RAIRO-Oper. Res. 51 (2017) 819–831
DOI: 10.1051/ro/2016068

RAIRO Operations Research
www.rairo-ro.org

APPROXIMATE LAGRANGIAN DUALITY AND SADDLE POINT OPTIMALITY IN SET OPTIMIZATION

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Abstract. In this paper, we establish approximate Lagrangian multiplier rule, Lagrangian duality and saddle point optimality for set optimization problem where the solutions are defined using set relations introduced by Kuroiwa (Kuroiwa D., The natural criteria in set-valued optimization. Sūrikaisekikenkyūsho Kōkyūroku **1031** (1998) 85–90).

Mathematics Subject Classification. 49J53, 90C46, 90C26.

Received February 1, 2015. Accepted October 5, 2016.

1. Introduction

Duality theory is an important aspect of optimization theory. Using duality theory a minimization problem can be related to a corresponding maximization problem in such a way that by solving the latter problem it is possible to get optimum value of the former problem. Lagrangian methods are a way to solve a constrained optimization problem by converting it into an unconstrained optimization problem. Under convexity assumptions a constrained optimum is characterized as a saddle point of the Lagrangian. Saddle points provide a link between primal and dual problems. For more details refer to [4, 18].

Let $F : X \rightrightarrows Y$ be a set-valued map from a nonempty set X to a real Hausdorff topological vector space Y and M be a nonempty subset of X. Consider the following set-valued optimization problem

$$\text{(P)} \qquad \begin{aligned} &\text{WMin} \quad F(x) \\ &\text{subject to} \quad x \in M. \end{aligned}$$

There are two well-known approaches for defining the solution concept of the problem (P). The first approach is in the vector sense, where a solution concept involves a single element of the image of the solution. In this approach an element $\bar{x} \in M$ is called weak minimal solution of (P) if there exists $\bar{y} \in F(\bar{x})$ such that

$$(F(M) - \bar{y}) \cap (-\text{int}K) = \emptyset,$$

where $F(M) = \bigcup_{x \in M} F(x)$, K is a closed convex pointed cone with nonempty interior that induces an ordering in Y and int$K$ denotes interior of K. The optimality conditions and duality theory using this approach has been

Keywords. Set optimization, approximate solutions, Lagrangian multiplier rule, Lagrangian duality, saddle point optimality.

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studied in various papers, for instance [1,2,15,18,20,21]. Corley [1,2] gave optimality conditions and Lagrangian duality theory for a set-valued maximization problem. Luc [18] introduced the notion of saddle points for the Lagrangian map of a set-valued optimization problem. Song [20] gave a cone separation theorem between two subsets of image space and used it to discuss the optimality conditions and duality for set-valued optimization problem in locally convex spaces. Recently, Long and Peng [17] studied Lagrangian rule and duality results for set-valued optimization problem with nearly cone-subconvexlike maps.

The second approach was introduced by Kuroiwa [9–13] which is a more appropriate solution concept for a set-valued optimization problems. Unlike the first approach, this approach requires the comparison of the whole set $F(\bar{x})$ if $\bar{x}$ is a solution of (P). This approach is usually referred to as set optimization approach and the first approach is referred to as vector approach. In [11], Kuroiwa presented existence and duality results for a set-valued optimization problem. Using this approach, Hernández and Rodríguez-Marín [6] established the duality results, Lagrange multiplier rule, necessary and sufficient conditions for the existence of saddle points of the Lagrangian map. To establish these results, using the second approach the authors, in [6] considered Lagrangian map involving continuous affine linear maps. This is in contrast to the traditional method where the Lagrangian map involves continuous linear maps. Recently, Lagrangian duality for set-valued optimization problems using complete-lattice theory was studied in [5,8,16].

In the recent years there has been a growing interest in the study of approximate solutions of optimization problems. However, the study has been done only for solutions of vector type. Usually approximate solutions are obtained on applying iterative algorithms to solve an optimization problem. Kutateladze [14] introduced the concept of approximate solutions which has been further used to obtain approximate Kuhn–Tucker type conditions and approximate duality theorems in [3]. Vályi [22] presented different types of nondominated approximate solutions of vector optimization problem and saddle point theorems corresponding to these solutions. Rong and Wu [19] extended the results given by Vályi [22] to the case of set-valued optimization problem. They gave ε-saddle point theorems and ε-duality theorems using ε-Lagrangian multipliers.

The main aim of this paper is to derive approximate Lagrange multiplier rule, duality and saddle point results for a constrained set-valued optimization problem using the set optimization approach. We consider a notion of weak approximate solutions in the sense of Kuroiwa [10]. Since in the set optimization approach the whole set is considered instead of just a point of the set, some strong assumptions are to be assumed while establishing Lagrangian and duality results. For instance in [6] the authors had to assume a separation condition to establish Lagrange multiplier. Moreover, the Lagrange map, used to establish Lagrange multiplier rule, duality and saddle point results involved affine maps rather than linear maps. In this paper we overcome both these issues but in the process we have to assume that the image set corresponding to the optimal solution has a strong minimum.

The rest of the paper is organized as follows. Section 2 is devoted to preliminaries which will be required in the sequel. In Section 3, we establish a Lagrange multiplier rule for the set-valued optimization problem. In Section 4, we formulate a Lagrangian dual problem and give the notion of approximate solutions for the dual problem. We then establish the weak and strong duality results in this section. In Section 5, we introduce the notion of approximate weak saddle point of the Lagrangian map and establish saddle point optimality conditions.

2. Preliminaries

The dual cone of K is defined as

$$K^{+} := \{\varphi \in Y^{*} : \varphi(k) \geq 0, \forall k \in K\},$$

where Y^{*} denotes the topological dual of Y.

