RAIRO-Oper. Res. 51 (2017) 921–930
DOI: 10.1051/ro/2016055

RAIRO Operations Research
www.rairo-ro.org

A NOTE ON OPTIMAL PORTFOLIO CORRESPONDING TO THE CVAR RATIO

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Abstract. Various reward-risk performance measures and ratios have been considered in reward-risk portfolio selection problems. This paper investigates the optimal portfolio corresponding to the CVaR (STARR) ratio. Considering the LP solvability of CVaR, a method is proposed for detecting the optimal portfolio by using the corresponding Mean-CVaR optimization problem. By applying LP tools, a method is suggested for producing the optimal portfolio as a by-product during the procedure of computing the efficient frontier of the Mean-CVaR problem.

Mathematics Subject Classification. 91G10, 90C05.

Received June 21, 2015. Accepted August 28, 2016.

1. Introduction

Reward-risk ratio optimization is an important mathematical approach in finance. In reward-risk analysis, a portfolio is preferred to another one if it has higher expected return and lower risk. For a given set of risky assets and a riskless asset, all investors will choose an optimal portfolio which is a linear combination of the riskless asset and a risky portfolio known as *optimal risky portfolio* (or briefly, *optimal portfolio*). The optimal risky portfolio is given by the portfolio that maximizes the performance measure. Generally, this measure is a ratio between the expected excess return and a risk measure of portfolio return (Biglova *et al.* [5]). Many reward-risk ratios, such as Sharpe ratio and CVaR (STARR) ratio, have been extensively used in financial portfolio management. Recently, Biglova *et al.* [5] provided an overview of various reward-risk performance measures and ratios. Stoyanov *et al.* [15] investigated the general reward-risk ratio optimization problems. For more reward-risk ratios and an empirical comparison, refer to [5, 10] and the references therein.

Markowitz [7] introduced his seminal work in modern portfolio theory for risky assets in the Mean-Variance (M-V) framework using variance as the measure of risk. Tobin [16] demonstrated that in a M-V portfolio selection problem, for a set of some risky assets and a riskless asset, each efficient portfolio can be represented by a combination of riskless asset and a unique risky portfolio. This fact is known as two-fund separation theorem. As a matter of fact, the optimal portfolio maximizes the Sharpe ratio [13, 14]

$$S(\mathbf{x}) = \frac{\bar{R}(\mathbf{x}) - r_c}{\varphi(\mathbf{x})}$$

Keywords. Reward-risk ratio optimization, CVaR ratio, optimal portfolio, linear programming, subderivative.

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between all feasible risky portfolios, where r_c is the return of the riskless asset and the reward $\bar{R}(\mathbf{x})$ and the risk $\varphi(\mathbf{x})$ of a portfolio $\mathbf{x}$ are measured by the mean and the standard deviation of the portfolio return, respectively. In this case, the optimal portfolio corresponds to the point of tangency, that is, the so-called *tangency portfolio*, between the line originated from the r_c on the Mean axis and the efficient frontier associated with the set of all feasible risky portfolios in the Mean-Standard deviation plane (see Fig. 1). Tütüncü [17] introduced a simple modification of Markowitz's critical line method (which is commonly used for generating the M-V efficient frontier) to determine the optimal (tangency) portfolio. Indeed, by using dual solutions, he has demonstrated how this modification can be used to produce the optimal portfolio as a by-product during the method. Keykhaei and Jahandideh [6] suggested a method which determines the location of the tangency portfolio in Mean-Risk portfolio selection problems for which the original portfolio selection problem

$$\min_{\mathbf{x}\in A} \varphi(\mathbf{x})$$

is equivalent to a Linear Programming (LP) problem

$$\min_{\mathbf{x}\in B} \bar{\varphi}(\mathbf{x}) = \mathbf{C}'\mathbf{x}.$$

In this case the tangency portfolio maximizes the ratio

$$S(\mathbf{x}) = \frac{\bar{R}(\mathbf{x}) - r_c}{\varphi(\mathbf{x})}$$

where φ is approximated by $\bar{\varphi}$. The goal in this paper is to detect the optimal portfolio which is the portfolio that maximizes the CVaR ratio

$$S(\mathbf{x}) = \frac{\bar{R}(\mathbf{x}) - r_c}{\varphi(\mathbf{x}) + r_c}$$

proposed by Martin *et al.* [8], where $\varphi(\mathbf{x})$ is measured by the CVaR associated to a portfolio $\mathbf{x}$.

In this paper, by using dual solutions, the detection of the optimal portfolio and its production as a by-product during the procedure of computing the efficient frontier corresponding to the set of feasible risky portfolios is demonstrated. Indeed, according to this method, it is sufficient to realize the optimal portfolio between two (extreme) efficient portfolios, but not all of them. This can be useful in large-scale portfolio selection problems.

The remainder of this paper is organized as follows. Section 2 describes the formulation of the CVaR portfolio optimization problem and applies the CVaR minimization algorithm presented in Rockafellar and Uryasev [11,12] for risky assets. Section 3 describes the portfolio selection approach for a set of risky assets and a cash account (as a riskless asset) and presents the analysis for the detection of the optimal portfolio. Section 4 investigates the efficient frontier and describes the procedure for producing the optimal portfolio as a by-product during the procedure of computing the efficient frontier of risky portfolios. An illustrative example is given in the last section.

