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GRAPH COLORING APPROACH WITH NEW UPPER BOUNDS FOR THE CHROMATIC NUMBER: TEAM BUILDING APPLICATION

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Abstract. In this paper, we focus on the coloration approach and estimation of chromatic number. First, we propose an upper bound of the chromatic number based on the orientation algorithm described in previous studies. This upper bound is further improved by developing a novel coloration algorithm. Second, we make a theoretical and empirical comparison of our bounds with Brooks's bound and Reed's conjecture for class of triangle-free graphs. Third, we propose an adaptation of our algorithm to deal with the team building problem respecting several hard and soft constraints. Finally, a real case study from healthcare domain is considered for illustration.

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1. Introduction

Graph coloring has been considered for many real-world problems looking for a partition of a set of objects into subsets containing only two-by-two compatible elements *e.g.* team building [2, 24, 29]. It is an NP-Hard problem and looks for the smallest number of colors (*i.e.* chromatic number) needed to color the vertices of a graph so that no two adjacent vertices share the same color [15]. Several works in literature deal with the problem of graph coloration [6, 7, 13, 14, 17] and more precisely on the chromatic number estimation [3, 12, 16, 18–21].

In this paper, we provide a novel coloration algorithm, generating a performant estimation to the chromatic number, and a comparison to existing studies in literature. In addition, an adaptation of our coloration algorithm to the team building problem and an illustration on the nurses team building problem from healthcare domain are provided.

Let $G(V,E)$ be a graph, where V is a set of n vertices and E is a set of m edges. A stable S is a subset of vertices that are not adjacent. Let $\chi(G)$ be the chromatic number. It represents the minimum number of colors such that: for all x, $y \in V$, $(x,y) \in E$, $\mathit{color}(x) \neq \mathit{color}(y)$. Let $\omega(G)$ and $\Delta(G)$ be the clique number and the

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maximum degree of graph G, respectively. The graph G is called K-colorable if $\chi(G) = K$. For any graph G, there are common results on the chromatic number and its bounds:

- $1 \leq \chi(G) \leq n$.
- If all vertices form a stable *i.e.* $m = 0$, then $\chi(G) = 1$
- If G is bipartite then $\chi(G) \leq 2$.
- If G is complete then $\chi(G) = n$.
- If G contains a clique of size K then at least K colors are needed for coloring it, so $\chi(G) \geq \omega(G)$.
- $\chi(G) \leq \Delta(G) + 1$ (theorem of Brooks [7]).
- $\chi(G) \leq \lceil \frac{\omega(G)+\Delta(G)+1}{2} \rceil$ (conjecture of Reed [27], validated by [3, 18–20] on some classes of graphs).

For the special class of triangle-free graphs that will be considered later in our paper, Reed's conjecture becomes:

$$\chi(G) \leq \left\lceil \frac{\Delta}{2} \right\rceil + 2 \tag{1.1}$$

The theorem of Brooks provides an upper bound of the chromatic number for any graph. However, Reed's conjecture provides better upper bound than the Brooks' theorem for special graphs such as line graphs of multigraphs [19] and quasi-line graphs [18].

We can see that the upper bound of Reed diverges when $\Delta(G)$ is large (*i.e.* G is not dense). As illustration, see the instances $1, 3, 11, 12$ and 14 in Table 2. An upper bound is provided recently in [13] based on the orientation algorithm. This upper bound is improved in this paper by a new coloration algorithm that we called block coloration algorithm (detailed in Sect. 2). The used algorithm runs in polynomial time and improves both upper bounds of Reed and Brooks (explained later in Sect. 3).

2. Block coloration algorithm

The developed coloration algorithm proceeds in 2 steps:

(1) First step: Construction of a directed graph, using the orientation algorithm in[13];
(2) Second step: Construction of vertices groups (block) by applying the Block coloration algorithm.

We recall the principle of the First Step (*i.e.* the orientation algorithm presented in [13]). We choose a not yet treated or not yet marked vertex with the highest degree, orient all the corresponding incident edges to the outside, mark all its neighboring vertices, and consider it as treated. This process is then re-iterated on the residual partial graph obtained by removing the treated vertex. At the end of this process, all vertices are marked or treated. However, if there are still undirected edges, the procedure is repeated on the partial subgraph generated by these edges.

