RAIRO-Oper. Res. 51 (2017) 669–684
DOI: 10.1051/ro/2016047

RAIRO Operations Research
www.rairo-ro.org

A STOCHASTIC PRODUCTION INVENTORY MODEL FOR DETERIORATING ITEMS WITH PRODUCTS' FINITE LIFE-CYCLE

Brojeswar Pal[1], Shib Sankar Sana[2] and Kripasindhu Chaudhuri[3]

Abstract. The article deals with a production inventory system for deteriorating items where the production rate of the system is a random variable within a finite range and the unit production cost depends on production lotsize as well as the rate of production. In the model, the maximum life-cycle of the products is finite and all the products are totally expired at the end of the life-time of the product. Shortages are allowed and partially backlogged. The backlogging rate is depended on length of the waiting time for the next replenishment. The main objective is to find out the optimal production lot-size such that the average expected cost per unit time of the inventory system is minimum. The different cases according to the value of the product's life-time, production run-time and cycle length of the system are discussed analytically and numerically. A numerical example and its sensitivity analysis along with its managerial insights are presented to illustrate the behavior of the proposed production-inventory model.

Mathematics Subject Classification. 90B05.

Received March 31, 2016. Accepted June 23, 2016.

1. Introduction

In the market, deterioration is one of the vital problem for the companies. It is important for any business to be aware about the deteriorating products for making more profit. Over the last couple of decades, researchers and practitioners have been working on deterioration that how the companies are managing deteriorating factors to increases their profit. They have also been facing more challenges to study on deterioration with products' finite life cycle and to find out the best strategies on marketing decision. A product life cycle refers to the time period from production to expire of the product. In practice, people are becoming more aware about marketing and they take decision consciously to buy their commodities. Generally, the deteriorating items (medicine, dairy products, vegetables, almost every food products, radioactive substances, blood, etc.) have finite life cycle. In a company, production planning is one of the most important parts for their business strategy. Careful production planning is necessary to ensure good deliveries and productive efficiencies. The production manager may face the difficult choice of whether to produce large quantities in bulk, store them and live with expensive financing and warehousing costs, or to remain flexible, manufacturing to meet orders but paying the penalty of lost economies

Keywords. Flexible manufacturing systems, cost benefit analysis, backlogged, deterioration, products life cycle.

[1] Department of Mathematics, The University of Burdwan, 713104 Burdwan, India. brojo_math@yahoo.co.in

[2] Department of Mathematics, Bhangar Mahavidyalaya, University of Calcutta, Bhangar, 743502 Kolkata, 24PGS (South), India. shib_sankar@yahoo.com

[3] Department of Mathematics, Jadavpur University, 700032 Kolkata, India. chaudhuriks@gmail.com

Article published by EDP Sciences

of scale. Now, it is main concern for the industries to find out marketing strategies with the inventories of deteriorating items with finite life cycle and varying production systems.

In deteriorating inventory literature, Dave and Patel [8] developed a deterministic inventory model for deteriorating items considering time proportion demand function. Sachan [23] presented an economic order-quantity model with constant deteriorating rate of the inventory. Shortages were also allowed for the situation of fixed cycle time and the time proportional demand were considered by him. Wee [33] studied the inventory management of deteriorating items with exponentially decreasing demand rate over a fixed time horizon allowing shortages. Sarkar *et al.* [27] presented an optimal payment time inventory system with deteriorating items under the condition that the supplier offers a permissible credit period for payment after the purchase of the goods and the retailer pays both the interest and the purchase price of the items to the supplier after the permissible credit period. Wu [37] proposed a deterministic inventory model with Weibull distributed deterioration under the circumstance of ramp type time dependent demand rate. He also allowed partial backlogging where backlogging rate depends on waiting time for the next replenishment. Teng *et al.* [32] developed an inventory model in which unsatisfied demand is partially backlogged at a negative exponential rate with the waiting time. Teng and Chang [31] introduced an economic production quantity models for deteriorating items considering price and stock-dependent demand. Inderfurth *et al.* [13] presented a deterministic lot-size model for a single product, focusing on a two-stage imperfect manufacturing system with reworking. Ghosh and Chaudhuri [9] studied an EOQ inventory model over a finite time-horizon for deteriorating item with quadratic time-dependent demand. They considered time-proportional rate of deterioration and shortages in every cycle. Hou and Lin [12] studied a deterministic economic order quantity inventory model taking into account inflation and time value of money. Wu *et al.* [38] investigated a inventory problem to determine an optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand. They also allowed shortages considering variable backlogging rate which is dependent on the waiting time for the next replenishment. Manna and Chaudhuri [17] formulated a production-inventory model for deteriorating items with linearly ramp type time dependent demand rate. They also assumed that the production rate is proportional to the demand rate and deterioration rate is time proportional. Wee *et al.* [35]developed an optimal imperfect quality inventory model for items with shortage back-ordering. Chang *et al.* [6] improved the model of Wu *et al.* [38] by changing the objective to maximizing the total profit and relaxed the restriction of zero ending inventory when shortages are not desirable. Ghosh *et al.* [10] presented an EOQ model of a perishable product with price dependent demand. They developed the model considering finite production, partial backlogging and lost sale where back-logging rate is dependent on the waiting time of the customers. Widyadana *et al.* [36] studied a deteriorating inventory problem with and without backorders. They solved the model without using derivatives and compared to the classical optimization method. Skouri *et al.* [29] extended the work of Manna and Chaudhuri [17] assuming a general function of time for the variable part of the demand rate considering with and without shortages. Sarkar [25] studied an EOQ model for finite replenishment rate where demand and deterioration rate are both time-dependent. He considered that the supplier offered trade credit to the retailer to buy more items with different discount rates on the purchasing costs. He and Wang [11] developed a production-inventory model for deteriorating items with demand disruption. They studied model into different scenarios according to disruption's time and magnitude. Lee and Dye [15] formulated a deteriorating inventory model with stock-dependent demand by allowing preservation technology cost as a decision variable in conjunction with replacement policy. Soni [30] extended the model of Chang *et al.* [6] considering permissible delay in payment and the demand rate as multivariate function of price and level of inventory. Shah *et al.* [28] developed an inventory system with non-instantaneous deteriorating item considering demand rate as a function of advertisement of an item and selling price. Lee and Kim [16] developed an integrated production-distribution model for both deteriorating and defective items under a single-vendor and single-buyer system. There are also several interesting and relevant papers related to inventory model with deterioration and shortages such as the works done by Abad [1], Wee and Yu [34], Papachristos and Skouri [21], Chung *et al.* [7], Papachristos and Konstantaras [22], Cárdenas–Barrón [2,3], Yang *et al.* [40], Chang [5], Pal *et al.* [18–20], Sana [24], Sarkar and Sarkar [26], Khanra *et al.* [14],Cárdenas–Barrón *et al.* [4], Wu *et al.* [39], etc.