2. Minimization of CVaR for risky assets

Value-at-Risk (VaR) has become a standard measure of risk used by many financial institutions. It measures the maximum amount that an investment may lose with a specified probability level over a certain time horizon. VaR has some drawbacks, such as non-subadditivity and non-convexity, see [2, 3]. Conditional value-at-risk (CVaR) introduced by Rockafellar and Uryasev [11, 12], is an alternative measure of risk. CVaR which is also called Expected Tail Loss (ETL), Expected Shortfall (ES), or Tail VaR, is the expected loss conditional that the loss is above the VaR. CVaR is convex and coherent in the sense of being positively homogeneous, subadditive, monotonic and translation invariant, see [9]. Rockafellar and Uryasev [11, 12] demonstrated that for linear loss functions, CVaR can be minimized by using LP algorithms.

Consider an investment opportunity in a set of $n \geq 2$ risky assets (in the scenes that their returns is not deterministic). Let r_j be the random rate of return (with mean $\bar{r}_j$) of the jth asset and x_j be the proportion allocated to the asset. Each portfolio is represented by its asset weights vector $\mathbf{x} = (x_1, \dots, x_n)' \in \mathbb{R}^n$. Then the portfolio $\mathbf{x}$ has the random return $R := R(\mathbf{x}) = \sum_{i=1}^{n} x_i r_i = \mathbf{x}'\mathbf{r}$ and the expected return $\bar{R} := \bar{R}(\mathbf{x}) = \sum_{i=1}^{n} x_i \bar{r}_i = \mathbf{x}'\bar{\mathbf{r}}$, where $\mathbf{r}$ and $\bar{\mathbf{r}}$ are the random vector and the mean vector of the asset returns. A portfolio $\mathbf{x}$ is *feasible* if it satisfies the following conditions:

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{b}, \\ \mathbf{C}\mathbf{x} &\geq \mathbf{d}, \end{aligned} \tag{2.1}$$

where $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{d} \in \mathbb{R}^p$, $\mathbf{A}$ is an $m \times n$, and $\mathbf{C}$ is a $p \times n$ matrix over $\mathbb{R}$.

As [11], the negative of the portfolio return $-R$ is considered as the loss function with the cumulative distribution function $F_R(\alpha) = P(-R \leq \alpha)$. The VaR_β of a portfolio $\mathbf{x}$ (with respect to a probability level $\beta \in (0, 1)$) is defined as the β-quantile of loss function as follows:

$$\mathrm{VaR}_\beta(\mathbf{x}) := \mathrm{VaR}_\beta(R) = \min\{\alpha \in \mathbb{R} \ : \ F_R(\alpha) \geq \beta\}.$$

In fact, this indicates that the probability that the possible portfolio loss exceeds VaR_β is less than $1 - \beta$.

The CVaR_β of a portfolio $\mathbf{x}$ is the mean of the β-tail distribution of $-R$ with distribution function:

$$F_R^\beta(\alpha) = \begin{cases} 0, & \text{if} \quad \alpha < \mathrm{VaR}_\beta(\mathbf{x}), \\ \dfrac{F_R(\alpha) - \beta}{1 - \beta}, & \text{if} \quad \alpha \geq \mathrm{VaR}_\beta(\mathbf{x}). \end{cases}$$

For continuous random variable R, as considered in this paper, CVaR_β is the conditional expectation of loss subject to $-R \geq \mathrm{VaR}_\beta$ (see Prop. 5 of [12]), that is,

$$\mathrm{CVaR}_\beta(\mathbf{x}) = \mathbb{E}[-R| - R \geq \mathrm{VaR}_\beta(\mathbf{x})].$$

CVaR has been considered as an alternative measure of risk for VaR because of its desirable properties. The convexity of CVaR has a key role in this investigation. As shown in [11, 12], CVaR can be minimized by using the LP algorithms.

Consider the following Mean-CVaR portfolio selection problem for the expected portfolio return ρ:

Problem (1):

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^n} \ & \mathrm{CVaR}_\beta(\mathbf{x}), \\ \text{s.t.} \ & \bar{\mathbf{r}}'\mathbf{x} = \rho, \\ & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{C}\mathbf{x} \geq \mathbf{d}. \end{aligned}$$

It was illustrated in [11] that by using a set of scenarios $\mathbf{R}_1, \dots, \mathbf{R}_N$ sampled from the joint distribution of the asset returns, the minimizing of $\mathrm{CVaR}_\beta(\mathbf{x})$ over a set of feasible portfolios S is equivalent to minimizing of the approximating expression

$$\tilde{F}(\mathbf{x}, \alpha) = \alpha + \frac{1}{N(1 - \beta)} \sum_{j=1}^{N} [-\mathbf{x}'\mathbf{R}_j - \alpha]^+$$

over $S \times \mathbb{R}$, that is,

$$\min_{\mathbf{x} \in S} \mathrm{CVaR}_\beta(\mathbf{x}) = \min_{(\mathbf{x}, \alpha) \in S \times \mathbb{R}} \tilde{F}(\mathbf{x}, \alpha). \tag{2.2}$$