Therefore, the following notations can be considered:

- $G^o(V, A)$ is the associated graph to $G(V, E)$, according to the orientation algorithm;
- The transition of the vertex v denoted $Tr(v)$ is the number of outgoing arcs;
- The incidence of the vertex v denoted $I(v)$ is the number of incoming arcs;

In this study, we define a lexicographical order on graph vertices L by contracting the sets of vertices that have the same incidence L_i, where L_i are built by classifying the graph vertices in the increasing order of their incidence degrees. L is the union of the subsets L_i, and the vertices of the same set L_i are classified in a decreasing order of their transition degrees.

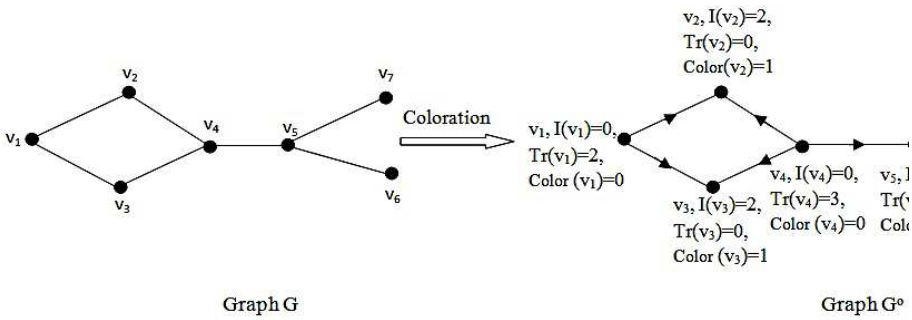

FIGURE 1. An illustration graphic of the block coloration algorithm.

Below the second step (*i.e.* block coloration algorithm):

Algorithm 1. Step 2: Block coloration.

Require: Graph $G^o(V, A)$, where A is a set of arcs
Ensure: C, set of colors C_i
  Begin
  $k \leftarrow 0$;
  $C_k \leftarrow \{L(1)\}$;
  for $v \leftarrow L(2)$ to $L(n)$ such that $n = |V|$ **do**
    $j \leftarrow 0$;
    $affect \leftarrow false$;
    while $affect = false$ **do**
      if $(x, v) \notin A$, $\forall x \in C_j$ **then**
        $C_j \leftarrow C_j \cup \{v\}$;
        $affect \leftarrow true$;
      else
        if $j = k$ **then**
          $k \leftarrow k + 1$, $j \leftarrow k$;
          $C_j \leftarrow \{v\}$;
          $affect \leftarrow true$;
        end if
      end if
      $j \leftarrow j + 1$;
    end while
  end for
  End.

It should be noted that $L_0 = \{v \in V$ such that $I(v) = 0\}$ and $C_0 = L_0$ at the end of the block coloration algorithm.

The complexity of our block coloration algorithm depends on the complexity of the First step which is polynomial [13]. The developed coloring algorithm gives a proper coloring which respects all adjacency constraints.

The Figure 1 illustrates an example of a graph and the result of our coloration algorithm.

From the lexicographical order, we have $L_0 = \{v_4, v_1, v_6, v_7\}$, $L_1 = \{v_2, v_3\}$, $L_2 = \{v_5\}$ which defined the treated set L such that $L \leftarrow L_0 \cup L_1 \cup L_2$, $L = \{v_1, v_4, v_6, v_7, v_2, v_3, v_5\}$. Furthermore, the block coloration algorithm gives $C_0 = \{v_1, v_4, v_6, v_7\}$ and $C_1 = \{v_2, v_3, v_5\}$.

To prove the efficiency of our algorithm regarding the respect of all constraints, three lemmas are provided and validated. Moreover, three consequences about the performance of our algorithm are deduced:

Lemma 2.1. *Vertices of set L_0 are not adjacent.*

Proof. By using the orientation algorithm (First step), all edges incident to the initial treated vertex are oriented to the outside so their incidence is equal to 0. The next vertex that can be treated is an unmarked vertex (not connected to a treated vertex). Hence, its incidence is also equal to 0. We continue the process until all vertices are either marked or treated. Hence, the set of treated vertices, which are represented by the set L_0 are not adjacent. □

Consequence 2.1. $\forall v \in V \setminus L_0, \exists \grave{v} \in L_0$ *such that* $(\grave{v}, v) \in A$.

Lemma 2.2. *If* $\exists v_1, v_2 \in V \setminus L_0$ *such that* $(v_1, v_2) \in E$ *then either* $Tr(v_1) > 0$ *or* $Tr(v_2) > 0$.