In this study, we introduce an inventory model for deteriorating items where the maximum life cycle of the products are finite. We consider that the production rate is a random variable and it follows a distribution function. The production cost is taken as a function of production lot-size and production rate. We also allow shortages with backlogging and assume that the backlogging rate is depended on waiting time for the next replenishment. We study the model with different cases according the value of the expected production run-time, product maximum life-time and cycle time of the system. Now, the objective of the model is to be optimize the expected average per unit time cost with respect to the production lot-size.

The rest of the paper is organized as follows: Section 2 illustrates fundamental assumptions and notations. Formulation of the model is discussed in Section 3. Section 4 analyzes Numerical analysis. Sensitivity analysis and its managerial insights are illustrated in Section 5 and finally conclusion of the paper is provided in Section 6.

2. Fundamental notation and assumption

The following notations and assumptions are adopted to depict the proposed model.

2.1. Notation

The following notations are used throughout the paper.

Q	Production lot-size.
$D(p)$	Demand rate per unit time.
t_p	Production run-time.
t_1	Production-inventory cycle time.
M	The maximum lifetime of the products.
δ	Backlogging rate of shortages.
θ	Deterioration rate of the products due to quality of the products, preservation system, handling expert of the products, pilferage, etc., which is estimated from previous knowledge.
h_c	Holding cost (\$) per unit per unit time.
b_c	The backlogging cost (\$) per unit per unit time due to shortages.
l_c	The lost sale cost (\$) per unit per unit time.
C_r	Purchasing cost (\$) per unit item.
$C_p(Q, P)$	Per unit production cost (\$).

2.2. Assumptions

The following assumptions are adopted to develop the model.

- Model is developed for single deteriorating item with finite life time of the products.
- Replenishment rate of supplier is instantaneously infinite, but it's size is finite, that means replacement of lot size is sufficiently large at any number, if needed.
- Production rate of the inventory system is random variable.
- Production cost per unit item depends on production lot-size and production rate.
- Shortages are allowed and partially backlogged.
- The backlogging rate is depended on the length of waiting time for next replenishment.

3. Formulation of the model

In this article, we formulate a production inventory model of deteriorating items where the life cycle of the products are finite. We consider that the rate of deterioration is constant. It is a fraction ($0 \leq \theta \leq 1$) of on-hand inventory due to quality of the products, preservation system, handling expert of the products, pilferage, etc., which is estimated from previous knowledge. The inventory system produces the finished products with random production rate P of the Q unit of the items where production lot size Q is a decision variable and the range of the random variable is finite. The production system runs upto $t_p = \frac{Q}{P}$ time and the one inventory cycle time is $t_1 = \frac{Q}{(1+\theta)D}$. Here, the demand of the products are consumed directly from production on the basis of 'first

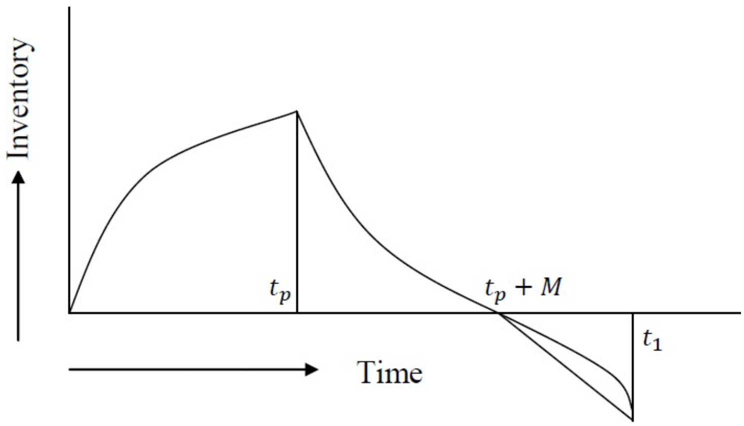

FIGURE 1. Inventory *versus* time for Section 3.1.1 of Case I.

product first out' and the rest amount is piled up upto the production run time, and we approximate the life cycle of whole item at end of production time as fixed (M). We consider that the per unit production cost is function of the production rate such as $C_p(Q,P) = \frac{L}{Q} + \alpha P$. Now, we study different cases according to the values of the maximum lifetime of the products (M), expected production run-time ($T_p = E[t_p]$) and the total cycle time of the inventory system (t_1).

3.1. Case I: When $\mathbf{t_1 \geq T_p + M}$

In this case, the sum of the maximum lifetime of the products and the expected production run-time is less than from the total cycle time. After the expected time T_p+M, the lifetime of the produced products is expired. So, the no inventory is available for sale between the time period T_p+M to t_1. Hence, the shortages are occurred during the time interval $(T_p + M, t_1)$ and all of the demand during the period $(T_p + M, t_1)$ is backlogged. The backlogged items are distributed in the next cycle time. We assume that the backlogged rate is dependent on length of the waiting time such as $\delta(t) = \mathrm{e}^{-\kappa(t_1-t)}$. According to the value of the demand rate, deterioration rate and production rate, the following subcases are occurred.

3.1.1. When $\mathsf{P} \geq (1+\theta)\mathsf{D}$

Here, production rate is higher than the $(1+\theta)$ times of demand rate. Hence, inventory level does not fall into shortages during production run-time. But, shortages are occurred after the time $t_p + M$ due to product's life-time factor and are continued up to t_1 time (see Fig. 1). So, in this stage, the governing differential equations are:

$$\frac{\mathrm{d}I_1(t)}{\mathrm{d}t} + \theta I_1(t) = P - D, \text{ with } \ I_1(0) = 0,\ 0 \leq t \leq t_p, \tag{3.1}$$

$$\frac{\mathrm{d}I_2(t)}{\mathrm{d}t} + \theta I_2(t) = -D, \text{ with } I_2(t_p) = I_1(t_p) \text{ and } I_2(t_p+M) = 0,\ t_p \leq t \leq t_p + M, \tag{3.2}$$

and

$$\frac{\mathrm{d}I_3(t)}{\mathrm{d}t} = -\delta(t)D, \text{ with } I_3(t_p+M) = 0,\ t_p + M \leq t \leq t_1. \tag{3.3}$$

Solving the above equations, using the boundary conditions, we have

$$I_1(t) = \frac{(1-\mathrm{e}^{-\theta t})(P-D)}{\theta},\ 0 \le t \le t_p, \tag{3.4}$$

$$I_2(t) = \frac{\mathrm{e}^{-t\theta}(\mathrm{e}^{(M+t_p)\theta} - \mathrm{e}^{t\theta})D}{\theta},\ t_p \le t \le \big(t_p + M\big) \tag{3.5}$$

and

$$I_3(t) = -\frac{D\mathrm{e}^{-\kappa t_1}(\mathrm{e}^{\kappa t} - \mathrm{e}^{(M+t_p)\kappa})}{\kappa},\ \big(t_p + M\big) \le t \le t_1. \tag{3.6}$$

