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EQUILIBRIUM BEHAVIOUR AND SOCIAL OPTIMIZATION IN MARKOVIAN QUEUES WITH IMPATIENT CUSTOMERS AND VARIANT OF WORKING VACATIONS

Gopinath Panda[1], Veena Goswami[2] and Abhijit Datta Banik[1]

Abstract. We study customers' equilibrium behaviour and social optimization in a single-server Markovian queue with impatient customers and variant of multiple working vacations, where the impatience is due to slow service rate. Under the variant of multiple working vacations, the server takes a working vacation as soon as the system gets empty. When an arriving customer joins the vacation system, it activates an impatience timer. If its patience timer expires before it gets service, the customer abandons the system, and never returns. The server is allowed to take at most J successive working vacations, if at the end of a working vacation the system remains empty. An arriving customer takes a decision on the basis of available information whether to join or to balk, which unifies wish for the service as well as reluctance to wait. We discuss equilibrium threshold strategies on the basis of linear reward-cost structure in the fully observable and fully unobservable cases. We present numerical results that establish the impact of the information level as well as various parameters on the equilibrium balking strategies and social benefits. The research outputs may be useful for decision makers to convey information to customers in net benefit maximization and for examining the corresponding social optimization problems.

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1. Introduction

Queuing systems with server vacations and impatient customers are studied extensively due to their potential applications in real life congestion problems such as impatient telephone switchboard customers, hospital emergency rooms making vital patient treatment decisions, inventory systems that store perishable goods [17], queues arising in telecommunication networks [3], call centers [7], cloud computing [15], wireless sensor network [14] and in several machine repair problems [22]. A comprehensive analysis of $M/M/1, M/G/1$ and $M/M/c$ queueing models with server vacations and customer impatience is presented in [1, 26], where the source of impatience is the unavailability of server(s). Several authors [18, 25] extended the $M/M/1$ model presented in [1] to include working vacations, whereas the corresponding finite buffer queueing model is studied by Laxmi *et al.* [11].

Keywords. Equilibrium balking strategies, customer impatience, multiple working vacations, social optimization.

[1] School of Basic Sciences, Indian Institute of Technology Bhubaneswar, Odisha, India. gopinath.panda@gmail.com; adattabanik@iitbbs.ac.in

[2] School of Computer Application, KIIT University, Bhubaneswar, Odisha, India. veena_goswami@yahoo.com

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Also, several extensions of the impatient queueing systems are found in the literature, where the impatience is during the server down period [6, 23], or during a slow service period [16].

Takagi [20] considered a new vacation policy called variant vacation, which is a generalization of single and multiple vacations. In this policy, the server is allowed to take a maximum of J consecutive vacations when it encounters an empty system at a vacation completion epoch. Several authors studied this policy in different queueing models. Recently, Yue *et al.* [24] presented the performance measures in an $M/M/1$ queue with impatient customers, using the above policy. An extension of this policy to the working vacation case, is called a variant of working vacations, where the server is allowed to take at most J consecutive working vacations at a vacation termination epoch, until the system is empty. Single as well as multiple working vacations are special cases of variant working vacation policy for $J = 1$ and $J = \infty$, respectively. Few works in this direction are discussed in [12, 28].

During the last few decades, there is an emerging trend to study queueing systems from an economic view point. It is important to consider customers' strategic behaviour in order to get the maximum benefit from a service system. A reward-cost structure is imposed on the system to reflect the customers' desires for service and their unwillingness to wait. Customers are allowed to make decision about their actions upon arrival in the system. They want to maximize their benefit against the other customers who have the same objective, which can be viewed as a game among the customers. These studies discover new managerial policies with the help of equilibrium threshold strategies and socially optimal strategies. Extensive bibliographical references in this area may be found in [10]. The analysis of customers' strategic behaviour in Markovian queues with impatient customers is carried in [9]. Further, several researchers explored the observable and unobservable Markovian queues to include vacations [8], set up times [5], breakdowns or repair [6], catastrophes [4], and retrials [21]. [13] studied equilibrium threshold strategies of both continuous ($M/M/1$) and discrete ($Geo/Geo/1$) time observable queues under a single vacation policy. [27] analyzed the balking behaviour of customers under four different information levels in the $M/M/1$ queue with working vacations. Recently, [19] discussed customers' equilibrium and socially optimal behaviour in an $M/M/1$ queue with multiple working vacations under three information levels such as the observable case, the partially observable case, and the unobservable case.

Although there are many papers in the literature on vacation queues with impatient customers, to the best of our knowledge, there is no study on the strategic behaviour of customers in queueing systems subject to a variant of multiple working vacations. In this paper, we consider an $M/M/1$ queueing systems with a variant of multiple working vacations and impatient customers, where customers become impatient during the working vacation phase of the server. During the working vacation period, the server renders service at a lower rate, rather than completely halting, and the present customers become impatient because of this slower service rate. Each customer, upon arrival to a working vacation phase, activates an independent and exponentially distributed patience timer, with parameter α. If the timer expires, and the customer has not been served and the system remains in working vacation, the customer leaves never to return.

The queueing model studied in this paper is motivated by some practical service systems, where the customers' impatience has a role in the performance of the system. Therefore, it is important to study the equilibrium and social behaviour of the customers in such systems. We investigate the behaviour of impatient customers under two different information levels: the fully observable case and the fully unobservable case. The corresponding strategies are investigated in each case. We obtain the steady state system-length distribution along with the mean sojourn times under these strategies. We also explore the effect of different parameters on the equilibrium thresholds via numerical experiments.

The rest of the paper is organized as follows. In Section 2, basic concepts and notation of the model in described. Section 3 describes the queueing model and its parameters. In Section 4, we analyze the model under the fully observable case and obtain the system length distribution by solving the balance equations. We determine the equilibrium threshold strategies as well as socially optimal behaviour of customers that decide to join or balk upon arrival. Similar performance measures are obtained in the fully unobservable case in Section 5. Section 6 presents a variety of numerical experiments in the form of figures and Section 7 concludes the paper.

2. Basic concepts and notation

In this section, we discuss some basic concepts related to the game among the strategic customers with the objective of increasing understandability.

Noncooperative game theory

Game theory deals with the problems on interacting decision-makers, considered as players. Noncooperative game theory is a branch of game theory for the resolution of conflicts among selfish players, each of which tries to optimize his own objective function.

Equilibrium benefit

Equilibrium benefit occurs when individual customers take actions aimed strictly at optimizing their own outcomes. It is natural to suppose that equilibrium benefit arises when customers act of their own free will.

Social benefit

The social benefit can be understood as the sum of the expected net payoff of all the customers. In order to achieve the maximal social benefit, the social optimum considers all the customers as a whole.

Pareto optimal

Pareto optimality refers to a condition under which a state of economic efficiency occurs. In such a condition, it is not possible to make an individual better off without making another individual worse off. A Pareto-optimal point which is also Nash equilibrium is a solution of the game.

Nash Equilibrium

Nash Equilibrium is an equilibrium point (solution of the competitive game) where each player's strategy is optimal given the strategies of all other players, *i.e.*, it is an equilibrium point where each player is unilaterally happy and does not want to deviate. So, no player in the game would take a different action as long as every other player remains the same.

Nash equilibrium and its relation to Pareto optimal points

Nash equilibrium is not necessarily Pareto optimal. Similarly, a Pareto optimal solution is not necessarily Nash equilibrium. In a Nash Equilibrium, the players have no desire to move because they will be worse off on doing so. In a Pareto Optimal condition, it is not possible to make any player better off without hurting another player at the same time. In other words, a Pareto optimal solution is efficient but a Nash equilibrium is strategically feasible.

Price of anarchy (PoA)

The ratio of optimal and equilibrium social benefit is known as the Price of anarchy (PoA). It measures the degree to which non-cooperation estimates cooperation. It is also, often used to measure the efficiency of a system degradation due to selfish behaviour of its customers.

FTC and ATC

Most of the decision problems stemming from queueing models exhibit one of the two phenomenons: avoid the crowd (ATC) or, follow the crowd (FTC). In ATC strategy, the player's tendency to select an action decreases with the tendency of the others in the population to choose it. This strategy is adopted when faced with the 'to queue or not to queue' decision problem. Customers always wish to avoid delay, and so, it is expected that a customer's willingness to join the queue is adversely affected by other customers decision to enter.

In the opposite case, that is, for FTC strategy, players try to imitate others. In systems that implement a form of dynamic service control, the service rate is usually increased on system congestion. Such systems might result in Follow the Crowd (FTC) behaviour, where customers that arrive during equilibrium conditions are highly encouraged to join the queue as a result of the increased arrival rate. These concepts are discussed in Hassin and Haviv [10].

The following notations are used in this paper.

λ	mean potential arrival rate
λ_e	mean effective arrival rate
μ	mean service rate during regular service
η	mean service rate during working vacation
α	mean patience rate
ϕ^{-1}	mean duration of working vacation
ρ	traffic intensity ($=\lambda/\mu$)
$N_s(t)$:	number of customers in the system at time t.
$\zeta(t)$	state of the server at time t,
	$\zeta(t) = 0$ – server is either idle or busy
	$\zeta(t) = k$ – server is on k^{th} working vacation, $1 \leq k \leq J$.
$L_e(1)$	system threshold when the server is idle or on regular service
$L_e(0)$	system threshold when the server is on working vacation
$\pi_{n,i}$	probability that there are n customers in the system when the server is in state i
R	reward received by a customer after service completion
C	waiting cost per time unit for a customer in the system
Δ	net benefit of a customer
Δ_s	social benefit of the system

3. Description of the model

Consider a single server Markovian queueing system with infinite buffer, wherein the arrival of customers follow a Poisson process with rate λ and the service times are exponentially distributed with rate μ. Customers are served individually with the classical first-come, first-served (FCFS) service discipline. Whenever the system becomes empty, the server switches to a working vacation mode. The duration of the working vacation (WV) period is exponentially distributed with rate ϕ. Before the completion of the vacation period if some customers are found in the system, then they are served with a lower service rate η. The service rate during working vacation is assumed to be exponentially distributed with rate η ($\eta < \mu$). If the server finds the system empty after completion of a vacation, then it takes another vacation and continues the process for at most J working vacations. If there is no customer in the system at the end of Jth WV, then the server remains idle, waiting for the arrival of customers to initiate service. This policy is known as a variant of working vacation (VWV) with exhaustive service. During service of a customer in WV, if the WV terminates before service completion, then the partial service received by the customer with rate η is lost and again its service starts in the regular service mode with rate μ. We represent this model as $M/M/1$ *variant working vacation queue with impatient customers.*

During working vacation period, a customer who joins the empty system will receive service immediately. But if the system is nonempty, the customer has to wait a random length of time for his service to begin. If the customer has to wait longer for service than his expectation, he will lose patience and leave the system without getting served. The patience time of a joining customer is the random amount of time he is willing to wait before his service starts and it is assumed to be exponentially distributed with rate α. If the patience time (or time to renege) terminates before vacation time or vacation service completion, then the customer reneges the system otherwise get served. We assume that reneging is possible only when the server is on a working vacation and is not allowed once a customer starts service. This is because of the slower service rate, for which customers have to wait longer to get service. If there are i customers in the system and the server is on working vacation mode, then the $(i-1)$ queueing customers may renege after their patience time expires. Due to the independence between the arrival and departure of an impatient customer without service, the average reneging rate is $(i-1)\alpha$. The interarrival times, vacation duration times, patience times and service times during regular service and working vacation are all mutually independent. For the existence of stationary distributions of the queueing system, we assume $\rho = \lambda/\mu < 1$.