Proof. By using the orientation algorithm and at iteration k all vertices are either marked or treated. If $|E| \neq |A|$, there exist not yet oriented edges incident to marked vertices. We consider the partial subgraph generated by the not yet oriented edges. Thus, if $\exists v_1, v_2 \in V \setminus L_0$ such that $(v_1, v_2) \in E$ then (v_1, v_2) has an orientation in any direction, either $(v_1, v_2) \in A$ or $(v_2, v_1) \in A$, and this according to their degree in the graph G. Therefore, $Tr(v_1) > 0$ or $Tr(v_2) > 0$. □

Consequence 2.2. $\forall v_1, v_2 \in V \setminus L_0$ *such that* $Tr(v_1) = Tr(v_2) = 0$ *then* $(v_1, v_2) \notin E$. *Thus,* v_1 *and* v_2 *can have the same color.*

Lemma 2.3. *All vertices of a set* C_i *are not adjacent.*

Proof. According to the condition "while" of the algorithm, two neighboring vertices of the same incidence can not be in the same set C_i. □

Consequence 2.3. *Each block of vertices* C_i *is a stable set.*

3. Two upper bounds for the chromatic number

In this section, we provide two Theorems (3.1) and (3.2) based on new upper bounds for the chromatic number $\chi(G)$, and two others showing the performance of these upper bounds.

Theorem 3.1. *For any graph G, we have*:

$$\chi(G) \leq \max_{v \in V}(I(v)) + 1.$$

Proof. There are three possible cases:

Case 1. We suppose that the vertex with maximum incidence is a vertex of an odd hole[1] with size $2k+1$, let $G_{\text{hole}}(V_{\text{hole}}, E_{\text{hole}})$ be the hole. It exists a unique $x \in V_{\text{hole}}$ such that $I(x) = 2$, then $I_G(x) = 2 + a$, where a corresponds to the outsides incidences. Thus, $\max_{v \in V}(I(v)) \geq I_G(x) = 2 + a$. This implies that $\chi = 3 \leq 2 + 1 + a$.

Case 2. We suppose that the vertex of maximum incidence is a vertex of an odd anti-hole[2] of size $2k+1$, let $G_{\text{anti-hole}}(V_{\text{anti-hole}}, E_{\text{anti-hole}})$ be the anti-hole. By using the orientation algorithm, it exists a unique $x \in V_{\text{anti-hole}}$ such that $I(x) = 2k-2$, then $I_G(x) = 2k - 2 + a$, where a corresponds to the outside incidences. Thus, $\max_{v \in V}(I(v)) + 1 \geq 1 + I_G(x) = 2k - 1 + a \geq \chi = k + 1$.

[1] A hole is a cycle, its length is greater or equal to 4.
[2] An anti-hole is the complement of a hole.

Case 3. We suppose that G is a Berge[3] graph; $\omega(G) = \chi(G)$. By using the theorem1 [13], $\omega(G) \leq \max_{v \in V}(I(v)) + 1$ and since $\omega(G) = \chi(G)$ then $\chi(G) \leq \max_{v \in V}(I(v)) + 1$. □

Theorem 3.2. *For any graph G, we have:*

$$\begin{cases} \chi(G) \leq |C| \\ |C| \leq \max_{v \in V}(I(v)) + 1 \end{cases} \tag{3.1}$$

Proof. The expression (1) is obvious. For the expression (2), here is the proof:

From the block coloration algorithm, the sets L_j are the sets of vertices of the same degree of incidence. Then $L_0 = \{v \in V \text{ such that } I(v) = 0\}$.

Let $\rho = \{j \leq \max_{v \in V}(I(v)); \exists\, x \in L_j \text{ such that } \Gamma(x) \cap L_j \neq \emptyset\}$, where $\Gamma(x) = \{y \in V | (y, x) \in E\}$.

If $\rho = \emptyset$, then the blocks C_j are constructed from the sets L_j as follows:

$$\begin{cases} C_0 & = L_0 \\ C_j & = L'_j \cup B_j | L'_j = L_j - \left[\left(\cup_{i=0}^{j-1} B_i\right) \cap L_j\right], \\ & B_j = \left\{x \in \cup_{i=j+1}^{n} L_i | \Gamma(x) \cap L'_j = \emptyset\right\}; \\ & B_0 = \emptyset, n \geq j \geq 1. \end{cases}$$

Where $n = |L| - 1$ and C_j nonempty. Thus, $|C| = i + 1 \leq n + 1 = |L|$, such that i represents the number of blocks C_j. Which gives $|C| \leq \max_{v \in V}(I(v)) + 1$.