Also, if $(\mathbf{x}^*, \alpha^*)$ minimizes the $\tilde{F}(\mathbf{x}, \alpha)$, then $\mathbf{x}^*$ minimizes the $\mathrm{CVaR}_\beta(\mathbf{x})$. Moreover, $\tilde{F}(\mathbf{x}, \alpha)$ is convex with respect to $(\mathbf{x}, \alpha)$ (see Thm. 2.2 of [11] and its following discussions). Finally Problem (1) can be reduced to the following LP problem:

Problem (2):

$$\begin{aligned}\min_{\mathbf{x}\in\mathbb{R}^n\ \alpha\in\mathbb{R}\ \mathbf{y}\in\mathbb{R}^N}\ & H(\alpha,\mathbf{y}) = \alpha + \gamma\sum_{j=1}^{N} y_j,\\ \text{s.t.}\ & \bar{\mathbf{r}}'\mathbf{x} = \rho,\\ & \mathbf{A}\mathbf{x} = \mathbf{b},\\ & \mathbf{C}\mathbf{x} \ge \mathbf{d}'\\ & y_j \ge -\mathbf{x}'\mathbf{R}_j - \alpha,\ y_j \ge 0,\ j = 1,\dots,N,\end{aligned}$$

where $\gamma = (N(1-\beta))^{-1}$.

Zhu *et al.* [18] proposed the following CVaR robust Mean-Variance portfolio optimization problem:

$$\min_{\mathbf{x}\in S}\{\mathrm{CVaR}_\beta(\mathbf{x}) + \lambda\mathbf{x}'\bar{\mathbf{Q}}\mathbf{x}\} = \min_{(\mathbf{x},\alpha)\in S\times\mathbb{R}}\left\{\tilde{F}(\mathbf{x},\alpha) + \lambda\mathbf{x}'\bar{\mathbf{Q}}\mathbf{x}\right\}, \tag{2.3}$$

where $\bar{\mathbf{Q}}$ is an estimate of the covariance matrix of asset returns and $\lambda \ge 0$ is the risk aversion parameter. Obviously, by letting $\lambda = 0$ the optimization problem in equation (2.2) can be obtained. Applying a smoothing approach, Zhu *et al.* [18] reformulated problem (2.3) into the following problem:

$$\min_{(\mathbf{x},\alpha)\in S\times\mathbb{R}}\left\{\alpha + \gamma\sum_{j=1}^{N}\rho_\epsilon(-\mathbf{x}'\mathbf{R}_j - \alpha) + \lambda\mathbf{x}'\bar{\mathbf{Q}}\mathbf{x}\right\}, \tag{2.4}$$

where $\rho_\epsilon(z)$ is defined as:

$$\begin{cases} z & \text{if} \quad z \ge \epsilon,\\ \dfrac{z^2}{4\epsilon} + \dfrac{1}{2}z + \dfrac{1}{4}\epsilon & \text{if} \quad -\epsilon \le z \le \epsilon,\\ 0 & \text{otherwise.}\end{cases}$$

In the formulation of Problem (2), generating a new sample adds an additional variable and constraint. For n risky assets and N mean return samples, Problem (2) has a total of $O(n+N)$ variables and $O(n+N)$ constraints. On the other hand, the smoothing formulation (2.4) has only $O(n)$ variables and $O(n)$ constraints. Therefore, an increase in the sample size N does not change the number of variables and constraints. The linearity of the objective function, as in Problem (2), is essential in this study. Note that the objective function in problem (2.4) includes a piecewise quadratic and a quadratic term. So, in the following, the LP Problem (2) is investigated.

By the Karush−Kuhn−Tucker (K-K-T) conditions, $(\mathbf{x}^*, \alpha^*, \mathbf{y}^*)$ is a (primal) solution of Problem (2), if and only if, there exist vectors $\lambda_\rho \in \mathbb{R}$, $\lambda_\mathbf{b} \in \mathbb{R}^m$, $\lambda_\mathbf{d} \in \mathbb{R}^p$ and $\lambda_\mathbf{y}, \lambda_0 \in \mathbb{R}^N$ such that:

$$\lambda_\rho\bar{\mathbf{r}} + \mathbf{A}'\lambda_\mathbf{b} + \mathbf{C}'\lambda_\mathbf{d} + \mathbf{R}\lambda_\mathbf{y} = 0, \tag{2.5}$$

$$1 - \mathbf{E}'\lambda_\mathbf{y} = 0, \tag{2.6}$$

$$\gamma\mathbf{E} - \lambda_\mathbf{y} - \lambda_0 = 0, \tag{2.7}$$

$$\bar{\mathbf{r}}'\mathbf{x}^* = \rho, \qquad \mathbf{A}\mathbf{x}^* = \mathbf{b}, \qquad \lambda_\mathbf{d}'(\mathbf{C}\mathbf{x}^* - \mathbf{d}) = 0, \tag{2.8}$$

$$\lambda'_{\mathbf{y}}(\mathbf{R}'\mathbf{x}^* + \alpha^*\mathbf{E} + \mathbf{y}^*) = 0, \qquad \lambda'_0\mathbf{y}^* = 0, \tag{2.9}$$

$$\mathbf{C}\mathbf{x}^* \geq \mathbf{d}, \qquad \mathbf{y}^* \geq -\mathbf{R}'\mathbf{x}^* - \alpha^*\mathbf{E}, \qquad \mathbf{y}^* \geq 0, \tag{2.10}$$

$$\lambda_{\mathbf{d}} \geq 0, \qquad \lambda_{\mathbf{y}} \geq 0, \qquad \lambda_0 \geq 0, \tag{2.11}$$

where $\mathbf{R} = [\mathbf{R}_1, \ldots, \mathbf{R}_N]$ and $\mathbf{E}$ is a N-column vector of ones.