We recall from [21] that a map F is nearly K-convexlike on M, if $\mathrm{cl}(F(M)+K)$ is a convex set where cl denotes the closure of a set. This notion extends the notion of K-convexlikeness as a K-convexlike set-valued valued map F on M is characterized by the convexity of $F(M)+K$.

We next recall an alternative theorem which will be required in the sequel.

Lemma 2.1 ([21]). *If F is nearly K-convexlike on X and* $\operatorname{int}K \neq \emptyset$, *then exactly one of the following statements holds*:

(i) *there exists $\bar{x} \in X$ such that $F(\bar{x}) \cap (-\operatorname{int}K) \neq \emptyset$;*
(ii) *there exists $\varphi \in K^+ \backslash \{0\}$ such that $\varphi(y) \geq 0$, for all $x \in X, y \in F(x)$.*

Let A be a subset of Y. We recall that an element $\hat{a} \in A$ is a strong minimum of A if $A \subseteq \hat{a} + K$.

Throughout the paper we assume that, $e \in \operatorname{int}K$ is a fixed element and $\varepsilon \geq 0$.

The vector criterion approach has been considered in literature [17, 19] to define the notions of approximate solutions. The set of all ε-weak minimal solutions and ε-weak maximal solutions of A are defined as

$$\varepsilon\text{-WMin}A := \{a_\varepsilon \in A : (A - a_\varepsilon + \varepsilon e) \cap (-\operatorname{int}K) = \emptyset\},$$

$$\varepsilon\text{-WMax}A := \{a_\varepsilon \in A : (A - a_\varepsilon - \varepsilon e) \cap \operatorname{int}K = \emptyset\}.$$

When $\varepsilon = 0$, we have the set of weak minimal solutions of A which we denote by WMin$A$.

We recall the following quasi order relations from [6, 7, 9]. Let $\mathcal{P}(Y)$ denote the collection of all nonempty subsets of Y. If $A, B \in \mathcal{P}(Y)$ then

$$A \leq^l B \text{ if and only if } B \subseteq A + K,$$

$$A <^l B \text{ if and only if } B \subseteq A + \operatorname{int}K.$$

We say that $A \in [B]^l$ if and only if $A \leq^l B$ and $B \leq^l A$. Clearly, $A \in [B]^l$ if and only if $A + K = B + K$.

We have a similar characterization for the strict order relation.

Lemma 2.2. *Let $A, B \in \mathcal{P}(Y)$. Then $A <^l B$ and $B <^l A$ if and only if $A + K = A + \operatorname{int}K = B + \operatorname{int}K = B + K$.*

We now use the set criterion approach to define approximate solutions.

Definition 2.3. Let $\mathcal{S}$ be a subfamily of $\mathcal{P}(Y)$. The set of all ε-l-weak minimal solutions and ε-l-weak maximal solutions of $\mathcal{S}$ are defined as

$$\varepsilon\text{-}l\text{-WMin}(\mathcal{S}) := \{A \in \mathcal{S} : B + \varepsilon e <^l A, B \in \mathcal{S} \Rightarrow A <^l B + \varepsilon e\},$$
$$\varepsilon\text{-}l\text{-WMax}(\mathcal{S}) := \{A \in \mathcal{S} : A <^l B - \varepsilon e, B \in \mathcal{S} \Rightarrow B - \varepsilon e <^l A\}.$$

Let Z be another real Hausdorff topological vector space and let C be a closed convex pointed cone in Z. Let $G : X \rightrightarrows Z$ be a set-valued map.

Consider the following set optimization problem

$$\text{(P)} \qquad \begin{aligned} &\text{WMin} \quad F(x) \\ &\text{subject to} \quad x \in M, \end{aligned}$$

where $M = \{x \in X : G(x) \cap (-C) \neq \emptyset\}$.

We say that problem (P) satisfies constraint qualification (CQ) if for every $\psi \in C^+ \backslash \{0\}$, there exists $\hat{x} \in X, \hat{z} \in G(\hat{x})$ such that $\psi(\hat{z}) < 0$.

Throughout the paper we assume that $F(x) \neq \emptyset$ and $G(x) \neq \emptyset$ for every $x \in X$.

We recall from [19] that an element $x_\varepsilon \in M$ is an ε-weak minimal solution of (P) if

$$F(x_\varepsilon) \cap \varepsilon\text{-WMin}F(M) \neq \emptyset.$$

This is equivalent to the fact that there exists $y_\varepsilon \in F(x_\varepsilon)$ such that

$$(F(M) - y_\varepsilon + \varepsilon e) \cap (-\operatorname{int}K) = \emptyset.$$

When $\varepsilon = 0$, this notion reduces to the notion of weak minimal solution.

In the set criterion sense, we say that $x_\varepsilon \in M$ is an ε-l-weak minimal solution of (P) if

$$F(x_\varepsilon) \in \varepsilon\text{-}l\text{-WMin}(\mathcal{F}),$$

where $\mathcal{F} := \{F(x) : x \in M\}$. This is equivalent to the fact that

$$F(x) + \varepsilon e <^l F(x_\varepsilon), x \in M \Rightarrow F(x_\varepsilon) <^l F(x) + \varepsilon e.$$

For notational convenience throughout the paper we denote ε-l-WMin$(\mathcal{F})$ by ε-l-WMin $(F(x) : x \in M)$ and use similar expressions later on. When $\varepsilon = 0$, this notion reduces to the notion of weakly l-minimal solution given in [7].

It can be easily seen that every ε-weak minimal solution of (P) is an ε-l-weak minimal solution of (P). However the converse is not true.

