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# ON MATCHING EXTENDABILITY OF LEXICOGRAPHIC PRODUCTS*

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**Abstract.** A graph $G$ of even order is *$\ell$-extendable* if it is of order at least $2\ell+2$, contains a matching of size $\ell$, and if every such matching is contained in a perfect matching of $G$. In this paper, we study the extendability of lexicographic products of graphs. We characterize graphs $G$ and $H$ such that their lexicographic product is not 1-extendable. We also provide several conditions on the graphs $G$ and $H$ under which their lexicographic product is 2-extendable.

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## 1. Introductory remarks

A *matching* in a graph $G$ is a set of pairwise non-adjacent edges and a matching $M$ is a *perfect* matching of $G$ if $V(M) = V(G)$. The *size of a matching* is given by the number of edges it contains. If every matching of size $\ell$ can be extended to a perfect matching in $G$, then $G$ is called *$\ell$-extendable*, where $|V(G)| \geq 2\ell+2$. In particular, 0-extendable means there exists a perfect matching in $G$. Note that, although extendability is originally defined for connected graphs in [8], we will work in a more general setting of not necessarily connected graphs.

The concept of $\ell$-extendability is widely analyzed in the literature. In 1980, Plummer [8] studied the properties of $\ell$-extendable graphs and showed that every 2-extendable graph is either bipartite or a brick. Necessary and sufficient conditions for a graph to be 1-extendable were given by Little, Grant and Holton [6]. There are also many other results related to $n$-extendability of graphs. However, there are few works on matching extension in different types of graph products. Győri and Plummer [4] showed that the Cartesian product of a $k$-extendable graph and an $\ell$-extendable graph is $(k+\ell+1)$-extendable. Győri and Imrich [3] showed that the strong product of a $k$-extendable graph and an $\ell$-extendable graph is $[(k+1)(\ell+1)/2]$-extendable. Liu and Yu [7] studied matching extendability from a prescribed vertex set in lexicographic products. Bai, Wu, Yang and Yu [1] studied

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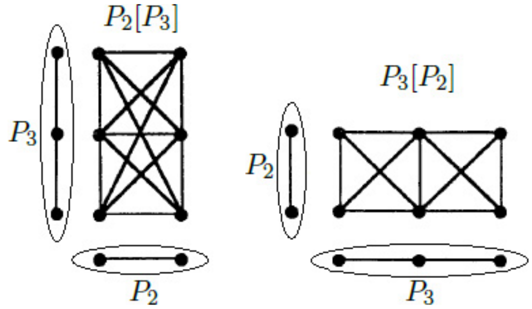


FIGURE 1. The lexicographic product of graphs is not commutative.

the lexicographic product of extendable graphs. In particular, they showed that the lexicographic product of a $k$-extendable graph and an $\ell$-extendable graph is $(k+1)(\ell+1)$-extendable. More results on graph products can be found in [5].

In this paper, we provide, to the best of our knowledge, the first results on the specific conditions making the lexicographic product of graphs 1 or 2-extendable. Furthermore, we also provide the characterization of 1-extendability in lexicographic products of graphs, which, to the best of our knowledge, does not exist in the literature.

The rest of this paper is organized as follows. We first give some definitions and preliminary results in Section 2. Section 3 is devoted to the characterization of 1-extendability in lexicographic products of graphs. In Section 4 we provide some conditions on the graphs $G$ and $H$ such that their lexicographic product is 2-extendable. Finally, in Section 5, we study the extendability of two special edges in the lexicographic product of an arbitrary graph with the empty graph when their lexicographic product is not 2-extendable in general.

## 2. Preliminaries

Throughout this paper, graphs are assumed to be finite and simple. For a graph $G$, we denote its vertex set by $V = V(G)$ and its edge set by $E = E(G)$. An *independent set* $I \subseteq V(G)$ is a set of pairwise non-adjacent vertices. A graph with $E = \emptyset$ (*i.e.* the graph is an independent set) will be called *empty graph*. For a set $S \subseteq V$ we let $G - S$ be the subgraph of $G$ induced by the set $V \setminus S$. For a subset $S \subseteq V$, we denote by $G_S$ the subgraph of $G$ induced by the set of vertices $S$. Connected components of $G$ will simply be called *components* of $G$. A component $C$ of $G$ is called *even* (*odd*, respectively), if the order of $C$ (*i.e.* the number of vertices in $C$) is even (odd, respectively). We denote by $o(G)$ the number of odd components of $G$. A (induced) path on $k$ vertices is denoted by $P_k$.

The *lexicographic product* $G[H]$ of two graphs $G$ and $H$ has vertex set $V(G) \times V(H)$ and two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent whenever $u_1v_1 \in E(G)$, or $u_1 = v_1$ and $u_2v_2 \in E(H)$. Note that the lexicographic product of two graphs may not be commutative. For example, $P_2[P_3]$ is not isomorphic to $P_3[P_2]$, as illustrated in Figure 1.

We can now state the following result of Tutte:

**Theorem 2.1** (Tutte [9])**.** *A graph $G$ has a perfect matching if and only if for every subset $S \subseteq V(G)$ we have $o(G - S) \leq |S|$.*

We will also need the following results.

**Lemma 2.2** (Little *et al.* [6], see also [10], Thm. 5.1.2)**.**
*Let $G$ be a 0-extendable graph,* i.e.*, a graph that has a perfect matching. Then $G$ is not 1-extendable if and only if there exists a subset $S \subseteq V(G)$ such that $S$ is not an independent set and $o(G - S) = |S|$.*

**Theorem 2.3** (Bai *et al.* [1]). *Let $G$ be $m$-extendable and $H$ be $n$-extendable. Then their lexicographic product $G[H]$ is $(m+1)(n+1)$-extendable.*

**Theorem 2.4** (Chan *et al.* [2]). *Let $\Gamma = G(S)$ be a Cayley graph over the Abelian group $G$ of even order. Then $\Gamma$ is 2-extendable if and only if it is not isomorphic to any of the following graphs.*

(i) $\mathbb{Z}_{2n}(1, 2n-1), n \geq 3$;
(ii) $\mathbb{Z}_{2n}(1, 2, 2n-1, 2n-2), n \geq 3$;
(iii) $\mathbb{Z}_{4n}(1, 4n-1, 2n), n \geq 2$;
(iv) $\mathbb{Z}_{4n+2}(2, 4n, 2n+1), n \geq 1$; *and*
(v) $\mathbb{Z}_{4n+2}(1, 4n+1, 2n, 2n+2), n \geq 1$.

## 3. 1-extendability of the lexicographic products

Let $G$ be a connected graph and let $H$ be an arbitrary graph. In this section we assume that $G$ is 0-extendable, and therefore the orders of both $G$ and $G[H]$ are even. It is an easy exercise to show that in this case $G[H]$ is also 0-extendable.

The main theorem of this section (Thm. 3.1) characterizes graphs $G$ and $H$ such that $G[H]$ is not 1-extendable. Its proof immediately follows from Theorem 3.2, Theorems 3.7 and 3.8.

**Theorem 3.1.** *Let $G$ be a 0-extendable graph and let $H$ be an arbitrary graph. Then $G[H]$ is not 1-extendable if and only if both of the following* (i) *and* (ii) *hold.*

(i) *$H$ is an empty graph.*
(ii) *There exists $S \subseteq V(G)$ such that $G - S$ has $|S|$ singleton components, and either $S$ is not an independent set, or $S$ is an independent set of $G$ and $G - S$ has at least one even component.*

This section is organized as follows. In Section 3.1, we show that $G[H]$ is 1-extendable if $H$ is non-empty or $G$ is 1-extendable (see Thm. 3.2). This implies that, while characterizing lexicographic products which are not 1-extendable, we can assume that $H$ is empty and $G$ is not 1-extendable. Consequently, we focus on lexicographic products with empty graphs in Section 3.2 where we show that it is enough to restrict our study to the lexicographic products with empty graphs on 2 vertices (see Thm. 3.7). Finally, using this restriction, we characterize lexicographic products with empty graphs which are not 1-extendable in Section 3.3 (see Thm. 3.8).

### 3.1. 1-extendability – case when $H$ is non-empty or $G$ is 1-extendable

We refer to a graph as a non-empty graph if it contains at least one edge, *i.e.*, the graph itself is not an independent set.

