s(t),A,T)}{\partial T} = 0$.

From equations (A.1) and (A.5) we have

$$\frac{\partial TP_1(s(t),A,T)}{\partial T} = \frac{R(T)}{T^2}\cdot \tag{A.11}$$

According to the condition for which $R(T)$ is strictly decreasing (*i.e.* $\frac{\mathrm{d}R(T)}{\mathrm{d}T} = (SR'' - HC'' - PC'')T < 0$) at point $T = T_1$ we have:

$$\frac{\partial^2 TP_1(s(t),\, A,\, T)}{\partial T^2}|_{T=T_1} = \frac{(SR'' - HC'' - PC'')}{T} < 0. \tag{A.12}$$

Therefore, $T_1 \in [t_d, \infty)$ is the global maximum solution of $TP_1(s(t), A, T)$.

Part (b). If $\Delta < 0$, then $R(t_d) < 0$. Since $R(T)$ is a strictly decreasing function of $T \in [t_d, \infty)$, $\forall T \in [t_d, \infty)$, $R(T) < 0$. Then from equation (A.11), $TP_1(s(t),\ A,\ T)$ is a strictly decreasing function of $T \in [t_d, \infty)$. Therefore, $TP_1(s(t), A, T)$ reaches its maximum value at $T = t_d$.

Appendix B. Proof of Lemma 5.2

The first order derivative of $TP_1(s(t), A, T)$ with respect to s_0 gives

$$\begin{aligned}\frac{\partial TP_1}{\partial s_0} = \frac{(1+A)^\lambda}{T_1}\Bigg\{ &\left(D_0 + \sum_{s\in\Omega}\gamma_s\, p_s - 2\mu s_0\right) t_d + \left(D_0 + \sum_{s\in\Omega}\gamma_s\, p_s\right)\int_{t_d}^{T_1} \mathrm{e}^{-\eta(t-t_d)}\mathrm{d}t \\ &+ 2s_0(\varepsilon\eta - \mu)\int_{t_d}^{T_1} (\mathrm{e}^{-\eta(t-t_d)})^2\mathrm{d}t - h\Bigg[\int_0^{t_d} \mu(t-t_d)\mathrm{d}t \\ &+ (\varepsilon\eta - \mu)\int_0^{t_d} (\mathrm{e}^{-\alpha t_d^2})\int_{t_d}^{T_1} (\mathrm{e}^{\alpha u^2})(\mathrm{e}^{-\eta(u-t_d)})\mathrm{d}u\mathrm{d}t \\ &+ (\varepsilon\eta - \mu)\int_{t_d}^{T_1} (\mathrm{e}^{-\alpha t^2})\int_{t}^{T_1} (\mathrm{e}^{\alpha u^2})(\mathrm{e}^{-\eta(u-t_d)})\mathrm{d}u\mathrm{d}t\Bigg] \\ &- c(\mathrm{e}^{-\alpha t_d^2})(\varepsilon\eta - \mu)\int_{t_d}^{T_1} (\mathrm{e}^{\alpha t^2})(\mathrm{e}^{-\eta(t-t_d)})\mathrm{d}t\Bigg\}. \end{aligned} \tag{B.1}$$

By solving $\frac{\partial TP_1}{\partial s_0} = 0$, s_0^{1*} yields

$$\begin{aligned} s_0^{1*} = \Bigg\{ &\left(D_0 + \sum_{s\in\Omega}\gamma_s\, p_s\right)\left[t_d + \int_{t_d}^{T_1} \mathrm{e}^{-\eta(t-t_d)}\mathrm{d}t\right] - h\Bigg[\int_0^{t_d} \mu(t-t_d)\mathrm{d}t \\ &\times (\varepsilon\eta - \mu)\int_0^{t_d} (\mathrm{e}^{-\alpha t_d^2})\int_{t_d}^{T_1} (\mathrm{e}^{\alpha u^2})(\mathrm{e}^{-\eta(u-t_d)})\mathrm{d}u\mathrm{d}t \\ &+ (\varepsilon\eta - \mu)\int_{t_d}^{T_1} (\mathrm{e}^{-\alpha t^2})\int_{t}^{T_1} (\mathrm{e}^{\alpha u^2})(\mathrm{e}^{-\eta(u-t_d)})\mathrm{d}u\mathrm{d}t\Bigg] \\ &- c(\mathrm{e}^{-\alpha t_d^2})(\varepsilon\eta - \mu)\int_{t_d}^{T_1} (\mathrm{e}^{\alpha t^2})(\mathrm{e}^{-\eta(t-t_d)})\mathrm{d}t\Bigg\}\Bigg/\Bigg\{-2(\varepsilon\eta - \mu)\int_{t_d}^{T_1} \mathrm{e}^{-2\eta(t-t_d)}\mathrm{d}t\ + 2\mu\Bigg\}. \end{aligned} \tag{B.2}$$