For any expected return ρ let $\Omega(\rho)$ denote the set of all primal-dual solutions $(\mathbf{x}^*, \alpha^*, \mathbf{y}^*, \Lambda)$ of Problem (2), satisfying (2.5)−(2.11), where $\Lambda = (\lambda_\rho, \lambda_{\mathbf{d}}, \lambda_{\mathbf{d}}, \lambda_{\mathbf{y}}, \lambda_0)$. $\lambda\rho$ is called an *optimal reward multiplier* which is not necessarily unique. Considering equation (2.2), if $(\mathbf{x}^*, \alpha^*, \mathbf{y}^*, \Lambda) \in \Omega(\rho)$, then $\mathbf{x}^*$ is the optimal solution of Problem (1) such that:

$$H(\alpha^*, \mathbf{y}^*) = \tilde{F}(\mathbf{x}^*, \alpha^*) = \mathrm{CVaR}_\beta(\mathbf{x}^*) \tag{2.12}$$

and α^* is the corresponding VaR (see Thm. 2 of [11]).

3. Portfolio selection framework

Now consider the investment opportunity in the set of risky assets and a cash account x_c with the certain rate of return r_c. In this case, each investor invests the portion x_c of his wealth in the cash account and invests the remainder $1-x_c$ in a feasible risky portfolio $\mathbf{x}$ satisfying conditions (2.1). Now the portfolio is $\dot{\mathbf{x}} = ((1-x_c)\mathbf{x}, x_c)$. The portfolio return of $\dot{\mathbf{x}}$ is $(1 - x_c)R + x_c r_c$. In the following, $\mathbf{x} := (\mathbf{x}, 0)$ and $\mathbf{x}_c := (\mathbf{0}, 1)$ means the totally risky and the totally cash investment, respectively. Note that since

$$\mathrm{CVaR}_\beta(R + r_c) = \mathrm{CVaR}_\beta(R) - r_c,$$

then the corresponding CVaR of the totally cash investment $\mathbf{x}_c = (\mathbf{0}, 1)$ is equal to $-r_c$ (for example see Cor. 12 of [12]). In this paper, actually, the risk of a typical portfolio $\dot{\mathbf{x}}$ is considered by $\phi_\beta(\dot{\mathbf{x}})$ which is defined as follows:

$$\phi_\beta(\dot{\mathbf{x}}) = \mathrm{CVaR}_\beta(\dot{\mathbf{x}}) + r_c. \tag{3.1}$$

So, a cash investment has zero risk. Obviously, the minimization of $\phi_\beta(\dot{\mathbf{x}})$ is equivalent to minimization of $\mathrm{CVaR}_\beta(\dot{\mathbf{x}})$, which leads to the minimization Problem (2). In fact, the portfolio optimization in the Mean-ϕ_β framework is aimed at. Let us consider totally risky investments. A portfolio is said to be *efficient* if it has the highest expected return among all feasible portfolios with the same risk and has the lowest risk among all feasible portfolios with the same expected return. Let $\rho_{\min}$ be the expected return of an efficient portfolio which has the minimum risk. Also let $\rho_{\max}$ be the highest obtainable expected return of feasible portfolios. Actually, any efficient portfolio has expected return $\rho \in [\rho_{\min}, \rho_{\max}]$. The graph which plots the risk of any efficient portfolio against its expected return is called the *efficient frontier* (see Fig. 1). Following [6], the function ϕ is defined as

$$\begin{aligned} \phi :\ [\rho_{\min}, \rho_{\max}] &\longrightarrow \mathbb{R} \\ \rho &\mapsto \phi_\beta(\mathbf{x}^*); \end{aligned} \tag{3.2}$$

where $(\mathbf{x}^*, \alpha^*, \mathbf{y}^*, \Lambda) \in \Omega(\rho)$. Indeed $\phi(\rho)$ is a approximate optimal risk (obtained from Problem (2)) corresponding to mean return ρ. Since CVaR is a convex function [9], ϕ is also a convex function which is not necessarily smooth. Indeed, for any ρ, it might be more than one subderivative of ϕ at ρ. Note that the scaler c is a subderivative of ϕ at ρ, if and only if

$$\phi(\rho_1) \geq \phi(\rho) + c(\rho_1 - \rho)$$

for any $\rho_1 \in [\rho_{\min}, \rho_{\max}]$. The subdifferential $\partial\phi(\rho)$ of ϕ at ρ is the set of all subderivative. Here we recall the Proposition 2.1 of Keykhaei and Jahandideh [6].

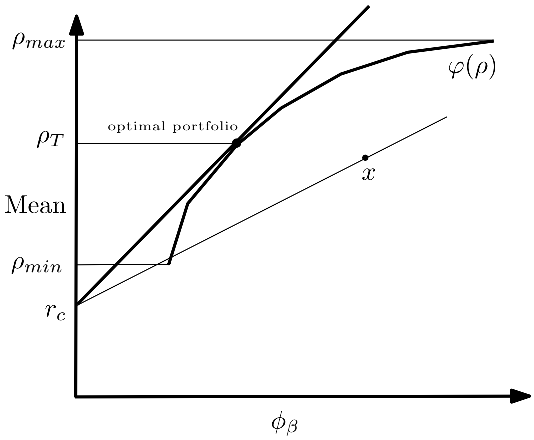

FIGURE 1. The efficient frontier of Mean-ϕ_β portfolio selection problem.

