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TRANSIENT ANALYSIS OF M/M/1 QUEUE WITH WORKING VACATION, HETEROGENEOUS SERVICE AND CUSTOMERS' IMPATIENCE

R. Sudhesh[1], A. Azhagappan[2] and S. Dharmaraja[3]

Abstract. This paper studies the time-dependent behavior of a single server queueing model with slow service during single and multiple working vacations, and customers' impatience due to slow service. The time-dependent system size probabilities for the underlying model are obtained in closed form. Further, time-dependent mean and variance are deduced. Finally, numerical illustrations are shown to visualize the system effect for various values of parameters.

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1. Introduction

During the last two decades, many researchers carried out works related to queues with vacations customer's impatience. Vacation queues where the server stop serving the customers completely during the entire vacation period. In a *multiple vacation policy*, if the server finds no customer waiting for service at the instant of server's vacation completion, it starts another vacation and continues this until a customer is waiting for service(refer [1,2,5,8,20]). In a *single vacation policy*, on the completion of vacation period, the server stays idle if there is no customer waiting in the queue(refer [1,5,15]). For both policies, if there is at least one customer waiting in the queue at the vacation completion instant, a busy period starts.

Customer's impatience is another important behavior in queueing models and it may occur due to the long wait in the queue. *Working vacation queue* is the queue in which the server serves the customers with a service rate lower than the service rate of busy period (refer [10,13,15,16,18]). This type of system has a wide application in many practical situations such as working rate of employees doing their official work in office as well as in home, recovery rate of a patient in hospital as well as in house, repair rate of a machine/vehicle during working days as well as in holidays, *etc*. Doshi [5] treats the situation that the server works on secondary customers as vacation(it may be machine breakdown, maintenance, *etc*.) where the server never provides any service to the primary customers waiting in the queue. But in our work, we treat the situation that the server works on

Keywords. Single server queue, multiple and single working vacation, impatient customers, continued fractions, confluent hypergeometric functions.

[1] Department of Mathematics, Bharathidasan Institute of Technology (BIT) Campus, Anna University, Tiruchirappalli, 620024 Tamilnadu, India. sudhesh@aucev.edu.in

[2] Department of Mathematics, St. Anne's College of Engineering and Technology, Anna University, Panruti, Cuddalore district, 607110 Tamilnadu, India. azhagappanmaths@gmail.com

[3] Department of Mathematics, Indian Institute of Technology Delhi, Haus Khas, 110016 New Delhi, India. dharmar@maths.iitd.ac.in

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secondary customer as working vacation where the server provides service with a slower service rate instead of not providing service.

Servi and Finn [13] introduced the M/M/1 queueing model with working vacations where a customer is served at a lower service rate instead of stopping the service completely. In a single processor computer and communication system, the arrived jobs (customers) are served with a particular service rate. The required maintenance work (working vacation) is divided into many small segments. When all jobs are absent, the processor starts a particular segment of maintenance work where it serves the arriving jobs with a lower service rate. During working vacation period, there may be loss of jobs (*i.e.*, impatience) due to slow service. We consider customers' impatience in our model due to slower service rate during the vacation period.

Choi *et al.* [3] analyzed the M/M/1 queue with two different classes of customers in which the higher priority class 1 customers have impatience of constant duration and class 2 customers have no impatience. Altman and Yechiali [1] presented a comprehensive analysis of the impatience behavior of single server queues such as M/M/1, M/G/1 queues and the multi-server M/M/c queue, for both the multiple and the single vacation cases and obtained various closed form results. Also they have showed that the proportion of customer abandonments under the single vacation regime is smaller than that under the multiple vacation discipline. Yechiali [17] derived various quality of service measures, mean sojourn time of a served customer, proportion of customers served, rate of lost customers due to disasters and rate of abandonments due to impatience for the M/M/c model when $c = 1$, $1 < c < \infty$ and $c = \infty$ cases. Li and Tian [10] derived the steady state probabilities of an M/M/1 queue with working vacations and vacation interruptions. Choudhury [4] analyzed a single server Markovian queueing system with customers impatience. Later the M/M/1 queue with single working vacation was studied by Tian *et al.* [15] where if the server returns from its working vacation period and finds no customer in the system, it does not take another vacation but waits for the next arrival.

Perel and Yechiali [12] derived the explicit expression for probability generating function of the number of customers in the system for single server, multiple server and infinite server queueing models. Sudhesh [14] derived time-dependent system size probabilities for the M/M/1 queue with system disasters and customer impatience using generating functions and continued fractions. Stationary solution of the M/M/1 queuing model with working vacation and impatient customers was obtained by Yue *et al.* [18]. Vijayalaxmi and Jyothsna [16] studied a renewal input finite buffer multiple working vacation queue with balking. Kalidass and Ramanath [8] derived explicit expressions for the time-dependent and steady-state probabilities of the M/M/1 queue with multiple vacations scheme and also obtained time-dependent mean and variance of the system. Yue *et al.* [19] obtained a closed form expression for the system size probability under steady-state in which the server is permitted to avail a maximum of K vacations when he/she finds no one waiting in the system after returning from each vacation.

Kim and Kim [9] obtained the loss probability, the waiting time distribution and the system size distribution for a single server queue with Markov modulated service rates and impatient customers. Ammar [2] obtained the time-dependent probabilities for the impatient behavior of the M/M/1 queueing model with multiple vacation where the impatience of customers is due to the absence of server upon arrival and also obtained the time-dependent mean and variance of the system. Recently Yue *et al.* [20] derived the probability generating functions of the number of customers in the system for both multiple and single vacation policies. Also they obtained performance measures like mean system size, proportion of customers served and average rate of abandonments due to the impatience. In the literature, so far no work is carried out to obtain the transient solution of an M/M/1 queueing model with slow service in multiple as well as single vacation and customers becoming impatient due to slow service during the vacation period. This gives us the motivation to carry out this research work.

The rest of the paper is partitioned as follows. In Section 2, some preliminary concepts are given for the better understanding to the reader. In Section 3, the transient system size probabilities of the M/M/1 queueing model with customers impatience and single working vacation is analyzed. The transient state system size probabilities are expressed explicitly in terms of the modified Bessel functions using the definition and properties of confluent hypergeometric functions. In Section 4, the time-dependent mean and variance are deduced. In Section 5, the impatient behavior of the M/M/1 queueing model with multiple working vacation is described and analyzed.