**Theorem 3.2.** *Let $G$ be a 0-extendable graph, then $G[H]$ is 1-extendable if $H$ is a non-empty graph or if $G$ is also 1-extendable.*

*Proof.* Assume first that $H$ is non-empty and pick an edge $e$ of $G[H]$.

**Case 1.** $e = (x, a)(x, b)$ for some $x \in V(G)$ and some adjacent vertices $a, b \in V(H)$. Let $M$ be an arbitrary perfect matching of $G$ and let $y \in V(G)$ be such that $xy \in M$. Now a perfect matching of $G[H]$ containing $e$ is:

$$\{(x,a)(x,b), (y,a)(y,b)\} \cup \bigcup_{c \in V(H) \setminus \{a,b\}} \{(x,c)(y,c)\} \cup \bigcup_{\substack{zw \in M \setminus \{xy\} \\ c \in V(H)}} \{(z,c)(w,c)\}.$$

**Case 2.** $e = (x,a)(y,b)$ for some adjacent vertices $x, y \in V(G)$ and some vertices $a, b \in V(H)$. If $xy$ is contained in some perfect matching $M$ of $G$, then a perfect matching of $G[H]$ containing $e$ is:

$$\{(x,a)(y,b), (x,b)(y,a)\} \cup \bigcup_{c \in V(H)\setminus\{a,b\}} \{(x,c)(y,c)\} \cup \bigcup_{\substack{zw \in M\setminus\{xy\} \\ c \in V(H)}} \{(z,c)(w,c)\}.$$

Assume now that $xy$ is not contained in any perfect matching of $G$. Pick an arbitrary perfect matching $M$ of $G$ and let $z, v \in V(G)$ be such that $xz, yv \in M$. Pick also an edge $cd \in E(H)$.

Let $M^1$ be an arbitrary perfect matching of the subgraph of $G[H]$ induced by the set of vertices

$$(\{z\} \times (V(H) \setminus \{c,d\})) \cup (\{x\} \times (V(H) \setminus \{a,b\})).$$

Note that such a perfect matching exists since we have all possible edges between the sets $\{z\} \times (V(H) \setminus \{c,d\})$ and $\{x\} \times (V(H) \setminus \{a,b\})$. Similarly, let $M^2$ be an arbitrary perfect matching of the subgraph of $G[H]$ induced by the set of vertices

$$(\{v\} \times (V(H) \setminus \{c,d\})) \cup (\{y\} \times (V(H) \setminus \{a,b\})).$$

Now a perfect matching of $G[H]$ containing edge $e$ is:

$$\{(x,a)(y,b), (x,b)(y,a), (z,c)(z,d), (v,c)(v,d)\} \cup$$

$$M^1 \cup M^2 \cup \bigcup_{\substack{wu \in M\setminus\{xz,yv\} \\ h \in V(H)}} \{(w,h)(u,h)\}.$$

This finishes the proof of Case 2, thus the case where $H$ is non-empty. It remains to show that $G[H]$ is 1-extendable if $G$ is 1-extendable (but $H$ is empty). Let $e = (x,a)(y,b)$ be the edge we want to extend into a perfect matching of $G[H]$. Furthermore, let $M$ be a perfect matching of $G$ containing the edge $xy$.

If $a = b$, then the following matching is a perfect matching of $G[H]$ containing $e$:

$$\bigcup_{\substack{wu \in M \\ h \in V(H)}} \{(w,h)(u,h)\}.$$

Otherwise (if $a \neq b$), the following matching is a perfect matching of $G[H]$ containing $e$:

$$\{e, (x,b)(y,a)\} \cup \bigcup_{\substack{wu \in M\setminus\{xy\} \\ h \in \{a,b\}}} \{(w,h)(u,h)\} \cup \bigcup_{\substack{wu \in M \\ h \in V(H)\setminus\{a,b\}}} \{(w,h)(u,h)\}. \qquad \square$$

### 3.2. Lexicographic products with empty graphs

We will assume from now on that $H$ is the empty graph on $n$ vertices, where $n \geq 2$. We will denote this graph by $E_n$. We identify the vertex set of $E_n$ by $\{0, 1, 2, \ldots, n-1\}$. For an arbitrary graph $G$ and for $v \in V(G)$ we abbreviate $v_i = (v, i) \in V(G[H])$ $\forall i \in \{0, 1, 2, \ldots, n-1\}$. Assume that $G$ is 0-extendable and $n \geq 3$. In this subsection, we show that $G[E_2]$ is 1-extendable if and only if $G[E_n]$ is 1-extendable. We will need the following definition.

**Definition 3.3.** Let $G$ be an arbitrary graph and let $S \subseteq V(G[E_2])$.

(i) $S$ is called *rectangular* if the following holds for all $v \in V(G)$ and $i \in \{0, 1\}$: $v_i \in S$ if and only if $v_{i+1} \in S$. Here, the addition in subscripts is modulo 2.

(ii) $S$ is called *almost rectangular* if there exists $v_i \in S$ such that $S \setminus \{v_i\}$ is rectangular.

**Lemma 3.4.** *Let $G$ be an arbitrary graph and $S \subseteq V(G[E_2])$. Then the following statements hold.*

(i) *If $S$ is rectangular, then the only odd components of $G[E_2] - S$ are singletons.*
(ii) *If $S$ is almost rectangular, then there is at most one non-singleton odd component of $G[E_2] - S$.*

*Proof.*

(i) Let $C$ be a component of $G[E_2] - S$ which is not a singleton. Pick $x_i \in C$ ($x \in V(G), i \in V(E_2)$). We will show that also $x_{i+1} \in C$. Observe that since $S$ is rectangular, we have that $x_{i+1} \notin S$. Since $C$ is not a singleton, there is a vertex $y_j \in C \setminus \{x_i\}$, which is adjacent to $x_i$. But then $x$ is adjacent to $y$ in $G$, and so $y_j$ is adjacent to $x_{i+1}$ in $G[E_2]$. This shows that $x_{i+1} \in C$ and we are done.
(ii) Let $v_i \in V(G[E_2])$ be such that $S \setminus \{v_i\}$ is rectangular, and let $C$ be a component of $G[E_2] - S$ which is not a singleton. If $v_{i+1} \notin C$, then, similarly as in (i) above, we find that $x_i \in C$ if and only if $x_{i+1} \in C$, and so $C$ is of even order. Therefore, the only possible odd component, which is not a singleton, is the component containing $v_{i+1}$. □

**Lemma 3.5.** *Let $G$ be an arbitrary graph and $S \subseteq V(G[E_2])$. Then there exists a rectangular $S_1 \subseteq V(G[E_2])$, such that $o(G[E_2] - S) - |S| \leq o(G[E_2] - S_1) - |S_1|$.*

*Proof.* Assume that $S$ is not rectangular and pick $v_i \in S$, such that $v_{i+1} \notin S$. Let $C$ be the component of $G[E_2] - S$, containing $v_{i+1}$, and let $C_1, \dots, C_t$ be all other components of $G[E_2] - S$.

**Case 1.** $C$ is even. In this case let $S' = S \cup \{v_{i+1}\}$. Note that the components of $G[E_2] - S'$ are $C_1, \dots, C_t$ and the components contained in $C \setminus \{v_{i+1}\}$. Therefore, we increase the cardinality of $S$ by 1, but we also increase the number of odd components at least by 1 (namely, $C \setminus \{v_{i+1}\}$ contains at least one new odd component). It follows that $|S| - o(G[E_2] - S) \geq |S'| - o(G[E_2] - S')$.

**Case 2.** $C$ is odd. In this case we let $S' = S \setminus \{v_i\}$. If $C$ is a singleton, then note that the components of $G[E_2] - S'$ are $C, \{v_i\}, C_1, \dots, C_t$. We decrease the cardinality of $S$ by 1, and increase the number of odd components by 1. Therefore, $|S| - o(G[E_2] - S) - 2 = |S'| - o(G[E_2] - S')$. If $C$ is not a singleton, then note that the components of $G[E_2] - S'$ are $C \cup \{v_i\}, C_1, \dots, C_t$. We decrease the cardinality of $S$ by 1, but also decrease the number of odd components by 1. Therefore, $|S| - o(G[E_2] - S) = |S'| - o(G[E_2] - S')$.