Proposition 3.1. *$\partial\phi(\rho)$ is equal with the set of optimal reward multipliers of Problem* (2)*.*

Now suppose the portion $\lambda \geq 0$ of our wealth can be invested in a feasible risky portfolio $\mathbf{x}$ and the remainder in the cash account, that is $\dot{\mathbf{x}} = \lambda\mathbf{x} + (1-\lambda)\mathbf{x}_c$. Using properties of CVaR (see [9]) we obtain:

$$\begin{aligned}\phi_\beta(\lambda\mathbf{x} + (1-\lambda)\mathbf{x}_c) &= \mathrm{CVaR}_\beta(\lambda\mathbf{x} + (1-\lambda)\mathbf{x}_c) + r_c\\ &= \lambda\mathrm{CVaR}_\beta(\mathbf{x}) - (1-\lambda)r_c + r_c\\ &= \lambda(\mathrm{CVaR}_\beta(\mathbf{x}) + r_c)\\ &= \lambda\phi_\beta(\mathbf{x}) + (1-\lambda)\phi_\beta(\mathbf{x}_c).\end{aligned}$$

Also, such a portfolio has the expected return $\lambda\bar{R} + (1-\lambda)r_c$. Thus, any portfolio which is a convex combination of a feasible risky portfolio and the cash account can be represented by a point on the line that connects the feasible portfolio and the cash account in the ϕ_β-Mean plane. In this case, the efficient frontier is a straight line which dominates the other lines, passing from the r_c on Mean axis and the tangency portfolio as the *optimal portfolio* on the efficient frontier of risky assets (see Fig. 1). Obviously, the slope of this line is a subderivative of ϕ at the mean return of the tangency portfolio in the Mean-ϕ_β framework and is the highest obtainable slope. In fact optimal portfolio maximizes the *CVaR ratio* (or *STARR ratio*)

$$S(\mathbf{x}) := \frac{\mathbb{E}[\mathbf{x}'\mathbf{r} - r_c]}{\mathrm{CVaR}_\beta(\mathbf{x}'\mathbf{r} - r_c)} = \frac{\mathbf{x}'\bar{\mathbf{r}} - r_c}{\phi_\beta(\mathbf{x})}, \tag{3.3}$$

proposed by Martin *et al.* [8], among all feasible portfolios (also, see [15]). Note that, since $\tilde{F}(\mathbf{x}, \alpha)$ is not necessarily strictly convex, it is possible that there exist more than one tangency point. In the following let $I_T = [\rho_{T_1}, \rho_{T_2}]$ (possibly the single point ρ_T) denote the interval of the expected returns of all optimal portfolios (see Fig. 2). Keykhaei and Jahandideh [6] presented the following lemma.

Lemma 3.2. *Let $(\mathbf{x}^*, \alpha^*, \mathbf{y}^*, \Lambda) \in \Omega(\rho)$, for which*

$$\lambda_\rho = \frac{\phi(\rho)}{\rho - r_c},$$

then $\mathbf{x}^*$ is an optimal portfolio.

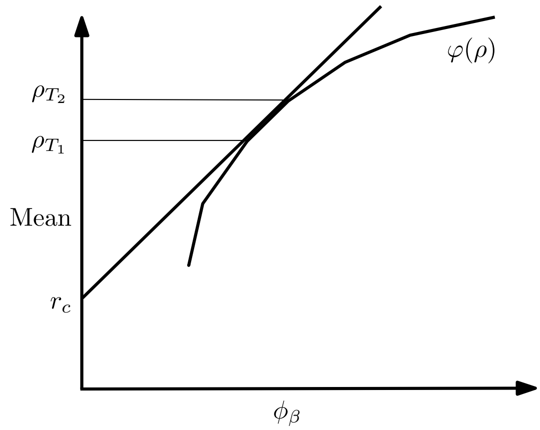

FIGURE 2. The Mean-ϕ_β efficient frontier with non-unique optimal portfolio.

In the following, some results are presented which could be useful to detect the optimal portfolio. For $(\mathbf{x}^*, \alpha^*, \mathbf{y}^*, \Lambda) \in \Omega(\rho)$ define

$$\theta(\Lambda) := r_c\tilde{\lambda}_\rho + \mathbf{b}'\lambda_\mathbf{b} + \mathbf{d}'\lambda_\mathbf{d},$$

where $\tilde{\lambda}_\rho = \lambda_\rho + 1$[2]

Theorem 3.3. *Let* $(\mathbf{x}^*, \alpha^*, \mathbf{y}^*, \Lambda) \in \Omega(\rho)$.

(I) *If* $\theta(\Lambda) = 0$, *then* $\mathbf{x}^*$ *is an optimal portfolio.*
(II) *If* $\rho \in (\rho_{T_2}, \rho_{\max}]$ $(\rho \in [\rho_{\min}, \rho_{T_1}))$, *then* $\theta(\Lambda) < 0$ $(\theta(\Lambda) > 0)$.