The time-dependent system size probabilities are derived explicitly. Also some special cases are deduced. In Section 6, the analytical results are visualized through numerical illustrations. In Section 7, this research work is concluded with the directions of proposed works in future.

2. Preliminaries

In this section, we introduce some of the preliminary concepts of modified Bessal functions, confluent hypergeometric functions and continued fractions that are required for the reader to understand this article in a better way.

2.1. Modified Bessal function

The modified Bessal function of the first kind of order p, denoted by $I_p(x)$, is defined as

$$I_p(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+p}}{k!\Gamma(k+p+1)},\ p > 0, \tag{2.1}$$

which is a solution of the following Bessal's modified equation

$$x^2y'' + xy' - (x^2 + p^2)y = 0,\ p \geq 0.$$

In particular, $I_p(x) = I_{-p}(x)$, for $p \geq 0$.

2.2. Confluent hypergeometric function

The confluent hypergeometric function (or Kummer function), denoted by ${}_1F_1(a;c;z)$, is defined as

$${}_1F_1(a;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{z^k}{k!}, \tag{2.2}$$

where z, a, c are complex and c is not a negative integer. Here $(a)_k$, known as Pochhammer symbol, is defined as

$$(a)_k = 1, for\ k = 0;\ = a(a+1)(a+2)\ldots(a+k-1), for\ k \geq 1.$$

The following identity is from Lorentzen and Waadeland [11]

$$\frac{{}_1F_1(a+1;c+1;z)}{{}_1F_1(a;c;z)} = \frac{c}{c-z+}\frac{(a+1)z}{c-z+1+}\frac{(a+2)z}{c-z+2+}\ldots, \tag{2.3}$$

which can be rewritten as

$$c\frac{{}_1F_1(a;c;z)}{{}_1F_1(a+1;c+1;z)} - (c-z) = \frac{(a+1)z}{c-z+1+}\frac{(a+2)z}{c-z+2+}\ldots$$

2.3. Continued fractions

Approximations using continued fractions provide a good representation for transcendental functions. They are very useful than the classical representation by power series. A systematic study of the theory of continued fraction is presented in [7]. A continued fraction is denoted by

$$\frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \ldots}}}$$

and is equivalently written as

$$\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\ldots,$$

where a_i, b_i, for $i = 1, 2, 3, \ldots$ are real or complex numbers. We apply this continued fraction technique to find the time-dependent solution of our model.

3. Single working vacation model

Consider an M/M/1 queueing model with single working vacation, slow service and impatient customers. Arrival of customers follows a Poisson process with rate λ and the service time follows an exponential distribution with rate μ_1. The server resumes vacation whenever the system is empty and serves the arriving customers at a lower service rate μ_0, following an exponential distribution. On completion of server's vacation period, the server waits dormant in the system if there is no customer to serve. Otherwise the server starts a normal busy period with an exponential service rate μ_1. The vacation time of the server follows an exponential distribution with parameter γ. So, in the single vacation model, the server may stay idle for some period whereas the multiple vacation model does not have any idle time for the server.

Assume that inter-arrival times, service times in the vacation period, service times in the busy period and vacation times are all independent. The service discipline is first-come-first-served (FCFS). Customers become impatient due to the slow service of the server under single vacation. That is, each individual customer activates an independent and exponentially distributed impatient timer with parameter ξ. The customer abandons the system and never returns if his/her service is not completed before his/her timer expires. The state transition of this model is as follows in the Figure 1.

Let $\{X(t), t \geq 0\}$ be the number of customers in the system and $J(t)$ be the status of the server at time t, which is defined as follows:

$$J(t) = \begin{cases} 0, & \text{if the server is in vacation and serves customers with slow service at time } t, \\ 1, & \text{if the server is busy and serves customers with normal service at time } t. \end{cases}$$

Then $\{X(t), J(t), t \geq 0\}$ is a continuous time Markov chain on the state space $S = \{0,0\} \cup \{0,1\} \cup \{n,j : n = 1,2,\ldots; j = 0 \text{ or } 1\}$. Let

$$P_{n,0}(t) = P\{X(t) = n, J(t) = 0\}, \; n = 0,1,2,\ldots,$$
$$P_{n,1}(t) = P\{X(t) = n, J(t) = 1\}, \; n = 0,1,2,\ldots$$

Then $P_{n,k}(t)$, $n = 0,1,2,\ldots, k = 0,1$, satisfy the forward Kolmogorov equations as follows:

$$P'_{0,0}(t) = -(\lambda+\gamma)P_{0,0}(t) + \mu_1 P_{1,1}(t) + (\mu_0+\xi)P_{1,0}(t), \tag{3.1}$$
$$P'_{1,0}(t) = -(\lambda+\mu_0+\xi+\gamma)P_{1,0}(t) + \lambda P_{0,0}(t) + (\mu_0+2\xi)P_{2,0}(t), \tag{3.2}$$
$$P'_{n,0}(t) = -(\lambda+\mu_0+n\xi+\gamma)P_{n,0}(t) + \lambda P_{n-1,0}(t) + (\mu_0+(n+1)\xi)P_{n+1,0}(t), n \geq 2, \tag{3.3}$$
$$P'_{0,1}(t) = -\lambda P_{0,1}(t) + \gamma P_{0,0}(t), \tag{3.4}$$
$$P'_{1,1}(t) = -(\lambda+\mu_1)P_{1,1}(t) + \lambda P_{0,1}(t) + \mu_1 P_{2,1}(t) + \gamma P_{1,0}(t), \tag{3.5}$$
$$P'_{n,1}(t) = -(\lambda+\mu_1)P_{n,1}(t) + \lambda P_{n-1,1}(t) + \mu_1 P_{n+1,1}(t) + \gamma P_{n,0}(t), \; n \geq 2, \tag{3.6}$$

with $P_{00}(0) = 1$, *i.e.*, the system is empty at time $t = 0$.