If $\rho \neq \emptyset$, then we set $h = \min\{j \in \rho\}$.

It is obvious that $|\rho| > 0$. Thus, we can consider that $1 \notin \rho$.

Let $V_1 = \{v \in L_h | I_{L_h}(v) \neq 0\}$. For any $v \in V_1, I_G(v) = h = I_{L_h}(v) + I_{\cup_{i=0}^{h-1} L_i}(v) + I_{\cup_{i=h+1}^{n} L_i}(v)$. Thus, $\forall v \in V_1$, either$\exists(h-j)$ sets that can receive v or $\exists(h-j)$ blocks that can be created.

Hence, $|C| \leq \max_{v \in V}(I(v)) + 1$. □

To evaluate the efficiency of our bounds, we made a formal comparison to classical bounds; theorem of Brooks and conjecture of Reed.

In the following theorem, we provide a first new upper bound for the chromatic number $\chi(G)$ based on the orientation algorithm. In addition, we show the performance of this upper bound and make a comparison with the upper bound of Brooks:

Theorem 3.3. *For any graph G, we have:*

$$\max_{v \in V}(I(v)) + 1 \leq \Delta(G) + 1.$$

Proof. As the vertex of maximum degree, $\Delta(G)$, is the first vertex which can be treated while orienting the incident edges to the outside. Thus, $\forall v \in V, d(v) \leq \Delta(G)$ this implies that $\max_{v \in V}(I(v)) \leq \Delta(G)$ then, $\max_{v \in V}(I(v)) + 1 \leq \Delta(G) + 1$. □

Knowing the fact that Reed provided a conjecture with a better upper bound than the upper bound of brooks [27], in the following theorem, we provide a second new upper bound for the chromatic number $\chi(G)$

[3]A Berge graph does not contain odd hole and odd anti-hole as induced subgraph.

TABLE 1. Comparison between $|C|$ and RC for instances of small sizes.

Instances	Δ	na	da	RC	$\lvert C\rvert$
Instances1	5	8.5	0.413	4	Average $= 2.88$ min $= 2$; max $= 4$
Instances2	6	9.5	0.387	5	Average $= 3.87$ min $= 2$; max $= 5$
Instances3	7	10.5	0.358	5	Average $= 3.83$ min $= 2$; max $= 5$
Instances4	8	11.5	0.325	6	Average $= 4.03$ min $= 2$; max $= 6$
Instances5	9	12.5	0.332	6	Average $= 3.86$ min $= 2$; max $= 6$
Instances6	10	13.5	0.321	7	Average $= 3.8$ min $= 2$; max $= 7$
Instances7	11	14.5	0.302	7	Average $= 4$ min $= 2$; max $= 7$
Instances8	12	15.5	0.323	8	Average $= 4.66$ min $= 2$; max $= 8$
Instances9	13	16.5	0.307	8	Average $= 4.63$ min $= 2$; max $= 8$

based on the block coloration algorithm (Second Step). Moreover, we show the performance of this second upper bound, which will be compared to the upper bound of Reed for a specific classes of graphs (triangle-free):

Theorem 3.4. *For any triangle-free graph, as $\Delta(G) \leq 4$:*

$$\chi(G) \leq |C| \leq \left\lceil \frac{\Delta}{2} \right\rceil + 2.$$

Proof.

If $\Delta(G) = 1$, then $|C| = 2 < \lceil \frac{\Delta(G)}{2} \rceil + 2 = 3$;
If $\Delta(G) = 2$, then

(1) For a Berge graph we have $|C| = 2 < \lceil \frac{\Delta(G)}{2} \rceil + 2 = 3$;
(2) For a not Berge graph we have $|C| = 3 = \lceil \frac{\Delta(G)}{2} \rceil + 2$.

If $\Delta(G) = 3$, then

(1) Either $|C| = 2 < \lceil \frac{\Delta(G)}{2} \rceil + 2 = 4$;
(2) Or $|C| = 3 < \lceil \frac{\Delta(G)}{2} \rceil + 2 = 4$.