Now, the inventory holding cost is

$$\begin{aligned} HC_1(Q) =& h_c\left(\int_0^{t_p} I_1(t)\mathrm{d}t + \int_{t_p}^{t_p+M} I_2(t)\mathrm{d}t\right) \\ =& h_c\left(\frac{\mathrm{e}^{-t_p\theta}\left(1+\mathrm{e}^{t_p\theta}(\theta t_p - 1)\right)(P-D) + \left(\left(-1+\mathrm{e}^{M\theta}\right) - M\theta\right)D}{\theta^2}\right). \end{aligned} \tag{3.7}$$

The total amount of backlogged cost at the end of cycle is

$$BC_1(Q) = b_c\int_{t_p+M}^{t_1} I_3(t)\mathrm{d}t = \frac{b_c}{\kappa^2}\left(1 - \mathrm{e}^{-\kappa(t_1-M-t_p)}\{1+(t_1-M-t_p)\kappa\}\right)D. \tag{3.8}$$

The total amount of lost sale cost during shortages time is

$$LC_1(Q) = l_c\int_{t_p+M}^{t_1}\{1-\delta(t)\}D\mathrm{d}t = \frac{l_c}{\kappa}\left(\mathrm{e}^{-\kappa(t_1-M-t_p)} + (t_1-M-t_p)\kappa - 1\right)D. \tag{3.9}$$

Therefore, Total cost of the inventory system = Total purchasing cost of raw material+Manufacturing cost + Inventory holding cost + Shortages cost + Lost sales during shortages.

The average cost per unit time during the time interval $[0, t_1]$ is

$$E\pi_1(Q) = \frac{(1-\theta)D}{Q}\left[Q(C_r+C_p) + \frac{h_c}{\theta^2}\left\{\mathrm{e}^{-t_p\theta}\left(1+\mathrm{e}^{t_p\theta}(\theta t_p-1)\right)(P-D)\right.\right. \tag{3.10}$$

$$\left.+\left(\left(\mathrm{e}^{M\theta}-1\right)-M\theta\right)\right\}D + \frac{b_c}{\kappa^2}\left\{1-\mathrm{e}^{-\kappa(t_1-M-t_p)}\{1+(t_1-M-t_p)\kappa\}\right\}D \tag{3.11}$$

$$\left.+\frac{l_c}{\kappa}\left\{\mathrm{e}^{-\kappa(t_1-M-t_p)} + (t_1-M-t_p)\kappa - 1\right\}D\right]. \tag{3.12}$$

Putting the value of t_1 and t_p in equation (3.10) and then simplifying, we have

$$\begin{aligned} E\pi_1(Q) = {}& D(1-\theta)\left(C_p + C_r + l_c + \frac{h_c}{\theta} - l_c\theta\right) \\ &- \frac{D^2(1-\theta)\left(\kappa\left(h_c\left(M\theta - \mathrm{e}^{M\theta}\right)\kappa + l_c\theta^2(1+M\kappa)\right) - b_c\theta^2\right)}{Q\theta^2\kappa^2} \\ &+ \frac{D^2(1-\theta)(h_c+l_c\theta)}{\theta}\frac{1}{P} - D(1-\theta)\frac{h_c}{Q\theta^2}P - \frac{D^2h_c(1-\theta)}{Q\theta^2}\mathrm{e}^{-\frac{Q\theta}{P}} + \frac{Dh_c(1-\theta)}{Q\theta^2}P\mathrm{e}^{-\frac{Q\theta}{P}} \\ &- \left(\frac{b_cD(1-\theta)^2}{\kappa} + \frac{D(1-\theta)(Dl_c\kappa + b_cD(1-M\kappa))}{Q\kappa^2}\right)\mathrm{e}^{\left(M+\frac{Q(-1+\theta)}{D}\right)\kappa}\mathrm{e}^{\frac{Q}{P}\kappa} \\ &+ D(1-\theta)\frac{b_cD\mathrm{e}^{\left(M+\frac{Q}{P}-\frac{Q(1-\theta)}{D}\right)\kappa}}{\kappa}\frac{1}{P}\mathrm{e}^{\frac{Q}{P}\kappa}. \end{aligned} \tag{3.13}$$

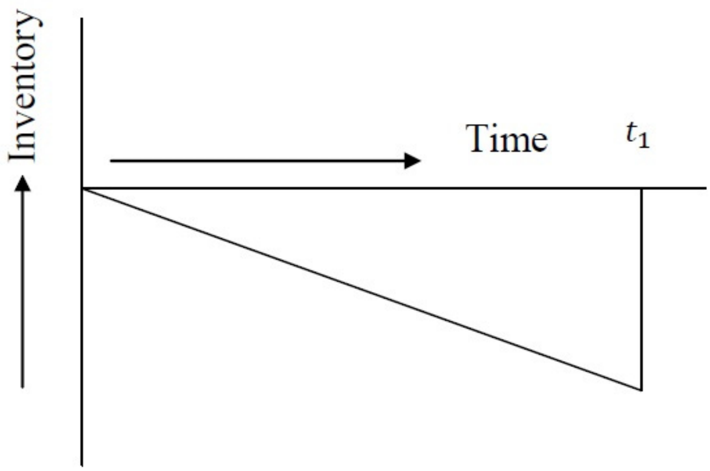

FIGURE 2. Inventory *versus* time for Section 3.1.2 of Case I.

3.1.2. When $0 \leq P < (1+\theta)D$

In this subcase, the production rate is less than $(1+\theta)D$. This is a rear situation but it may happen due to highly deteriorating products and supply disruption of raw materials. The system faces shortages from the beginning and it never be recovered within the current cycle period. Here, the demand rate generally is not affected by the shortages situation because the availability of the products is not sufficient in the market for highly deterioration. Here, the cycle length of the system continue up to time $t_1 = \frac{Q}{D}$ as the system cycle length is not affected by the deterioration for shortages (see Fig. 2).

So, in this stage, the governing differential equations are:

$$\frac{\mathrm{d}I_1(t)}{\mathrm{d}t} = -(D-P), \text{ with } I_1(0) = 0,\ 0 \leq t \leq t_1. \tag{3.14}$$

Solving the above equations using the boundary conditions, we have

$$I_1(t) = -(D-P)t,\ 0 \leq t \leq t_1. \tag{3.15}$$

The shortages cost is

$$BC_2(Q) = \frac{b_c}{2}\left((D-P)t_1^2\right). \tag{3.16}$$

The total cost of the inventory system = Total purchasing cost of raw material + Manufacturing cost + Shortages cost.