Example 2.4. Consider problem (P) with $X = \mathbb{R}, Y = \mathbb{R}^2, Z = \mathbb{R}, K = \mathbb{R}^2_+$ and $C = \mathbb{R}_+$. Let $\varepsilon = 0.1$ and $e = (1, 1)$. Let $F : X \rightrightarrows Y$ be defined as

$$F(x) = \begin{cases} \{(x + a, 1 + a - x) : a > 0\}, & \text{if } x \in \,]0, 1], \\ [(0.2, 1.2), (1.2, 0.2)], & \text{if } x = 0, \\ \{(2, 2)\}, & \text{otherwise.} \end{cases}$$

Let $G : X \rightrightarrows Z$ be defined as

$$G(x) = \begin{cases} [-1, 1], & \text{if } x \in [0, 1], \\ [1, 2], & \text{otherwise.} \end{cases}$$

Clearly, $M = [0, 1]$ and the set of ε-l-weak minimal solutions is $[0, 1]$ but the set of ε-weak minimal solutions is $]0, 1]$.

In fact, if x_ε is an ε-weak minimal solution of (P) then, by definition, it is clear that there does not exist any $x \in M$ such that $F(x) + \varepsilon e <^l F(x_\varepsilon)$.

This fact can be further generalized to the following lemma.

Lemma 2.5. *If* ε-WMin$F(M) \neq \emptyset$, *then* x_ε *is an* ε-l-*weak minimal* (ε-l-*weak maximal*) *solution of* (P) *if and only if there is no* $x \in M$ *such that* $F(x) + \varepsilon e <^l F(x_\varepsilon)(F(x_\varepsilon) <^l F(x) - \varepsilon e)$.

Remark 2.6. The above lemma also holds under the assumption that WMin$F(x_\varepsilon)$ is nonempty instead of the condition that ε-WMin$F(M)$ is nonempty. Hence, the above lemma holds even if $F(x_\varepsilon)$ has a strong minimum.

3. Lagrangian multiplier rule

In this section we establish Lagrangian multiplier rule for set optimization problem (P). In fact, we establish a link of the set optimization problem (P) with an unconstrained set-valued problem.

Let $\mathscr{L}(Z, Y)$ denote the space of all continuous linear maps from Z to Y and let

$$\mathscr{L}_+(Z, Y) := \{T \in \mathscr{L}(Z, Y) : T(C) \subseteq K\},$$

where $T(C) := \cup_{c \in C} T(c)$.

Let $L : X \times \mathscr{L}_+(Z, Y) \rightrightarrows Y$ be the set-valued Lagrangian map defined as

$$L(x, T) = F(x) + T(G(x)).$$

In [10], Kuroiwa considered the set-valued Lagrangian map $L : X \times Z \times \mathscr{L}_+(Z,Y) \rightrightarrows Y$ defined as $L(x,z,T) = F(x) + T(z)$ where $z \in G(x)$. Hernández and Rodríguez-Marín [6] considered the set-valued Lagrangian map $L : X \times \mathscr{A}_*(Z,Y) \rightrightarrows Y$ defined as $L(x,T) = F(x) + T(G(x))$ where

$$\mathscr{A}_*(Z,Y) = \{T \in \mathscr{A}(Z,Y) : T(z) = \bar{T}(z) + m \text{ such that } \bar{T} \in \mathscr{A}(Z,Y) \text{ and } m \in -K\}$$

and $\mathscr{A}(Z,Y)$ is the set of all continuous affine linear maps from Z to Y.

Before establishing the main result of this section we state the the Lagrangian multiplier rule (Thm. 4.1) given by Hernández and Rodríguez-Marín [6].

We recall from [6] that problem (P) satisfies generalized Slater's constraint qualification if there exists $x \in X$ such that $G(x) \cap (-\mathrm{int}C) \neq \emptyset$.

Theorem 3.1. *Let (F,G) be $(K \times C)$-convexlike on X, $\mathrm{int}C \neq \emptyset$ and assume that problem* (P) *satisfies generalized Slater's constraint qualification. Let x_0 be a l-weak minimal solution of* (P), $\mathrm{WMin}F(x_0) \neq \emptyset$ *and $F(x_0)$ be K-bounded. Let $B = \bigcap_{y_0 \in F(x_0)} (y_0 - K)$ and let $H = \{(\varphi,\psi) \in K^+ \backslash \{0\} \times C^+ : (\varphi,\psi)$ separates $(F,G)(X)$ and $B \times (-C)\}$. If $H \neq \emptyset$ then H contains an element (φ_0,ψ_0) such that $\inf\{(\varphi_0,\psi_0)(y,z) : (y,z) \in (F,G)(X)\} = (\varphi_0,\psi_0)(y_0,z_0)$ for some $(y_0,z_0) \in (F,G)(x_0)$. Then there exists $A_0 \in \mathscr{A}_*(Z,Y)$ such that*

$$-A_0(G(x_0) \cap (-C)) \subseteq K, \tag{3.1}$$

and x_0 is a l-weak minimal solution of the problem

$$\begin{aligned} \mathrm{WMin} \quad & F(x) + A_0(G(x)) \\ \text{subject to} \quad & x \in X. \end{aligned}$$

For $T \in \mathscr{L}_+(Z,Y)$ we consider the following set optimization problem

$$(\mathrm{UP})_T \qquad \begin{aligned} \mathrm{WMin} \quad & L(x,T) \\ \text{subject to} \quad & x \in X. \end{aligned}$$

We recall that $x_\varepsilon \in X$ is an ε-l-weak minimal solution of $(\mathrm{UP})_T$ if

$$L(x,T) + \varepsilon e <^l L(x_\varepsilon,T), x \in X \Rightarrow L(x_\varepsilon,T) <^l L(x,T) + \varepsilon e. \tag{3.2}$$

The following Lagrangian multiplier rule establishes that an ε-l-weak minimal solution of (P) is an ε-l-weak minimal solution of the unconstrained problem $(\mathrm{UP})_T$ for some $T \in \mathscr{L}_+(Z,Y)$.