If $\Delta(G) = 4$, then

(1) Either $|C| = 2 < \lceil \frac{\Delta(G)}{2} \rceil + 2 = 4$;
(2) Or $|C| = 3 < \lceil \frac{\Delta(G)}{2} \rceil + 2 = 4$.
(3) Or $|C| = 4 = \lceil \frac{\Delta(G)}{2} \rceil + 2$ (case of chvàtal graphs).

It is concluded that $|C| \leq \lceil \frac{\Delta}{2} \rceil + 2$ for $\Delta(G) \leq 4$. □

In absence of formal proof for $\Delta \geq 5$, we propose an empirical comparison based on instances that are generated randomly.

Note that na, da and RC are respectively, the average number of vertices, average density and bound of Reed's conjecture for triangle-free graphs, which is $\lceil \frac{\Delta}{2} \rceil + 2$.

TABLE 2. Comparison between $|C|$ and RC for instances of large sizes.

Instance	n	m	$\Delta(G)$	RC	$\|C\|$	Instance	n	m	$\Delta(G)$	RC	$\|C\|$
Instance1	100	235	80	42	9	Instance26	200	860	43	24	22
Instance2	100	230	75	40	40	Instance27	200	1100	39	22	20
Instance3	100	240	85	45	9	Instance28	200	1226	37	21	20
Instance4	100	254	99	52	10	Instance29	200	1400	40	22	21
Instance5	100	368	16	10	10	Instance30	200	1500	43	24	21
Instance6	100	380	20	12	11	Instance31	400	720	280	142	9
Instance7	100	520	30	17	13	Instance32	400	849	390	197	12
Instance8	100	550	36	20	15	Instance33	400	830	360	182	10
Instance9	100	620	33	19	16	Instance34	400	800	320	162	10
Instance10	100	647	39	22	16	Instance35	400	1000	37	21	17
Instance11	150	275	70	37	7	Instance36	400	1500	46	25	22
Instance12	150	280	76	40	7	Instance37	400	3800	76	40	39
Instance13	150	320	100	52	11	Instance38	400	6500	80	42	40
Instance14	150	300	90	47	8	Instance39	400	7000	82	43	40
Instance15	150	400	40	22	13	Instance40	400	7500	88	46	41
Instance16	150	580	33	19	17	Instance41	500	690	310	157	8
Instance17	150	610	38	21	19	Instance42	500	750	380	192	9
Instance18	150	789	41	22	22	Instance43	500	780	380	192	10
Instance19	150	1200	40	22	20	Instance44	500	800	400	202	9
Instance20	150	1330	42	23	20	Instance45	500	810	395	200	10
Instance21	200	206	199	102	5	Instance46	500	824	405	205	10
Instance22	200	600	32	18	16	Instance47	500	5500	76	40	33
Instance23	200	690	35	20	16	Instance48	500	6300	81	43	37
Instance24	200	790	42	23	20	Instance49	500	7800	78	41	38
Instance25	200	800	48	26	20	Instance50	500	8000	85	47	40

In Table 1, we present 9 sets of instances. Each row in this table represents the average of 6 sets of 5 instances (a total of 30 instances), having the same characteristics (same Δ and same number of vertices).

In Table 2, we present the experimental results on randomly generated instances of triangle-free graphs, where the vertices number n and Δ are sufficiently large.

We note that the numerical tests confirm the dominance of our bound in front of Reed's bound.

4. Block coloration with limited cardinality for the team building problem

Team building has attracted the attention of researchers in different domains; healthcare [1, 4], sports [11], education [29], military [10], *etc.* Team Building is the problem of composing high-performing teams respecting a set of hard and soft constraints *e.g.* predefined team size, skills and complementarity between team members. It is an NP-Hard problem [4].

The modeling of the team building problem has been achieved in literature using graph theory [2, 23, 29]. In [29] the authors modeled the team building problem as a weighted graph and provided a heuristic based on vertex contraction technique to solve the considered problem. In [2, 23], the authors provided greedy algorithms for graph coloring to deal with the team building problem where the vertices represent the teams members and the edges represent the adjacency relationships.

Teams have generally predefined cardinalities. Thus, to deal with the team building problem, we propose a modified version of our block coloration algorithm where the stables have fixed cardinalities (identical or non-identical).

To create stables with fixed identical cardinalities, p, or fixed non identical cardinalities, $p[j]$, the following condition "$(x, v) \notin A$, $\forall x \in C_j$" in the block coloration algorithm becomes "$(x, v) \notin A$, $\forall x \in C_j$ and $C_j \leq p$ (or $C_j \leq p[j]$)", where $p[j]$ depends on the characteristics of block C_j".