The state of the system at time t is defined by the random variables, $N_s(t)$ and $\zeta(t)$. The process $\{(N_s(t), \zeta(t)) : t \geq 0\}$ is a continuous time Markov chain (CTMC) with state space $\Omega = \{(n,i) : \; n \geq 0, \; 0 \leq i \leq J\}$. The non-zero state transitions are given below

$$\begin{aligned}
&\widehat{q}_{(n,i),(n+1,i)} = \lambda, \quad n \geq 0, \; 0 \leq i \leq J, && \widehat{q}_{(n+1,0),(n,0)} = \mu, \quad n \geq 1,\\
&\widehat{q}_{(n+1,i),(n,i)} = \eta + (n-1)\alpha, \quad n \geq 0, \; 1 \leq i \leq J, && \widehat{q}_{(1,0),(0,1)} = \mu,\\
&\widehat{q}_{(n,i),(n,0)} = \phi, \quad 1 \leq i \leq J, \; n \geq 1, && \widehat{q}_{(0,i),(0,i+1)} = \phi, \quad 1 \leq i \leq J-1,\\
&\widehat{q}_{(0,J),(0,0)} = \phi.
\end{aligned}$$

Using these transition rates, the steady state balance equations for the model are given by

$$\lambda\pi_{0,0} = \phi\pi_{0,J}, \tag{3.1a}$$

$$(\lambda + \mu)\pi_{n,0} = \lambda\pi_{n-1,0} + \mu\pi_{n+1,0} + \phi\sum_{i=1}^{J}\pi_{n,i}, \; n \geq 1, \tag{3.1b}$$

$$(\lambda + \phi)\pi_{0,1} = \eta\pi_{1,1} + \mu\pi_{1,0}, \tag{3.1c}$$

$$(\lambda + \phi)\pi_{0,i} = \eta\pi_{1,i} + \phi\pi_{0,i-1}, \; 2 \leq i \leq J, \tag{3.1d}$$

$$(\lambda + \phi + \eta + (n-1)\alpha)\pi_{n,i} = \lambda\pi_{n-1,i} + (\eta + n\alpha)\pi_{n+1,i}, \quad 1 \leq i \leq J, \; n \geq 1, \tag{3.1e}$$

where $\{\pi_{n,i} : i = 0, 1, \ldots, J \text{ and } n \geq 0\}$ is the equilibrium distribution of the CTMC $\{(N_s(t), \zeta(t))\}$. The normalization equation for solving the above set of balance equations is given by

$$\sum_{n=0}^{\infty}\pi_{n,0} + \sum_{n=0}^{\infty}\sum_{i=1}^{J}\pi_{n,i} = 1. \tag{3.2}$$

The transition rate diagram is presented in Figure 1.

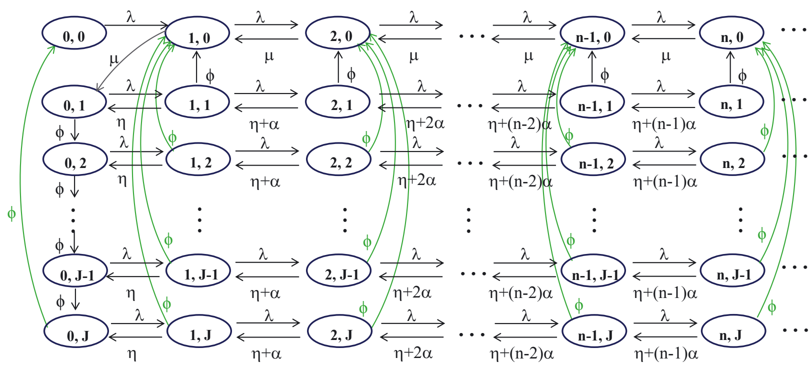

FIGURE 1. Transition rate diagram of the original model.

Our interest is to study the strategic behaviour of customers, regarding their dilemma whether to join or to balk the system upon their arrival. Two different information levels are available to the customers at their arrival instant: the fully observable case: customers obtain the information about the system-length $N_s(t)$ and the state of the server $\zeta(t)$ at their arrival instant; and the fully unobservable case: Customers have neither the information about the system-length $N_s(t)$ nor the state of the server $\zeta(t)$ at their arrival instant. Customers' decision to join or balk depends on the information available to them upon their arrival. To model the decision process, we assume that after service completion, every customer receives a reward of R units. On the other hand, customers have to pay a waiting cost of C units per time unit that is continuously accumulated from the time he arrives at the system till he leaves after being served. A linear cost-reward function is used to study a customer's expected net benefit (Δ) after service completion, defined as $\Delta = R - C\,T(n,i)$ or ($\Delta = R - C\,E(W)$), where $T(n,i)$ ($E(W)$) represents the mean sojourn time of an arriving customer in observable (unobservable) queue. Customers are risk neutral and want to maximize their expected net benefit by making decisions only at their arrival instants. We assume all customers to be identical and homogeneous. They make take decisions to maximize their own benefit keeping in mind that others do the same. However, the decisions of individual customers affect the system delay, which in turn affect the benefit of all customers. Thus, the decision problem can be modelled as a non-cooperative and symmetric game among the customers. We assume that their decisions are irrevocable in the sense that retrial of balking customers is not allowed. In this situation, we will derive the symmetric equilibrium strategies that are best responses against themselves, that is no customer has an incentive to deviate from such a strategy unilaterally. Let $F(x, y)$ be the payoff of a customer that selects a strategy x when others follow the strategy y. An equilibrium (symmetric Nash) strategy is a best response against itself, that is, if all customers agree to follow it no one can benefit by altering it. Mathematically, a strategy s_e is an equilibrium strategy if $F(s_e, s_e) \geq F(s, s_e)$, for every $s \in S$. A strategy s_1 is said to dominate strategy s_2 if $F(s_1, s) \geq F(s_2, s)$, for every $s \in S$ and for at least one s the inequality is strict. A strategy s^* is said to be weakly dominant if it dominates all other strategies in S.

The basic assumption concerning reward cost structure is that the customer is tempted to participate even if the system is empty at the arrival instant. In other words, for an arriving customer who enters the empty system, the reward for service is more than the service cost. So we assume

$$R > \frac{C}{\mu} + \frac{C}{\phi+\eta+\alpha}\left(\frac{\lambda+\phi+\eta}{\lambda+\phi+\eta+\alpha} + \frac{\phi}{\mu}\right), \tag{3.3}$$

for the equilibrium analysis of the balking strategies in all the cases. Otherwise, no customers will ever join the system which becomes empty for the first time. The detailed description of the derivation of condition (3.3) is presented in Appendix A.

4. Fully observable queue with variant working vacations

In this section, we will analyze the fully observable case in which customers are informed about the exact state of the system (n, i) upon arrival. In this case, a pure threshold strategy is specified by a pair $(L_e(0), L_e(1))$ and the balking strategy has the form 'While arriving at time t, observe $(N_s(t), \zeta(t))$; enter if $N_s(t) \leq L_e(\zeta(t))$ and balk otherwise', where $L_e(1)$ is the threshold when an arriving customer finds the system on regular service mode or idle and $L_e(0)$ is the threshold when it is on working vacation. As the queue builds faster in case of working vacation than in case of regular service, so the relation between the thresholds will be $L_e(0) < L_e(1)$. In this case, a weakly dominant equilibrium strategy always exists, *i.e.*, the tagged customer maximizes its expected net benefit, irrespective of the other customers' decision. This strategy is characterized by a pair of thresholds, one for each server state.

Theorem 4.1. *In the fully observable $M/M/1$ queue with impatient customers and a variant of multiple working vacations, there exist equilibrium thresholds*

$$(L_e(0), L_e(1)) = \left(\lfloor x_e \rfloor, \left\lfloor \frac{\mu R}{C} \right\rfloor - 1\right) \tag{4.1}$$

such that the strategy 'observe $(N_s(t), \zeta(t))$; enter if $N_s(t) \leq L_e(\zeta(t))$ and balk otherwise' is a unique Nash equilibrium in the class of threshold strategies (weakly dominant), where x_e is the unique root of

$$R - C\left(A_x + \sum_{k=1}^{x} A_{k-1} \prod_{i=k}^{x} B_i\right) = 0,$$

where $A_n = \frac{1}{\phi+\eta+(n+1)\alpha}\left(\frac{\xi_n}{\xi_{n+1}} + \frac{(n+1)\phi}{\mu}\right)$ and $B_m = \left(\frac{\eta+m\alpha}{\phi+\eta+(m+1)\alpha}\right)$ with $\xi_n = \lambda+\phi+\eta+n\alpha$ for $n = 0, 1, 2, \ldots$, and $\lfloor x_e \rfloor$ represents the integral part of x_e.

Proof. Consider a tagged customer that encounters the system at the state $(n, 0)$ upon arrival and decides to join. The tagged customer will leave the system only after its service completion (no reneging in regular service mode), for which he has to wait for a sum of $(n+1)$ independent and exponentially distributed service times with rate μ. Here, the distribution of the remaining service time of the customer in service, if any, is identical to the service time distribution of the other customers because of the memoryless property of the exponential distribution. Let $T_{n,i}$ denote the expected sojourn time of a customer that does not renege the system, given that the state upon his arrival is $(n, i),\ n \geq 0,\ i = 0, 1, \ldots, J$. Then, we have

$$T_{n,0} = \frac{n+1}{\mu}, \quad n = 0, 1, \ldots, \tag{4.2}$$

The expected net benefit (Δ), if a tagged customer who observes n customers ahead of him upon arrival and decides to join the system, is

$$\Delta_{fo}(n, i) = R - CT_{n,i}.$$

For $i = 0$, the net benefit will be

$$\Delta_{fo}(n, 0) = R - CT_{n,0} = R - C\ \frac{n+1}{\mu}.$$

On the other hand, if the tagged customer encounters the system at state (n, i) upon arrival and decides to join, then his departure from the system will either be due to its service completion or due to reneging. In this setting, the equilibrium benefit of the tagged customer can be calculated only after his service completion with reward R. So the assumption here is that the tagged customer remains in the system until his service completion. Hence, if the tagged customer won't never renege from the system, then his mean sojourn time $T_{n,i}$ can be calculated as a result of three different transitions. Let $P_{(n+1,i)(n+1,0)}$ be the transition probability from state $(n+1, i)$ to state $(n+1, 0)$, $P_{(n+1,i)(n,i)}$ be the transition probability from state $(n+1, i)$ to state (n, i), and $P_{(n+1,i)(n+2,i)}$ be the transition probability from state $(n+1, i)$ to state $(n+2, i)$. We compute $T_{n,i}$ following the similar approach used in [2]. For $n \geq 1$, we get

$$\begin{aligned} T_{n,i} = {} & P_{(n+1,i)(n+1,0)} \left(\frac{1}{\xi_{n+1}} + T_{n,0} \right) + P_{(n+1,i)(n,i)} \left(\frac{1}{\xi_{n+1}} + T_{n-1,i} \right) \\ & + P_{(n+1,i)(n+2,i)} \left(\frac{1}{\xi_{n+1}} + T_{n,i} \right),\ 1 \leq i \leq J. \end{aligned} \tag{4.3}$$