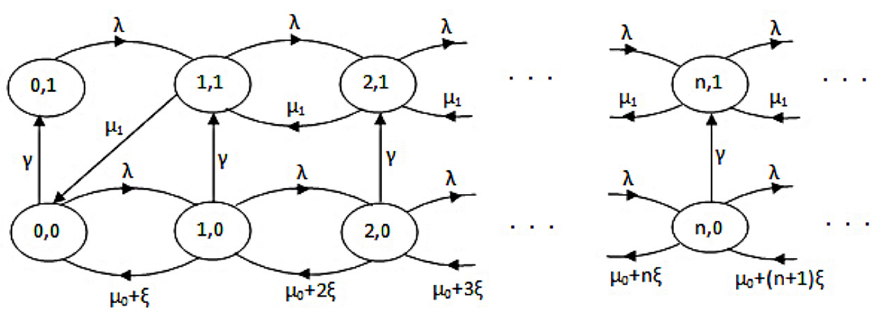

Figure 1. State transition diagram for single working vacation queuing model.

3.1. Transient solution

In this section, we derive the time-dependent system size probabilities for the model under consideration. In Theorem 3.1, the transient probabilities $P_{n,1}(t)$, for $n \geq 1$, are derived using probability generating function method in terms of modified Bessel function. In Theorem 3.2, the transient probabilities $P_{n,0}(t)$, for $n \geq 0$, are derived using continued fraction method in terms of confluent hypergeometric function. The following theorem expresses $P_{n,1}(t)$ in terms of $P_{n,0}(t)$ and $P_{0,1}(t)$, for $n = 1, 2, 3, \ldots$

Theorem 3.1. *The probabilities $P_{n,1}(t)$ are obtained, for $n = 1, 2, 3, \ldots$, from (3.5) and (3.6) in terms of modified Bessel functions as*

$$P_{n,1}(t) = \gamma \int_0^t \mathrm{e}^{-(\lambda+\mu_1)(t-u)} \sum_{m=0}^{\infty} P_{m,0}(u)\beta_1^{n-m} \left[I_{n-m}(\alpha_1(t-u)) - I_{n+m}(\alpha_1(t-u))\right] \mathrm{d}u$$
$$+ \mu_1 \int_0^t \mathrm{e}^{-(\lambda+\mu_1)(t-u)} P_{0,1}(u)\beta_1^{n+1} \left[I_{n-1}(\alpha_1(t-u)) - I_{n+1}(\alpha_1(t-u))\right] \mathrm{d}u, \tag{3.7}$$

where $P_{0,1}(t)$ is obtained from (3.4) as

$$P_{0,1}(t) = \gamma \int_0^t P_{0,0}(u)\mathrm{e}^{-\lambda(t-u)}\mathrm{d}u \tag{3.8}$$

and $I_n(t)$ is the modified Bessel function of the first kind of order n, $\alpha_1 = 2\sqrt{\lambda\mu_1}$ and $\beta_1 = \sqrt{\frac{\lambda}{\mu_1}}$. The proof of Theorem 3.1 is given in Appendix A. The following theorem expresses $P_{n,0}(t)$ in terms of $P_{0,0}(t)$, for $n = 1, 2, 3, \ldots$ and gives $P_{0,0}(t)$ explicitly.

Theorem 3.2. *The probabilities $P_{n,0}(t)$ are obtained for $n = 1, 2, 3, \ldots$ from (3.1), (3.2) and (3.3) using the continued fraction method and confluent hypergeometric functions as*

$$P_{n,0}(t) = Q_n(t) * P_{0,0}(t), \tag{3.9}$$

where

$$P_{0,0}(t) = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{i=0}^{r} \binom{k}{r}\binom{r}{i}\left(\frac{\mu_0+\xi}{\lambda\gamma}\right)^i \lambda^r\gamma^k \mathrm{e}^{-(\lambda+\gamma)t}\frac{t^k}{k!} * \mathrm{e}^{-\lambda t}\frac{t^{r-i-1}}{(r-i-1)!}$$
$$* Q_1^i(t) * \frac{\lambda}{{\beta_1}^{r-i+1}}\mathrm{e}^{-(\lambda+\mu_1)t}\left[I_{r-i-1}(\alpha_1(t) - I_{r-i+1}(\alpha_1(t))\right]^{*(r-i)}$$
$$* \left[\sum_{m=0}^{\infty}\frac{\lambda}{\beta_1^{m+1}}\mathrm{e}^{-(\lambda+\mu_1)t}\left[I_{m-1}(\alpha_1(t) - I_{m+1}(\alpha_1(t))\right] * Q_m(t)\right]^{*(k-r)}, \tag{3.10}$$

$$Q_n(t) = \lambda^n \sum_{k=0}^{\infty}(-\lambda)^k \frac{\prod_{j=1}^{k}(\mu_0+(n+j)\xi)}{\xi^k k!} a_{n+k}(t) * \sum_{i=1}^{\infty}(\lambda)^i b_i(t), \tag{3.11}$$

$$a_k(t) = \frac{1}{\xi^{2k-1}k!}\sum_{r=1}^{k}\prod_{j=1}^{k}(\mu_0+j\xi)\,\frac{(-1)^{r-1}}{(r-1)!(k-r)!}\mathrm{e}^{-(\gamma+\mu_0+r\xi)t},\ k = 1, 2, 3, \ldots, \tag{3.12}$$

$$b_k(t) = \sum_{i=1}^{k}(-1)^{i-1}a_i(t) * b_{k-i}(t),\ k = 2, 3, 4, \ldots; b_1(t) = a_1(t), \tag{3.13}$$

where $*$ denotes the convolution and $*(k-r)$ denotes the $(k-r)$ fold convolution. The proof of Theorem 3.2 is given in Appendix B.

4. Moments

As we have derived the time-dependent solution in the last section, it is very important to analyze the mean and variance of the number of customers in the system at time t. Therefore, in this section, we derive the time-dependent mean and variance of the number of customers in the system at time t.