In the team building problem, the resources can be classified into a set of categories according to their skills. Ideally, team members have complementary skills, and each resource belongs to one of these categories. For what follows, we consider T the set of h categories of resources T_t ($T = \cup_{t=1}^{h} T_t$). Usually, among the h available categories, a set of b categories ($b \leq h$) are considered critical owing to the necessity to have members from these categories in each team, *e.g.* head of department, supervisor. This presents a hard constraint in the team building problem (see constraint 2).

The objective of our work is to build coherent teams C_i respecting the following constraints:

(1) $\forall C_i$: $|C_i| \leq p$ (or $|C_i| \leq p[i]$); this means that each team must have at most p elements;
(2) $\forall C_i$: $C_i \cap (\cup_{t=1}^{b} T_t) \neq \emptyset$ where T_t represents the set of vertices (elements) of the same category t; this means that each team must have at least one element from $\cup_{t=1}^{b} T_t$;
(3) The number of teams, $q \leq \sum_{t=1}^{b} |T_t|$; this means that the number of teams is less than the number of elements in $\cup_{t=1}^{b} T_t$;
(4) Hard constraints correspond to edges of type $(T_t, T_t), t = 1, \ldots, b$; *i.e.* edges between vertices of the same category;
(5) Soft constraints correspond to all edges except those of type $(T_t, T_t), t = 1, \ldots, b$;

Using the above constraints and modifications on our block coloration algorithm, we get an algorithm adapted to stables with limited cardinality that is adapted to the team building problem. At the end of this block coloration algorithm with limited cardinality, the configuration that we obtained is not necessarily complete[4], this is due to the fact that two vertices $v_1, v_2 \in V | (v_1, v_2) \in (\times_{t=1}^{b} T_t), (v_1, v_2) \notin E$ can be in the same team, which can give the teams C_i such that $\forall v \in C_i, v \notin (\cup_{t=1}^{b} T_t)$. A complete configuration is deductible with possible permutation.

Proposition 4.1. *For any initial solution configuration, provided by the block coloration algorithm of limited cardinality, a complete configuration is deductible with possible permutation.*

Proof. We suppose that the initial solution is not complete, thus $\exists i \in \{1, \ldots, k\}$ such that C_i does not satisfy the condition (2), and both conditions (3) and (4) are satisfied, then $\exists$ at least $j \in \{1, \ldots, k\}, j \neq i$ such that $C_j \cap (\cup_{t=1}^{b} T_t) = b$, thus, for $v \in C_i$ and $y \in C_j$ such that $y \in (\cup_{t=1}^{b} T_t)$, If ($\forall v_1 \in C_j - \{y\}$, $(v_1, v) \notin E$) and ($\forall v_2 \in C_i - \{v_1\}, (v_2, y) \notin E$), then $C_j \leftarrow (C_j - \{y\}) \cup \{v\}$ and $C_i \leftarrow (C_i - \{v\}) \cup \{y\}$, else if $\exists h \in \{1, \ldots, k\}, h \neq i, j$ such that for $y_2 \in C_h | y_2 \in (\cup_{t=1}^{b} T_t)$, ($\forall v_3, v_1, v_2 \in C_h - \{y_2\}, C_j - \{y\}, C_i - \{v\}$, $(y, v_2), (v, v_1), (y, v_3) \notin E$), then $C_h \leftarrow (C_h - \{y_2\}) \cup \{y\}$, $C_j \leftarrow (C_j - \{y\}) \cup \{v\}$ and $C_i \leftarrow (C_i - \{v\}) \cup \{y_2\}$, else Cancel soft edges between the vertex v and the set C_j, and pose $C_j \leftarrow (C_j - \{y\}) \cup \{v\}$ and $C_i \leftarrow (C_i - \{v\}) \cup \{y\}$. □

To improve team performance, two strategies are proposed for choosing the vertex v that can leave its block:

Strategy 1. Give preferential treatment to a vertex of lower transition. This vertex presents a high probability in a move without giving conflict.
Strategy 2. Let f be a preference function defined on closed interval which associates a weight w for any preferably edge, such that $f(x_i, x_j) = w \geq 1 | (x_i, x_j) \in E$. It is to be noted that if the degree of conflict is very low then $w = 1$. Our aim is to build coherent team (*i.e.* teams with low weight). Thus, we choose the vertex v with the lowest weight transition.