Observe that after the above steps either $v_i, v_{i+1} \in S'$ or $v_i, v_{i+1} \notin S'$. If $S'$ is rectangular then we set $S_1 = S'$ and we are done. If $S'$ is not rectangular, then we repeat the above steps. After finitely many steps, we will end up with a rectangular set $S_1$ such that $o(G[E_2] - S) - |S| \leq o(G[E_2] - S_1) - |S_1|$. □

**Lemma 3.6.** *Let $G$ be an arbitrary graph and $S \subseteq V(G[E_2])$ such that $S$ is not an independent set. Then there exists either a rectangular or an almost rectangular set $S_1 \subseteq V(G[E_2])$, such that $S_1$ is not an independent set and $o(G[E_2] - S) - |S| \leq o(G[E_2] - S_1) - |S_1|$.*

*Proof.* Pick an edge $x_j y_\ell$ in $G_S$. If $x_{j+1}, y_{\ell+1} \in S$, then the same procedure as in the proof of Lemma 3.5 will yield a rectangular set $S_1$ such that $\{x_j, y_\ell\} \in S_1$ and $o(G[E_2] - S) - |S| \leq o(G[E_2] - S_1) - |S_1|$. If $x_{j+1} \notin S$ and $y_{\ell+1} \in S$, then we again apply the procedure in the proof of Lemma 3.5, but with $v_i \neq x_j$. This will yield an almost rectangular set $S_1$ such that $\{x_j, y_\ell\} \in S_1$, $x_{j+1} \notin S_1$, and $o(G[E_2] - S) - |S| \leq o(G[E_2] - S_1) - |S_1|$. If $x_{j+1} \in S$ and $y_{\ell+1} \notin S$, then we similarly get an almost rectangular set $S_1$ such that $\{x_j, y_\ell\} \in S_1$, $y_{\ell+1} \notin S_1$ and $o(G[E_2] - S) - |S| \leq o(G[E_2] - S_1) - |S_1|$.

Finally, assume that $x_{j+1} \notin S$ and $y_{\ell+1} \notin S$. We apply the procedure in the proof of Lemma 3.5 with $v_i \neq x_j$ and $v_i \neq y_\ell$. This will yield a set $S'$ such that

- $x_{j+1}, y_{\ell+1} \notin S'$,

- $S' \setminus \{x_j, y_\ell\}$ is rectangular,
- $o(G[E_2] - S) - |S| \leq o(G[E_2] - S') - |S'|$.

Let $C$ be a component of $G[E_2] - S'$ containing $x_{j+1}$, and let $C_1, \ldots, C_n$ be all other components of $G[E_2] - S'$. As $x_{j+1}$ and $y_{\ell+1}$ are adjacent, we have that $y_{\ell+1} \in C$. Since $S' \setminus \{x_j, y_\ell\}$ is rectangular, this implies that $C$ is even. Let $S_1 = S' \cup \{x_{j+1}\}$ and note that $S_1$ is almost rectangular and is not an independent set. Note also that the components of $G[E_2] - S_1$ are $C_1, \ldots, C_n$ and components contained in $C \setminus \{x_{j+1}\}$. Therefore, we increase the cardinality of $S'$ by 1, but we also increase the number of odd components by 1 (namely, $C \setminus \{x_{j+1}\}$ contains at least one new odd component). It follows that $o(G[E_2] - S') - |S'| \leq o(G[E_2] - S_1) - |S_1|$. □

**Theorem 3.7.** *Let $G$ be a 0-extendable graph and let $n \geq 3$ be an integer. Then $G[E_2]$ is 1-extendable if and only if $G[E_n]$ is 1-extendable.*

*Proof.* Assume that $G[E_2]$ is 1-extendable and pick an edge $e = x_i y_j$ of $G[E_n]$. If $i = j$ then let $\ell = i+1$. If $i \neq j$ then let $\ell = j$. Note that the subgraph $G'$ of $G[E_n]$ induced by the vertices $\{z_i, z_\ell \mid z \in V(G)\}$ is isomorphic to $G[E_2]$. As $G[E_2]$ is 1-extendable, there is a perfect matching $M^{G'}$ of the subgraph $G'$, which contains edge $e$. Pick an arbitrary perfect matching $M$ of $G$. Now a perfect matching of $G[E_n]$ containing $e$ is:

$$M^{G'} \cup \{z_k w_k \mid zw \in M, k \in \{0, 1, \ldots, n-1\} \setminus \{i, \ell\}\}.$$

Assume now that $G[E_2]$ is not 1-extendable. By Lemma 2.2 there exists a subset $S \subseteq V(G[E_2])$ such that $S$ is not an independent set and $|S| = o(G[E_2] - S)$. By Lemma 3.6, there exists either a rectangular or an almost rectangular set $S_1 \subseteq V(G[E_2])$ such that $S_1$ is not an independent set and $0 = o(G[E_2] - S) - |S| \leq o(G[E_2] - S_1) - |S_1|$. As $G[E_2]$ is 0-extendable, we have, by Theorem 2.1, that $o(G[E_2] - S_1) = |S_1|$.

**Case 1.** $S_1$ is rectangular. Recall that by Lemma 3.4(i), all odd components of $G[E_2] - S_1$ are singletons. Let $v^1, v^2, \ldots, v^m \in V(G)$ be such that $S_1 = \{v_i^j \mid j \in \{1, \ldots, m\}, i \in \{0, 1\}\}$. Furthermore, let $u^1, u^2, \ldots, u^m \in V(G)$ be such that $u_i^j$ (where $j \in \{1, \ldots, m\}$ and $i \in \{0, 1\}$) are the singleton components of $G[E_2] - S_1$. Now define set $R \subseteq V(G[E_n])$:

$$R = \{v_i^j \mid j \in \{1, \ldots, m\}, i \in \{0, 1, \ldots, n-1\}\}.$$

Note that $R$ is not an independent set and that $u_i^j$ (where $j \in \{1, \ldots, m\}$ and $i \in \{0, 1, \ldots, n-1\}$) are the singleton components of $G[E_n]$. By Lemma 2.2, $G[E_n]$ is not 1-extendable.

**Case 2.** $S_1$ is almost rectangular. Recall that by Lemma 3.4(ii), there is at most one odd component of $G[E_2] - S_1$ that is not a singleton. In fact, we will show that there must be one odd component that is not a singleton. Let $v^0, v^1, v^2, \ldots, v^m \in V(G)$ be such that $S_1 = \{v_\ell^0\} \cup \{v_i^j \mid j \in \{1, \ldots, m\}, i \in \{0, 1\}\}$. Without loss of generality we can assume that $\ell = 0$. Assume for a moment that $\{v^1, \ldots, v^m\}$ is not an independent set of the graph $G$ and let $S_2 = S_1 \setminus \{v_0^0\}$. Note that $S_2$ is rectangular and not independent. Furthermore, $o(G[E_2] - S_2) = o(G[E_2] - S_1) - 1$, as the odd component of $G[E_2] - S_1$ containing $v_1^0$ will be an even component in $G[E_2] - S_2$. Therefore, $0 \leq o(G[E_2] - S_1) - |S_1| = o(G[E_2] - S_2) - |S_2|$, and we could proceed as in Case 1 above, with set $S_2$ instead of set $S_1$. This shows that we can assume that $\{v^1, \ldots, v^m\}$ is an independent set of graph $G$. However, as $S_1$ is not an independent set, there is an edge between $v^0$ and $v^i$ in graph $G$ (for some $i$ such that $1 \leq i \leq m$). Without loss of generality we can assume that $i = 1$. Let $C$ be the component of $G[E_2] - S_1$ containing $v_1^0$. We know from the proof of Lemma 3.4(ii) that $C$ is the only possible odd component that is not a singleton. Let $u^1, u^2, \ldots, u^m \in V(G)$ be such that $u_i^j$ (where $j \in \{1, \ldots, m\}$ and $i \in \{0, 1\}$) are the singleton components of $G[E_2] - S_1$ that are different from $C$. If $C$ is singleton, then let $S_2 = S_1 \setminus \{v_0^0\}$. Observe that $u_i^j$ (where $j \in \{1, \ldots, m\}$ and $i \in \{0, 1\}$) and $v_0^0, v_1^0$ are singleton components of $G[E_2] - S_2$. By Theorem 2.1, $G[E_2]$ is not 0-extendable, a contradiction. Therefore, $C$ is not a singleton.