4.1. Mean

Let $E(X(t))$ be the average number of customers in the system at time t.

$$E(X(t)) = \sum_{n=1}^{\infty} n\left(P_{n,0}(t) + P_{n,1}(t)\right),\ E(X(0)) = 0.$$

From (3.2), (3.3), (3.5) and (3.6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E(X(t)) = \lambda - \mu_0 \sum_{n=1}^{\infty} P_{n,0}(t) - \xi \sum_{n=1}^{\infty} nP_{n,0}(t) - \mu_1 \sum_{n=1}^{\infty} P_{n,1}(t).$$

Integrating the above equation, we get

$$E(X(t)) = \lambda t - \mu_0 \sum_{n=1}^{\infty} \int_0^t P_{n,0}(u)\mathrm{d}u - \xi \sum_{n=1}^{\infty} n \int_0^t P_{n,0}(u)\mathrm{d}u - \mu_1 \sum_{n=1}^{\infty} \int_0^t P_{n,1}(u)\mathrm{d}u. \tag{4.1}$$

4.2. Variance

Let $Var(X(t))$ be the variance of the number of customers in the system at time t.

$$Var(X(t)) = E(X^2(t)) - (E(X(t)))^2,$$

where

$$E(X^2(t)) = \sum_{n=1}^{\infty} n^2\left(P_{n,0}(t) + P_{n,1}(t)\right).$$

From (3.2), (3.3), (3.5) and (3.6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E(X^2(t)) = \lambda + 2\lambda E(X(t)) - \mu_0 \sum_{n=1}^{\infty}(2n-1)P_{n,0}(t) - \xi \sum_{n=1}^{\infty}(2n^2-n)P_{n,0}(t) - \mu_1 \sum_{n=1}^{\infty}(2n-1)P_{n,1}(t).$$

On integration, we obtain

$$\begin{aligned} E(X^2(t)) = \lambda t + 2\lambda \int_0^t E(X(u))\mathrm{d}u - \mu_0 \sum_{n=1}^{\infty}(2n-1)\int_0^t P_{n,0}(u)\mathrm{d}u - \xi \sum_{n=1}^{\infty}(2n^2-n)\int_0^t P_{n,0}(u)\mathrm{d}u \\ - \mu_1 \sum_{n=1}^{\infty}(2n-1)\int_0^t P_{n,1}(u)\mathrm{d}u, \end{aligned} \tag{4.2}$$

where $P_{n,0}(t)$ and $P_{n,1}(t)$, for $n = 1, 2, 3, \ldots$, are given by (3.9) and (3.7) respectively.

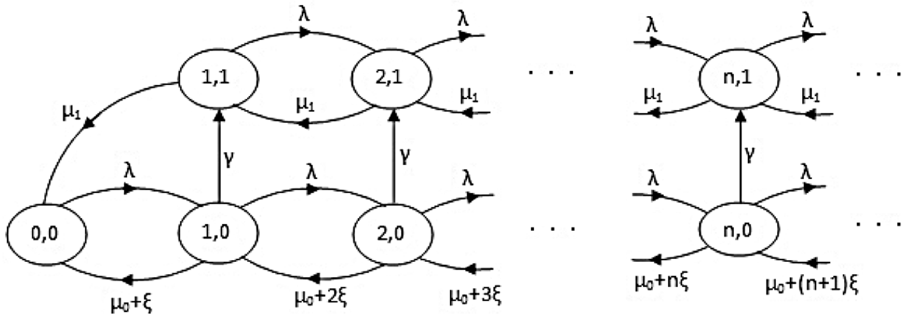

Figure 2. State transition diagram for multiple working vacation queueing model.

5. Multiple working vacation model

Consider an M/M/1 queueing model with multiple working vacation, slow service and impatience of customers due to slow service. Customers arrive according to a Poisson process with rate λ and the service times are exponentially distributed with rate μ_1. The server takes vacation at the end of each busy period and serves the customers who arrive with a lower service rate μ_0 ($<\mu_1$) which follows an exponential distribution. In the multiple vacation queueing model, if the server finds no customers in the system upon returning from the vacation, the server takes another vacation and continues this till a customer shows up in the queue for service. On the other hand, if the server finds a customer at the instant of vacation completion, the server starts a busy period of service time exponentially distributed with rate μ_1 until the system becomes empty. The server vacation times follow an exponential distribution with parameter γ. The busy period with rate μ_1 starts at the time instant when at least one customer waits in the queue when the server returns to the system after the completion of vacation period and ends at the time instant when the system becomes empty for the first time after this service. The busy period with rate μ_0 starts at the time instant when a customer arrives to the system when the server is in working vacation and ends at the time instant of the completion of that particular segment of vacation period.

Assume that inter-arrival times, service times in the busy period and vacation times are all independent. The service discipline is First-Come-First-Served (FCFS). During the vacation period, arriving customers become impatient due to slow service. That is, each individual customer activates an independent impatience timer, exponentially distributed with parameter ξ. If the customer's service has not been completed before his/her timer expires, he/she abandons the system and never returns. The state transition diagram of a multiple working vacation queueing model is given in Figure 2.

Let $\{X(t), t \geq 0\}$ be the number of customers in the system at time t and $J(t)$ be the status of the server at time t, which is defined as follows:

$$J(t) = \begin{cases} 0, & \text{if the server is in vacation and serves customers with slow service at time } t, \\ 1, & \text{if the server is busy and serves customers with normal service at time } t. \end{cases}$$

Then $\{X(t), J(t), t \geq 0\}$ is a continuous time Markov chain on the state space $S = \{0, 0\} \cup \{n, j : n = 1, 2, \ldots; j = 0 \text{ or } 1\}$. Let

$$P_{n,0}(t) = P\{X(t) = n, J(t) = 0\}, \ n = 0, 1, 2, \ldots,$$
$$P_{n,1}(t) = P\{X(t) = n, J(t) = 1\}, \ n = 1, 2, 3, \ldots$$

Then $P_{n,j}(t)$, $n=0,1,2,\ldots,j=0,1$, satisfy the forward Kolmogorov equations as follows:

$$P'_{0,0}(t) = -\lambda P_{0,0}(t) + \mu_1 P_{1,1}(t) + (\mu_0+\xi)P_{1,0}(t), \tag{5.1}$$
$$P'_{1,0}(t) = \lambda P_{0,0}(t) - (\lambda+\mu_0+\xi+\gamma)P_{1,0}(t) + (\mu_0+2\xi)P_{2,0}(t), \tag{5.2}$$
$$P'_{n,0}(t) = \lambda P_{n-1,0}(t) - (\lambda+\mu_0+n\xi+\gamma)P_{n,0}(t) + (\mu_0+(n+1)\xi)P_{n+1,0}(t), n\geq 2, \tag{5.3}$$
$$P'_{1,1}(t) = -(\lambda+\mu_1)P_{1,1}(t) + \mu_1 P_{2,1}(t) + \gamma P_{1,0}(t), \tag{5.4}$$
$$P'_{n,1}(t) = -(\lambda+\mu_1)P_{n,1}(t) + \lambda P_{n-1,1}(t) + \mu_1 P_{n+1,1}(t) + \gamma P_{n,0}(t),\ n\geq 2, \tag{5.5}$$

with $P_{0,0}(0)=1$, *i.e.*, the system is empty at time $t=0$.