Theorem 3.2. *Let (F,G) be nearly $(K \times C)$-convexlike on X, $\mathrm{int}C \neq \emptyset$ and assume problem* (P) *satisfies* CQ. *If x_ε is an ε-l-weak minimal solution of* (P) *and $F(x_\varepsilon)$ has a strong minimum then there exists $T_\varepsilon \in \mathscr{L}_+(Z,Y)$ such that*

$$-T_\varepsilon(G(x_\varepsilon) \cap (-C)) \subseteq (\mathrm{int}K \cup \{0\}) \backslash (\varepsilon e + \mathrm{int}K), \tag{3.3}$$

and x_ε is an ε-l-weak minimal solution of $(\mathrm{UP})_{T_\varepsilon}$.

Proof. Let $\hat{y}_\varepsilon$ be a strong minimum of $F(x_\varepsilon)$. Define $Q : X \rightrightarrows Y \times Z$ as

$$Q(x) = (F(x) + \varepsilon e - \hat{y}_\varepsilon, G(x)).$$

Clearly, Q is nearly $(K \times C)$-convexlike on X.

We will show that $Q(X) \cap (-\mathrm{int}(K \times C)) = \emptyset$. If $(y',z') \in Q(X) \cap (-\mathrm{int}(K \times C))$ then there exists $x' \in X$ such that $y' \in (F(x') + \varepsilon e - \hat{y}_\varepsilon) \cap (-\mathrm{int}K)$ and $z' \in G(x') \cap (-\mathrm{int}C)$. Hence, $x' \in M$ and there exists $y \in F(x')$ such that

$$y + \varepsilon e - \hat{y}_\varepsilon \in -\mathrm{int}K.$$

Let $y_\varepsilon \in F(x_\varepsilon)$. As $\hat{y}_\varepsilon$ is a strong minimum of $F(x_\varepsilon)$, we have $\hat{y}_\varepsilon = y_\varepsilon - k$ for some $k \in K$. Hence,

$$y + \varepsilon e - y_\varepsilon + k \in -\mathrm{int}K,$$

which implies that

$$y_\varepsilon \in y + \varepsilon e + \mathrm{int}K \subseteq F(x') + \varepsilon e + \mathrm{int}K.$$

Therefore,

$$F(x_\varepsilon) \subseteq F(x') + \varepsilon e + \mathrm{int}K, \tag{3.4}$$

which contradicts the fact that x_ε is an ε-l-weak minimal solution of (P).

Hence, by Lemma 2.1 there exists $(\varphi, \psi) \in (K^+ \times C^+)\backslash\{(0,0)\}$ such that

$$\varphi(y + \varepsilon e - \hat{y}_\varepsilon) + \psi(z) \geq 0, \tag{3.5}$$

for all $x \in X, y \in F(x)$ and $z \in G(x)$. If $\varphi = 0$ then $\psi(z) \geq 0$, for all $x \in X$ and $z \in G(x)$. Since (P) satisfies CQ, there exists $\hat{x} \in X$ such that $\psi(\hat{z}) < 0$ for some $\hat{z} \in G(\hat{x})$ which is a contradiction. Thus, $\varphi \in K^+\backslash\{0\}$. Define $T_\varepsilon : Z \to Y$ as

$$T_\varepsilon(z) = \psi(z)\bar{k},$$

where $\bar{k} \in \mathrm{int}K$ is such that $\varphi(\bar{k}) = 1$. Clearly, $T_\varepsilon \in \mathscr{L}_+(Z, Y)$.

Suppose x_ε is not an ε-l-weak minimal solution of $(\mathrm{UP})_{T_\varepsilon}$, then there exists $\bar{x} \in X$ such that

$$F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)) \subseteq F(\bar{x}) + T_\varepsilon(G(\bar{x})) + \varepsilon e + \mathrm{int}K.$$

Hence, for $z_\varepsilon \in G(x_\varepsilon) \cap (-C)$, we have

$$\hat{y}_\varepsilon + T_\varepsilon(z_\varepsilon) \in \bar{y} + T_\varepsilon(\bar{z}) + \varepsilon e + \mathrm{int}K,$$

for some $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$. Therefore,

$$\hat{y}_\varepsilon + T_\varepsilon(z_\varepsilon) - \bar{y} - T_\varepsilon(\bar{z}) - \varepsilon e \in \mathrm{int}K,$$

which implies that

$$\varphi(\bar{y} + \varepsilon e - \hat{y}_\varepsilon) + \varphi(T_\varepsilon(\bar{z}) - T_\varepsilon(z_\varepsilon)) < 0.$$

Since $z_\varepsilon \in -C$ we have

$$\varphi(T_\varepsilon(z_\varepsilon)) = \psi(z_\varepsilon) \leq 0.$$

Hence,

$$\varphi(\bar{y} + \varepsilon e - \hat{y}_\varepsilon) + \psi(\bar{z}) < 0,$$

which contradicts (3.5).

For $z_\varepsilon \in G(x_\varepsilon) \cap (-C)$ we have $\psi(z_\varepsilon) \leq 0$ and $T_\varepsilon(z_\varepsilon) = \psi(z_\varepsilon)\bar{k}$ which implies that

$$-T_\varepsilon(z_\varepsilon) \in \mathrm{int}K \cup \{0\}.$$

In particular taking $x = x_\varepsilon, y = \hat{y}_\varepsilon, z = z_\varepsilon$ in (3.5) we get

$$\psi(z_\varepsilon) \geq -\varphi(\varepsilon e),$$

which implies that $-T_\varepsilon(z_\varepsilon) \notin \varepsilon e + \mathrm{int}K$. Thus

$$-T_\varepsilon(G(x_\varepsilon) \cap (-C)) \subseteq (\mathrm{int}K \cup \{0\})\backslash(\varepsilon e + \mathrm{int}K). \qquad \square$$

Remark 3.3. In the set criterion approach, the order relation is between two sets, unlike the vector approach where the relation is between a point of a set and another set. Hence, in the proof of the above theorem in (3.4), we require the entire set $F(x_\varepsilon)$ to lie in the set $F(x') + \varepsilon e + \text{int}K$, unlike the vector approach where only one element of $F(x_\varepsilon)$ is required to be in the set $F(x') + \varepsilon e + \text{int}K$. Due to this reason we assume a strong condition, namely, the existence of a strong minimum of $F(x_\varepsilon)$.