The block coloration algorithm of limited cardinality allows to respect the maximum cardinality teams while respecting the wishes of the employees (vertices). Hence, it guarantees all the conditions except the condition (2). However, with a possible permutation, the algorithm generates a solution that respects all the conditions except the condition (5), it should be noted that the permutation never exceeds $\sum_{t=1}^{b} |T_t|$.

[4]A complete configuration is a set of teams having at least one vertex $v \in V$ such that $v \in (\cup_{t=1}^{b} T_t)$.

Our approach is generic. Strategy 1 or strategy 2 is selected according to the relevant application, in the case where the solution found by the block coloration algorithm of limited cardinality is not complete. In the following, we provide an illustration of our algorithm to the team building problem from the healthcare domain; nurses team building in a surgical service of a clinic from Tunisia.

4.1. Illustration to Nurses team building

Nurses team building and rostering problems have attracted much research attention [1, 4, 8, 9]. It is defined as the Operational Research problem composing nurses teams and assigning teams to shifts, (*e.g.* {day, night} [5] or {early, late, night} [30]), typically with a set of hard and soft constraints.

In the following, we give an illustration of our algorithm on the problem of nurses team building. The assignment of teams to shifts is also provided.

The surgery service of the Clinic of Soukra, in Tunis (Tunisia) is working with two shifts {day, night} per 24 hours and 4 nursing teams. Nurses are classified into 7 classes according to their skills and experiences:

- Student Nurse (SN): from 0 to 2 y of experiences.
- Nurse (N): from 0 to 5−10 y of experiences. The main resource who made the nursing work.
- Principal Nurse (PN): more that 5 y of experience. Responsible of the coordination between the members of the nursing team.
- Secondary Overseer Nurse (SON): principal Nurse with several managerial and social skills. He (she) is always available and replace the Principal Nurse when he (she) is absent.
- Overseer Nurse (ON): more than 5 y of experience as Secondary Overseer Nurse.
- Night General Overseer (NGO): overseer Nurse with managerial skills.
- General Overseer (GO): overseer Nurse with a global vision on pathologies, high management level, admission management, polyvalent, good relationship.

The General Overseer controls all nursing teams services and works from $7a.m.$ to $4p.m.$. For each service, nursing teams are controlled by an Overseer Nurse who is available each day from $6a.m.$ to $3p.m.$. The nursing teams are composed of 3 possible categories of employees:

- T_1 represents the set of Secondary Overseer Nurses;
- T_2 represents the set of Principal Nurses;
- T_3 represents the set of Nurses and Student Nurses.

The related soft and hard constraints (*e.g.* mix of nurses qualifications for a particular shift, maximum working hours allowed for nurses, individual nurses preferences for shifts and for working with other persons) are expressed as follows:

- number of shifts per day (*e.g.* two shifts {day, night});
- number of nurses per shift (*e.g.* for the medical service, 4 at day (from $7a.m.$ to $7p.m$) and 3 to 4 at night (from $7p.m.$ to $7a.m$)), this is expressed by $|C_i| \leq p[i]$;
- number q of teams ($q = 4$), $q \leq |T_1| + |T_2|$;
- one Principal Nurse (PN) at least at night, that results in $\forall C_i : C_i \cap (T_2) \neq \emptyset$ and $\nexists v_1, v_2 \in T_2$ such that v_1 and v_2 can be in the same team;
- one Secondary Overseer Nurse (SON) works the day shift and can replace a Principal Nurse (PN), it implies that $\forall C_i : C_i \cap (T_1 \cup T_2) \neq \emptyset$ and $\nexists v_1, v2 \in T_1 \times T_2$ such that (v_1, v_1) and (v_2, v_2) can be in the same team. It also implies that $\exists v_1, v_2, v_3 \in T_1 \times T_2 \times T_3$ such that (v_1, v_2), (v_1, v_3), (v_2, v_3) and (v_3, v_3) can be in the same team where $|T_1| \leq |T_2|$.

Using the adjacency matrix in Table 3, a graph G is constructed by taking nurses as vertices. Our Graph contains $|V| = 14$ vertices (4 teams, $p = [4, 4, 3, 3]$, *i.e.* a team of 4 nurses at day shift and a team of 3 nurses at night, a team works every other day).

TABLE 3. Adjacency matrix (conflicts between nurses).