The mean sojourn time of the tagged customer in state $(n+1, i)$ prior to the first transition from the state $(n+1, i)$ to other states is $\frac{1}{\xi_{n+1}}$. The transition probabilities in equation (4.3) can be calculated as

$$\begin{aligned} P_{(n+1,i)(n+1,0)} &= \frac{\phi}{\xi_{n+1}},\ P_{(n+1,i)(n+2,i)} = \frac{\lambda}{\xi_{n+1}}, \\ P_{(n+1,i)(n,i)} &= \frac{(n+1)\alpha}{\xi_{n+1}} \left(\frac{n}{n+1} \right) + \frac{\eta}{\xi_{n+1}} = \frac{n\alpha + \eta}{\xi_{n+1}}, \quad \text{for} \quad n \geq 0. \end{aligned}$$

Thus, $T_{n,i},\ 1 \leq i \leq J$ be given as

$$\begin{aligned} T_{n,i} &= \frac{\phi}{\xi_{n+1}} \left(\frac{1}{\xi_{n+1}} + T_{n,0} \right) + \frac{n\alpha + \eta}{\xi_{n+1}} \left(\frac{1}{\xi_{n+1}} + T_{n-1,i} \right) + \frac{\lambda}{\xi_{n+1}} \left(\frac{1}{\xi_{n+1}} + T_{n,i} \right) \\ &= A_n + B_n T_{n-1,i},\ \text{for } n = 1, 2, \ldots. \end{aligned} \tag{4.4}$$

For $n = 0$, we get

$$T_{0,i} = \frac{1}{\phi + \eta + \alpha} \left(\frac{\xi_0}{\xi_1} + \frac{\phi}{\mu} \right) = A_0. \tag{4.5}$$

Recursively iterating (4.4) and using (4.5), we get

$$
\begin{aligned}
T_{n,i} &= A_n + \sum_{k=1}^{n} A_{k-1} \prod_{j=k}^{n} B_j \\
&= \frac{1}{\phi+\eta+(n+1)\alpha}\left(\frac{\xi_n}{\xi_{n+1}} + \frac{(n+1)\phi}{\mu}\right) \\
&\quad + \sum_{k=1}^{n} \frac{1}{\phi+\eta+k\alpha}\left(\frac{\xi_{k-1}}{\xi_k} + \frac{k\phi}{\mu}\right) \prod_{l=k}^{n}\left(\frac{\eta+l\alpha}{\phi+\eta+(l+1)\alpha}\right),\ n\ge 0, 1\le i\le J.
\end{aligned} \tag{4.6}
$$

It can be easily checked that $T_{n,i}$ is strictly increasing for n. The expected net benefit of a customer who joins the system when he observes state (n,i) is $\Delta_{fo}(n,i) = R - CT_{n,i}$. The customer prefers to join if $\Delta_{fo}(n,i) > 0$; prefers to balk if $\Delta_{fo}(n,i) < 0$ and indifferent between balking and joining if $\Delta_{fo}(n,i) = 0$. Now solving $\Delta_{fo}(n,i) = 0$, we get the threshold values $L_e(i)$ for $i = 0,1$. Here

$$
L_e(1) = \left\lfloor \frac{\mu R}{C} \right\rfloor - 1, \quad \text{and} \quad L_e(0) = \lfloor x_e \rfloor,
$$

where x_e is the unique solution of

$$
\frac{R}{C} = \frac{1}{\phi+\eta+(n+1)\alpha}\left(\frac{\xi_n}{\xi_{n+1}} + \frac{(n+1)\phi}{\mu}\right) + \sum_{k=1}^{x} \frac{1}{\phi+\eta+k\alpha}\left(\frac{\xi_{k-1}}{\xi_k} + \frac{k\phi}{\mu}\right)\prod_{l=k}^{x}\left(\frac{\eta+l\alpha}{\phi+\eta+(l+1)\alpha}\right)\cdot \qquad \square
$$

In the next result, we have presented the stationary distributions of the system.

Theorem 4.2. *Consider a fully observable $M/M/1$ queue with balking, reneging and variant working vacations, in which the customers follow the threshold policy $(L_e(0), L_e(1))$. The stationary probabilities $\{\pi_{n,i} : (n,i) \in \Omega_{fo}\}$ are given by*

$$
\begin{aligned}
\pi_{n,i} &= h_n \Psi^{i-1} \pi_{L_e(0)+1,1}, \quad 0 \le n \le L_e(0)+1,\ 1 \le i \le J, \\
\pi_{n,0} &= u_n\ \pi_{L_e(0)+1,1}, \quad 0 \le n \le L_e(1)+1,
\end{aligned}
$$

where $\Psi = \frac{\phi h_0}{(\lambda+\phi)h_0 - \eta h_1}$ and h_n, u_n and $\pi_{L_e(0)+1,1}$ are given below;

$$
h_{n-1} = \begin{cases} \frac{\phi+\eta+L_e(0)\alpha}{\lambda}, & n = L_e(0)+1, \\ \frac{\lambda+\phi+\eta+(n-1)\alpha}{\lambda} h_n - \frac{\eta+n\alpha}{\lambda} h_{n+1}, & n = L_e(0), \ldots, 1, \end{cases} \tag{4.7}
$$

$$
u_n = \begin{cases} \lambda^{-1}\phi h_0 \Psi^{J-1}, & n = 0, \\ \mu^{-1}\{(\lambda+\phi)h_0 - \eta h_1\}, & n = 1, \\ (1+\rho)u_{n-1} - \rho u_{n-2} - \frac{\phi h_{n-1}(1-\Psi^J)}{\mu(1-\Psi)}, & n = 2, \ldots, L_e(0)+2, \\ (1+\rho)u_{n-1} - \rho u_{n-2}, & n = L_e(0)+3, \ldots, L_e(1)+1. \end{cases} \tag{4.8}
$$

$$
\pi_{L_e(0)+1,1} = \left(\frac{1-\Psi^J}{1-\Psi}\sum_{n=0}^{L_e(0)+1} h_n + \sum_{n=0}^{L_e(1)+1} u_n\right)^{-1}. \tag{4.9}
$$

with the initial value $h_{L_e(0)+1} = 1$.

Proof. If all the customers follow the same threshold strategy, enter if $N_s(t) \le L_e(\zeta(t))$ and balk otherwise, then $\{(N_s(t), \zeta_s(t))\}$ forms a bivariate Markov chain with finite state space $\Omega_{fo} = \{(k,0) : 0 \le k \le L_e(1)+1\} \cup \{(k,i) : 0 \le k \le L_e(0)+1,\ 1 \le i \le J\}$. The transition diagram is depicted in Figure 2. Let $\pi_{k,0}$ $(\pi_{k,i})$ represent the

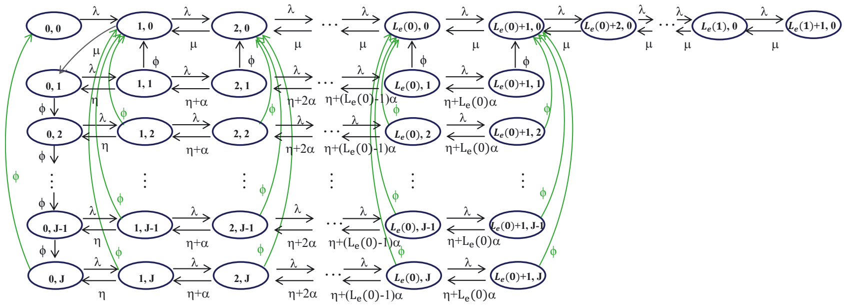

FIGURE 2. Transition rate diagram of the fully observable model with threshold $(L_e(1), L_e(0))$.

probability that there are k customers in the system just prior to the arrival of kth customer when the server is idle or busy (on ith working vacation). These stationary probability distributions can be obtained from the following system of balance equations.

$$\lambda\pi_{0,0} = \phi\pi_{0,J}, \tag{4.10a}$$

$$(\lambda+\mu)\pi_{n,0} = \lambda\pi_{n-1,0} + \mu\pi_{n+1,0} + \phi\sum_{i=1}^{J}\pi_{n,i},\ 1 \le n \le L_e(0)+1, \tag{4.10b}$$

$$(\lambda+\mu)\pi_{n,0} = \lambda\pi_{n-1,0} + \mu\pi_{n+1,0},\ L_e(0)+2 \le n \le L_e(1), \tag{4.10c}$$

$$\mu\pi_{L_e(1)+1,0} = \lambda\pi_{L_e(1),0}, \tag{4.10d}$$

$$(\lambda+\phi)\pi_{0,1} = \eta\pi_{1,1} + \mu\pi_{1,0} \tag{4.10e}$$

$$(\lambda+\phi)\pi_{0,i} = \eta\pi_{1,i} + \phi\pi_{0,i-1},\ 2 \le i \le J \tag{4.10f}$$

$$(\lambda+\phi+\eta+(n-1)\alpha)\pi_{n,i} = \lambda\pi_{n-1,i} + (\eta+n\alpha)\pi_{n+1,i},\ 1 \le n \le L_e(0),\ 1 \le i \le J \tag{4.10g}$$

$$(\phi+\eta+L_e(0)\alpha)\pi_{L_e(0)+1,i} = \lambda\pi_{L_e(0),i},\ 1 \le i \le J, \tag{4.10h}$$

with the normalization condition

$$\sum_{n=0}^{L_e(1)+1}\pi_{n,0} + \sum_{n=0}^{L_e(0)+1}\sum_{i=1}^{J}\pi_{n,i} = 1. \tag{4.11}$$

From (4.10h) and (4.10g), using backward substitution scheme, we get

$$\pi_{n,i} = h_n\pi_{L_e(0)+1,i},\ 0 \le n \le L_e(0)+1,\ 1 \le i \le J,$$

where the unknowns $h_n, 0 \le n \le L_e(0)+1$ are given by (4.7). Using the equation (4.10f) recursively, we get

$$\pi_{L_e(0)+1,i} = \Psi^{i-1}\pi_{L_e(0)+1,1},\ 2 \le i \le J, \tag{4.12}$$

Using (4.12), the steady state probabilities $\pi_{n,i},\ 0 \le n \le L_e(0)+1,\ 1 \le i \le J$, are expressed as

$$\pi_{n,i} = h_n\Psi^{i-1}\pi_{L_e(0)+1,1}.$$

Again, applying the successive backward substitution scheme in the equations (4.10a)–(4.10c), we have

$$\pi_{n,0} = u_n\ \pi_{L_e(0)+1,1},\quad 0 \le n \le L_e(1)+1,$$

where the unknown u_n is given by (4.8). Now, all the stationary probabilities are expressed in terms of the only unknown $\pi_{L_e(0)+1,1}$, which can be evaluated by the use of normalization condition (4.11). □

The unconditional average number of customers in the system when the server is in regular service period (system operating under regular conditions, *i.e.*, customers are getting service with rate μ) is $E(L_1) = \sum_{n=1}^{L_e(1)+1} n\pi_{n,0}$ and the unconditional average number of customers in the system when the server is on working vacations (system operating under specific conditions, *i.e.*, customers are getting service with rate η) is $E(L_0) = \sum_{n=1}^{L_e(0)+1}\sum_{i=1}^{J} n\pi_{n,i}$. The unconditional average number of customers in the system is given by

$$E(L) = E(L_1) + E(L_0) = \sum_{n=1}^{L_e(1)+1} n\pi_{n,0} + \sum_{n=1}^{L_e(0)+1}\sum_{i=1}^{J} n\pi_{n,i}.$$

Because of the PASTA property, the probability of blocking is equal to

$$P_{block} = \pi_{L_e(1)+1,0} + \sum_{i=1}^{J} \pi_{L_e(0)+1,i}.$$

4.1. Equilibrium and socially optimal balking strategy

The fully observable queues acts as a finite buffer $M/M/1$ queueing system where the buffer capacity is $L_e(0)$ during working vacations and $L_e(1)$ during regular service or idle mode. There is a finite service charge (waiting cost) C that has to be paid by every customer after completing service. Then we have such a customer, who strictly prefers to enter if $\Delta_{fo}(n,i) > 0$ and is indifferent between entering and balking if it equals zero and does not enter the system if $\Delta_{fo}(n,i) < 0$. So the customer arriving at time t decides to enter if and only if $N_s(t) \le L_e(j)$, where $L_e(0), L_e(1)$ are obtained by solving the equations $\Delta_{fo}(n,0) = 0,\ \Delta_{fo}(n,i) = 0$, see Theorem 4.1.