5.1. Transient solution

In this section, we derive the time-dependent system size probabilities of the multiple working vacation model. The following theorem gives the expression for $P_{n,1}(t)$ in terms of $P_{n,0}(t)$, for $n=1,2,3,\ldots$

Theorem 5.1. *The probabilities $P_{n,1}(t)$ are obtained for $n=1,2,3,\ldots$ from (5.4) and (5.5) in terms of modified Bessel functions as*

$$P_{n,1}(t) = \gamma\int_0^t \mathrm{e}^{-(\lambda+\mu_1)(t-u)}\sum_{m=1}^{\infty} P_{m,0}(u)\beta_1^{n-m}\left[I_{n-m}(\alpha_1(t-u)) - I_{n+m}(\alpha_1(t-u))\right]\mathrm{d}u, \tag{5.6}$$

where $I_n(t)$ is the modified Bessel function of the first kind of order n, $\alpha_1 = 2\sqrt{\lambda\mu_1}$ and $\beta_1 = \sqrt{\frac{\lambda}{\mu_1}}$.

The proof of Theorem 5.1 is given in Appendix C. Theorem 5.1 is a corollary of the Theorem 3.1, *i.e.*, when $P_{0,1}(t)=0$, the results in Theorem 3.1 get reduced into the results of Theorem 5.1. The following theorem expresses $P_{n,0}(t)$ in terms of $P_{0,0}(t)$, for $n=1,2,3,\ldots$ and gives $P_{0,0}(t)$ explicitly.

Theorem 5.2. *The probabilities $P_{n,0}(t)$ are obtained for $n=1,2,3,\ldots$ from (5.1), (5.2) and (5.3) using the continued fraction method as in (3.9), $Q_n(t)$ as in (3.11) and*

$$P_{0,0}(t) = \mu_1\sum_{k=0}^{\infty}\sum_{r=0}^{k}\gamma^k\binom{k}{r}\left(\frac{\mu_0+\xi}{\gamma}\right)^r \mathrm{e}^{-\lambda t}\frac{t^k}{k!} * Q_1^r(t)$$
$$* \left[\sum_{m=1}^{\infty}\beta_1^{1-m}\left[I_{m-1}(\alpha_1(t) - I_{m+1}(\alpha_1(t))\right]\mathrm{e}^{-(\lambda+\mu_1)t} * Q_m(t)\right]^{*(k-r)}. \tag{5.7}$$

The proof of Theorem 5.2 is given in Appendix D. Theorem 5.2 is a particular case of Theorem 3.2, *i.e.* when $\gamma=0$ is substituted in (B.1), the results of Theorem 3.2 get reduced into the results of Theorem 5.2.

Remark 5.3. When $\mu_0=\mu_1$, there is no difference between the service rates in busy periods and vacations.

5.2. Special cases

case(1): When $\xi=0$, $\mu_0=0$ and $\mu_1=\mu$, then $P_{n,0}(t)$ becomes

$$P_{n,0}(t) = \lambda^n \mathrm{e}^{-(\lambda+\gamma)t}\frac{t^n}{(n-1)!} * P_{0,0}(t),$$

which coincides with (13) of [8].

case(2): When $\mu_0 = 0$ and $\mu_1 = \mu$, then $P_{0,0}(t)$ and $Q_n(t)$ become

$$P_{0,0}(t) = \mu \sum_{k=0}^{\infty} \sum_{r=0}^{k} \gamma^k \binom{k}{r} \left(\frac{\xi}{\gamma}\right)^r \mathrm{e}^{-\lambda t} \frac{t^k}{k!} * Q_1^r(t) * \left[\sum_{m=1}^{\infty} \beta^{1-m} \left[I_m(\alpha(t) - I_{m+2}(\alpha(t))\right] \mathrm{e}^{-(\lambda-\mu)t} * Q_m(t) \right]^{*(k-r)},$$

$$Q_n(t) = \lambda^n \sum_{k=0}^{\infty} (-\lambda)^k \binom{n+k}{k} a_{n+k}(t) * \sum_{i=1}^{\infty} (\lambda)^i b_i(t),$$

which coincide respectively with the results (4.13) and (4.17) in [2].

Remark 5.4. The mean and variance of the number of customers in the system at time t for multiple working vacation model is derived and obtained the same results as in (4.1) and (4.2) respectively.

6. Numerical illustration

For the Multiple Working Vacation Queueing Model, the time-dependent system size probabilities are plotted for $\lambda = 1, \mu_0 = 1.25, \mu_1 = 1.5, \xi = 0.1, \gamma = 0.1$ in Figures 3 and 4. Assume that there is no customer initially

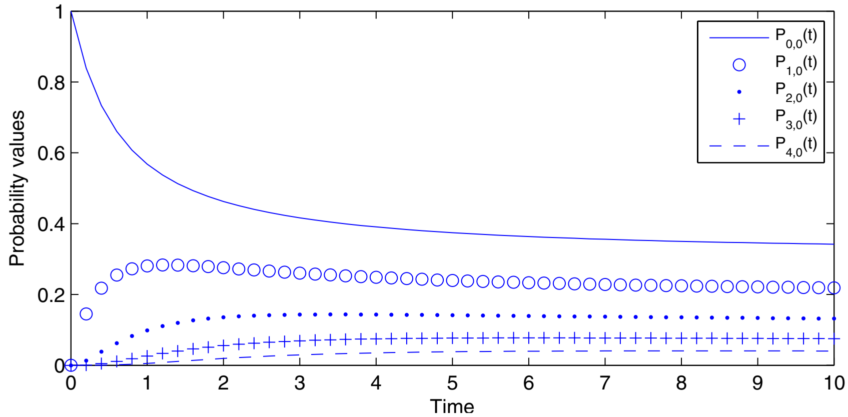

FIGURE 3. Transient-state system size probabilities for multiple working vacation queueing model.