Let $w^1, w^2, \ldots, w^\ell \in V(G)$ be such that $C = \{v_1^0\} \cup \{w_i^j \mid j \in \{1, \ldots, \ell\}, i \in \{0,1\}\}$. We will now show that $\ell$ is odd. Assume that $\ell$ is even. Observe that since $v_1^0 \in C$, $v_1^0$ is adjacent only to the vertices in $C \cup S_1$. Therefore, $v_0^0$ is also only adjacent to the vertices in $C \cup S_1$. This shows that $\{u^0\}, \ldots, \{u^m\}, \{v^0, w^1, \ldots, w^\ell\}$ are odd components of $G - \{v^1, v^2, \ldots, v^m\}$. This result and Theorem 2.1 together imply that $G$ is not 0-extendable, a contradiction. This shows that $\ell$ is odd.

Now define set $R \subseteq V(G[E_n])$:

$$R = \{v_0^0\} \cup \{v_i^j \mid j \in \{1, \ldots, m\}, i \in \{0, 1, \ldots, n-1\}\}.$$

Note that $R$ is not an independent set and $u_i^j$ ($j \in \{1, \ldots, m\}$, $i \in \{0, 1, \ldots, n-1\}$) are the singleton components of $G[E_n] - R$. Moreover, component containing $v_1^0$ is

$$C_1 = \{w_i^j \mid j \in \{1, 2, \ldots, \ell\}, i \in \{0, 1, \ldots, n-1\}\} \cup \{v_i^0 \mid i \in \{1, \ldots, n-1\}\}.$$

As $\ell$ is odd, $C_1$ is also an odd component of $G[E_n] - R$. By Lemma 2.2, $G[E_n]$ is not 1-extendable. □

### 3.3. 1-extendability - case when $H$ is empty

In this section, we will characterize 0-extendable graphs $G$ such that lexicographic product $G[E_n]$ is not 1-extendable, where $E_n$ is the empty graph on $n$ vertices. By Theorem 3.7 we can assume that $n = 2$, and by Theorem 3.2 we can assume that $G$ is not 1-extendable.

**Theorem 3.8.** *Let $G$ be a 0-extendable graph. Then $G[E_2]$ is not 1-extendable if and only if there exists $S \subseteq V(G)$, such that $G - S$ has $|S|$ singleton components, and either $S$ is not an independent set, or $S$ is an independent set of $G$ and $G - S$ has at least one even component.*

*Proof.* Assume first that there exists $S \subseteq V(G)$ such that $G - S$ has $|S|$ singleton components. Let $m = |S|$ and let $u^1, u^2, \ldots, u^m$ be these singleton components. Define $R = \{s_i \mid s \in S, i \in \{0,1\}\} \subseteq V(G[E_2])$. Observe that $u_i^j$, $j \in \{1, \ldots, m\}, i \in \{0,1\}$ are singleton components of $G[E_2] - R$. If $S$ is not an independent set, then $R$ is also not an independent set; therefore, $G[E_2]$ is not 1-extendable by Lemma 2.2. If $S$ is an independent set of $G$, then let $C$ be an even component of $G - S$. Pick $x \in C$ which has a neighbor in $S$. Let $R_1 = R \cup \{x_0\} \subset V(G[E_2])$. Note that $u_i^j$, $j \in \{1, \ldots, m\}, i \in \{0,1\}$ are singleton components of $G[E_2] - R_1$, and that $\{w_0, w_1 \mid w \in C\} \setminus \{x_0\}$ is an odd component of $G[E_2] - R_1$. Again, by Lemma 2.2, $G[E_2]$ is not 1-extendable.

Assume next that $G[E_2]$ is not 1-extendable. By Lemma 2.2 there exists a subset $R \subseteq V(G[E_2])$ such that $R$ is not an independent set and $|R| = o(G[E_2] - R)$. By Lemma 3.6, there exists either a rectangular or an almost rectangular set $R_1 \subseteq V(G[E_2])$ such that $R_1$ is not an independent set and $0 = o(G[E_2] - R) - |R| \leq o(G[E_2] - R_1) - |R_1|$. As $G[E_2]$ is 0-extendable, we have, by Theorem 2.1, that $o(G[E_2] - R_1) = |R_1|$.

**Case 1.** $R_1$ is rectangular. Recall that, by Lemma 3.4 (i), all odd components of $G[E_2] - R_1$ are singletons. Let $v^1, v^2, \ldots, v^m \in V(G)$ be such that $R_1 = \{v_i^j \mid j \in \{1, \ldots, m\}, i \in \{0,1\}\}$. Furthermore, let $u^1, u^2, \ldots, u^m \in V(G)$ be such that $u_i^j$ ($j \in \{1, \ldots, m\}, i \in \{0,1\}\}$) are the singleton components of $G[E_2] - R_1$. Now let $S = \{v^1, v^2, \ldots, v^m\} \subseteq V(G)$. Note that $S$ is not an independent set and $u^1, u^2, \ldots, u^m$ are singleton components of $G - S$. Since $G$ is 0-extendable, $G - S$ has no other odd components.

**Case 2.** $R_1$ is almost rectangular. Let $v^0, v^1, v^2, \ldots, v^m \in V(G)$ be such that $R_1 = \{v_\ell^0\} \cup \{v_i^j \mid j \in \{1, \ldots, m\}, i \in \{0,1\}\}$. Without loss of generality, we can assume that $\ell = 0$. Note also that, by using the same arguments as in the proof of Theorem 3.7 (see Case 2), we can assume that there is an edge between $v^0$ and $v^i$ in $G$, (for some $i$ such that $1 \leq i \leq m$), and that $\{v^1, v^2, \ldots, v^m\}$ is an independent set of $G$. Otherwise we define $R_2 = R_1 \setminus \{v_0^0\}$ and proceed as in Case 1 above. Without loss of generality we can also assume that $i = 1$.

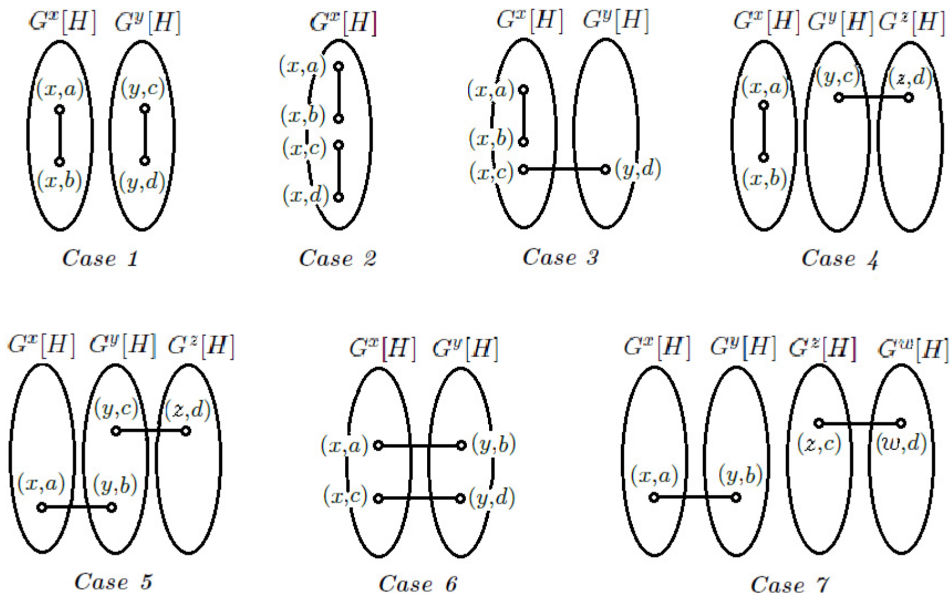


FIGURE 2. All possible cases for two non-adjacent edges in $G[H]$ with respect to the subgraphs their endpoints belong to.

Let $C$ be a component of $G[E_2] - S_1$ containing $v_1^0$. As in the proof of Theorem 3.7 (Case 2), we show that $C$ is an odd component of $G[E_2] - R_1$ that is not a singleton. Let $u^1, u^2, \ldots, u^m \in V(G)$ be such that $u_i^j$ ($j \in \{1, \ldots, m\}, i \in \{0, 1\}$) are the singleton components of $G[E_2] - R_1$.