Remark 3.4. We now investigate the above theorem for the case $\varepsilon = 0$. Under the assumptions of the above theorem if x_0 is an l-weak minimal solution of (P) and $F(x_0)$ has a strong minimum, then there exists $T_0 \in \mathscr{L}_+(Z, Y)$ such that

$$T_0(G(x_0) \cap (-C)) = \{0\}$$

and x_0 is an l-weak minimal solution of $(\text{UP})_{T_0}$. Even though we have to assume the existence of a strong minimal solution, we do not impose any separation condition as imposed in Theorem 4.1 of [6]. The proof is simplified and we have established the existence of a continuous linear map T instead of the existence of a continuous affine linear map in the formulation of the unconstrained optimization problem. Moreover, we have established the theorem under a generalized notion of cone convexlikeness.

We now give an example to illustrate the above result.

Example 3.5. Consider problem (P) with $X = \mathbb{R}, Y = \mathbb{R}^2, Z = \mathbb{R}, K = \mathbb{R}^2_+$ and $C = \mathbb{R}_+$. Let $0 \leq \varepsilon < 1, e = (1, 1)$ and let $F : X \rightrightarrows Y$ be defined as

$$F(x) = \begin{cases} [(1,1),(0,2)], & \text{if } x \geq 0, x \neq 1, \\ [0,1] \times [0,1], & \text{if } x = 1, \\ \{(1,1)\}, & \text{if } x < 0. \end{cases}$$

Let $G : X \rightrightarrows Z$ be defined as

$$G(x) = \begin{cases} [-1,1], & \text{if } x \in [0,2], \\ \{1,2\}, & \text{if } x > 2, \\ \{-x\}, & \text{if } x < 0. \end{cases}$$

Clearly, M and the set of ε-l-weak minimal solutions is $[0, 2]$, (F, G) is nearly $(K \times C)$-convexlike on X and problem (P) satisfies constraint qualification CQ. For $\hat{x}_\varepsilon = 1$, we observe that $F(\hat{x}_\varepsilon)$ has a strong minimum at $(0, 0)$. Define the linear map $T_\varepsilon : Z \to Y$ as $T_\varepsilon(z) = (\varepsilon^2 z, \varepsilon^2 z)$. Then

$$T_\varepsilon(G(\hat{x}_\varepsilon) \cap (-C)) = \{(-t, -t) : 0 \leq t \leq \varepsilon^2\}.$$

Clearly, T_ε satisfies (3.3) and $\hat{x}_\varepsilon$ is an ε-l-weak minimal solution of $(\text{UP})_{T_\varepsilon}$.

Remark 3.6. We reinforce that we have imposed the condition that $F(x_\varepsilon)$ has a strong minimum in the above theorem. However, in the above example we observe that for $\bar{x}_\varepsilon = 2$, $F(\bar{x}_\varepsilon)$ does not have a strong minimum T_ε satisfies (3.3) and $\bar{x}_\varepsilon$ is an ε-l-weak minimal solution of $(\text{UP})_{T_\varepsilon}$.

4. Lagrangian duality

The aim of this section is to present a dual problem (D) corresponding to the problem (P) and establish weak and strong duality results.

We define a dual map $\Phi : \mathscr{L}_+(Z, Y) \rightrightarrows \mathcal{P}(Y)$ as

$$\Phi(T) := \varepsilon\text{-}l\text{-WMin}(L(x, T) : x \in X),$$

and associate the following dual problem corresponding to (P)

$$\text{(D)} \qquad \text{WMax} \quad \Phi(T)$$
$$\text{subject to} \quad T \in \mathscr{L}_+(Z,Y).$$

Since $\Phi(T) \in \mathcal{P}(\mathcal{P}(Y))$ the notion of ε-l-weak maximal solution of the dual problem needs to be modified accordingly.

Definition 4.1. A pair $(x_\varepsilon, T_\varepsilon) \in X \times \mathscr{L}_+(Z,Y)$ is a feasible pair of (D) if $L(x_\varepsilon, T_\varepsilon) \in \Phi(T_\varepsilon)$.

We denote the set of feasible pairs of (D) by N_ε.

Definition 4.2. An element $T_\varepsilon \in \mathscr{L}_+(Z,Y)$ is said to be an ε-l-weak maximal solution of (D) if

$$\varepsilon\text{-}l\text{-WMax}(L(x_\varepsilon, T_\varepsilon) : (x_\varepsilon, T_\varepsilon) \in N_\varepsilon) \neq \emptyset.$$

This is equivalent to the fact that $T_\varepsilon \in \mathscr{L}_+(Z,Y)$ is an ε-l-weak maximal solution of (D) if there exists $x_\varepsilon \in X$ such that $(x_\varepsilon, T_\varepsilon) \in N_\varepsilon$ and

$$L(x_\varepsilon, T_\varepsilon) <^l L(\bar{x}_\varepsilon, \bar{T}_\varepsilon) - \varepsilon e, (\bar{x}_\varepsilon, \bar{T}_\varepsilon) \in N_\varepsilon \Rightarrow L(\bar{x}_\varepsilon, \bar{T}_\varepsilon) - \varepsilon e <^l L(x_\varepsilon, T_\varepsilon).$$

We observe that if x_ε is an ε-l-weak minimal solution of $(\text{UP})_{T_\varepsilon}$ then $L(x_\varepsilon, T_\varepsilon) \in \Phi(T_\varepsilon)$, that is, $(x_\varepsilon, T_\varepsilon) \in N_\varepsilon$.