The average total cost per unit time during the time interval $[0, t_1]$ is

$$E\pi_2(Q) = \frac{(1-\theta)D}{Q}\left[Q(C_r + C_p) + \frac{b_c}{2}\left\{(D-P)t_1^2\right\}\right]. \tag{3.17}$$

Putting the value of $t_1 = \frac{Q}{D}$ in equation (3.17) and then simplifying, we have

$$E\pi_2(Q) = C_r D + \frac{b_c Q}{2} + \frac{D\Gamma}{Q} + \left(D\alpha - \frac{b_c PQ}{2D}\right)P. \tag{3.18}$$

Using the equations (3.13) and (3.18), the expected average total cost per unit time is

$$E\Pi_1(Q) = \int_L^{(1+\theta)D} E\pi_2(Q)f(P)\,\mathrm{d}P + \int_{(1+\theta)D}^{U} E\pi_1(Q)f(P)\,\mathrm{d}P. \tag{3.19}$$

Now, the objective of the system is to optimize the per unit time expected average cost function with respect to production lot-size, *i.e.*,

$$\text{Minimize } \; E\Pi_1(Q) \text{ subject to } t_1 \geq \big(T_p + M\big). \tag{3.20}$$

3.1.3. Solution considering uniform probability distribution

Let us consider the probability density functions of production rate (P) be

$$f(P) = \begin{Bmatrix} \frac{1}{U-L}, & L \leq P \leq U, \\ 0, & \text{elsewhere} \end{Bmatrix}$$

Using the uniform density function of the production rate and simplifying the expected average cost function (3.19) per unit time (See the Appendix A), we have the expected average cost function per unit time is

$$E\Pi_1(Q) = A_1 + \frac{A_2}{Q} + A_3Q + A_4Q^2 + \mathrm{e}^{\kappa M}\left(B_1 + \frac{B_2}{Q} + B_3Q + B_4Q^2 + B_5Q^3\right)\mathrm{e}^{-\kappa\frac{Q(1-\theta)}{D}},$$

where

$$\begin{aligned}
A_1 &= \frac{D}{2(L-U)\theta}\Big[2h_cU(1+\theta) + D^2\alpha\theta^2(1+\theta)^2 - 2D(1+\theta)(h_c + C_r\theta + h_c\theta) + \theta\Big\{2C_rL \\
&\quad + \alpha\left(L^2 - U^2(1+\theta)\right)\Big\}\Big] + \frac{D^2 l_c(1+\theta)(\log U - \log D(1+\theta))}{L-U} \\
&\quad + \frac{D(h_c(1+\theta) + \theta(C_r + l_c + C_r\theta))}{\theta}, \\
A_2 &= \frac{D}{(U-L)\theta^2}\Big[D^2h_c(1+\theta)^2 + \Gamma\theta^2(U(1+\theta) - L(2+\theta)) + D(1+\theta)\Big\{\mathrm{e}^{M\theta}h_c(U-L) \\
&\quad + (Ll_cM - l_cMU + \Gamma)\theta^2 - h_c(U - LM\theta + MU\theta)\Big\}\Big] + \frac{D^2(1+\theta)(b_c - l_c\kappa)}{\kappa^2}, \\
A_3 &= \frac{2b_cDLU - b_cL^2U - 2D^3h_c(1+\theta) + D^2U\left(2h_c - b_c(1-\theta^2)\right)}{4D(L-U)U} \\
&\quad + \frac{Dh_c(1+\theta)(\log U - \log D(1+\theta))}{2(U-L)}, \\
A_4 &= -\frac{h_c\theta\left(D^2(1+\theta)^2 - 2DU(1+\theta)^2 + U^2(1+2\theta)\right)}{12(U-L)U^2(1+\theta)}, \\
B_1 &= \frac{D^2(l_c + b_cM)(1+\theta)(\log U - \log D(1+\theta))}{U-L} - \frac{b_cD(U - D(1+\theta))}{(U-L)\kappa}, \\
B_2 &= \frac{D^2(1+\theta)(U - D(1+\theta))(l_c\kappa + b_c(M\kappa - 1))}{(U-L)\kappa^2}, \\
B_3 &= \frac{D(U - D(1+\theta))(b_c + l_c\kappa + b_cM\kappa)}{2(U-L)U} - \frac{b_cD(\log U - \log D(1+\theta))}{U-L}, \\
B_4 &= \frac{\{U - D(1+\theta)\}\kappa[l_c\{U + D(1+\theta)\}\kappa + b_c\{U(M\kappa - 4) + D(1+\theta)(2 + M\kappa)\}]}{12(U-L)U^2(1+\theta)}, \\
B_5 &= -\frac{b_c\{U - D(1+\theta)\}^2\{U + 2D(1+\theta)\}\kappa^2}{36D(U-L)U^3(1+\theta)^2}.
\end{aligned}$$

Differentiating the per unit time average expected cost function with respect to the production lot (Q), we get

$$\frac{\mathrm{d}E\Pi_1(Q)}{\mathrm{d}Q} = A_3 - \frac{A_2}{Q^2} + 2A_4Q + \Bigg[\left(B_3 - \frac{B_1(1-\theta)\kappa}{D}\right) - \frac{B_2(1-\theta)\kappa}{DQ} - \frac{B_2}{Q^2} + \left(2B_4 - \frac{B_3(1-\theta)\kappa}{D}\right)Q + \left(3B_5 - \frac{B_4(1-\theta)\kappa}{D}\right)Q^2 - \frac{B_5(1-\theta)\kappa}{D}Q^3\Bigg]\mathrm{e}^{M\kappa - \frac{Q(1-\theta)\kappa}{D}}$$

$$\frac{d^2E\Pi_1(Q)}{\mathrm{d}Q^2} = 2A_4 + \frac{2A_2}{Q^3} + \Bigg[\left(2B_4 - \frac{2B_3(1-\theta)\kappa}{D} + \frac{B_1(1-\theta)^2\kappa^2}{D^2}\right) + \frac{2B_2}{Q^3} + \frac{2B_2(1-\theta)\kappa}{DQ^2} + \frac{B_2(1-\theta)^2\kappa^2}{D^2Q} + \left(6B_5 - \frac{4B_4(1-\theta)\kappa}{D} + \frac{B_3(1-\theta)^2\kappa^2}{D^2}\right)Q - \left(6\frac{B_5(1-\theta)\kappa}{D} - \frac{B_4(1-\theta)^2\kappa^2}{D^2}\right)Q^2 + \frac{B_5(1-\theta)^2\kappa^2}{D^2}Q^3\Bigg]\mathrm{e}^{M\kappa - \frac{Q(1-\theta)\kappa}{D}}.$$

Let Q^* be the solution of the equation $\frac{\mathrm{d}E\Pi_1(Q)}{\mathrm{d}Q} = 0$. The solution Q^* is optimal if it satisfies the second order condition $\frac{d^2E\Pi_1(Q)}{\mathrm{d}Q^2} > 0$ at $Q = Q^*$. We show the conditions numerically as it is not possible to prove the condition analytically.