The optimal decision of a tagged customer to join or balk upon arrival, is independent of the decision taken by other customers. Due to the FCFS discipline, the expected net benefit of a customer is not affected by the strategies of future customers. Also, in the fully observable case, customers' expected net benefit is not affected by knowing the strategies of past customers. So, the threshold strategy presented in Theorem 4.1 is individually optimized, irrespectively of what the other customers do. Such a strategy is known as weakly dominant as it is the best response against any strategies of the others.

Because of the PASTA property, the probability that an arrival finds the system at state $(L_e(0)+1, i)$ or $(L_e(1)+1, 0)$ and then balks, is equal to $\pi_{L_e(1)+1,0} + \pi_{L_e(0)+1,1} + \pi_{L_e(0)+1,2} + \cdots + \pi_{L_e(0)+1,J}$. The effective arrival rate of the system will be

$$\lambda_e = \lambda\left(\sum_{n=0}^{L_e(1)} \pi_{n,0} + \sum_{n=0}^{L_e(0)}\sum_{i=1}^{J}\pi_{n,i}\right).$$

Hence, the social benefit per time unit $\Delta_s(L_e(0), L_e(1))$, when all customers follow the same equilibrium threshold strategy $(L_e(0), L_e(1))$, can be expressed as

$$\Delta_s(L_e(0), L_e(1)) = \lambda_e R - C\left(\sum_{n=1}^{L_e(1)+1} n\pi_{n,0} + \sum_{n=1}^{L_e(0)+1}\sum_{i=1}^{J} n\pi_{n,i}\right).$$

5. Fully unobservable queue with variant working vacations

In this section, we will study the fully unobservable case, wherein arriving customers neither have the information about the state of the server nor the system-length, while deciding whether to join or balk. In this case, customers' decision is equivalent to selecting a joining probability $f,\ 0 \le f \le 1$, which is a mixed strategy. If all arriving customers follow the strategy f, then customers enter the system with effective arrival rate λf. The joining decisions of individual customers (assumed indistinguishable) affect the system delay and thus the benefit of other customers. Hence, the situation can be considered as a symmetric game among the customers and the optimal decision of a customer has to take into account the strategies of the other customers. We are interested to investigate the customer's individually and socially optimal behaviour. Now, we have an $M/M/1$ queue with a variant working vacations and impatience customers, where the arrival rate is given by λf, and the distributions of the service times, the vacation time and patience time are specified as in the original model. For the derivation of the stationary probabilities, we assume the criterion $\lambda f < \mu$. The state space is $\Omega_{fu} = \{(n,i) : 0 \leq i \leq J, n \geq 0\}$ and the transition rate diagram is illustrated in Figure 3. The stationary distributions of the system can be obtained from the balance equations given below.

$$\lambda f \pi_{0,0} = \phi \pi_{0,J} \tag{5.1a}$$

$$(\lambda f + \mu)\pi_{n,0} = \lambda f \pi_{n-1,0} + \mu \pi_{n+1,0} + \phi \sum_{i=1}^{J} \pi_{n,i}, \quad n \geq 1, \tag{5.1b}$$

$$(\lambda f + \phi)\pi_{0,1} = \mu \pi_{1,0} + \eta \pi_{1,1}, \tag{5.1c}$$

$$(\lambda f + \phi)\pi_{0,i} = \phi \pi_{0,i-1} + \eta \pi_{1,i}, \quad 2 \leq i \leq J, \tag{5.1d}$$

$$(\lambda f + \phi + \eta + (n-1)\alpha)\pi_{n,i} = \lambda f \pi_{n-1,i} + (\eta + n\,\alpha)\pi_{n+1,i}, \quad 1 \leq i \leq J,\ n \geq 1. \tag{5.1e}$$

Lemma 5.1. *In the fully unobservable $M/M/1$ queue with impatient customers and a variant of working vacations wherein all customers follow a mixed balking strategy f, the stationary probability distributions are given by*

$$\pi_{0,0} = \frac{\phi}{\lambda f}\ \widehat{\Psi}^{J-1}\ \pi_{0,1}, \tag{5.2}$$

$$\pi_{1,0} = \frac{\phi}{\mu}\ \widehat{\Psi}^{-1}\ \pi_{0,1}, \tag{5.3}$$

$$\pi_{n,0} = \left(1 + \frac{\lambda f}{\mu}\right)\ \pi_{n-1,1} - \frac{\lambda f}{\mu}\ \pi_{n-2,0} - \frac{\phi}{\mu} \sum_{i=1}^{J} \pi_{n-1,i}, \quad n \geq 2, \tag{5.4}$$

$$\pi_{0,i} = \widehat{\Psi}^{i-1}\ \pi_{0,1}, \quad 1 \leq i \leq J, \tag{5.5}$$

$$\pi_{1,i} = \frac{\widehat{\Psi}^{i-2}}{\eta}\Big(\lambda f \widehat{\Psi} - \phi(1 - \widehat{\Psi})\Big)\pi_{0,1}, \quad 1 \leq i \leq J, \tag{5.6}$$

$$\pi_{n,i} = \frac{\lambda f + \phi + \eta + (n-2)\alpha}{\eta + (n-1)\alpha}\ \pi_{n-1,i} - \frac{\lambda f}{\eta + (n-1)\alpha}\ \pi_{n-2,i}, \quad n \geq 2,\ 1 \leq i \leq J, \tag{5.7}$$

where $\widehat{\Psi} = \frac{\phi K_1(1)}{(\eta-\alpha)K_2(1)}$.

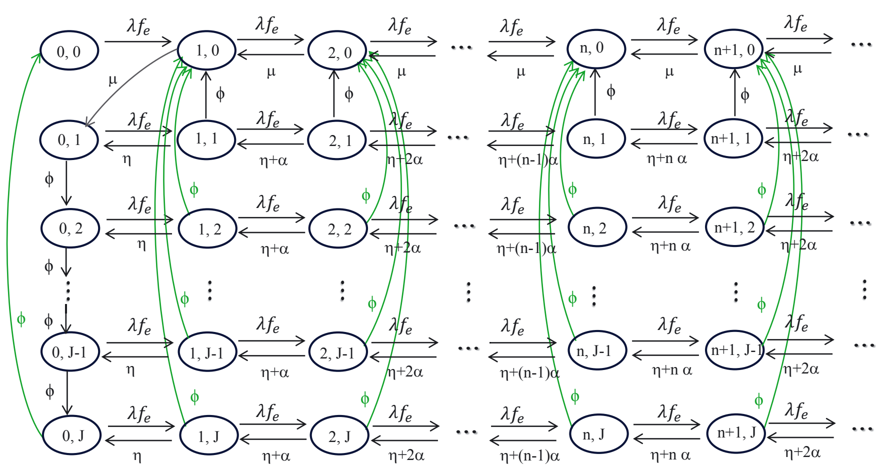

FIGURE 3. Transition rate diagram for the mixed strategy f_e in the fully unobservable model.

Proof. The details of the proof is presented in Appendix B. □

The probability that the system is busy (or on working vacation), denoted by p_0 (p_1) and are given by

$$p_0 = \sum_{n=0}^{\infty} \pi_{n,0} = \frac{\phi\pi_{0,1}}{\mu-\lambda f}\left(\frac{\delta\Big(\lambda f-(\eta-\alpha)(1-\widehat{\Psi})\Big)}{(\phi+\alpha)} + \frac{\mu\widehat{\Psi}^{J-1}}{\lambda f}\right),$$

$$p_1 = \sum_{n=0}^{\infty}\sum_{i=1}^{J} \pi_{n,i} = \delta\pi_{0,1}.$$

The conditional average number of the customers in the busy (or on working vacation) system, denoted by $E[N^-|0]$ ($E[N^-|WV]$) and are given by

$$E[N^-|0] = \frac{\frac{\delta(\lambda f-(\eta-\alpha)(1-\widehat{\Psi}))}{\phi+\alpha}\left\{\frac{\mu}{\mu-\lambda f}+\frac{\lambda f-\eta-\phi}{\phi+2\alpha}\right\} + \frac{\mu\widehat{\Psi}^{J-1}}{\mu-\lambda f} + \frac{\delta\lambda f}{\phi+2\alpha}}{\frac{\delta\Big(\lambda f-(\eta-\alpha)(1-\Psi)\Big)}{(\phi+\alpha)} + \frac{\mu\widehat{\Psi}^{J-1}}{\lambda f}},$$

$$E[N^-|WV] = \frac{\lambda f-(\eta-\alpha)(1-\widehat{\Psi})}{\phi+\alpha}\cdot$$

The average number of the customers in the system in the fully unobservable queue is

$$\begin{aligned} E(N_s) &= \sum_{n=1}^{\infty} n\pi_{n,0} + \sum_{n=1}^{\infty}\sum_{i=1}^{J} n\pi_{n,i} \\ &= \frac{\delta\Big(\lambda f-(\eta-\alpha)(1-\widehat{\Psi})\Big)\pi_{0,1}}{\phi+\alpha}\left\{1+\frac{\phi}{\mu-\lambda f}\left(\frac{\mu}{\mu-\lambda f}+\frac{\lambda f-\eta-\phi}{\phi+2\alpha}\right)\right\} \\ &\quad + \frac{\phi\pi_{0,1}}{\mu-\lambda f}\left(\frac{\mu\widehat{\Psi}^{J-1}}{\mu-\lambda f}+\frac{\delta\lambda f}{\phi+2\alpha}\right)\cdot \end{aligned} \tag{5.8}$$

5.1. Equilibrium and socially optimal balking strategy

In this case, a mixed strategy for an arriving customer is specified by the joining probability f. The equilibrium behaviour of the customers is reported as follows:

Lemma 5.2. *In the fully unobservable $M/M/1$ queue with variant working vacations and impatience customers, the expected mean sojourn time of a customer who decides to join the system, is strictly increasing for $f \in [0,1]$.*

Proof. The expected mean sojourn time of an arriving customer who decides to join the system can be obtained from (5.8) using Little's law

$$E(W) = (g_1(f)+g_2(f))\pi_{0,1}, \tag{5.9}$$

where

$$g_1(f) = \frac{\delta\Big(\lambda f-(\eta-\alpha)(1-\widehat{\Psi})\Big)}{(\phi+\alpha)\,\lambda f}\left\{1+\frac{\phi}{\mu-\lambda f}\left(\frac{\mu}{\mu-\lambda f}+\frac{\lambda f-\eta-\phi}{\phi+2\alpha}\right)\right\},$$

$$g_2(f) = \frac{\phi}{(\mu-\lambda f)\,\lambda f}\left(\frac{\mu\widehat{\Psi}^{J-1}}{\mu-\lambda f}+\frac{\delta\lambda f}{\phi+2\alpha}\right)\cdot$$