Proof. Let $(\mathbf{x}^*, \alpha^*, \mathbf{y}^*, \Lambda) \in \Omega(\rho)$. Adding equation (2.5) multiplied by $-\mathbf{x}^{*\prime}$, equation (2.6) multiplied by α^*, and equation (2.7) multiplied by $\mathbf{y}^{*\prime}$, gives

$$\alpha^* + \gamma\mathbf{y}^{*\prime}\mathbf{E} = \lambda_\rho\rho + \mathbf{b}'\lambda_\mathbf{b} + \mathbf{d}'\lambda_\mathbf{d}, \tag{3.4}$$

by considering (2.8) and (2.9). If $\theta(\Lambda) = 0$, then by (3.4)

$$\lambda_\rho = \frac{\alpha^* + \gamma\mathbf{y}^{*\prime}\mathbf{E} + r_c}{\rho - r_c}.$$

So, by (2.12), (3.1) and (3.2) we have $\lambda_\rho = \phi(\rho)/(\rho - r_c)$. Then $\mathbf{x}^*$ is an optimal portfolio by Lemma 3.2. Considering Proposition 3.1, for any $\rho \in (\rho_{T_2}, \rho_{\max}]$ we have

$$\lambda_\rho > \frac{\phi(\rho)}{\rho - r_c}.$$

Thus, by (3.4),

$$\lambda_\rho(\rho - r_c) > \alpha^* + \gamma\mathbf{y}^{*\prime}\mathbf{E} + r_c = \lambda_\rho\rho + \mathbf{b}'\lambda_\mathbf{b} + \mathbf{d}'\lambda_\mathbf{d} + r_c,$$

then $\theta(\Lambda) < 0$. The reminder of the claim can be proved similarly. □

The proof of the following corollary is similar to that of Corollary 3.3 of [6] and therefore is omitted

Corollary 3.4. *Let* $(\mathbf{x}^*, \alpha^*, \mathbf{y}^*, \Lambda) \in \Omega(\rho_{\max})$ $(\Omega(\rho_{\min}))$. *If* $\theta(\Lambda) > 0$ $(\theta(\Lambda) < 0)$, *then* $\mathbf{x}^*$ *is an optimal portfolio.*

[2]Keykhaei and Jahandideh [6] set $\theta(\Lambda) = r_c\lambda_\rho + \mathbf{b}'\lambda_\mathbf{b} + \mathbf{d}'\lambda_\mathbf{d}$.

4. Efficient frontier analysis

Employing the method of Aneja and Nair [1], Keykhaei and Janahdideh [6] proposed a method for LP solvable portfolio selection problems that computes all extreme efficient portfolios and the efficient frontier of risky assets. Also, they demonstrated how a tangency portfolio can be produced as a by-product during the process. Here, a modification of the method of Keykhaei and Jahandideh is introduced. Considering Problem 2, the procedure first finds two efficient extreme points $z^{(1)}$ ($z^{(2)}$) corresponding to $\rho = \rho_{\max}$ ($\rho = \rho_{\min}$). To obtain the i'th efficient extreme point ($i \geq 3$), using the two efficient extreme points $z^{(r)} = (z_1^{(r)}, z_2^{(r)})$ and $z^{(s)} = (z_1^{(s)}, z_2^{(s)})$ considered by the algorithm, the following problem (associated with Problem 2) is used:

Problem (i):

$$
\begin{aligned}
\min_{\mathbf{X}=(\mathbf{x},\alpha,\mathbf{y})} \quad & a_i Z_2(\mathbf{X}) - b_i Z_1(\mathbf{X}) := a_i[\alpha + \gamma \sum_{j=1}^{N} y_j] - b_i \bar{\mathbf{r}}'\mathbf{x}, \\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\
& \mathbf{C}\mathbf{x} \geq \mathbf{d}' \\
& \mathbf{y} + \mathbf{R}'\mathbf{x} + \alpha \mathbf{E} \geq 0, \\
& \mathbf{y} \geq 0,
\end{aligned}
$$

where $a_i = |z_1^{(r)} - z_1^{(s)}|$ and $b_i = |z_2^{(r)} - z_2^{(s)}|$. If there are alternative optima, a portfolio with highest return is chosen. Now, Problem (i) produces a new efficient extreme point, if there exists a solution which improves the objective function in comparison to $z^{(r)}$ and $z^{(s)}$. The algorithm terminates when no extreme point is available.

Let $\mathit{\Omega}_i$ denotes the set of all primal-dual solutions $(\mathbf{X}^*, \hat{\mathit{\Lambda}})$ of Problem (i), where $\hat{\mathit{\Lambda}} = (\hat{\lambda}_{\mathbf{b}}, \hat{\lambda}_{\mathbf{d}}, \hat{\lambda}_{\mathbf{y}}, \hat{\lambda}_0)$ in which $\hat{\lambda}_{\mathbf{b}}$, $\hat{\lambda}_{\mathbf{d}}$, $\hat{\lambda}_{\mathbf{y}}$ and $\hat{\lambda}_0$ are the Lagrangian multipliers corresponding to the first constraint, the second and so on, respectively. Note that applying the SIMPLEX method enables the extraction of the dual solutions from its tableau, for example see [4]. Moreover, most of LP solvers in mathematical softwares such as MATLAB, compute dual solutions as well as primal solutions.