Let $w^1, w^2, \ldots, w^\ell \in V(G)$ be such that $C = \{v_1^0\} \cup \{w_i^j \mid j \in \{1, \ldots, \ell\}, i \in \{0, 1\}\}$. As in the proof of Theorem 3.7 (Case 2), we show that $\ell$ is odd. Now let $S = \{v^1, v^2, \ldots, v^m\}$. Recall that $S$ is an independent set of $G$. Observe also that $u^1, u^2, \ldots, u^m$ are singleton components of $G - S$ and that $(w^1, w^2, \ldots, w^\ell, v^0)$ is an even component of $G - S$. □

## 4. 2-EXTENDABILITY OF LEXICOGRAPHIC PRODUCTS

In the following, we denote by $G^x[H]$ the subgraph of $G[H]$ induced by the set of vertices in $\{x\} \times V(H)$ for some $x \in V(G)$. Likewise, we refer by $G^{xy}[H]$ to the subgraph of $G[H]$ induced by the set of vertices in $(\{x\} \times V(H)) \cup (\{y\} \times V(H))$ for some $x, y \in V(G)$. Moreover, we denote by $M^k$ a perfect matching of $G^k[H]$ for some $k \in V(G)$.

**Observation 4.1.** *Let $x, y, z, w$ be four distinct vertices of $G$ and $a, b, c, d$ be four (not necessarily distinct) vertices of $H$ such that the two edges $e_1, e_2 \in E(G[H])$ whose endpoints are among these vertices are non-adjacent. Then $e_1$ and $e_2$ can belong to one of the following cases (illustrated in Fig. 2):*

Case 1. $e_1 = (x, a)(x, b)$ *and* $e_2 = (y, c)(y, d)$.
Case 2. $e_1 = (x, a)(x, b)$ *and* $e_2 = (x, c)(x, d)$.
Case 3. $e_1 = (x, a)(x, b)$ *and* $e_2 = (x, c)(y, d)$.
Case 4. $e_1 = (x, a)(x, b)$ *and* $e_2 = (y, c)(z, d)$.
Case 5. $e_1 = (x, a)(y, b)$ *and* $e_2 = (y, c)(z, d)$.
Case 6. $e_1 = (x, a)(y, b)$ *and* $e_2 = (x, c)(y, d)$.
Case 7. $e_1 = (x, a)(y, b)$ *and* $e_2 = (z, c)(w, d)$.

**Theorem 4.2.** *If $G$ is a connected graph and $H$ is a 1-extendable graph, then $G[H]$ is 2-extendable.*

*Proof.* Theorem 2.3 implies that if $G$ is 0-extendable and $H$ is 1-extendable, then $G[H]$ is 2-extendable. Hence, we will consider here the case where $G$ is not 0-extendable. The following 7 cases refer to Figure 2. In each case, we will exhibit a perfect matching $M^P$ of $G[H]$ containing the edges $e_1$ and $e_2$.

**Case 1.** Since $H$ is 1-extendable, $e_1$ can be extended to a perfect matching $M^x$ in $G^x[H]$. Likewise, $e_2$ can be extended to a perfect matching $M^y$ in $G^y[H]$. Then $M^P$ is:

$$M^x \cup M^y \cup \bigcup_{k \in V(G) \setminus \{x,y\}} M^k.$$

**Case 2.** Since $G$ is connected, $x$ has a neighbor $x' \in V(G)$, *i.e.*, $xx' \in E(G)$. Notice that $G^{xx'}[H]$ is isomorphic to $P_2[H]$, which is 2-extendable since $H$ is 1-extendable and $P_2$ is 0-extendable. Let $M^1$ be a perfect matching of $G^{xx'}[H]$ containing $e_1$ and $e_2$. Hence, $M^P$ is:

$$M^1 \cup \bigcup_{k \in V(G) \setminus \{x,x'\}} M^k.$$

**Case 3 and Case 6.** In both cases, $G^{xy}[H]$ is isomorphic to $P_2[H]$, which is 2-extendable. Let $M^1$ be a perfect matching of $G^{xy}[H]$ containing $e_1$ and $e_2$. Hence, $M^P$ is:

$$M^1 \cup \bigcup_{k \in V(G) \setminus \{x,y\}} M^k.$$

**Case 4.** Since $H$ is 1-extendable, $e_1$ can be extended to a perfect matching in $H$. Let $M^1$ be a perfect matching of $G^x[H]$ containing $e_1$. Besides, notice that $G^{yz}[H]$ is isomorphic to $P_2[H]$, which is 2-extendable (and hence 1-extendable). Let $M^2$ be a perfect matching of $G^{yz}[H]$ containing $e_2$. Then $M^P$ is:

$$M^1 \cup M^2 \cup \bigcup_{k \in V(G) \setminus \{x,y,z\}} M^k.$$

**Case 5.** Pick a vertex $a' \in V(H)$ such that $aa' \in E(H)$. Such a neighbor of $a$ always exists since $H$ is (also) 0-extendable ($H$ cannot have isolated vertices). Likewise, pick a vertex $d' \in V(H)$ such that $dd' \in E(H)$. Since $H$ is 1-extendable, $aa'$ can be extended to a perfect matching $M^1$ in $H$. Pick an edge $ef \in M^1$ such that $e, f \notin \{a, a'\}$. Such an edge always exists since $H$ is 1-extendable and therefore has at least 4 vertices. Let $M^{2'}$ be a perfect matching of $G^z[H]$ containing the edge $dd'$ and let $M^2 = M^{2'} \setminus \{dd'\}$. Since $H$ is 1-extendable, there exist $u, v \in \{V(H) \setminus \{b, c\}\}$. Let $G'_{xy}[H]$ be the subgraph of $G[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a, a', e, f\})) \cup (\{y\} \times (V(H) \setminus \{b, c, u, v\})).$$

Let $M^3$ be a perfect matching of $G'_{xy}[H]$. Then $M^P$ is:

$$\{e_1, e_2, (x, a')(y, u), (z, d')(y, v), (x, e)(x, f)\} \cup$$
$$M^2 \cup M^3 \cup \bigcup_{k \in V(G) \setminus \{x,y,z\}} M^k.$$

**Case 7.** Notice that both $G^{xy}[H]$ and $G^{zw}[H]$ are isomorphic to $P_2[H]$, which is 2-extendable. Let $M^1$ be a perfect matching of $G^{xy}[H]$ containing $e_1$ and $M^2$ be a perfect matching of $G^{zw}[H]$ containing $e_2$. Then $M^P$ is:

$$M^1 \cup M^2 \cup \bigcup_{k \in V(G) \setminus \{x,y,z,w\}} M^k.$$

This concludes the proof. $\square$

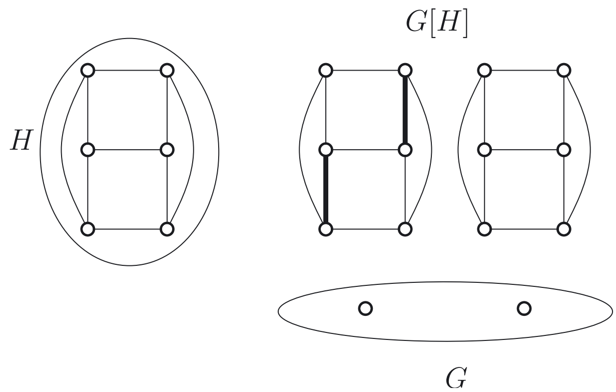


FIGURE 3. $H$ is 1-extendable but $G$ is not connected and the two bold edges can not be extended to a perfect matching of $G[H]$.

The example in Figure 3 shows that if the assumption that $G$ is connected is omitted, then Theorem 4.2 does not hold anymore.

We will now consider the case where $G$ is 1-extendable. The following proposition will be useful in our proof. To prove it, we will need the notion of a Cayley graph. Let $G$ denote a finite group with identity 1 and let $S$ denote an inverse-closed subset of $G \setminus \{1\}$. The *Cayley graph* $\mathrm{Cay}(G; S)$ of the group $G$ with respect to the *connection set* $S$ is the graph with vertex set $G$, in which $g \in G$ is adjacent with $h \in G$ if and only if $h = gs$ for some $s \in S$. Observe that $\mathrm{Cay}(G; S)$ is regular with valency $k = |S|$ and is connected if and only if $S$ generates $G$.