We now prove the weak duality theorem and establish the relationship between the primal and the dual problem.

Theorem 4.3. *Let $u_\varepsilon \in M$ and $(x_\varepsilon, T_\varepsilon) \in N_\varepsilon$. If*

$$F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)) \not<^l F(u_\varepsilon) + \varepsilon e, \tag{4.1}$$

then

$$F(u_\varepsilon) + \varepsilon e \not<^l F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)).$$

Proof. As $u_\varepsilon \in M$, there exists $z_\varepsilon \in G(u_\varepsilon) \cap (-C)$ which implies $T_\varepsilon(z_\varepsilon) \in -K$ as $T_\varepsilon \in \mathscr{L}_+(Z,Y)$. Hence,

$$K \subseteq T_\varepsilon(z_\varepsilon) + K \subseteq T_\varepsilon(G(u_\varepsilon)) + K, \tag{4.2}$$

and

$$\text{int}K \subseteq T_\varepsilon(z_\varepsilon) + \text{int}K \subseteq T_\varepsilon(G(u_\varepsilon)) + \text{int}K. \tag{4.3}$$

Suppose (4.1) holds and to the contrary

$$F(u_\varepsilon) + \varepsilon e <^l F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)).$$

Then, using (4.3) we get

$$F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)) \subseteq F(u_\varepsilon) + T_\varepsilon(G(u_\varepsilon)) + \varepsilon e + \text{int}K.$$

Hence,

$$F(u_\varepsilon) + T_\varepsilon(G(u_\varepsilon)) + \varepsilon e <^l F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)). \tag{4.4}$$

Since $(x_\varepsilon, T_\varepsilon) \in N_\varepsilon$, therefore

$$L(x_\varepsilon, T_\varepsilon) \in \Phi(T_\varepsilon).$$

Hence, from (4.4) it follows that

$$F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)) <^l F(u_\varepsilon) + T_\varepsilon(G(u_\varepsilon)) + \varepsilon e. \tag{4.5}$$

Using Lemma 2.2 it follows from (4.4) and (4.5) that

$$F(u_\varepsilon) + T_\varepsilon(G(u_\varepsilon)) + \varepsilon e + K = F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)) + \text{int}K. \tag{4.6}$$

Now using (4.2) and (4.6) we have

$$\begin{aligned} F(u_\varepsilon) + \varepsilon e &\subseteq F(u_\varepsilon) + \varepsilon e + K \\ &\subseteq F(u_\varepsilon) + T_\varepsilon(G(u_\varepsilon)) + \varepsilon e + K \\ &= F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)) + \text{int}K, \end{aligned}$$

which contradicts (4.1). □

Remark 4.4. For $\varepsilon = 0$, if $u_0 \in M$ and $(x_0, T_0) \in N_\varepsilon$ and

$$F(x_0) + T_0(G(x_0)) \not<^l F(u_0),$$

then

$$F(u_0) \not<^l F(x_0) + T_0(G(x_0)).$$

This result is similar to Proposition 2.1 of [10] with the Lagrangian given by $L(x, z, T) = F(x) + T(z)$ where $z \in G(x)$.

Corollary 4.5. *If $u_\varepsilon \in M$ and $(x_\varepsilon, T_\varepsilon) \in N_\varepsilon$ such that*

$$F(u_\varepsilon) \leq^l F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)), \tag{4.7}$$

then u_ε is an ε-l-weak minimal solution of (P) *and T_ε is an ε-l-weak maximal solution of* (D).

Proof. Suppose there exists $\bar{u}_\varepsilon \in M$ such that

$$F(\bar{u}_\varepsilon) + \varepsilon e <^l F(u_\varepsilon).$$

Using (4.7), we have

$$F(\bar{u}_\varepsilon) + \varepsilon e <^l F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)).$$

Since $(x_\varepsilon, T_\varepsilon) \in N_\varepsilon$, using Theorem 4.3, we get

$$F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)) <^l F(\bar{u}_\varepsilon) + \varepsilon e.$$

Again using (4.7), we have

$$F(u_\varepsilon) <^l F(\bar{u}_\varepsilon) + \varepsilon e.$$

Therefore, u_ε is an ε-l-weak minimal solution of (P).

Suppose there exists a feasible pair $(\bar{x}_\varepsilon, \bar{T}_\varepsilon)$ such that

$$L(x_\varepsilon, T_\varepsilon) <^l L(\bar{x}_\varepsilon, \bar{T}_\varepsilon) - \varepsilon e.$$

Using (4.7), we obtain

$$F(u_\varepsilon) + \varepsilon e <^l F(\bar{x}_\varepsilon) + \bar{T}_\varepsilon(G(\bar{x}_\varepsilon)).$$

Again by Theorem 4.3, we have

$$F(\bar{x}_\varepsilon) + \bar{T}_\varepsilon(G(\bar{x}_\varepsilon)) <^l F(u_\varepsilon) + \varepsilon e,$$

which together with (4.7) implies that

$$L(\bar{x}_\varepsilon, \bar{T}_\varepsilon) - \varepsilon e <^l L(x_\varepsilon, T_\varepsilon),$$

and thus, T_ε is an ε-l-weak maximal solution of (D). □

Remark 4.6. For $\varepsilon = 0$, if $u_0 \in M$ and $(x_0, T_0) \in N_\varepsilon$ and (4.7) holds, that is, $F(u_0) \leq^l F(x_0) + T_0(G(x_0))$ then u_0 is an l-weak minimal solution of (P) and T_0 is an l-weak maximal solution of (D). This result improves the Corollary 3.1 of [6]. We have proved the result under the assumption of the quasi order relation (4.7) whereas in [6] to prove the same result authors assume the condition

$$F(u_0) \in [F(x_0) + T_0(G(x_0))]^l.$$

In the following example we show that $F(u_0) \leq^l F(x_0) + T_0(G(x_0))$ holds and hence Corollary 3.1 can be applied but the reverse relation $F(x_0) + T_0(G(x_0)) \leq^l F(u_0)$ does not hold. Hence, Corollary 3.1 of [6] is not applicable.