To show $E(W)$ is an increasing function, it means to show that both $g_1(f)$ and $g_2(f)$ are increasing functions for $f \in [0,1]$. We express $g_1(f)$ as the product of two functions as $g_1(f) = h_1(f)h_2(f)$ to reduce the complexity in our proof, where $h_1(f) = \frac{\delta}{\lambda f}\frac{\lambda f-(\eta-\alpha)(1-\widehat{\Psi})}{(\phi+\alpha)}$ and $h_2(f) = 1 + \frac{\phi}{\mu-\lambda f}\left(\frac{\mu}{\mu-\lambda f} + \frac{\lambda f-\eta-\phi}{\phi+2\alpha}\right)$. Differentiating $h_1(f),\ h_2(f)$ with respect to f, we get

$$\begin{aligned} h_1'(f) &= \frac{\delta(\eta-\alpha)(1-\widehat{\Psi})}{\lambda(\phi+\alpha)}\frac{1}{f^2},\\ h_2'(f) &= \frac{\phi\lambda}{(\mu-\lambda f)^2}\left[\frac{2\mu}{\mu-\lambda f} + \frac{\mu-\eta}{\phi+2\alpha} - \frac{\phi}{\phi+2\alpha}\right]. \end{aligned}$$

We have $h_1'(f) > 0$ for $\eta > \alpha$ and $h_2'(f) > 0$, since $\mu > \lambda f, \mu > \eta$. Also it can be easily checked that $h_1(f),\ h_2(f) > 0$ under the same conditions. Using the product rule of derivative for $g_1(f) = h_1(f)h_2(f)$, that is, $g_1'(f) = h_1(f)h_2'(f) + h_1'(f)h_2(f)$, we get $g_1(f)$ to be an increasing function. Again differentiating $g_2(f)$ with respect to f and simplifying, we have

$$\begin{aligned} g_2'(f) &= \frac{\phi}{(\mu-\lambda f)^2\ \lambda^2 f^2}\left[\lambda f(\mu-\lambda f)\left(\frac{\mu\widehat{\Psi}^{J-1}\lambda}{(\mu-\lambda f)^2} + \frac{\delta\lambda}{\phi+2\alpha}\right) - (\mu\lambda - 2\lambda^2 f)\left(\frac{\mu\widehat{\Psi}^{J-1}}{\mu-\lambda f} + \frac{\delta\lambda f}{\phi+2\alpha}\right)\right]\\ &= \frac{\phi}{(\mu-\lambda f)^2\ \lambda f^2}\left[\frac{\mu\widehat{\Psi}^{J-1}(3\lambda f-\mu)}{(\mu-\lambda f)} + \frac{\delta\lambda^2 f^2}{\phi+2\alpha}\right]. \end{aligned}$$

It is seen that $g_2(f)$ is an increasing function, because of $g_2'(f) > 0$. As both $g_1(f)$ and $g_2(f)$ are increasing for $f \in [0,1]$, hence their sum $E(W)$ is an increasing function. $\square$

Theorem 5.3. *In the fully unobservable $M/M/1$ queue with variant working vacations and impatient customers and $\lambda f < \mu$, $\eta < \mu$, a unique Nash equilibrium mixed strategy 'enter with probability f_e' exists, where f_e is given by*

$$f_e = \begin{cases} f_{\mathrm{e}}^*, & \text{if } R \in \left(\frac{C}{\mu} - C\frac{1-\widehat{\Psi}^J}{\widehat{\Psi}^J}\frac{\eta-\alpha}{\mu\phi}\frac{2\alpha(\mu+\phi)+\phi(\mu-\eta)}{(\phi+\alpha)(\phi+2\alpha)}, C\frac{\gamma\left(\mu-\lambda+\frac{\phi\mu}{\mu-\lambda}+\frac{\phi(\lambda-\eta-\phi)}{\phi+2\alpha}\right)+(\phi+\alpha)\left(\frac{\phi\mu\,\widehat{\Psi}^{J-1}}{\mu-\lambda}+\frac{\phi\delta\,\lambda}{\phi+2\,\alpha}\right)}{\gamma\lambda\phi+\delta\lambda(\phi+\alpha)(\mu-\lambda)+\mu\phi(\phi+\alpha)\widehat{\Psi}^{J-1}}\right)\\[2ex] 1, & \text{if } R \in \left[C\frac{\gamma\left(\mu-\lambda+\frac{\phi\mu}{\mu-\lambda}+\frac{\phi(\lambda-\eta-\phi)}{\phi+2\alpha}\right)+(\phi+\alpha)\left(\frac{\phi\mu\,\widehat{\Psi}^{J-1}}{\mu-\lambda}+\frac{\phi\delta\,\lambda}{\phi+2\,\alpha}\right)}{\gamma\lambda\phi+\delta\lambda(\phi+\alpha)(\mu-\lambda)+\mu\phi(\phi+\alpha)\widehat{\Psi}^{J-1}}, \infty\right) \end{cases}$$

where f_{e}^* is the unique solution of equation $R - CE(W) = 0$.

Proof. If a tagged customer decides to join the system upon arrival, then his expected net benefit will be

$$\begin{aligned} \varDelta_{fu}(f) = R - CE(W) = R &- C\frac{\phi\pi_{0,1}}{(\mu-\lambda f)\ \lambda f}\left(\frac{\mu\widehat{\Psi}^{J-1}}{\mu-\lambda f} + \frac{\delta\lambda f}{\phi+2\alpha}\right)\\ &- C\frac{\delta\Big(\lambda f-(\eta-\alpha)(1-\widehat{\Psi})\Big)}{(\phi+\alpha)\ \lambda f}\left\{1 + \frac{\phi}{\mu-\lambda f}\left(\frac{\mu}{\mu-\lambda f} + \frac{\lambda f-\eta-\phi}{\phi+2\alpha}\right)\right\}\pi_{0,1}. \end{aligned} \tag{5.10}$$

It can be easily shown that $\varDelta_{fu}(f)$ is strictly decreasing for $f \in [0,1]$. Now

$$\varDelta_{fu}(0) = R - C\left[\frac{1}{\mu} - \frac{1-\widehat{\Psi}^J}{\widehat{\Psi}^J}\frac{\eta-\alpha}{\mu\phi}\frac{2\alpha(\mu+\phi)+\phi(\mu-\eta)}{(\phi+\alpha)(\phi+2\alpha)}\right],$$

and

$$\begin{aligned}\Delta_{fu}(1) &= R - C\frac{\gamma\left(\mu-\lambda+\frac{\phi\mu}{\mu-\lambda}+\frac{\phi(\lambda-\eta-\phi)}{\phi+2\alpha}\right)+(\phi+\alpha)\left(\frac{\phi\mu\widehat{\Psi}^{J-1}}{\mu-\lambda}+\frac{\phi\delta\lambda}{\phi+2\alpha}\right)}{\gamma\lambda\phi+\delta\lambda(\phi+\alpha)(\mu-\lambda)+\mu\phi(\phi+\alpha)\widehat{\Psi}^{J-1}}\\ &= R - C\frac{\gamma(\mu-\lambda)+\frac{\phi\mu}{\mu-\lambda}\left[\gamma+(\phi+\alpha)\widehat{\Psi}^{J-1}\right]+\frac{\phi}{\phi+2\alpha}\left[\gamma(\lambda-\eta-\phi)+\delta\lambda(\phi+\alpha)\right]}{\gamma\lambda\phi+\delta\lambda(\phi+\alpha)(\mu-\lambda)+\mu\phi(\phi+\alpha)\widehat{\Psi}^{J-1}},\end{aligned}$$

where $\gamma = \delta\left(\lambda-(\eta-\alpha)(1-\widehat{\Psi})\right)$. □

If $R \in \left(\frac{C}{\mu} - C\frac{1-\widehat{\Psi}^J}{\widehat{\Psi}^J}\frac{\eta-\alpha}{\mu\phi}\frac{2\alpha(\mu+\phi)+\phi(\mu-\eta)}{(\phi+\alpha)(\phi+2\alpha)}, C\frac{\gamma\left(\mu-\lambda+\frac{\phi\mu}{\mu-\lambda}+\frac{\phi(\lambda-\eta-\phi)}{\phi+2\alpha}\right)+(\phi+\alpha)\left(\frac{\phi\mu\widehat{\Psi}^{J-1}}{\mu-\lambda}+\frac{\phi\delta\lambda}{\phi+2\alpha}\right)}{\gamma\lambda\phi+\delta\lambda(\phi+\alpha)(\mu-\lambda)+\mu\phi(\phi+\alpha)\widehat{\Psi}^{J-1}}\right)$, then equation (5.10) has a unique solution in $(0,1)$, which is denoted by f_e^*. On the other hand if

$$R \in \left[C\frac{\gamma\left(\mu-\lambda+\frac{\phi\mu}{\mu-\lambda}+\frac{\phi(\lambda-\eta-\phi)}{\phi+2\alpha}\right)+(\phi+\alpha)\left(\frac{\phi\mu\widehat{\Psi}^{J-1}}{\mu-\lambda}+\frac{\phi\delta\lambda}{\phi+2\alpha}\right)}{\gamma\lambda\phi+\delta\lambda(\phi+\alpha)(\mu-\lambda)+\mu\phi(\phi+\alpha)\widehat{\Psi}^{J-1}}, \infty\right),$$

then $\Delta_{fu}(f)$ is positive for every f in $(0,1)$. In this situation, the tagged customer's best decision is to enter the system, *i.e.*, $f_e = 1$. Hence, 'joining the system' is the unique Nash equilibrium strategy, which is a dominant one.

The social benefit per time unit, when all joining customers follow a mixed strategy f, is given by

$$\begin{aligned}\Delta_s(f) = \lambda f(R - CE(W)) &= \lambda f R - C\frac{\phi\pi_{0,1}}{\mu-\lambda f}\left(\frac{\mu\widehat{\Psi}^{J-1}}{\mu-\lambda f}+\frac{\delta\lambda f}{\phi+2\alpha}\right)\\ &\quad - C\frac{\delta\left(\lambda f-(\eta-\alpha)(1-\widehat{\Psi})\right)\pi_{0,1}}{\phi+\alpha}\left\{1+\frac{\phi}{\mu-\lambda f}\left(\frac{\mu}{\mu-\lambda f}+\frac{\lambda f-\eta-\phi}{\phi+2\alpha}\right)\right\}.\end{aligned}$$

The first derivatives of $\Delta_s(f)$ with respect to f are, respectively

$$\Delta_s'(f) = \lambda R - C\lambda E(W) - C\lambda f E'(W), \quad \text{and} \quad \Delta_s''(f) = -C\lambda E'(W) - C\lambda f E''(W).$$

Since $\Delta_s''(f) < 0$ for $f \in [0,1]$, so $\Delta_s(f)$ is strictly concave down in $[0,1]$. Let the socially optimal joining probability be f^* and the corresponding socially optimal joining rate be $\lambda^* = \lambda f^*$. Since $\Delta_s(f)$ is strictly concave down for f in $[0,1]$, there exist a unique maximum of $\Delta_s(f)$, which can be obtained by solving the equation $\Delta_s'(f) = 0$. Thus, the social benefit increases with λ, attains a maximum value at λ^* and then it decreases with further increase in λ. The graphical representation of this behaviour is illustrated in Figure 17.