3.2. Case II: When $\mathbf{t_1 \leq (T_p + M)}$

In this section, the sum of the maximum lifetime of the products and the production run-time is greater than from the total cycle time of the inventory system. So, the shortages of the demand are not occurred for product's lifetime. But, the system may fall into shortages due to the reason of production rate which is a random variable. According to the value of the demand rate, deterioration rate and production rate, the following subcases are occurred.

3.2.1. *When* $\mathsf{P} \geq (1+\theta)\mathsf{D}$

In the subcase, we insert the condition on production rate such that the system does not fall into shortages of demand during production run-time (see the model diagram Fig. 3). So, in this stage, the governing differential equations are:

$$\frac{\mathrm{d}I_1(t)}{\mathrm{d}t} + \theta I_1(t) = P - D, \text{ with } I_1(0) = 0,\ 0 \leq t \leq t_p \tag{3.21}$$

and

$$\frac{\mathrm{d}I_2(t)}{\mathrm{d}t} + \theta I_2(t) = -D, \text{ with } I_2(t_p) = I_1(t_p) \text{ and } I_2(t_1) = 0,\ t_p \leq t \leq t_1. \tag{3.22}$$

Solving the above equations using the boundary conditions, we have

$$I_1(t) = \frac{(1-\mathrm{e}^{-t\theta})(P-D)}{\theta},\ 0 \leq t \leq t_p \tag{3.23}$$

and

$$I_2(t) = \frac{(\mathrm{e}^{\theta(t_1-t)} - 1)D}{\theta},\ t_p \leq t \leq t_1. \tag{3.24}$$

The inventory holding cost is

$$\begin{aligned} HC_2(Q,P) &= h_c\left(\int_0^{t_p} I_1(t)\mathrm{d}t + \int_{t_p}^{t_1} I_2(t)\mathrm{d}t\right) \\ &= h_c\left(\frac{\mathrm{e}^{-\theta t_p}\left(P + P\mathrm{e}^{\theta t_p}(\theta t_p - 1) + \left(\mathrm{e}^{\theta t_1} - \theta t_1\mathrm{e}^{\theta t_p} - 1\right)D\right)}{\theta^2}\right). \end{aligned} \tag{3.25}$$

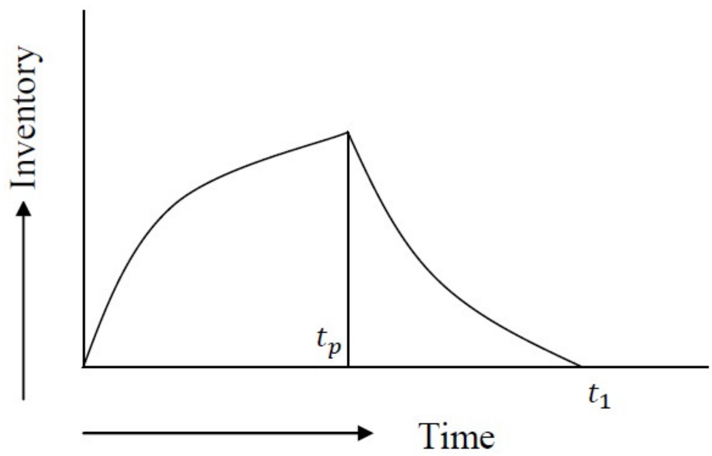

FIGURE 3. Inventory *versus* time for Section 3.2.1 of Case II.

The total cost of the inventory system = Total purchasing cost of raw material + Manufacturing cost + Inventory holding cost.

The average total cost per unit time during the time interval $[0, t_1]$ is

$$E\pi_3(Q) = \frac{(1-\theta)D}{Q}\left[Q(C_r + C_p) + \frac{h_c}{\theta^2}\left\{\mathrm{e}^{-\theta t_p}\left(P + \mathrm{e}^{\theta t_p}(\theta t_p - 1) + \left(\mathrm{e}^{\theta t_1} - \theta t_1 \mathrm{e}^{\theta t_p} - 1\right) D\right)\right\}\right]. \tag{3.26}$$

Putting the value of t_1 and t_p in eq. (24) and then simplifying, we have

$$E\pi_3(Q) = D(C_r + h_c)(1-\theta) + \frac{DL(1-\theta)}{P} + \frac{D\left(Q\alpha - \frac{h_c}{\theta^2}\right)(1-\theta)}{Q}P$$
$$+ \frac{D^2\left(\mathrm{e}^{\frac{Q(1-\theta)\theta}{D}} - 1\right)h_c(1-\theta)}{Q\theta^2}\mathrm{e}^{-\frac{Q\theta}{P}} + \frac{Dh_c(1-\theta)}{Q\theta^2}P\mathrm{e}^{-\frac{Q\theta}{P}}. \tag{3.27}$$

3.2.2. *When* $0 \leq \mathsf{P} < (1+\theta)\mathsf{D}$

Similar as Section 3.1.2 of Case I. Hence using equation (3.18), the average total cost per unit time during the time interval $[0, t_1]$ is

$$E\pi_2(Q) = C_r D + \frac{b_c Q}{2} + \frac{D\Gamma}{Q} + \left(D\alpha - \frac{b_c P Q}{2D}\right)P. \tag{3.28}$$

Using the equations (3.27) and (3.28), the expected average total cost per unit time is

$$E\Pi_2(Q) = \int_L^{(1+\theta)D} E\pi_4(Q) f(P)\,\mathrm{d}P + \int_{(1+\theta)D}^{U} E\pi_3(Q) f(P)\,\mathrm{d}P. \tag{3.29}$$

Now, the objective of the system is to optimize the per unit time expected average cost function with respect to production lot-size, *i.e.*,

$$\text{Minimize } \; E\Pi_2(Q) \text{ subject to } t_1 \leq \big(T_p + M\big). \tag{3.30}$$

3.2.3. Solution considering uniform probability distribution

Considering the probability density functions of production rate (P) as

$$f(P) = \begin{Bmatrix} \frac{1}{U-L}, & L \le P \le U, \\ 0, & \text{elsewhere} \end{Bmatrix}.$$