Example 4.7. Consider problem (P) with $X = \mathbb{R}, Y = Z = \mathbb{R}^2$ and $K = C = \mathbb{R}^2_+$. Let $0 \leq \varepsilon < 1, e = (1, 1)$ and let $F : X \rightrightarrows Y$ be defined as

$$F(x) = \begin{cases} [(x, 0), (0, x)], & \text{if } x \in (0, 2], \\ \{(-10, 0), (0, 0)\}, & \text{if } x = 0, \\ \{(0, 0)\}, & \text{otherwise,} \end{cases}$$

and let $G : X \rightrightarrows Z$ be defined as

$$G(x) = \begin{cases} \{(-x, 0)\}, & \text{if } x \in [0, 1], \\ [(1, 1), (2, 2)], & \text{otherwise.} \end{cases}$$

Here, M and the set of ε-l-weak minimal solutions is $[0, 1]$. Let $T_\varepsilon : Z \to Y$ be defined as $T_\varepsilon(z_1, z_2) = (z_1, z_2)$ and let $x_\varepsilon = 1$. Then $(x_\varepsilon, T_\varepsilon) \in N_\varepsilon$, $u_\varepsilon = 0 \in M$ and (4.7) holds, where

$$F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)) = [(-1, 1), (0, 0)].$$

Hence, by Corollary 4.5, u_ε is an ε-l-weak minimal solution of (P) and T_ε is an ε-l-weak maximal solution of (D). Clearly,

$$F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)) \not\leq^l F(u_\varepsilon).$$

Hence, the Corollary 3.2 of [6] cannot be applied here for $\varepsilon = 0$.

We next give the strong duality result for the problems (P) and (D).

Theorem 4.8. *If x_ε is an ε-l-weak minimal solution of* (P), $G(x_\varepsilon) \subseteq C$ *and the assumptions in Theorem* 3.2 *hold then there exists $T_\varepsilon \in \mathscr{L}_+(Z, Y)$ such that T_ε is an ε-l-weak maximal solution of* (D).

Proof. Let x_ε be an ε-l-weak minimal solution of (P). Then by Theorem 3.2, there exists $T_\varepsilon \in \mathscr{L}_+(Z, Y)$ such that x_ε is an ε-l-weak minimal solution of $(\mathrm{UP})_{T_\varepsilon}$. Hence, $(x_\varepsilon, T_\varepsilon)$ is a feasible pair of (D). Using $G(x_\varepsilon) \subseteq C$ we have

$$F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)) \subseteq F(x_\varepsilon) + K,$$

that is,

$$F(x_\varepsilon) \leq^l F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)).$$

Now using Corollary 4.5, it follows that T_ε is an ε-l-weak maximal solution of (D). □

Remark 4.9. As mentioned in Remark 3.3, in set criteria we need to compare two sets unlike the vector approach where only one element in the intersection is sufficient. Due to this we impose a strong condition $G(x_\varepsilon) \subseteq C$ instead of the assumption of the form $G(x_\varepsilon) \cap C \neq \emptyset$, in the vector approach. This condition has also been considered in Theorem 3.3 in [6].

5. ε-l-WEAK SADDLE POINTS

In this section we introduce the concept of ε-l-weak saddle points of the set-valued Lagrangian map L and give saddle point optimality conditions.

Definition 5.1. An ordered pair $(x_\varepsilon, T_\varepsilon) \in X \times \mathscr{L}_+(Z,Y)$ is said to be an ε-l-weak saddle point of the Lagrangian map L, if

(i) $L(x_\varepsilon, T_\varepsilon) \in \varepsilon$-$l$-WMin$(L(x, T_\varepsilon) : x \in X)$;
(ii) $L(x_\varepsilon, T_\varepsilon) \in \varepsilon$-$l$-WMax$(L(x_\varepsilon, T) : T \in \mathscr{L}_+(Z,Y))$.

Let $L(x_\varepsilon, \mathscr{L}_+(Z,Y)) := \cup_{T\in\mathscr{L}_+(Z,Y)} L(x_\varepsilon, T)$.

We now give sufficient saddle point optimality conditions for the primal and dual problems.

Theorem 5.2. *Let $(x_\varepsilon, T_\varepsilon)$ be an ε-l-weak saddle point of L such that*

$$\varepsilon\text{-WMax}L(x_\varepsilon, \mathscr{L}_+(Z,Y)) \neq \emptyset, \tag{5.1}$$

and WMin$L(x_\varepsilon, T_\varepsilon) \neq \emptyset$. *Then*

(i) $G(x_\varepsilon) \cap (-C) \neq \emptyset$;
(ii) $-T_\varepsilon(z_\varepsilon) \notin \varepsilon e + \mathrm{int}K$, *for every* $z_\varepsilon \in G(x_\varepsilon)$;
(iii) $(x_\varepsilon, T_\varepsilon) \in N_\varepsilon$.

Further, if $T_\varepsilon(G(x_\varepsilon)) \subseteq K$ then x_ε is an ε-l-weak minimal solution of (P) *and T_ε is an ε-l-weak maximal solution of* (D).