**Proposition 4.3.** *Let $G$ be a cycle of even length and let $H$ be an arbitrary graph of order at least 2. Pick independent edges $e_1 = (x, a)(y, b)$ and $e_2 = (z, c)(w, d)$ of $G[H]$ such that $x \neq y$ and $z \neq w$. Then there exists a perfect matching of $G[H]$ containing $e_1$ and $e_2$.*

*Proof.* Let $E_H$ be the empty graph on the vertices of the graph $H$. Observe that $G[E_H]$ is a Cayley graph of the Abelian group $\mathbb{Z}_\ell \times \mathbb{Z}_n$, where $\ell$ is the length of the cycle $G$ and $n = |V(H)|$. Therefore, $G[E_H]$ is 2-extendable by Theorem 2.4, and hence there is a perfect matching $M$ of $G[E_H]$ containing edges $e_1$ and $e_2$. Observe that $M$ is also a perfect matching of $G[H]$, which proves the result. □

**Theorem 4.4.** *If $G$ is a 1-extendable connected graph and $H$ is an arbitrary graph of order at least 2, then $G[H]$ is 2-extendable.*

*Proof.* The work in [1] implies that if $H$ is 0-extendable and $G$ is 1-extendable, then $G[H]$ is 2-extendable. Hence, we will consider here the case where $H$ is not 0-extendable. Consult Figure 2 for the 7 cases in the proof and let us denote by $M^P$ a perfect matching of $G[H]$ containing $e_1$ and $e_2$.

**Case 1.** Since $G$ is 1-extendable and connected, the minimum degree of $G$ is at least 2. Let $x' \neq y$ be a neighbor of $x$ and let $M^G$ be a perfect matching of $G$ that contains $xx'$. Furthermore, let $y'$ be a neighbor of $y$ such that $yy' \in M^G$ and $M^1$ be a perfect matching of the subgraph of $G^{xx'}[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a, b\})) \cup (\{x'\} \times (V(H) \setminus \{a, b\})).$$

Let $M^2$ be a perfect matching of the subgraph of $G^{yy'}[H]$ induced by the following set of vertices:

$$(\{y\} \times (V(H) \setminus \{c, d\})) \cup (\{y'\} \times (V(H) \setminus \{c, d\})).$$

Let $M^{G'} = M^G \setminus \{(x,x')(y,y')\}$. Then $M^P$ is:

$$\{e_1, e_2, (x',a)(x',b), (y',c)(y',d)\} \cup M^1 \cup M^2 \cup \bigcup_{\substack{zw \in M^{G'} \\ h \in V(H)}} \{(z,h)(w,h)\}.$$

**Case 2.** Pick a vertex $x'$ such that $xx' \in E(G)$. Let $M^1$ be a perfect matching of the subgraph of $G^{xx'}[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a,b,c,d\})) \cup (\{x'\} \times (V(H) \setminus \{a,b,c,d\})).$$

Since $G$ is 1-extendable, $xx'$ can be extended to a perfect matching $M^G$ in $G$. Then $M^P$ is:

$$\{e_1, e_2, (x',a)(x',b), (x',c)(x',d)\} \cup M^1 \cup \bigcup_{\substack{zw \in M^G \setminus \{xx'\} \\ h \in V(H)}} \{(z,h)(w,h)\}.$$

**Case 3.** We have two subcases here:
*Subcase* 3.1: $d \notin \{a,b\}$. Let $M^1$ be a perfect matching of the subgraph of $G^{xy}[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a,b,c\})) \cup (\{y\} \times (V(H) \setminus \{a,b,d\})).$$

Let $M^G$ be a perfect matching of $G$ containing $xy$ and $M^{G'} = M^G \setminus \{xy\}$. Then $M^P$ is:

$$\{e_1, e_2, (y,a)(y,b)\} \cup M^1 \cup \bigcup_{\substack{zw \in M^{G'} \\ h \in V(H)}} \{(z,h)(w,h)\}.$$

*Subcase* 3.2: $d \in \{a,b\}$. Assume without loss of generality that $d = a$. Let $M^G$ be a perfect matching of $G$ containing $xy \in E(G)$. Pick a neighbor $y'$ of $y$ such that $yy' \in M^G$. Such a neighbor always exists since $G$ is a 1-extendable connected graph and hence has minimum degree 2. Let $M^1$ be a perfect matching of the subgraph of $G^{xy}[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a,b,c\})) \cup (\{y\} \times (V(H) \setminus \{b,c,d\})).$$

Moreover, let $u \neq y$ be a neighbor of $y'$ such that $y'u \in M^G$ and let $M^2$ be a perfect matching of the subgraph of $G^{y'u}[H]$ induced by the following set of vertices:

$$(\{y'\} \times (V(H) \setminus \{a,b,c\})) \cup (\{u\} \times (V(H) \setminus \{a,b,c\})).$$

Let $M^{G'} = M^G \setminus \{xy, y'u\}$. Then $M^P$ is:

$$\{e_1, e_2, (y,c)(y',a), (y,b)(y',b), (y',c)(u,c), (u,a)(u,b)\} \cup$$
$$M^1 \cup M^2 \cup \bigcup_{\substack{zw \in M^{G'} \\ h \in V(H)}} \{(z,h)(w,h)\}.$$

**Case 4.** Let $M^G$ be a perfect matching of $G$ containing $yz \in E(G)$. Pick the vertex $x' \notin \{y,z\}$ such that $xx' \in M^G$. Let $M^1$ be a perfect matching of the subgraph of $G^{xx'}[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a,b\})) \cup (\{x'\} \times (V(H) \setminus \{a,b\})).$$

Let $M^2$ be a perfect matching of the subgraph of $G^{yz}[H]$ induced by the following set of vertices:

$$(\{y\} \times (V(H) \setminus \{c\})) \cup (\{z\} \times (V(H) \setminus \{d\})).$$

Let $M^{G'} = M^G \setminus \{xx', yz\}$. Then $M^P$ is:

$$\{e_1, e_2, (x', a)(x', b)\} \cup M^1 \cup M^2 \cup \bigcup_{\substack{zw \in M^{G'} \\ h \in V(H)}} \{(z, h)(w, h)\}.$$

**Case 5.** Let $M^{G_1}$ be a perfect matching of $G$ containing $xy$ and let $M^{G_2}$ be a perfect matching of $G$ containing $yz$. Since the symmetric difference of two perfect matchings consists of isolated vertices and even cycles, the edges $xy$ and $yz$ are contained in an even cycle $Z$ of $G$, in which every second edge is contained in $M^{G_2}$. By Proposition 4.3, there is a perfect matching $M^Z$ containing $e_1$ and $e_2$ in the subgraph of $G[H]$ induced by the set of vertices in $V(Z) \times V(H)$. Let $M^{G'} = M^G \setminus E(Z)$. Then $M^P$ is:

$$M^Z \cup \bigcup_{\substack{zw \in M^{G'} \\ h \in V(H)}} \{(z, h)(w, h)\}.$$

**Case 6.** Let $M^1$ be a perfect matching of the subgraph of $G^{xy}[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a, c\})) \cup (\{y\} \times (V(H) \setminus \{b, d\})).$$

Let $M^G$ be a perfect matching of $G$ containing $xy \in E(G)$ and let $M^{G'} = M^G \setminus \{xy\}$. Then $M^P$ is:

$$\{e_1, e_2\} \cup M^1 \cup \bigcup_{\substack{zw \in M^{G'} \\ h \in V(H)}} \{(z, h)(w, h)\}.$$

**Case 7.** Let $M^G$ be a perfect matching of $G$ containing $xy$. We have the following subcases:
*Subcase* 7.1: If $zw \in M^G$, then let $M^1$ be a perfect matching of the subgraph of $G^{xy}[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a\})) \cup (\{y\} \times (V(H) \setminus \{b\})).$$

Furthermore, let $M^2$ be a perfect matching of the subgraph of $G^{xy}[H]$ induced by the following set of vertices:

$$(\{z\} \times (V(H) \setminus \{c\})) \cup (\{w\} \times (V(H) \setminus \{d\})).$$