At point $s_0 = s_0^{1*}$

$$\frac{\partial^2 TP_1}{\partial s_0^2}|_{s_0=s_0^{1*}} = (1+A)^\lambda\left\{-2\mu t_d + 2(\varepsilon\eta - \mu)\int_{t_d}^{T_1} \mathrm{e}^{-2\eta(t-t_d)}\mathrm{d}t\right\}. \tag{B.3}$$

Since we have defined $(\varepsilon\eta - \mu) < 0$, $\frac{\partial^2 TP_1}{\partial s_0^2}|_{s_0 = s_0^{1*}} < 0$. Thus s_0^{1*} is the global optimum solution for fixed A and $T_1 \in [t_d, \infty)$.

Appendix C. Proof of Lemma 5.3

The first order partial derivative of $TP_2(s(t),\, A,\, T)$ with respect to T is given by

$$\frac{\partial TP_2}{\partial T} = -\frac{h}{2}(D_0 + \sum_{s\in\Omega} \gamma_s\, p_s - \mu s_0)(1+A)^\lambda + \frac{B\cdot A + O}{T^2}\cdot \tag{C.1}$$

Motivated by equation (C.1) the auxiliary function $S(T)$, $T \in (0, t_d]$ is defined as:

$$S(T) = -\frac{h}{2}(D_0 + \sum_{s\in\Omega} \gamma_s\, p_s - \mu s_0)(1+A)^\lambda T^2 + B\cdot A + O. \tag{C.2}$$

From equation (46) it follows that:

$$S(0) = O + B\cdot A > 0 \tag{C.3}$$

$$S(t_d) = O + B\cdot A - \frac{h}{2}\left(D_0 + \sum_{s\in\Omega} \gamma_s\, p_s - \mu s_0\right)(1+A)^\lambda t_d^2 = \Delta(A, s(t)). \tag{C.4}$$

The first order derivative of $S(T)$ with respect to$T \in (0, t_d]$ gives:

$$\frac{\mathrm{d}S(T)}{\mathrm{d}T} = -h(D_0 + \sum_{s\in\Omega} \gamma_s\, p_s - \mu s_0)(1+A)^\lambda T < 0. \tag{C.5}$$

Then $S(T)$ is a strictly decreasing function with respect to $T \in (0, t_d]$.

Part (a). If $\Delta \leqslant 0$, Then $S(T)$ is strictly decreasing function from $S(0) > 0$ to $S(t_d) \leqslant 0$. Thus there is a unique value of T (say T_2) where $S(T_2) = 0$.

From equations (C.1) and (C.2) we have

$$\frac{\partial TP_2(s(t),\, A,\, T)}{\partial T} = \frac{S(T)}{T^2}\cdot \tag{C.6}$$

At point we $T = T_2$ have:

$$\frac{\partial^2 TP_2(s(t),\, A,\, T)}{\partial T^2}|_{T=T_2} = -2\frac{B\cdot A + O}{T_2^3} < 0. \tag{C.7}$$

Therefore, $T_2 \in (0, t_d]$ is the global maximum solution of $TP_2(s(t), A, T)$.

Part (b). If $\Delta > 0$, then $S(t_d) > 0$. Then from equation (C.3), $TP_2(s(t),\, A,\, T)$ is a strictly increasing function of $T \in (0, t_d]$, therefore the value of $T \in (0, t_d]$ and reaches its maximum value at $T = t_d$.

Appendix D. Proof of Lemma 5.4

The first order derivative of $TP_2(s(t), A, T)$ with respect to s_0 gives

$$\frac{\partial TP_2}{\partial s_0} = \frac{(1+A)^\lambda}{T_2}[D_0 + \sum_{s\in\Omega} \gamma_s\, p_s - 2\mu s_0 + \frac{h}{2}\mu T_2 + c\mu]. \tag{D.1}$$

By solving $\frac{\partial TP_2}{\partial s_0}=0$, s_0^{2*} yields:

$$s_0^{2*}=\frac{D_0+\sum\limits_{s\in\Omega}\gamma_s\,p_s+\frac{h}{2}\mu T_2+c\mu}{2\mu}\cdot \tag{D.2}$$

At point $s_0=s_0^{2*}$ we have:

$$\frac{\partial^2 TP_2}{\partial s_0^2}|_{s_0=s_0^{2*}}=-2\mu(1+A)^\lambda<0. \tag{D.3}$$

Thus s_0^{2*} is the global optimum solution for fixed A and $T_2\in(0,t_d]$.

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