Lemma 4.1. *Let* $(\mathbf{X}^*, \hat{\mathit{\Lambda}}) \in \mathit{\Omega}_i$ *in which* $\mathbf{X}^* = (\mathbf{x}, \alpha, \mathbf{y})$. *Then* $(\mathbf{x}, \lambda_\rho, \lambda_{\mathbf{b}}, \lambda_{\mathbf{d}}, \lambda_{\mathbf{y}}, \lambda_0) \in \mathit{\Omega}(\rho)$, *where* $\rho = Z_1(\mathbf{X}^*) = \bar{\mathbf{r}}'\mathbf{x}$, $\lambda_\rho = b_i/a_i$ *and* $(\lambda_{\mathbf{b}}, \lambda_{\mathbf{d}}, \lambda_{\mathbf{y}}, \lambda_0) = (1/a_i)\hat{\mathit{\Lambda}}$.

Proof. Using K-K-T necessary optimality condition for $(\mathbf{X}^*, \hat{\mathit{\Lambda}})$ and the fact that $a_i > 0$, the relations (2.5)−(2.11) can be obtained for $(\mathbf{x}, \lambda_\rho, \lambda_{\mathbf{b}}, \lambda_{\mathbf{d}}, \lambda_{\mathbf{y}}, \lambda_0)$. This completes the proof. □

For $(\mathbf{X}^*, \hat{\mathit{\Lambda}}) \in \mathit{\Omega}_i$ let[3]

$$
\hat{\theta}(\hat{\mathit{\Lambda}}) := (a_i + b_i) r_c + \mathbf{b}'\hat{\lambda}_{\mathbf{b}} + \mathbf{d}'\hat{\lambda}_{\mathbf{d}}.
$$

Theorem 4.2. *Let* $(\mathbf{X}^*, \hat{\mathit{\Lambda}}) \in \mathit{\Omega}_i$.

(I) *If* $\hat{\theta}(\hat{\mathit{\Lambda}}) = 0$, *then* $\mathbf{x}$ *in* $\mathbf{X}^*$ *is an optimal portfolio.*
(II) *If* $Z_1(\mathbf{X}^*) \in (\rho_{T_2}, \rho_{\max}]$ $(Z_1(\mathbf{X}^*) \in [\rho_{\min}, \rho_{T_1}))$, *then* $\hat{\theta}(\hat{\mathit{\Lambda}}) < 0$ $(\hat{\theta}(\hat{\mathit{\Lambda}}) > 0)$.

Proof. By Lemma 4.1, $\hat{\mathit{\Lambda}}$ introduces the dual solution $\mathit{\Lambda} = (1/a_i)(b_i, \hat{\lambda}_{\mathbf{b}}, \hat{\lambda}_{\mathbf{d}}, \hat{\lambda}_{\mathbf{y}}, \hat{\lambda}_0)$ for Problem 2. Then $\theta(\mathit{\Lambda}) = (1/a_i)\hat{\theta}(\hat{\mathit{\Lambda}})$. Now Theorem 3.3 completes the proof. □

[3]Keykhaei and Jahandideh [6] set $\hat{\theta}(\hat{\mathit{\Lambda}}) = b_i r_c + \mathbf{b}'\hat{\lambda}_{\mathbf{b}} + \mathbf{d}'\hat{\lambda}_{\mathbf{d}}$.

In the following, how an optimal portfolio can be recognized during the procedure of detecting extreme efficient portfolios for the set of feasible portfolio of risky assets (2.1) is discussed. At the first step, the sign of $\theta(\Lambda)$ at $z^{(1)}$ and $z^{(2)}$ corresponding to $\rho = \rho_{\max}$ and $\rho = \rho_{\min}$, respectively, is examined. If $\theta(\Lambda) \geq 0$ ($\theta(\Lambda) \leq 0$), then $z^{(1)}$ ($z^{(2)}$) is a tangent point. Otherwise, If an efficient extreme point in which $\hat{\theta}(\hat{\Lambda}) = 0$ is met, then this point corresponds to an optimal portfolio. In the last case, if two adjacent efficient extreme points for which $\hat{\theta}(\hat{\Lambda})$ takes different sign are recognized, then the point which maximizes the ratio (3.3) corresponds to an optimal portfolio. In fact, this method allows for the comparison of the value of (3.3) only between two points. This property is quite useful, especially for large-scale portfolio selection problems which produce a large number of efficient extreme points.

5. Illustrative example

In this section a numerical example is presented by using historical price data of $n = 30$ stocks[4] chosen from S&P 100. The daily (closing) prices from 9/14/2009 to 2/10/2012 are used. Two-week period (ten business days) investment is considered. $N = 600$ two-week rates of return for every stock i is used and calculated as follows:

$$r_{it} = \frac{p_i^{t+10}}{p_i^t} - 1,$$

where p_i^t denotes the price of the ith stock at day t. The mean rate of return for stuck i is computed as $\bar{r}_i = \mathbb{E}(r_i) = \frac{1}{T}\sum_{i=1}^{T} r_{it}$. The rate of return for any cash account is set to 0.75 % over two weeks. The feasible region is set as $S = \{\mathbf{x} \in \mathbb{R}^n \ : \ \sum_{i=1}^{n} x_i = 1; \ x_i \geq 0, \ i = 1, \ldots, n\}$. The primal-dual solutions are computed using SIMPLEX method in MATLAB. The efficient extreme points obtained by Aneja−Nair method and their corresponding amount of $\hat{\theta}(\hat{\Lambda})$ are represented in Table 1 (Also, see Fig. 3). A solution of Problem (i) is considered as a new efficient extreme point if, in comparison to $z^{(r)}$ and $z^{(s)}$, it improved the objective function more than 10^{-5}. Note that $\theta(\Lambda)$ is only computed for $z^{(1)}$ and $z^{(2)}$ and $\hat{\theta}(\hat{\Lambda})$ is computed for the other efficient extreme points. The i'th efficient point is shown by $z^{(i)}$ and is obtained from two closest points with the lower labels at its left and its right sides. For example for $\beta = 0.99$, $z^{(6)}$ is obtained from $z^{(4)}$ and $z^{(3)}$. As can be seen, the number of efficient extreme points increases when β decreases. For each level of β, the optimal portfolio

Table 1. Efficient extreme points corresponding to three levels $\beta = 0.99, 0.95, 0.9$.

$\beta = 0.99$	$z^{(1)*}$	$z^{(5)}$	$z^{(4)}$	$z^{(7)}$	$z^{(6)}$	$z^{(8)}$	$z^{(3)}$	...	$z^{(17)}$	$z^{(2)}$
mean (%)	1.713	1.682	1.513	1.438	1.435	1.334	1.3	...	0.97	0.928
risk (%)	11.083	10.729	9.101	8.47	8.448	7.758	7.538	...	6.172	6.157
$\theta(\Lambda); \hat{\theta}(\hat{\Lambda})(10^{-5})$	228.37	3.07	10.56	2.12	7.23	5.11	32.11	...	3.91	6092.63
$\beta = 0.95$	$z^{(1)}$	$z^{(6)}$	$z^{(7)}$	$z^{(5)}$	$z^{(9)*}$	$z^{(8)}$	$z^{(10)}$	...	$z^{(30)}$	$z^{(2)}$
mean(%)	1.713	1.682	1.623	1.593	1.586	1.517	1.484	...	0.921	0.864
risk(%)	8.702	8.388	7.814	7.538	7.478	6.887	6.603	...	4.555	4.532
$\theta(\Lambda); \hat{\theta}(\hat{\Lambda})(10^{-5})$	−1005.16	−0.76	−0.45	−0.28	0.22	0.45	0.23	...	4.8	4513.77
$\beta = 0.9$	$z^{(1)}$	$z^{(6)}$	...	$z^{(4)}$	$z^{(16)*}$	$z^{(15)}$	$z^{(17)}$	...	$z^{(37)}$	$z^{(2)}$
mean(%)	1.713	1.66	...	1.5	1.488	1.459	1.449	...	0.935	0.897
risk(%)	7.462	6.931	...	5.497	5.4	5.198	5.13	...	3.671	3.668
$\theta(\Lambda); \hat{\theta}(\hat{\Lambda})(10^{-5})$	−2341.29	−2.26	...	−1.48	−0.001	0.229	0.228	...	2.62	3673.28

[4]The chosen stocks are: AAPL, ABT, AEP, ALL, APA, AXP, BAX, BMY, CL, COP, DELL, EMR, ETR, FDX, GOOG, HNZ, IBM, JPM, KO, MCD, NKE, NWSA, OXY, RTN, SLB, TXN, UPS, V, WFC and XOM.

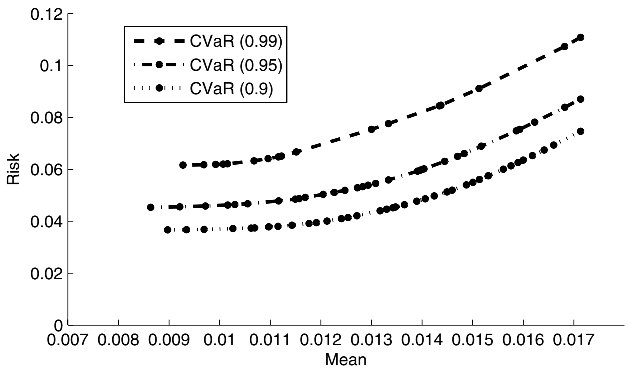

FIGURE 3. The Mean-ϕ_β efficient frontiers corresponding to Table 1 for three levels $\beta = 0.99, 0.95, 0.9$. Black disks denote the efficient extreme portfolios.

is denoted by $*$. For $\beta = 0.99$, the portfolio with the highest obtainable mean return has positive $\theta(\Lambda)$. So, at the first step the optimal portfolio can be introduced. For other values of β, the optimal portfolio is chosen between two successive portfolios with different signs of $\hat{\theta}(\hat{\Lambda})$. Note that, if the interest is only on the optimal portfolio, then the procedure can be stopped after realization of the optimal portfolio $z^{(r)*}$. Also, if borrowing is not allowed, the points $z^{(1)} - z^{(r)*}$ which are successively presented in the table and $z^{(0)} = (0.0075, 0)$ construct the set of efficient extreme points corresponding to the Mean-ϕ_β portfolio selection problem consisting of the aforementioned n risky assets (with the feasible region (2.1)) and a cash account with $r_c = 0.75$ %.

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