Moreover, let $M^{G'} = M^G \setminus \{(x, y)(z, w)\}$. Then $M^P$ is:

$$\{e_1, e_2\} \cup M^1 \cup M^2 \cup \bigcup_{\substack{tt' \in M^{G'} \\ h \in V(H)}} \{(t, h)(t', h)\}.$$

*Subcase* 7.2: If $zw \notin M^G$, then let $v$ be a neighbor of $z$ such that $vz \in M^G$. Since the symmetric difference of two perfect matchings consists of isolated vertices and even cycles, the edges $vz$ and $zw$ are on an even cycle $Z$ of $G$, in which every second edge is contained in $M^G$.
If $\{x, y\} \in V(Z)$, then let $M^1$ be a perfect matching of the subgraph of $G[H]$ induced by the set of vertices in $V(Z) \times V(H)$ and which contains $e_1$ and $e_2$ (recall that this subgraph is 2-extendable due to Prop. 4.3). Let $M^{G'} = M^G \setminus E(Z)$. Then $M^P$ is:

$$M^1 \cup \bigcup_{\substack{tt' \in M^{G'} \\ h \in V(H)}} \{(t, h)(t', h)\}.$$

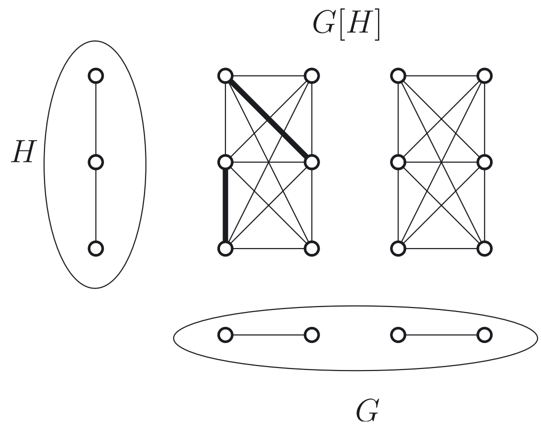


FIGURE 4. $G$ is not connected but 1-extendable, and the two bold edges can not be extended to a perfect matching of $G[H]$.

If $\{x, y\} \notin V(Z)$, then let $M^2$ be a perfect matching of $V(Z) \times V(H)$ containing $e_2$. Moreover, let $M^{G'} = M^G \setminus (E(Z) \cup \{e_1\})$. Then $M^P$ is:

$$\{e_1\} \cup M^2 \cup \bigcup_{\substack{tt' \in M^{G'} \\ h \in V(H)}} \{(t, h)(t', h)\}. \qquad \square$$

The example in Figure 4 shows that the connectivity requirement on $G$ is necessary in Theorem 4.4; in other words, if $G$ is a disconnected 1-extendable graph, then $G[H]$ might not be 2-extendable.

We will now consider the case where both $G$ and $H$ are 0-extendable. First, observe that when $H = P_2$, which is 0-extendable, and $G = P_4$, which is also 0-extendable, then $G[H]$ is not 2-extendable: let $E(H) = \{ab\}$ and $E(G) = \{xy, yz, zw\}$. Then $e_1 = (x, b)(y, a)$ and $e_2 = (y, b)(z, a)$ cannot be extended to a perfect matching in $G[H]$. In the following, we prove that the case where $|V(H)| = 2$ is the only case where $G[H]$ is not 2-extendable when both $G$ and $H$ are 0-extendable.

**Theorem 4.5.** *Let $G$ and $H$ be 0-extendable graphs and let $|V(H)| \geq 4$. Then $G[H]$ is 2-extendable.*

*Proof.* We treat separately each one of the 7 cases depicted in Figure 2 and described at the beginning of Section 4. In particular we provide a perfect matching $M^P$ of $G[H]$ containing $e_1$ and $e_2$ for each case.

**Case 1.** Let $M^1$ be an arbitrary perfect matching of the subgraph of $G[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a, b\})) \cup (\{y\} \times (V(H) \setminus \{c, d\})).$$

Then $M^P$ is:

$$\{e_1, e_2\} \cup M^1 \cup \bigcup_{k \in \{V(G) \setminus \{x,y\}\}} M^k.$$

**Case 2.** Let $M^1$ be an arbitrary perfect matching of the subgraph of $G[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a, b, c, d\})) \cup (\{y\} \times (V(H) \setminus \{a, b, c, d\})).$$

Then $M^P$ is:

$$\{e_1, e_2\} \cup M^1 \cup \{(y,a)(y,b), (y,c)(y,d)\} \cup \bigcup_{k \in V(G)\setminus\{x,y\}} M^k.$$

**Case 3.** We have the following two subcases:

*Subcase* 3.1: $d \neq a$ and $d \neq b$. Note that we may have $c = d$ in this case. Let $M^1$ be an arbitrary perfect matching of the subgraph of $G[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a,b,c\})) \cup (\{y\} \times (V(H) \setminus \{a,b,d\})).$$

Then $M^P$ is:

$$\{e_1, e_2, (y,a)(y,b)\} \cup M^1 \cup \bigcup_{k \in V(G)\setminus\{x,y\}} M^k.$$

*Subcase* 3.2: $d = a$ or $d = b$. Assume without loss of generality that $d = a$. Consider the case where there exists $c' \in V(H)$ such that $cc' \in E(H)$ and $c' \notin \{a,b\}$. Let $M^1$ be an arbitrary perfect matching of the subgraph of $G[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a,b,c,c'\})) \cup (\{y\} \times (V(H) \setminus \{a,b,c,c'\})).$$

Then $M^P$ is:

$$\{e_1, e_2, (x,c')(y,b), (y,c)(y,c')\} \cup M^1 \cup \bigcup_{k \in V(G)\setminus\{x,y\}} M^k.$$

If the neighborhood of $c$ is a subset of $\{a,b\}$, then consider a perfect matching $M$ of $H$. If $ca \in M$, then there exists $b' \in V(H)$ such that $bb' \in M$. Let $M^1$ be an arbitrary perfect matching of the subgraph of $G[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a,b,b',c\})) \cup (\{y\} \times (V(H) \setminus \{a,b,b',c\})).$$

Then $M^P$ is:

$$\{e_1, e_2, (x,b')(y,c), (y,b)(y,b')\} \cup M^1 \cup \bigcup_{k \in V(G)\setminus\{x,y\}} M^k.$$

If $cb \in M$, then let $M^1$ be an arbitrary perfect matching of the subgraph of $G[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a,b,c\})) \cup (\{y\} \times (V(H) \setminus \{a,b,c\}))$$

Then $M^P$ is:

$$\{e_1, e_2, (y,b)(y,c)\} \cup M^1 \cup \bigcup_{k \in V(G)\setminus\{x,y\}} M^k.$$

**Case 4.** If $x$ has a neighbor $w$ such that $w \notin \{y,z\}$, then $e_1$ can be extended to a perfect matching $M^1$ in $G^{xw}[H]$ since $G^{xw}[H]$ is 1-extendable. Likewise, $e_2$ can be extended to a perfect matching $M^2$ in $G^{yz}[H]$ since $G^{yz}[H]$ is 1-extendable. Then $M^P$ is:

$$M^1 \cup M^2 \cup \bigcup_{k \in V(G)\setminus\{x,y,z,w\}} M^k.$$

If the neighborhood of $x$ is a subset of $\{y,z\}$, then consider a perfect matching $M$ of $G$ and assume without loss of generality that $xy \in M$. Therefore, there exists $w \in V(G)$ such that $zw \in M$. Consider a vertex $d' \in V(H)$ such that $dd' \in E(H)$. Since $G^{zw}[H]$ is 1-extendable, $(z,d)(z,d')$ can be extended to a perfect matching $M^3$ in $G^{zw}[H]$. Let $c' \in V(H) \setminus \{c\}$ and $M^4$ be a perfect matching of the subgraph of $G^{xy}[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a,b\})) \cup (\{y\} \times (V(H) \setminus \{c,c'\})).$$

Then $M^P$ is:

$$\{e_1, e_2, (y,c')(z,d')\} \cup M^3 \cup M^4 \cup \bigcup_{k \in V(G) \setminus \{x,y,z,w\}} M^k.$$