6. Numerical results and discussion

In this section, we illustrate through some numerical experiments the effect of the system parameters on the behaviour of the customers under two different strategies explained above. In particular, we are interested in the values of the equilibrium thresholds and socially optimal behaviour for the fully observable model and the values of equilibrium joining probabilities as well as the equilibrium social benefits for the fully unobservable model. Using maple 17 software, the data is obtained for various system parameters for the models discussed.

Firstly, we consider the fully observable model and explore the sensitivity of the equilibrium pure threshold policies $(L_e(0), L_e(1))$ with respect to the system parameters R, C, μ, α, η and ϕ. Figure 4 presents the impact of α on thresholds $L_e(0)$ and $L_e(1)$. When the patience rate α varies, the threshold $L_e(1)$ remains fixed since the rate α is irrelevant to the customer's decision when it has full state information. But, the threshold $L_e(0)$

increases with the patience rate α. It indicates that an arriving customers' joining behaviour has positive externality to future customers, when the patience rate is relatively high. This shows that availability of full information to customers about the system upon arrival helps them to make their decision.

Figure 5 shows that as service reward R increases, there is initially a linear increase in the equilibrium threshold ($L_e(0) < L_e(1)$) and as reward rate increases beyond a certain value, $L_e(0)$ surpasses $L_e(1)$. It is intuitive that an arriving customer is more likely to join the system if the reward rate is high. This can help managers make decisions about the reward that should be offered in order to encourage impatient customers to stay in the system.

Figure 6 exhibits that the degree of customer's unwillingness to join the queue increases as the cost of waiting increases. The variance of equilibrium threshold based on the service rate during working vacation η is depicted in Figure 7. During working vacation as η increases, so does the customer's willingness to join the queue. Also, η obviously becomes irrelevant when the server performs regular service. The behaviour of equilibrium thresholds with respect to varying values of service rate μ is studied in Figure 8. As μ increases, there is an increase in both the equilibrium thresholds. This can be understood on considering that a higher value of μ signals that the server can serve a higher number of customers per time unit. Figure 9 displays the impact that varying working vacation ϕ has on the equilibrium thresholds. When working vacation duration is shorter, customers have a greater incentive to join the system. Thus, the equilibrium threshold in working vacation $L_e(0)$ increases gradually as ϕ varies, while $L_e(1)$ remains constant.

Figure 11 presents variance of social benefit with respect to reneging rate α. As expected, social benefit increases as α increases. Figure 11 shows the behaviour of social benefit with respect to a variance in the number of working vacation phases. At the beginning, the social benefit is maximum for $J = 1$ (single working vacation case) and then gradually decreases with increase in the value of J and then remains unchanged for further increase in J. A judicious choice of J will help the decision makers to get the optimal benefit out of the impatient customers. The drop in the social benefit may be regarded as the value of revealing information on server status.

In the second numerical experiment, we turn our attention to the fully unobservable queueing model where customers have no information about the system state. Here, we have studied the sensitivity analysis of equilibrium joining probabilities (f_e) under various system parameters and explore some notable behaviour. Figures 12–14 study the effect of equilibrium joining probabilities under service rate during working vacation, number of phases of working vacation and reward rate, respectively. It is observed that f_e increases with increase

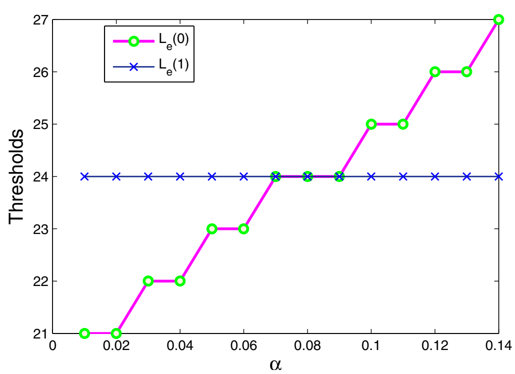

FIGURE 4. Equilibrium thresholds *vs.* α for $\lambda = 1, \mu = 5, \eta = 2, \phi = 1, J = 10, R = 10, C = 2$.

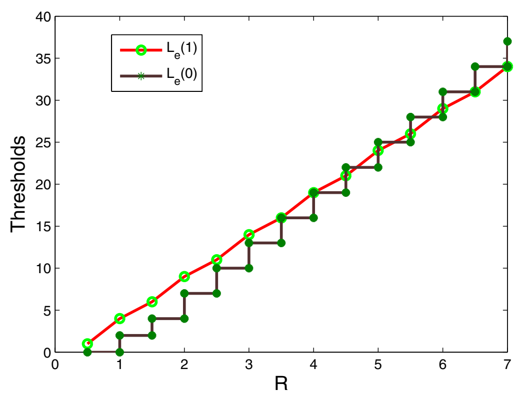

FIGURE 5. Equilibrium thresholds *vs.* R for $\lambda = 1, \mu = 5, \eta = 2, \phi = 1, \alpha = 0.1, J = 10, C = 1$.

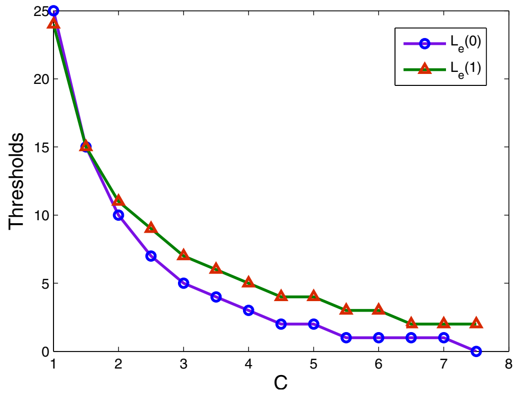

FIGURE 6. Equilibrium threshold *vs.* C for $\lambda = 1, \mu = 5, \eta = 2, \phi = 1, \alpha = 0.1, J = 15, R = 5$.

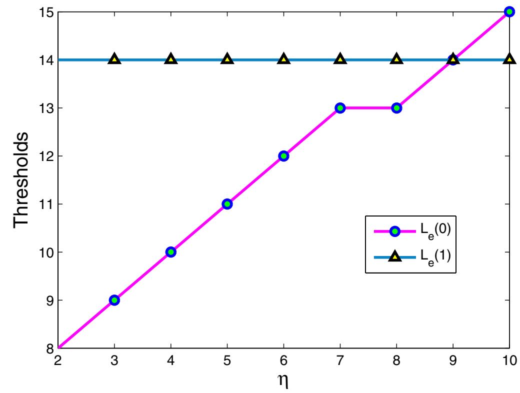

FIGURE 7. Equilibrium threshold *vs.* η for $\lambda = 1, \mu = 10, \phi = 1, \alpha = 0.1, J = 10, R = 3, C = 2$.

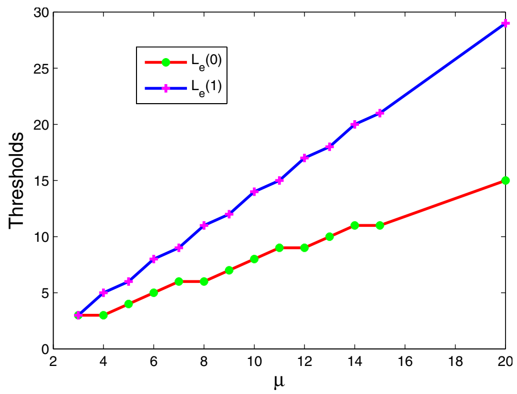

FIGURE 8. Equilibrium threshold *vs.* μ for $\lambda = 1, \eta = 2, \phi = 1, \alpha = 0.1, J = 10, R = 3, C = 2$..

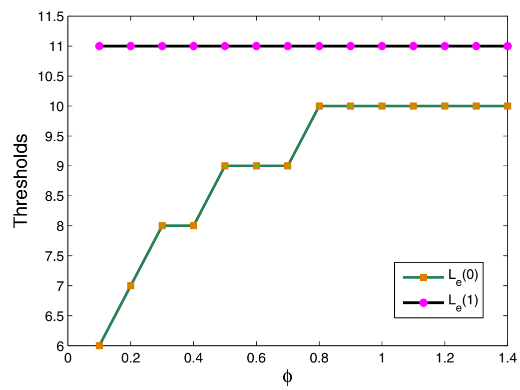

FIGURE 9. Equilibrium threshold *vs.* ϕ for $\lambda = 1, \mu = 5, \eta = 2, \alpha = 0.1, J = 10, R = 5, C = 2$..

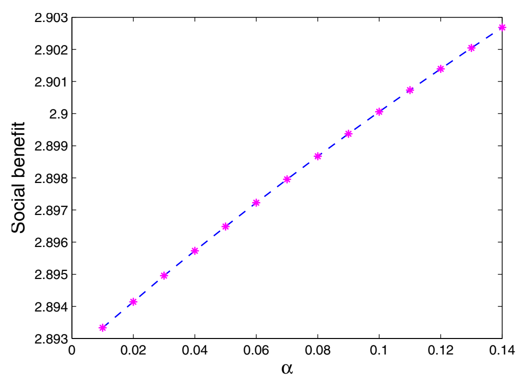

FIGURE 10. Social benefit *vs.* α for $\lambda = 1, \mu = 5, \eta = 2, \phi = 1, J = 10, R = 5, C = 2$.

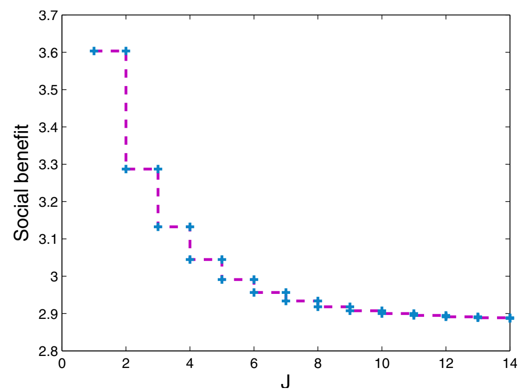

FIGURE 11. Social benefit *vs.* J for $\lambda = 1, \mu = 5, \eta = 2, \phi = 1, \alpha = 0.1, R = 5, C = 2$.

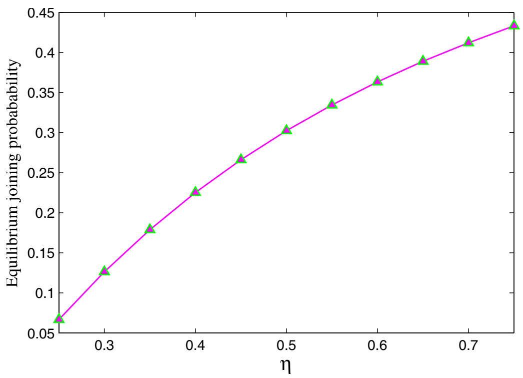

FIGURE 12. Dependence of f_e^* on η for $\lambda = 0.4, \mu = 1, \phi = 0.1, \alpha = 0.2, J = 10, R = 4, C = 2$.