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}
v_1	0	1	0	0	0	1	1	0	0	0	0	0	0	0
v_2	1	0	1	0	0	0	0	0	0	0	0	0	0	0
v_3	0	1	0	1	1	0	0	0	1	0	1	0	0	0
v_4	0	0	1	0	1	0	0	0	0	0	0	0	1	1
v_5	0	0	1	1	0	0	0	0	0	0	0	0	1	1
v_6	1	0	0	0	0	0	0	0	1	0	0	0	0	0
v_7	1	0	0	0	0	0	0	0	1	0	0	0	0	0
v_8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
v_9	0	0	1	0	0	1	1	0	0	1	1	0	0	0
v_{10}	0	0	0	0	0	0	0	0	1	0	0	0	0	0
v_{11}	0	0	1	0	0	0	0	0	1	0	0	0	0	0
v_{12}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
v_{13}	0	0	0	1	1	0	0	0	0	0	0	0	0	0
v_{14}	0	0	0	1	1	0	0	0	0	0	0	0	0	0

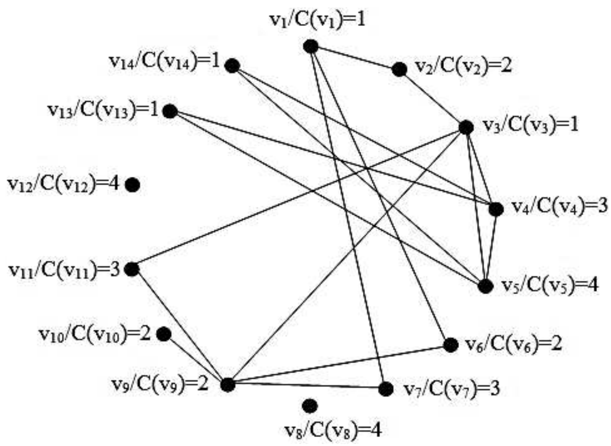

FIGURE 2. Graph G shows the model and result of our modified block coloration algorithm.

If a specific constraint is present between two nurses, corresponding vertices are joined using an edge. Thus, they are considered adjacent and are colored using different colors.

Let $\{v_1, v_2\}$ be the set of Secondary Overseer Nurses (SON), $\{v_3, v_4, v_5\}$ the set of Principal Nurses (PN), $\{v_6, \ldots, v_{12}\}$ be the set of Nurses (N) and $\{v_{13}, v_{14}\}$ the set of Student Nurses (SN).

The result of our coloration algorithm of limited cardinality for the data presented in Table 3 is shown in Figure 2.

The teams composition given by block coloration algorithm of limited cardinality represents a complete configuration, as shown in Table 4.

Teams 1 and 2, containing particularly Secondary Overseer Nurses (SON), are assigned to day shifts (D), and Teams 3 and 4 are assigned to night shifts (N).

Each nurse is already assigned to one of the four teams and works according to the four-day basic pattern reported in Table 5. The pattern is cyclically repeated.

TABLE 4. Teams obtained by the modified block coloration algorithm.

Team 1	Team 2	Team 3	Team 4
v_3, v_1, v_{13}, v_{14} 1 SON 1 PN 2 SN	v_{10}, v_9, v_2, v_6 1 SON 3 N	v_{11}, v_7, v_4 1 PN 2 N	v_5, v_8, v_{12} 1 PN 2 N

TABLE 5. Four-day pattern for Nurses teams.

Day	1	2	3	4
Team 1	D	–	D	–
Team 2	–	D	–	D
Team 3	N	–	N	–
Team 4	–	N	–	N

5. Conclusion

In this paper, we proposed two efficient upper bounds for the chromatic number. The first one is based on the orientation method cited in [13] and the second one is the result of the block coloration algorithm. It has been found that the second bound is more accurate.The efficiency of these bounds is proved theoretically and empirically in comparison with the bound of Brooks [7] for any graph, and the conjecture of Reed [27] for triangle-free graphs.

To provide stables with predefined cardinality to deal with the team building problem, a modified block coloration algorithm is generated based on our graph coloring algorithm. For illustration, this new algorithm is applied to nurses team building for the surgery service of a private hospital (Clinic of Soukra in Tunis, Tunisia).

For the future research, we plan to adapt our graph coloration algorithm to deal with weighted graphs and apply the new algorithm to nurses scheduling problem, and the assignment of patients to nurses.

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