**Case 5.** Consider a perfect matching $M$ of $H$ and let $a', d' \in V(H)$ be such that $aa' \in M, dd' \in M$. Pick an edge $ef \in M$ such that $e, f \notin \{a, a'\}$. Let $M^{1'}$ be a perfect matching of $G^z[H]$ and $d'$ be a vertex such that $(z,d)(z,d') \in M^{1'}$. Moreover, let $M^1 = M^{1'} \setminus \{(z,d)(z,d')\}$.
Note that there exist $u, v \in V(H) \setminus \{b,c\}$ since $|V(H)| \geq 4$. Let $G'_{xy}[H]$ be the subgraph of $G[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a,a',e,f\})) \cup (\{y\} \times (V(H) \setminus \{b,c,u,v\})).$$

Let $M^2$ be a perfect matching of $G'_{xy}[H]$. Then $M^P$ is:

$$\{e_1, e_2, (x,a')(y,u), (z,d')(y,v), (x,e)(x,f)\} \cup$$
$$M^1 \cup M^2 \cup \bigcup_{k \in V(G) \setminus \{x,y,z\}} M^k.$$

**Case 6.** Let $M^1$ be an arbitrary perfect matching of the subgraph of $G[H]$ induced by the following set of vertices:

$$(\{x\} \times (V(H) \setminus \{a,c\})) \cup (\{y\} \times (V(H) \setminus \{b,d\})).$$

Then $M^P$ is:

$$\{e_1, e_2\} \cup M^1 \cup \bigcup_{k \in V(G) \setminus \{x,y\}} M^k.$$

**Case 7.** Notice that both $G^{xy}[H]$ and $G^{zw}[H]$ are isomorphic to $P_2[H]$, which is 1-extendable. Let $M^1$ be a perfect matching of $G^{xy}[H]$ containing $e_1$ and $M^2$ be a perfect matching of $G^{zw}[H]$ containing $e_2$. Then $M^P$ is:

$$M^1 \cup M^2 \cup \bigcup_{k \in V(G) \setminus \{x,y,z,w\}} M^k. \qquad \square$$

## 5. Special 2-extensions of lexicographic products

Let $G$ be a 0-extendable but not 1-extendable graph. Let us refer by a *problematic edge* to an edge $e$ that cannot be extended to a perfect matching of $G$. We are interested in the case where $G[E_2]$ is not 2-extendable but the two copies of a problematic edge of $G$ can be extended to a perfect matching of $G[E_2]$.

Let us first point out the fact that we can restrict our attention to the case where $G[E_2]$ is 1-extendable since otherwise even one copy of a problematic edge in $G$ cannot be extended to a perfect matching in $G[E_2]$. This result, stated in the next lemma, follows from the proof of Theorem 3.8:

**Lemma 5.1.** *Let $G$ be a 0-extendable but not 1-extendable graph. If $G[E_2]$ is not 1-extendable then every copy of every problematic edge of $G$ is also problematic in $G[E_2]$.*

The following theorem characterizes the case where both copies of a problematic edge in $G$ can be extended to a perfect matching in $G[E_2]$.

**Theorem 5.2.** *Let $G$ be a 0-extendable but not 1-extendable graph with a problematic edge $e$, and assume $G[E_2]$ is 1-extendable. Then the two copies $\{e_1, e_2\}$ of $e$ in $G[E_2]$ can not be extended to a perfect matching of $G[E_2]$ if and only if there exists a subset of vertices $S \subset V(G)$ such that $e \in E(G_S)$ and $G - S$ has $|S| - 1$ singletons and one odd component that is not a singleton.*

*Proof.* Throughout the proof, let the problematic edge $e$ be $uv$ and its two copies in $G[E_2]$ be $u_1v_1$ and $u_2v_2$.

Assume that there is a subset $S \subset V$ such that $e \in G_S$ and $G-S$ has $|S|-1$ singletons and one odd component that is not a singleton. Let $S_1$ and $S_2$ be the two copies of $S$ in $G[E_2]$ and let $S' = S_1 \cup S_2 \setminus \{u_1, v_1, u_2, v_2\}$. Now, let $G' = G[E_2] \setminus \{u_1, v_1, u_2, v_2\}$. Since $G' - S'$ has at least $2|S| - 2$ odd components (singletons) and $|S'| = 2|S| - 4$, we have that $G'$ has no perfect matching (by Thm. 2.1). Therefore, $\{e_1, e_2\}$ cannot be extended to a perfect matching in $G[E_2]$.

Now assume that $\{e_1, e_2\}$ cannot be extended to a perfect matching in $G[E_2]$ and let us show that there is a subset $S \subset V(G)$ such that $e \in E(G_S)$ and $G - S$ has $|S| - 1$ singletons and one odd component that is not a singleton. By our assumption, the graph $G' = G[E_2] \setminus \{u_1, v_1, u_2, v_2\}$ (which is the lexicographic product of $G \setminus \{u, v\}$ with $E_2$) does not admit a perfect matching. Therefore, there is a subset $S' \subset V(G')$ such that $o(G'-S') > |S'|$. By Lemma 3.5, there exists a rectangular $\bar{S} \subset V(G')$ such that $o(G'-S')-|S'| \leq o(G'-\bar{S})-|\bar{S}|$ and therefore $|\bar{S}| < o(G' - \bar{S})$. Moreover, we know by Lemma 3.4 that all odd components of $G' - \bar{S}$ are singletons, say $x_1^1, \ldots, x_1^\ell$ and $x_2^1, \ldots, x_2^\ell$, respectively, in each copy of $G \setminus \{u, v\}$. Hence $2\ell > |\bar{S}|$. Now, let $S_1 = \bar{S} \cap V(G_1 \setminus \{u_1, v_1\})$ where $G_1 \setminus \{u_1, v_1\}$ is one copy of $G \setminus \{u, v\}$ in $G'$, and consider the set $S = S_1 \cup \{u, v\}$ in $G$. Clearly, $x^1, \ldots, x^\ell$ are also singletons in $G - S$. As $2\ell > |\bar{S}| = 2|S_1| = 2|S| - 4$, we have that $\ell \geq |S| - 1$. On the other hand, since we assumed that $G[E_2]$ is 1-extendable (see Lem. 5.1 and the related discussion), in particular $e_2$ is extendable to a perfect matching in $G[E_2]$. It follows that $\bar{S} \cup \{u_1, v_1\}$ is not a Tutte set in $G[E_2] \setminus \{u_2, v_2\}$, that is $2\ell = o((G[E_2] \setminus \{u_2, v_2\}) - (\bar{S} \cup \{u_1, v_1\})) \leq |\bar{S} \cup \{u_1, v_1\}| = |\bar{S}| + 2 = 2|S_1| + 2 = 2(|S| - 2) + 2$, and thus $\ell \leq |S| - 1$. Therefore, we have exactly $\ell = |S| - 1$ singletons in $G - S$. In addition, there is necessarily one more odd component (which is not a singleton) in $G - S$ since $G$ is 0-extendable and hence has an even number of vertices. This concludes the proof. $\square$

Note that if $G[E_2]$ is not 1-extendable, then one can find $S \subset V(G)$ such that $G-S$ has exactly $|S|$ singletons; this corresponds to the special case of Theorem 3.1 where $S$ is not an independent set (contains a problematic edge).

After establishing several results on the 2-extendability of lexicographic products, we can suggest some open research questions in the same direction. Recall that the work in [1] proves that the lexicographic product of an $m$-extendable graph and an $n$-extendable graph is $(m + 1)(n + 1)$-extendable. Theorem 4.5 proves that when both $G$ and $H$ are 0-extendable, $G[H]$ is not only 1-extendable, but also 2-extendable if $|V(H)| \geq 4$. This naturally raises the following question: can a result stronger than $(m + 1)(n + 1)$-extendability be obtained by imposing a restriction on $|V(H)|$? Likewise, we plan to investigate whether stronger forms of Theorem 4.2 and Theorem 4.4 can be obtained by keeping the connectivity condition: if $G$ is a connected graph and $H$ is an $m$-extendable graph, is $G[H]$ $(m + 1)$-extendable? If $G$ is a connected $m$-extendable graph and $H$ an arbitrary graph of order at least 2, is $G[H]$ $(m + 1)$-extendable?

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