Proof. We first show that $G(x_\varepsilon) \cap (-C) \neq \emptyset$. Suppose that $G(x_\varepsilon) \cap (-C) = \emptyset$. Hence, any $z_\varepsilon \in G(x_\varepsilon)$ is such that $z_\varepsilon \notin -C$. Since $C = C^{++}$ we have that there exists $\psi \in C^+$ such that $\psi(z_\varepsilon) > 0$. Let $\bar{k} \in \mathrm{int}K$ be fixed and define $T_n \colon Z \to Y$ as $T_n(z) := n\psi(z)\bar{k}$, for every $z \in Z$ and $n \in \mathbb{N}$. Clearly, $T_n \in \mathscr{L}_+(Z,Y)$ and $T_n(z_\varepsilon) \in T_n(G(x_\varepsilon)) \cap \mathrm{int}K$. For any $T_\varepsilon \in \mathscr{L}_+(Z,Y)$ and $y \in F(x_\varepsilon)$

$$\begin{aligned} n\psi(z_\varepsilon)\bar{k} - T_\varepsilon(z_\varepsilon) - \varepsilon e &\in F(x_\varepsilon) + T_n(z_\varepsilon) - (y + T_\varepsilon(z_\varepsilon)) - \varepsilon e \\ &\subseteq L(x_\varepsilon, T_n) - (y + T_\varepsilon(z_\varepsilon)) - \varepsilon e \\ &\subseteq L(x_\varepsilon, \mathscr{L}_+(Z,Y)) - (y + T_\varepsilon(z_\varepsilon)) - \varepsilon e. \end{aligned}$$

For sufficiently large n we have

$$(L(x_\varepsilon, \mathscr{L}_+(Z,Y)) - (y + T_\varepsilon(z_\varepsilon)) - \varepsilon e) \cap \mathrm{int}K \neq \emptyset.$$

As $y + T_\varepsilon(z_\varepsilon) \in L(x_\varepsilon, T_\varepsilon)$, hence

$$(L(x_\varepsilon, \mathscr{L}_+(Z,Y)) - L(x_\varepsilon, T_\varepsilon) - \varepsilon e) \cap \mathrm{int}K \neq \emptyset$$

for every $T_\varepsilon \in \mathscr{L}_+(Z,Y)$, which contradicts (5.1). Hence, $G(x_\varepsilon) \cap (-C) \neq \emptyset$ and $x_\varepsilon \in M$.

Since $(x_\varepsilon, T_\varepsilon)$ is an ε-l-weak saddle point of L, therefore by definition

$$L(x_\varepsilon, T_\varepsilon) \in \varepsilon\text{-}l\text{-WMax}\{L(x_\varepsilon, T) : T \in \mathscr{L}_+(Z,Y)\}$$

where

$$\begin{aligned} \varepsilon\text{-}l\text{-WMax}\{L(x_\varepsilon, T) : T \in \mathscr{L}_+(Z,Y)\} := \{&L(x_\varepsilon, T_\varepsilon) : \\ &L(x_\varepsilon, T_\varepsilon) <^l L(x_\varepsilon, T) - \varepsilon e, T \in \mathscr{L}_+(Z,Y) \Rightarrow L(x_\varepsilon, T) - \varepsilon e <^l L(x_\varepsilon, T_\varepsilon)\}. \end{aligned}$$

Hence by Lemma 2.5 we observe that, for every $T \in \mathscr{L}_+(Z,Y)$,

$$F(x_\varepsilon) + T(G(x_\varepsilon)) - \varepsilon e \nsubseteq F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)) + \mathrm{int}K.$$

In particular, for $T \equiv 0$ we have

$$F(x_\varepsilon) - \varepsilon e \nsubseteq F(x_\varepsilon) + T_\varepsilon(G(x_\varepsilon)) + \mathrm{int}K,$$

which further implies that $-T(z_\varepsilon) \notin \varepsilon e + \mathrm{int}K$, for every $z_\varepsilon \in G(x_\varepsilon)$. Also, by the definition of ε-l-weak saddle point we have

$$L(x_\varepsilon, T_\varepsilon) \in \varepsilon\text{-}l\text{-WMin}(L(x, T_\varepsilon) : x \in X).$$

Thus, $(x_\varepsilon, T_\varepsilon) \in N_\varepsilon$.

Further, if $T_\varepsilon(G(x_\varepsilon)) \subseteq K$, then the result follows from Corollary 4.5. □

Remark 5.3. If we take $\varepsilon = 0$, we obtain sufficient l-weak saddle point optimality conditions for the primal and dual problems. This result generalizes the corresponding result given by, Hernández and Rodríguez-Marín [6] in terms of l-saddle points.

We now give sufficient conditions for existence of an ε-l-weak saddle point of the Lagrangian map L. The proof follows on the lines of Theorem 5.2 in [6].

Theorem 5.4. *If $\varepsilon > 0$ and $(x_\varepsilon, T_\varepsilon) \in N_\varepsilon$ is such that $T_\varepsilon(G(x_\varepsilon)) \subseteq K$ then $(x_\varepsilon, T_\varepsilon)$ is an ε-l-weak saddle point of L.*

6. Conclusions

The set criterion approach introduced by Kuroiwa [10, 13] is more relevant in the study of set-valued optimization problem. During the last few years the focus has shifted to the study of set-valued optimization using this approach. Moving further in this direction, we have considered approximate Lagrangian function involving continuous linear maps and established Lagrangian multiplier rule and duality results. We established Lagrange multiplier rule under the assumption of the existence of a strong minimum. It would be of interest to investigate if such a strong minimality condition could be relaxed to establish Lagrange multiplier rule. We also introduced the notion of approximate weak saddle point of the Lagrangian map and established saddle point optimality.

Acknowledgements. The authors would like to thank the anonymous referees for providing many helpful suggestions which improved the paper. Research of C.S. Lalitha was supported by R & D Doctoral Research Programme Fund for University faculty.

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