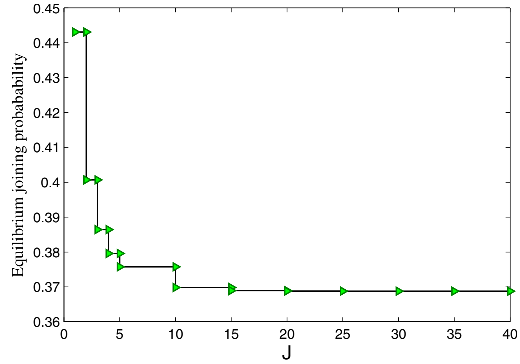

FIGURE 13. f_e^* as a function of J for $\lambda = 0.2, \mu = 1, \eta = 0.5, \phi = 0.1, \alpha = 0.2, R = 4, C = 2$.

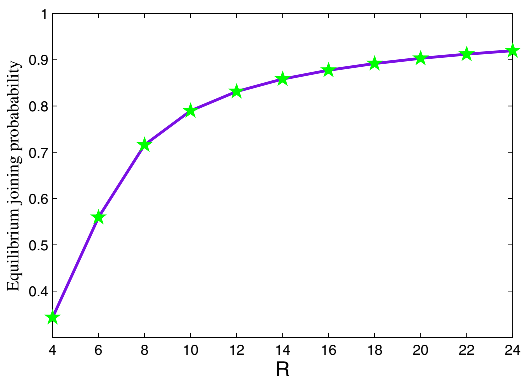

FIGURE 14. f_e^* as a function of R for $\lambda = 0.5, \mu = 1, \eta = 0.5, \phi = 0.1, \alpha = 0.2, J = 10, C = 2$.

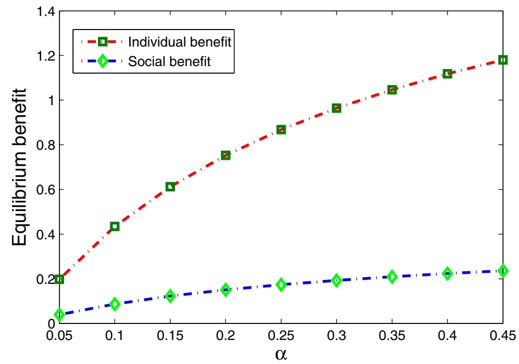

FIGURE 15. Net benefits *vs.* α for $\lambda = 0.5, \mu = 1, \eta = 0.5, \phi = 0.1, f = 0.4, J = 10, R = 5, C = 2$.

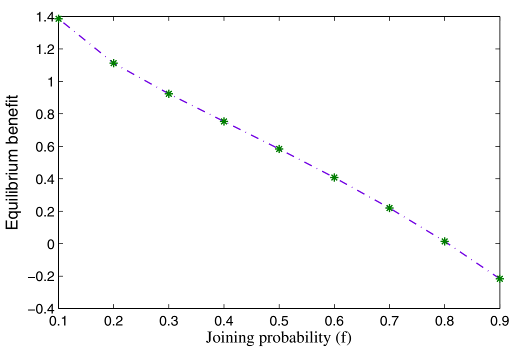

FIGURE 16. Net benefits *vs.* f for $\lambda = 0.5, \mu = 1, \eta = 0.5, \phi = 0.1, \alpha = 0.2, R = 5, C = 2, J = 10$.

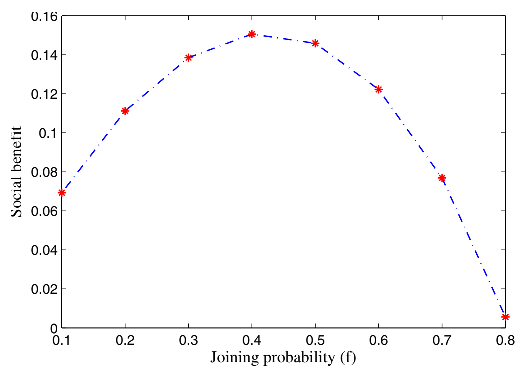

FIGURE 17. Social benefit *vs.* f for $\lambda = 0.5, \mu = 1, \eta = 0.5, \phi = 0.1, \alpha = 0.2, R = 5, C = 2, J = 10$.

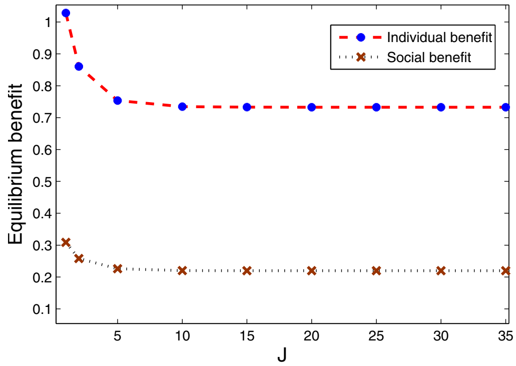

Figure 18. Net benefits *vs.* J for $\lambda = 0.5, \mu = 1, \eta = 0.5, \phi = 0.1, \alpha = 0.3, f = 0.6, R = 5, C = 2$

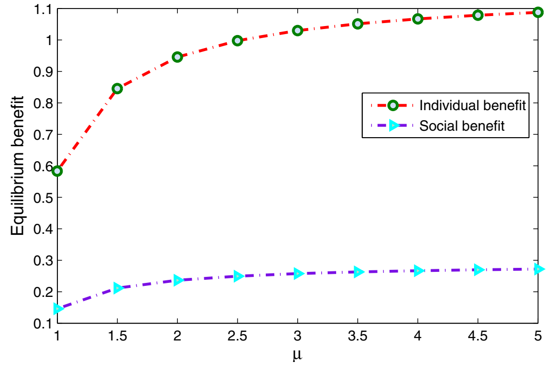

Figure 19. Net benefits *vs.* μ for $\lambda = 0.5, \eta = 0.5, \phi = 0.1, \alpha = 0.2, f = 0.5, J = 10, R = 5, C = 2$

in η and R and decreases with increase in J. The enhancement of reward for service completion attracts more customers to join the system.

Next, we consider the individual and social equilibrium benefits for different model parameters ($\lambda = 0.5; \mu = 1.0; \eta = 0.5; \phi = 0.1; \alpha = 0.2; J = 10; R = 5; C = 2$) in the fully unobservable queue. The equilibrium benefits are directly proportional to α and μ, which are illustrated in Figures 15 and 19, respectively. In both the plots, the individual equilibrium benefit is higher than the social equilibrium benefit. In Figure 16, the individual equilibrium benefit decreases linearly with increase in f. This is because of the longer waiting time of the customers in crowd. Figure 17 describes the behaviour of equilibrium social benefit for various joining probabilities, f. Here the equilibrium social benefit first gradually increases and attains an optimum value, then decreases with further increase in f. This downward concavity of the social benefit function is theoretically proved in the last part of the Section 5.1 and is numerically demonstrated here. This behaviour is due to the fact that the system is rarely congested when joining probability is small, which improves the social benefit. Figure 18 compares the individual and social equilibrium benefits to number of phases of working vacation, J. We observe that individual benefit is more than social benefit. As customers are not aware of the server state as no information is given to them upon their arrival, when J increase, it has no impact on arriving customers.

7. Conclusion

In this paper, we have analyzed the customer's strategic behaviour in a Markovian queue with variant working vacations and impatient customers, where customers decide whether to join or balk the system upon arrival. We have identified two different cases with respect to the level of information provided to customers at their arrival instant. The fully observable and fully unobservable models with respect to system performance measures under equilibrium balking and social optimal customer behaviour have been discussed. We observed that for the fully observable queue, customers' optimal balking threshold in working vacation state exceeds the one in the normal busy state. For the fully unobservable queue, we observed that individual net benefit and social benefit increase as patience timer α increases, but individual net benefit is more in comparison to social benefit.

We have discussed some numerical results to explore the impact of the information level on the equilibrium balking behaviour and to relate the customers' equilibrium balking and social optimal strategies. This work may be extended to study the equilibrium and social optimal behaviour in partial observable and partial unobservable cases. Another extension would be to include the case of general interarrival times or generally distributed service times. A study of such behaviour will allow managers to take such cases into consideration and may require to give customers with enough information on congestion and system's operations strategy.

Appendix A. Derivation of equation (3.3)

When an arriving customer joins an empty system, he may encounter the following possible states $(0,0),(0,1),\ldots,(0,J)$. The mean sojourn time of an arriving customer if he decides to join the system on state $(0,0)$ is $T_{0,0}=\frac{1}{\mu}$. Similarly, if he decides to join the system state $(0,i)$ and won't never renege, his mean sojourn time becomes $T_{0,i}=\frac{1}{\phi+\eta+\alpha}\left(\frac{\xi_0}{\xi_1}+\frac{\phi}{\mu}\right)$, for $i\le i\le J$, see (4.5). So, the mean sojourn time of an arriving customer who encounters the empty system and joins can be obtained by combining both the cases. Thus, substituting the value of ξ_i, the overall mean sojourn time of an arriving customer to an empty system becomes $\frac{1}{\mu}+\frac{1}{\phi+\eta+\alpha}\left(\frac{\lambda+\phi+\eta}{\lambda+\phi+\eta+\alpha}+\frac{\phi}{\mu}\right)$. The customer will decide to join only if his service reward exceeds his net waiting (queue+service) cost, that is, $R>\frac{C}{\mu}+\frac{C}{\phi+\eta+\alpha}\left(\frac{\lambda+\phi+\eta}{\lambda+\phi+\eta+\alpha}+\frac{\phi}{\mu}\right)$.

Appendix B. Proof of Lemma 5.1

The stationary probabilities of the system are calculated using the state balance equations (5.1). Define the probability generating functions (pgfs) as

$$G_0(z)=\sum_{n=0}^{\infty}\pi_{n,0}z^n,\quad 0\le z\le 1,\quad G_i(z)=\sum_{n=0}^{\infty}\pi_{n,i}z^n,\quad 0\le z\le 1,\quad i=1,2,\ldots,J,$$

where the normalization condition $\sum_{n=0}^{\infty}\pi_{n,0}+\sum_{n=0}^{\infty}\sum_{i=1}^{J}\pi_{n,i}=1$ becomes $G_0(1)+\sum_{i=1}^{J}G_i(1)=1$ and $G_i^{'}(z)=\sum\limits_{n=1}^{\infty}nz^{n-1}\pi_{n,i},\ i=0,1,2,\ldots,J$. Multiplying (5.1d) and (5.1e) by the appropriate powers of z^n and then summing over all possible values of n (with $2\le i\le J$), we get

$$\alpha z(1-z)G_i^{'}(z)-\Big\{(1-z)(\lambda fz-\eta+\alpha)z+\phi z\Big\}G_i(z)-(\eta-\alpha)(1-z)\pi_{0,i}+\phi z\pi_{0,i-1}=0. \tag{B.1}$$

Equation (B.1) can be rearranged to

$$G_i^{'}(z)-\left\{\frac{\lambda fz-\eta+\alpha}{\alpha z}+\frac{\phi}{\alpha(1-z)}\right\}G_i(z)-\frac{\eta-\alpha}{\alpha z}\pi_{0,i}+\frac{\phi}{\alpha(1-z)}\pi_{0,i-1}=0, \tag{B.2}$$

for $\alpha,z\ne 0$ and $z\ne 1$. To solve the first order linear differential equation (B.2), we obtain an integrating factor (I.F.) as

$$\mathrm{e}^{\int\left(-\frac{\lambda f}{\alpha}+(\frac{\eta}{\alpha}-1)\frac{1}{z}-\frac{\phi}{\alpha}\frac{1}{1-z}\right)dz}=\mathrm{e}^{-\frac{\lambda f}{\alpha}z}\ z^{\frac{\eta}{\alpha}-1}\ (1-z)^{\frac{\phi}{\alpha}}. \tag{B.3}$$