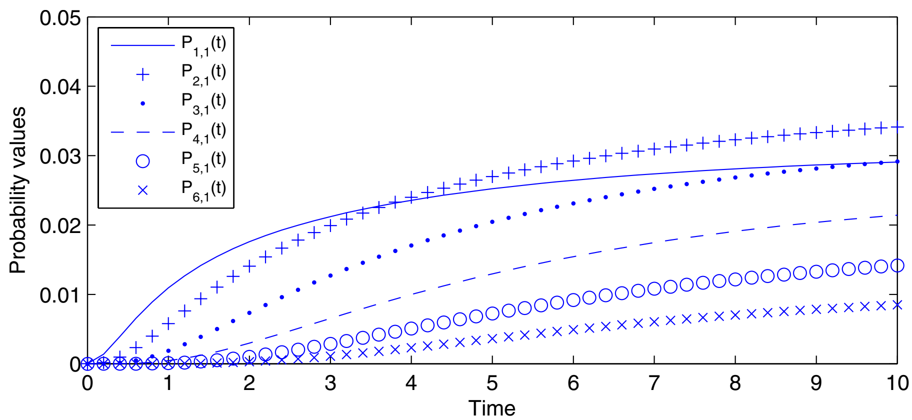

FIGURE 4. Transient-state system size probabilities for multiple working vacation queueing model.

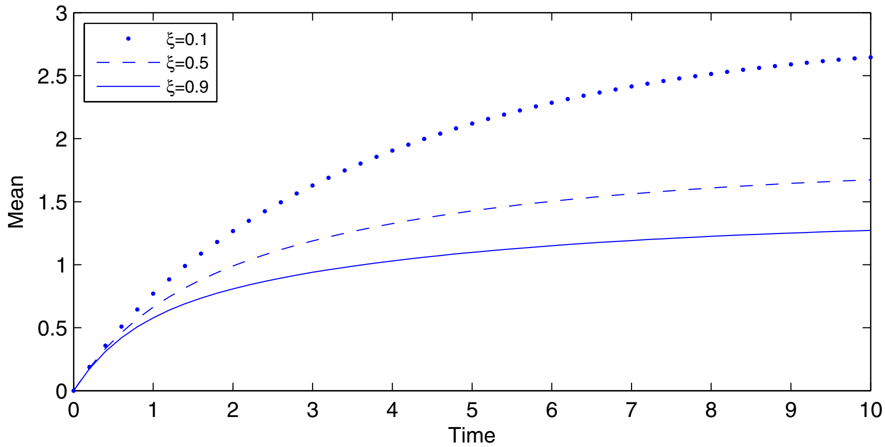

FIGURE 5. Mean queue length for multiple working vacation queueing model with different values of ξ.

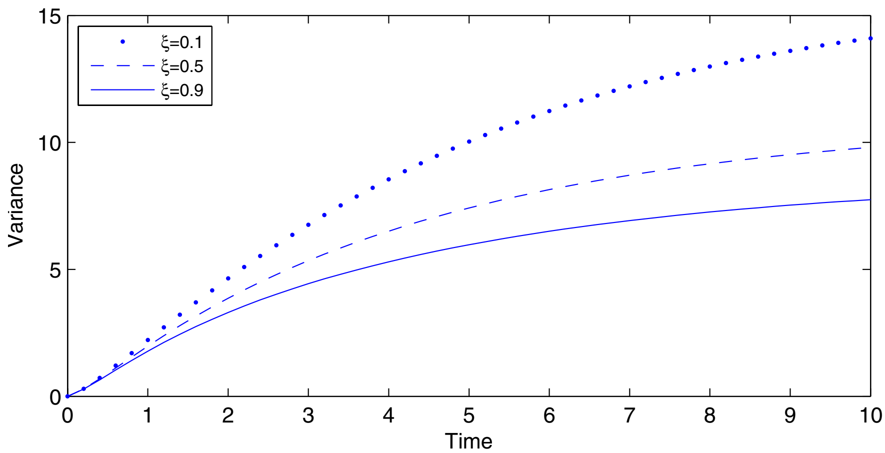

FIGURE 6. Variance for multiple working vacation queueing model with different values of ξ.

in the system. It is observed that the probability curves in Figure 3 (except $P_{0,0}(t)$) and Figure 4 increase initially and attain steady-state as time t increases. Figure 5 shows that the increase in impatient rate ξ leads to decrease in the mean number of customers in the queue. The variance also decreases as the increase of customer's impatience rate ξ which is shown in Figure 6.

For the single working vacation queueing model, the transient-state system size probabilities are plotted for $\lambda = 1, \mu_0 = 1.25, \mu_1 = 1.5, \xi = 0.1, \gamma = 0.1$ in Figures 7 and 8. In Figure 7, the probability curves increase initially and then decrease before reaching a steady-state for large values of t. In Figure 8, the probability curves increase initially and reach a steady-state for large values of t. Figures 9 and 10 show that the increase in impatient rate ξ leads to decrease in the mean and variance number of customers in the queue.

When comparing these figures (Figs. 5, 6, 9 and 10), it is evident that whenever the impatience rate of customers increases, they move out of the system rapidly and hence the average number of customers in the system automatically gets reduced. Also the mean and variance of the number of customers in the system for multiple working vacation case is less than that of single working vacation case.

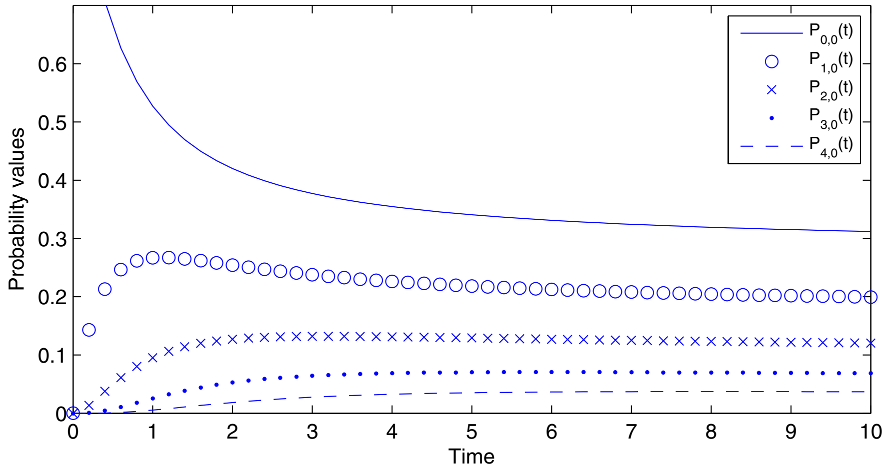

FIGURE 7. Transient-state system size probabilities for single working vacation queueing model.