The expected average cost function (3.29) per unit time is (see Appendix A for expected value)

$$E\Pi_2(Q) = E_1 + E_2 Q + E_3 Q^2 + \frac{E_4}{Q} + \left(F_1 + F_2 Q + F_3 Q^2 + \frac{F_4}{Q}\right) \mathrm{e}^{\frac{\theta Q}{(1+\theta)D}},$$

where

$$\begin{aligned}
E_1 &= \frac{D}{2(L-U)\theta}\left[D^2\alpha\theta^2(1+\theta)^2 - 2D(1+\theta)(h_c + C_r\theta + h_c\theta) + 2h_c(U + L\theta)\right. \\
&\quad \left.+\theta\left\{\alpha\left(L^2 - U^2(1+\theta)\right) - 2C_r(U(1+\theta) - L(2+\theta))\right\}\right] + \frac{D^2 h_c(1+\theta)(\log U - \log D(1+\theta))}{(U-L)\theta}, \\
E_2 &= \frac{2b_c DLU - b_c L^2 U - 2D^3 h_c(1+\theta) + D^2 U\left(2h_c - b_c\left(1-\theta^2\right)\right)}{4D(L-U)U} + \frac{Dh_c(1+\theta)(\log U - \log D(1+\theta))}{2(U-L)}, \\
E_3 &= -\frac{h_c\theta\left(D^2(1+\theta)^2 - 2DU(1+\theta)^2 + U^2(1+2\theta)\right)}{12(U-L)U^2(1+\theta)}, \\
E_4 &= \frac{D\left(D^2 h_c(1+\theta)^2 + D(1+\theta)\left(\Gamma\theta^2 - h_c U\right) + \Gamma\theta^2(U(1+\theta) - L(2+\theta))\right)}{(U-L)\theta^2}, \\
F_1 &= -\frac{D^2 h_c(1+\theta)(\log U - \log D(1+\theta)])}{(U-L)\theta}, F_2 = \frac{Dh_c(U + D(1+\theta))}{2(U-L)U}, \\
F_3 &= -\frac{h_c\theta\left(U^2 - D^2(1+\theta)^2\right)}{12(U-L)U^2(1+\theta)},\ F_4 = \frac{D^2 h_c(1+\theta)(U + D(1+\theta))}{(U-L)\theta^2}.
\end{aligned}$$

Differentiating the per unit time average expected cast function with respect to the production lot (Q), we get

$$\begin{aligned}
\frac{\mathrm{d}E\Pi_2(Q)}{\mathrm{d}Q} &= E_2 + 2E_3 Q - \frac{E_4}{Q^2} + \left[\left(F_2 + \frac{F_1\theta}{(1+\theta)D}\right) + \left(2F_3 + \frac{F_2\theta}{(1+\theta)D}\right)Q + \frac{F_3\theta Q^2}{(1+\theta)D}\right. \\
&\quad \left. + \left(\frac{F_4\theta}{(1+\theta)QD} - \frac{F_4}{Q^2}\right)\right]\mathrm{e}^{\frac{Q\theta}{D+D\theta}}, \\
\frac{d^2 E\Pi_2(Q)}{\mathrm{d}Q^2} &= 2E_3 + \frac{2E_4}{Q^3} + \left[2F_3 + \frac{F_1\theta^2}{(D(1+\theta))^2} + \frac{2F_2\theta}{D(1+\theta)} + \left(\frac{F_2\theta^2}{(D(1+\theta))^2} + \frac{4F_3\theta}{D(1+\theta)}\right)Q + \frac{F_3 Q^2\theta^2}{(D+D\theta)^2}\right. \\
&\quad \left. + \frac{2F_4}{Q^3} + \frac{F_4\theta^2}{Q(D+D\theta)^2} - \frac{2F_4\theta}{Q^2(D(1+\theta))}\right]\mathrm{e}^{\frac{Q\theta}{D(1+\theta)}}.
\end{aligned}$$

Let Q^* be the solution of the equation $\frac{\mathrm{d}E\Pi_2(Q)}{\mathrm{d}Q} = 0$. The solution Q^* is optimal if it satisfies the second order condition $\frac{d^2 E\Pi_2(Q)}{\mathrm{d}Q^2} > 0$ at $Q = Q^*$. We show the conditions numerically as it is not possible to prove the condition analytically.

4. Numerical example

Here, we illustrate our model numerically to find out the insight behaviors of the proposed model. We consider the values of the parameters in appropriate units as follows: Demand rate $D = 600$ unit per unit time, inventory

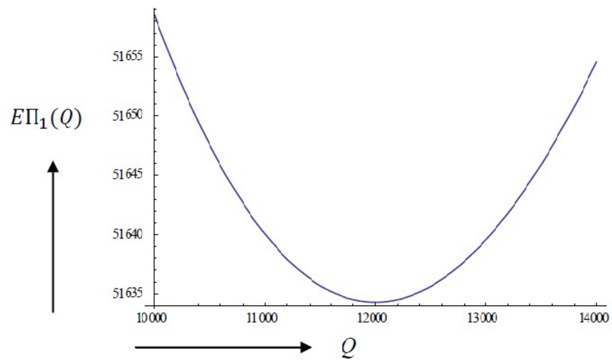

FIGURE 4. Expected per unit time average cost *versus* production lot-size for Case I.

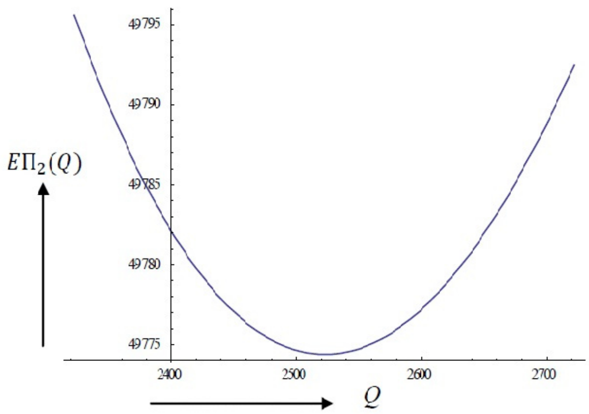

FIGURE 5. Expected per unit time average cost *versus* production lot-size for Case I.

holding cost $h_c = \$2.5$ per unit per unit time, purchasing cost $C_r = \$15$ per unit, back-ordering cost $b_c = \$5$ per unit per unit time, lost sale cost $l_c = \$4$ per unit per unit time, deterioration rate $\theta = 0.12$, production cost parameters $\Gamma = 10\,000$ and $\alpha = 0.01$, products' maximum life cycle time $M = 5$ unit, parameter of back-ordering rate $\kappa = 0.4$ and parameters of uniform distribution $U = 10\,000$ and $L = 10$. Then, the optimal results for Case I are optimal production lot size $Q^* = 10\,789.80$ unit, optimal expected production run-time $T_p = 7.46$ unit, optimal cycle time $t_1 = 16.06$ unit $> (T_p + M = 12.46)$ and the optimal per unit time expected average cost $E\Pi_1 = \$51\,799.30$. The above results are optimal as $\frac{d^2 E\Pi_1(Q)}{dQ^2} = 13.07 \times 10^{-6} > 0$ at the value of $Q^* = 10\,789.80$ unit. It is clear from the Figure 4 that the objective function $E\Pi_1$ is convex and unimodal function.