Multiplying both sides of equation (B.2) by the I.F., we get

$$\frac{d}{dz}\left[\mathrm{e}^{-\frac{\lambda f}{\alpha}z}z^{\frac{\eta}{\alpha}-1}(1-z)^{\frac{\phi}{\alpha}}\ G_i(z)\right]=-\frac{1}{\alpha}\left(\frac{\phi\pi_{0,i-1}}{1-z}-\frac{(\eta-\alpha)\pi_{0,i}}{z}\right)\mathrm{e}^{-\frac{\lambda f}{\alpha}z}\ z^{\frac{\eta}{\alpha}-1}\ (1-z)^{\frac{\phi}{\alpha}}. \tag{B.4}$$

Integrating both sides of equation (B.4) from 0 to z, we get

$$G_i(z)=\frac{-\phi\pi_{0,i-1}K_1(z)+(\eta-\alpha)\pi_{0,i}K_2(z)}{\alpha\mathrm{e}^{-\frac{\lambda f}{\alpha}z}\ z^{\frac{\eta}{\alpha}-1}\ (1-z)^{\frac{\phi}{\alpha}}}, \tag{B.5}$$

where

$$K_1(z)=\int_0^z\mathrm{e}^{-\frac{\lambda f}{\alpha}x}\ x^{\frac{\eta}{\alpha}-1}\ (1-x)^{\frac{\phi}{\alpha}-1}\mathrm{d}x,$$
$$K_2(z)=\int_0^z\mathrm{e}^{-\frac{\lambda f}{\alpha}x}\ x^{\frac{\eta}{\alpha}-2}\ (1-x)^{\frac{\phi}{\alpha}}\mathrm{d}x.$$

Since $G_i(1) = \sum_{n=0}^{\infty} \pi_{n,i} < \infty$ and $z = 1$ is a root of the denominator of equation (B.5), so $z = 1$ must be the root of the numerator. Thus, we obtain

$$-\phi K_1(1)\pi_{0,i-1} + (\eta - \alpha)K_2(1)\pi_{0,i} = 0, \quad i = 2, 3 \ldots, J. \tag{B.6}$$

Similarly, the pgf $G_1(z)$ can be found from (5.1e) for $i = 1$ and (5.1c)

$$G_1'(z) - \left\{\frac{\lambda f z - \eta + \alpha}{\alpha z} + \frac{\phi}{\alpha(1-z)}\right\} G_1(z) - \frac{\eta - \alpha}{\alpha z}\pi_{0,1} + \frac{\mu}{\alpha(1-z)}\pi_{1,0} = 0, \tag{B.7}$$

and then on simplification

$$G_1(z) = \frac{-\mu\pi_{1,0}K_1(z) + (\eta - \alpha)\pi_{0,1}K_2(z)}{\alpha e^{-\frac{\lambda f}{\alpha}z} z^{(\frac{\eta}{\alpha}-1)}(1-z)^{\frac{\phi}{\alpha}}}. \tag{B.8}$$

Again using the same argument that $G_1(1) = \sum_{n=0}^{\infty} \pi_{n,1} < \infty$ and $z = 1$ is a root of the denominator of equation (B.8), so $z = 1$ must be a root of the numerator of (B.8). Thus, we have

$$\pi_{1,0} = \frac{(\eta - \alpha)K_2(1)}{\mu K_1(1)}\pi_{0,1}. \tag{B.9}$$

Using (B.6) recursively and incorporating (B.9), we get

$$\pi_{0,i} = \left(\frac{\phi K_1(1)}{(\eta - \alpha)K_2(1)}\right)^{i-1} \pi_{0,1} = \widehat{\Psi}^{i-1}\pi_{0,1}, \quad 1 \le i \le J. \tag{B.10}$$

The stationary probabilities $\pi_{0,0}$ and $\pi_{1,i}$ can be easily calculated from (5.1a), and (5.1c)–(5.1d) respectively. The expressions are given by

$$\pi_{0,0} = \frac{\phi}{\lambda f}\pi_{0,1}\widehat{\Psi}^{J-1}, \tag{B.11}$$

$$\pi_{1,i} = \frac{\widehat{\Psi}^{i-2}}{\eta}\Big(\lambda f\widehat{\Psi} - \phi(1 - \widehat{\Psi})\Big)\pi_{0,1}, \quad 1 \le i \le J. \tag{B.12}$$

Using (5.1e) and (5.1b), we obtain

$$\pi_{n+1,i} = \frac{\lambda f + \phi + \eta + (n-1)\alpha}{\eta + n\alpha}\pi_{n,i} - \frac{\lambda f}{\eta + n\alpha}\pi_{n-1,i}, \quad n \ge 1, \quad 1 \le i \le J, \tag{B.13}$$

$$\pi_{n+1,0} = \left(1 + \frac{\lambda f}{\mu}\right)\pi_{n,1} - \frac{\lambda f}{\mu}\pi_{n-1,0} - \frac{\phi}{\mu}\sum_{i=1}^{J}\pi_{n,i}, \quad n \ge 1. \tag{B.14}$$

Summing $\pi_{0,i}$ over i from 1 to J, yields

$$\sum_{i=1}^{J}\pi_{0,i} = \frac{1 - \widehat{\Psi}^J}{1 - \widehat{\Psi}}\pi_{0,1}.$$

Applying L'Hopital's rule in (B.1) and (B.8), we obtain

$$\lim_{z\to 1} G_i(z) = \lim_{z\to 1}\frac{(\eta - \alpha)(1-z)\pi_{0,i} - \phi z\pi_{0,i-1}}{\alpha[(\frac{\eta}{\alpha} - 1)(1-z) - \frac{\phi}{\alpha}z]}, \quad 2 \le i \le J,$$

$$\lim_{z\to 1} G_1(z) = \lim_{z\to 1}\frac{(\eta - \alpha)(1-z)\pi_{0,1} - \mu z\pi_{1,0}}{\alpha[(\frac{\eta}{\alpha} - 1)(1-z) - \frac{\phi}{\alpha}z]}.$$

which after simplification reduce to

$$G_i(1) = \pi_{0,i-1} = \widehat{\Psi}^{i-2}\pi_{0,1}, \quad 1 \le i \le J, \tag{B.15}$$

and

$$\sum_{i=1}^{J} G_i(1) = \sum_{i=1}^{J} \widehat{\Psi}^{i-2}\pi_{0,1} = \frac{1-\widehat{\Psi}^J}{\widehat{\Psi}(1-\widehat{\Psi})}\pi_{0,1} = \delta\ \pi_{0,1}. \tag{B.16}$$

Equation (B.2) can be rewritten as

$$G_i^{'}(z) = \frac{\{(1-z)(\lambda f z - \eta + \alpha) + \phi z\}\, G_i(z) + (\eta - \alpha)(1-z)\pi_{0,i} - \phi z \pi_{0,i-1}}{\alpha z(1-z)}.$$

Applying L'Hopital's rule

$$\lim_{z\to 1} G_i^{'}(z) = \lim_{z\to 1} \frac{\{\lambda f(1-2z) + \eta - \alpha + \phi\}\, G_i(z) - [(\eta-\alpha)\pi_{0,i} + \phi\pi_{0,i-1}]}{\alpha(1-2z) - \{(\lambda f - \eta + \alpha)(1-z) + \phi z\}}.$$

Using (B.15), we get

$$G_i^{'}(1) = \frac{\{\lambda f - \eta + \alpha\}\, G_i(1) + (\eta-\alpha)\pi_{0,i}}{\phi + \alpha}, \quad 1 \le i \le J,$$

$$\sum_{i=1}^{J} G_i^{'}(1) = \frac{\delta\Big(\lambda f - (\eta-\alpha)(1-\widehat{\Psi})\Big)\pi_{0,1}}{(\phi+\alpha)}. \tag{B.17}$$

Similarly, we can also obtain

$$G_i^{''}(1) = \frac{2(\lambda f - \eta - \phi)G_i^{'}(1) + 2\lambda f G_i(1)}{\phi + 2\alpha}, \quad 1 \le i \le J,$$

$$\sum_{i=1}^{J} G_i^{''}(1) = \frac{2\delta\ \pi_{0,1}}{\phi + 2\alpha}\left(\lambda f + \frac{(\lambda f - \eta - \phi)(\lambda f - (\eta-\alpha)(1-\widehat{\Psi}))}{\phi+\alpha}\right).$$

Multiplying (5.1a) and (5.1b) by the appropriate powers of z^n and then summing over all possible values of n, we get

$$G_0(z) = \frac{\phi z \sum_{i=1}^{J} G_i(z) - [\mu\pi_{1,0} + \phi \sum_{i=1}^{J-1} \pi_{0,i}]z - \mu(1-z)\pi_{0,0}}{(\lambda f z - \mu)(1-z)}. \tag{B.18}$$

Using L'Hopital's rule, we obtain

$$\lim_{z\to 1} G_0(z) = \lim_{z\to 1} \frac{\phi \sum_{i=1}^{J} G_i(z) + \phi z \sum_{i=1}^{J} G_i^{'}(z) - [\mu\pi_{1,0} + \phi \sum_{i=1}^{J-1} \pi_{0,i}] + \mu\pi_{0,0}}{\lambda f(1-z) + (\mu - \lambda f)},$$

which gives

$$G_0(1) = \frac{\phi \sum_{i=1}^{J} G_i^{'}(1) + \mu\pi_{0,0}}{\mu - \lambda f} = \frac{\phi\pi_{0,1}}{\mu - \lambda f}\left[\frac{\delta\Big(\lambda f - (\eta-\alpha)(1-\widehat{\Psi})\Big)}{(\phi+\alpha)} + \frac{\mu\widehat{\Psi}^{J-1}}{\lambda f}\right]. \tag{B.19}$$

Now, we deduce $G_0^{'}(1)$ from (B.18), using L'Hopital's rule as

$$\begin{aligned} G_0^{'}(1) &= \frac{\lambda f}{\mu-\lambda f}G_0(1) + \frac{\phi}{\mu-\lambda f}\sum_{i=1}^{J} G_i^{'}(1) + \frac{\phi}{2(\mu-\lambda f)}\sum_{i=1}^{J} G_i^{''}(1) \\ &= \frac{\phi\pi_{0,1}}{\mu-\lambda f}\left[\frac{\delta(\lambda f-(\eta-\alpha)(1-\widehat{\Psi}))}{\phi+\alpha}\left\{\frac{\mu}{\mu-\lambda f}+\frac{\lambda f-\eta-\phi}{\phi+2\alpha}\right\}+\frac{\mu\widehat{\Psi}^{J-1}}{\mu-\lambda f}+\frac{\delta\lambda f}{\phi+2\alpha}\right]\cdot \end{aligned} \tag{B.20}$$

Using the normalization condition $G_0(1)+\sum_{i=1}^{J} G_i(1)=1$, we get

$$\pi_{0,1} = \frac{\lambda f\ (\phi+\alpha)\ (\mu-\lambda f)}{\delta\lambda f\phi(\mu-(\eta-\alpha)(1-\widehat{\Psi}))+\delta\lambda f\alpha(\mu-\lambda f)+\mu\phi(\phi+\alpha)\widehat{\Psi}^{J-1}}\cdot \tag{B.21}$$

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