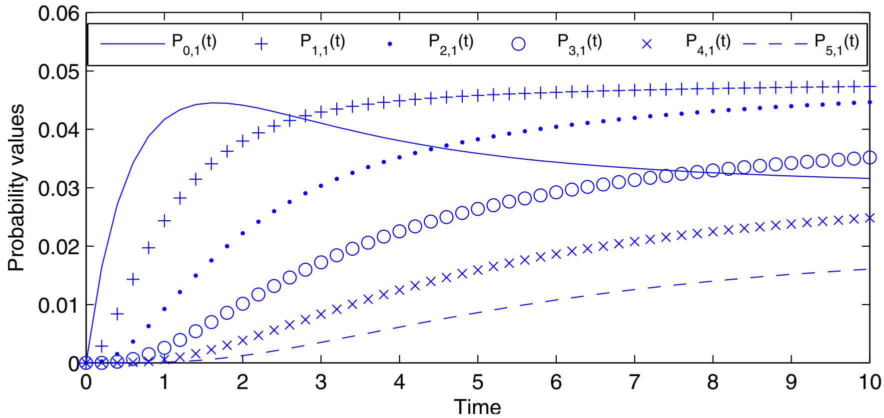

FIGURE 8. Transient-state system size probabilities for single working vacation queueing model.

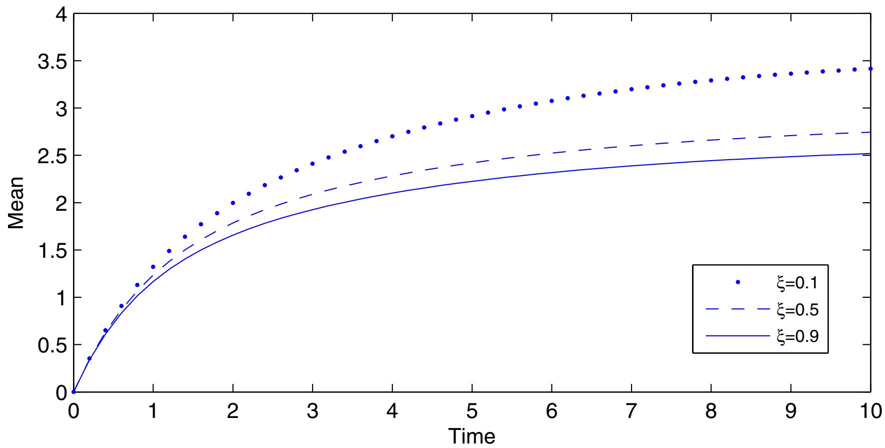

FIGURE 9. Mean queue length for single working vacation queueing model with different values of ξ.

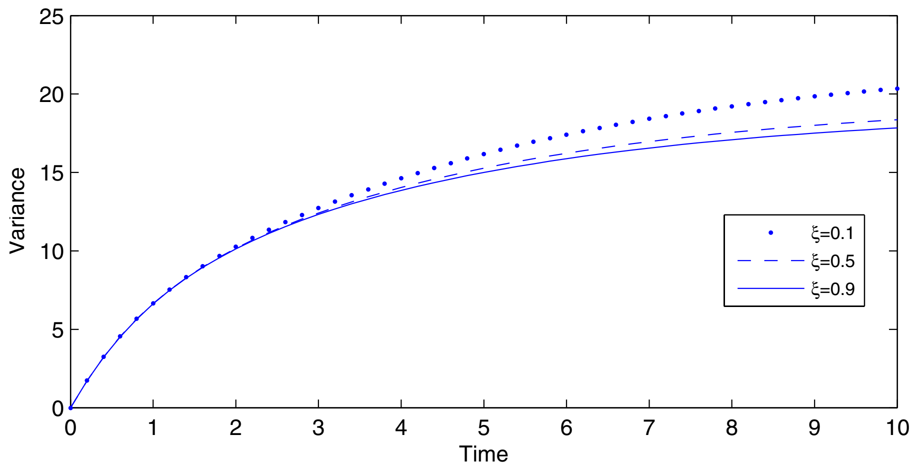

FIGURE 10. Variance for single working vacation queueing model with different values of ξ.

7. Conclusion and future work

This paper deals with the impatient behavior of an M/M/1 queueing model with the server providing heterogeneous service in multiple and single working vacation. Explicit expressions for the transient-state system size probabilities are derived using generating functions and continued fractions. Numerical illustrations help to visualize the analytical results.

In future, this work will be extended to the models with two heterogeneous servers and multi-servers in the context of synchronous or asynchronous vacation and also this work will be extended to the multi-server models in which some of the servers take vacation simultaneously.

Appendix A.

Proof of Theorem 3.1. Let $G_s(z,t) = \sum_{n=1}^{\infty} P_{n,1}(t)z^n$, $G_s(z,0) = 0$, where s denotes the single vacation case.

From (3.5) and (3.6), we get

$$\frac{\partial G_s(z,t)}{\partial t} = \left[-(\lambda+\mu_1) + \lambda z + \frac{\mu_1}{z}\right] G_s(z,t) + \mu_1\left(1-\frac{1}{z}\right) P_{0,1}(t) + \gamma \sum_{n=0}^{\infty} P_{n,0}(t)z^n. \tag{A.1}$$

Solving the equation (A.1), we obtain

$$\begin{aligned} G_s(z,t) = {} & \gamma \int_0^t \left[\sum_{n=1}^{\infty} P_{n,0}(u)z^n\right] \mathrm{e}^{-(\lambda+\mu_1)(t-u)} \mathrm{e}^{(\lambda z+\frac{\mu_1}{z})(t-u)} \mathrm{d}u \\ & + \mu_1\left(1-\frac{1}{z}\right) \int_0^t P_{0,1}(u) \mathrm{e}^{-(\lambda+\mu_1)(t-u)} \mathrm{e}^{(\lambda z+\frac{\mu_1}{z})(t-u)} \mathrm{d}u. \end{aligned} \tag{A.2}$$