Here, the optimal results for Case II are optimal production lot size $Q^* = 2508.08$ unit, optimal expected production run-time $T_p = 1.73$ unit, optimal cycle time $t_1 = 3.73$ unit $\leq (T_p + M = 6.73)$ and the optimal per unit time expected average cost $E\Pi_2 = \$49810.90$. The above results are optimal as $\frac{d^2 E\Pi_2(Q)}{dQ^2} = 0.000992328 > 0$ at the value of $Q^* = 2508.08$ unit. Here, Figure 5 shows clearly that the objective function $E\Pi_2$ is a convex function. Similar as before, the required optimal solution is unique here.

Table 1. Sensitivity analysis of key parameters.

Parameter		Case I: $t_1 > t_p + M$					Case II: $t_1 \leq t_p + M$				
		Q	t_p	$(t_p + M)$	t_1	$E\Pi_1$	Q	t_p	$(t_p + M)$	t_1	$E\Pi_2$
M	2.0	7599.33	5.25	7.25	11.31	51 014.00	2508.08	1.73	3.73	3.73	49 810.90
	3.5	7490.02	5.17	8.68	11.15	51 223.90	2508.08	1.73	5.23	3.73	49 810.90
	6.5	14 951.90	10.34	16.84	22.25	52 561.40	2508.08	1.73	8.23	3.73	49 810.90
	8.0	19 960.60	13.80	21.80	29.70	53 425.50	2508.08	1.73	9.73	3.73	49 810.90
θ	0.08	11 861.30	8.20	13.20	18.30	50 956.00	2493.81	1.72	6.72	3.85	48 140.50
	0.10	11 054.80	7.64	12.64	16.75	51 277.70	2500.59	1.73	6.73	3.79	48 976.30
	0.14	11 015.20	7.62	12.62	16.10	52 428.70	2516.23	1.74	6.74	3.68	50 644.30
	0.16	11 788.30	8.15	13.15	16.94	53 117.70	2525.02	1.75	6.75	3.63	51 476.50
D	400	10 065.00	6.96	11.96	22.47	33 966.90	1921.31	1.33	6.33	4.29	34 001.90
	500	10 600.70	7.33	12.33	18.93	42 823.20	2223.17	1.54	6.54	3.97	41 912.50
	700	11 049.30	7.64	12.64	14.09	60 892.00	2780.37	1.92	6.92	3.55	57 709.40
	800	11 777.90	8.14	13.14	13.15	70 112.20	3042.82	2.10	7.10	3.34	65 615.90
h_c	1.5	7115.74	4.92	9.92	10.59	50 035.00	3100.15	2.14	7.14	4.61	48 697.50
	2.0	9186.07	6.35	11.35	13.67	50 946.00	2755.30	1.90	6.91	4.10	49 285.80
	3.0	11 598.40	8.02	13.02	17.26	52 635.50	2319.20	1.60	6.60	3.45	50 290.10
	3.5	12 122.90	8.38	13.38	18.04	53 465.00	2168.58	1.50	6.50	3.23	50 734.20
b_c	3	11 749.90	8.12	13.12	17.48	51 083.20	2537.36	1.75	6.75	3.78	49 738.70
	4	11 307.60	7.82	12.82	16.83	51 442.80	2522.60	1.74	6.74	3.75	49 774.90
	6	10 089.70	6.98	11.98	15.01	52 151.20	2493.77	1.72	6.72	3.71	49 846.60
	7	8868.84	6.13	11.13	13.20	52 493.50	2479.69	1.71	6.71	3.69	49 882.20
U	8000	9864.20	8.25	13.25	14.68	45 813.30	2582.12	2.16	7.16	3.84	43 197.90
	9000	10 282.00	7.78	12.78	15.30	48 770.40	2541.26	1.92	6.92	3.78	46 497.30
	11 000	11 338.10	7.22	12.22	16.87	54 880.20	2480.56	1.58	6.58	3.69	53 134.40
	12 000	11 882.30	7.02	12.02	17.68	58 000.30	2457.33	1.45	6.45	3.66	56 465.20
L	1	10 679.80	9.84	14.84	15.89	51 816.90	2507.66	2.31	7.31	3.73	49 794.70
	5	10 728.50	8.16	13.16	15.97	51 809.00	2507.85	1.91	6.91	3.73	49 801.90
	100	11 977.70	5.57	10.57	17.82	51 631.20	2510.37	1.17	6.17	3.74	49 975.70
	500	17 213.40	5.43	10.43	25.62	51 203.70	2477.19	0.78	5.78	3.67	50 791.60

5. Sensitivity analysis with managerial insights

From the Table 1, we observe the sensitivity of the parameters which help the decision makers to take appropriate decisions on their marketing strategy. Now, we discuss the sensitivity of the parameters as follows:

- With the increasing value of the maximum life-time of the product M, the optimal production lot size, optimal cycle time of the inventory system, expected production run-time and per unit time expected average cost increase for both Cases, I and II.
- The optimal cycle time of the inventory system, expected production run-time and per unit time expected average cost increase for both the cases when the deterioration rate θ is increasing. But, the optimal production lot-size for Case I decreases first then increases when θ increases.
- When the demand rate D is decreased, the optimal production lot-size, optimal cycle time of the inventory system, expected production run-time and per unit time expected average cost decrease for both Cases I and II.
- The optimal production lot-size, optimal cycle time of the inventory system, expected production run-time and per unit time expected average cost increase with the higher value of inventory per unit per unit time holding cost h_c for the Case I but the optimal production lot-size, optimal cycle time of the inventory system and expected production run-time decrease and the per unit time expected average cost increase with the higher value of inventory per unit per unit time holding cost h_c for the Case II.

- When the shortages cost b_c per unit per unit time increases, the optimal production lot-size, optimal cycle time of the inventory system and expected production run-time decrease but the per unit time expected average costs increase for both the cases.
- With the increasing value of the upper limit of the uniform distribution (U), the optimal production lot size, the optimal cycle time of the inventory system and per unit time expected average cost increase but expected production run-time decreases for Case I. The optimal production lot-size, optimal cycle time of the inventory system and expected production run-time decrease but the per unit time expected average cost increase with the higher value of U for the Case II.
- When the lower limit of the uniform distribution L increases, the optimal production lot size and the optimal cycle time of the inventory system increase but expected production run-time and per unit time expected average cost decrease for Case I. Again, the optimal production lot-size, optimal cycle time of the inventory system and expected production run-time decrease but the per unit time expected average cost increases with the higher value of L for the Case II.
- The proposed model helps a manager of a production industry to control production and inventory levels, production run time and starting of production to achieve minimum expected average cost of the system under several values of the parameters related to the system costs.

6. Conclusion

We have developed a production inventory model with deteriorating items of finite life cycle. One of the important factor of business is to be aware about of the product's life cycle. It has a significant impact on the decisions making of a company. The production rate of the system is considered as a random variable which follows a probability distribution function. We assume that the per unit production cost depends on production lot size and production rate. Shortages with partially backlogging are also allowed in the model and the backlogged rate is depended on the length of the waiting time. We have studied the model with two cases: Case I, where the sum of the maximum lifetime of the products and the production run-time is less than from the total cycle time and Case II, where the sum of the maximum lifetime of the products and the production run-time is greater than from the total cycle time. In the Case I, we have discussed that the shortages may occur due to product's life cycle. Here, we analyzed the model with respect to the production lot size such that the average expected per unit time cost of the inventory system is minimum. We also investigate the model through numerical example and discuss the sensitivity of the main parameters.

The major contribution of the model is to study a stochastic production inventory model considering the production rate of the system is a random variable and the maximum life-cycle of the products is finite. Different cases according the value of the products's life-time, production run-time and one cycle length of the system have been studied in the model.

The proposed model considers only one cycle production-inventory model for continuous variables those are major limitations of this model. These limitations may be waived in future considering supply chain system comprising of supplier of raw materials, manufacturer, retailers and the end customers for discrete variables under bargaining strategies. In future, the more realistic features of demand such as uncertainty in demand, price and stock-dependent demand may be included in the proposed model. The present model can be extended further implementing financial strategies such as quantity discount, cash discount, trade-credit financing, restriction on storage capacity, production lot size, limitation on capital investment and others into the model.

Appendix A.

The expected value with in range $[D(1+\theta), U]$ of P and $\frac{1}{P}$ are

$$\int_{(1+\theta)D}^{U} Pf(P)\mathrm{d}P = \frac{U^2 - D^2(1+\theta)^2}{2(U-L)}$$

and

$$\int_{(1+\theta)D}^{U} \frac{f(P)}{P}\mathrm{d}P = \frac{\log U - \log[D(1+\theta)]}{U-L}$$

respectively. Again, we know

$$\mathrm{e}^{-\frac{Q\theta}{P}} = 1 - \frac{\theta Q}{P} + \frac{\theta^2 Q^2}{2P^2} - \frac{\theta^3 Q^3}{6P^3} + \frac{\theta^4 Q^4}{24P^4} + O[Q]^5.$$

As $0 < \theta < 1$ is small, we approximate the function $\mathrm{e}^{-\frac{Q\theta}{P}}$ up to fourth term and get

$$\mathrm{e}^{-\frac{Q\theta}{P}} = 1 - \frac{\theta Q}{P} + \frac{\theta^2 Q^2}{2P^2} - \frac{\theta^3 Q^3}{6P^3}.$$

Hence, the expected value with in range $[D(1+\theta), U]$ of $\mathrm{e}^{-\frac{\theta Q}{P}}$ and $P\mathrm{e}^{-\frac{\theta Q}{P}}$ are

$$\begin{aligned}\int_{(1+\theta)D}^{U} \mathrm{e}^{-\frac{Q\theta}{P}} f(P)\mathrm{d}P &= \frac{6DQ^2\theta^2(1+\theta) - 12D^3(1+\theta)^3 + 12D^2Q\theta(1+\theta)^2\log D(1+\theta)] - Q^3\theta^3}{12D^2(U-L)(1+\theta)^2} \\ &\quad - \frac{6Q^2U\theta^2 - Q^3\theta^3 + 12QU^2\theta\log U - 12U^3}{12(U-L)U^2}\end{aligned}$$

and

$$\begin{aligned}\int_{(1+\theta)D}^{U} P\mathrm{e}^{-\frac{Q\theta}{P}} f(p)\mathrm{d}P &= \frac{1}{6D(L-U)U(1+\theta)}\Big[(U - D(1+\theta))\Big\{Q^3\theta^3 - 3D^2U(1+\theta)^2 \\ &\quad - 3DU(1+\theta)(U-2Q\theta)\Big\} - 3DQ^2U\theta^2(1+\theta)(\log U - \log[D(1+\theta)])\Big],\end{aligned}$$

respectively.

Again,

$$\mathrm{e}^{\frac{Q}{P}\kappa} = 1 + \frac{\kappa Q}{P} + \frac{\kappa^2 Q^2}{2P^2} + \frac{\kappa^3 Q^3}{6P^3} + \frac{\kappa^4 Q^4}{24P^4} + O[Q]^5.$$

As $0 < \kappa < 1$ is small, $\frac{\kappa}{P}$ is very small. So, we approximate the function $\mathrm{e}^{\frac{Q\kappa}{P}}$ up to fourth term and get

$$\mathrm{e}^{\frac{Q}{P}\kappa} = 1 + \frac{\kappa Q}{P} + \frac{\kappa^2 Q^2}{2P^2} + \frac{\kappa^3 Q^3}{6P^3}.$$

Hence, the expected value with in range $[D(1+\theta), U]$ of $\mathrm{e}^{\frac{Q}{P}\kappa}$ and $\frac{\mathrm{e}^{\frac{Q}{P}\kappa}}{P}$ are

$$\begin{aligned}\int_{(1+\theta)D}^{U} \mathrm{e}^{\frac{Q}{P}\kappa} f(P)\mathrm{d}P &= \frac{6DQ^2(1+\theta)\kappa^2 + Q^3\kappa^3 - 12D^3(1+\theta)^3 - 12D^2Q(1+\theta)^2\kappa\log D(1+\theta)]}{12D^2(U-L)(1+\theta)^2} \\ &\quad - \frac{6Q^2U\kappa^2 + Q^3\kappa^3 - 12QU^2\kappa\log U - 12U^3}{12(U-L)U^2}\end{aligned}$$

and

$$\begin{aligned}\int_{(1+\theta)D}^{U} \frac{f(P)\mathrm{e}^{\frac{Q}{P}\kappa}}{P}\mathrm{d}P &= \frac{Q\kappa\left(36D^2(1+\theta)^2 + 9DQ(1+\theta)\kappa + 2Q^2\kappa^2\right) - 36D^3(1+\theta)^3\log D(1+\theta)]}{36D^3(U-L)(1+\theta)^3} \\ &\quad - \frac{Q\kappa\left(36U^2 + 9QU\kappa + 2Q^2\kappa^2\right) - 36U^3\log U}{36(U-L)U^3},\end{aligned}$$

respectively.

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