It is well known that, if $\alpha_1 = 2\sqrt{\lambda\mu_1}$ and $\beta_1 = \sqrt{\frac{\lambda}{\mu_1}}$, then

$$\mathrm{e}^{\left[(\lambda z+\frac{\mu_1}{z})t\right]} = \sum_{n=-\infty}^{\infty} (\beta_1 z)^n I_n(\alpha_1 t),$$

where $I_n(t)$ is the modified Bessel function of the first kind of order n. Comparing the coefficients of z^n on both sides of (A.2) for $n > 0$, we get

$$P_{n,1}(t) = \gamma \int_0^t \sum_{m=0}^{\infty} P_{m,0}(u) \mathrm{e}^{-(\lambda+\mu_1)(t-u)} \beta_1^{n-m} I_{n-m}(\alpha_1(t-u)) \mathrm{d}u \tag{A.3}$$

$$+ \mu_1 \int_0^t P_{0,1}(u) \mathrm{e}^{-(\lambda+\mu_1)(t-u)} \beta_1^{n} \left[I_n(\alpha_1(t-u)) - \beta_1 I_{n+1}(\alpha_1(t-u)) \right] \mathrm{d}u. \tag{A.4}$$

The above holds for $n = -1, -2, -3, \ldots$, with the left hand side replaced by zero. Using $I_{-n}(x) = I_n(x)$ for $n = 1, 2, 3, \ldots$,

$$0 = \gamma \int_0^t \sum_{m=0}^{\infty} P_{m,0}(u) \mathrm{e}^{-(\lambda+\mu_1)(t-u)} \beta_1^{-n-m} I_{n+m}(\alpha_1(t-u)) \mathrm{d}u \tag{A.5}$$

$$+ \mu_1 \int_0^t P_{0,1}(u) \mathrm{e}^{-(\lambda+\mu_1)(t-u)} \beta_1^{-n} \left[I_n(\alpha_1(t-u)) - \beta_1 I_{n-1}(\alpha_1(t-u)) \right] \mathrm{d}u. \tag{A.6}$$

From (A.4) and (A.6), we obtain the expression (3.7), for $n = 1, 2, 3, \ldots$ Also we obtain (3.8) from (3.4). Thus we have expressed $P_{n,1}(t)$ in terms of $P_{n,0}(t)$ and $P_{0,1}(t)$, for $n = 1, 2, 3, \ldots$ and $P_{0,1}(t)$ in terms of $P_{0,0}(t)$. $\square$

Appendix B.

Proof of Theorem 3.2. Let $\hat{P}_{n,1}(s)$ be the Laplace transform of $P_{n,1}(t)$. Taking Laplace transform on (5.1)–(5.3), we get

$$\hat{P}_{0,0}(s) = \frac{1}{(s+\lambda+\gamma) - \mu_1 \frac{\hat{P}_{1,1}(s)}{\hat{P}_{0,0}(s)} - (\mu_0+\xi) \frac{\hat{P}_{1,0}(s)}{\hat{P}_{0,0}(s)}} \cdot \tag{B.1}$$

$$\frac{\hat{P}_{1,0}(s)}{\hat{P}_{0,0}(s)} = \frac{\lambda}{(s+\lambda+\mu_0+\xi+\gamma) - (\mu_0+2\xi) \frac{\hat{P}_{2,0}(s)}{\hat{P}_{1,0}(s)}} \cdot \tag{B.2}$$

$$\frac{\hat{P}_{n,0}(s)}{\hat{P}_{n-1,0}(s)} = \frac{\lambda}{(s+\lambda+\mu_0+n\xi+\gamma) - (\mu_0+(n+1)\xi) \frac{\hat{P}_{n+1,0}(s)}{\hat{P}_{n,0}(s)}} \cdot \tag{B.3}$$

Solving (B.2) and (B.3) and on repeated applications of the identity (2.3), we obtain

$$\hat{P}_{n,0}(s) = \left(\frac{\lambda}{\xi}\right)^n \frac{1}{\prod_{i=1}^{n} \left(\frac{s+\gamma+\mu_0}{\xi} + i \right)} \frac{{}_1F_1\left(\frac{\mu_0}{\xi} + n + 1; \frac{s+\gamma+\mu_0}{\xi} + n + 1; -\frac{\lambda}{\xi} \right)}{{}_1F_1\left(\frac{\mu_0}{\xi} + 1; \frac{s+\gamma+\mu_0}{\xi} + 1; -\frac{\lambda}{\xi} \right)} \hat{P}_{0,0}(s).$$

$$\hat{P}_{n,0}(s) = \hat{Q}_n(s) \hat{P}_{0,0}(s). \tag{B.4}$$

Laplace inversion of (B.4) yields (3.9) which is the system size probability in vacation state. Using (5.6) and (B.4), for $n = 1$, in (B.1) and after some mathematical manipulations, we obtain

$$\hat{P}_{0,0}(s) = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{i=0}^{r}\binom{k}{r}\binom{r}{i}\left(\frac{\mu_0+\xi}{\lambda\gamma}\right)^{i}\frac{\lambda^r\gamma^k}{(s+\lambda+\gamma)^{k+1}(s+\lambda)^{r-i}}$$
$$\times \hat{Q_1^i}(s)\left(\frac{p_1-\sqrt{p_1^2-\alpha_1^2}}{\alpha_1\beta_1}\right)^{r-i}\left[\sum_{m=0}^{\infty}\left(\frac{p_1-\sqrt{p_1^2-\alpha_1^2}}{\alpha_1\beta_1}\right)^{m}\hat{Q}_m(s)\right]^{k-r}, \tag{B.5}$$

where $p_1 = s + \lambda + \mu_1$. The Laplace inversion of the above equation gives the empty system size probability $P_{0,0}(t)$ which is shown in (5.7). From (B.4), we obtain

$$\hat{Q}_n(s) = \left(\frac{\lambda}{\xi}\right)^n \frac{1}{\prod_{i=1}^{n}\left(\frac{s+\gamma+\mu_0}{\xi}+i\right)}\frac{{}_1F_1\left(\frac{\mu_0}{\xi}+n+1;\frac{s+\gamma+\mu_0}{\xi}+n+1;-\frac{\lambda}{\xi}\right)}{{}_1F_1\left(\frac{\mu_0}{\xi}+1;\frac{s+\gamma+\mu_0}{\xi}+1;-\frac{\lambda}{\xi}\right)}\cdot \tag{B.6}$$

Using the definition of confluent hypergeometric function, we have

$$\frac{{}_1F_1\left(\frac{\mu_0}{\xi}+n+1;\frac{s+\gamma+\mu_0}{\xi}+n+1;-\frac{\lambda}{\xi}\right)}{\prod_{i=1}^{n}\left(\frac{s+\gamma+\mu_0}{\xi}+i\right)} = \sum_{k=0}^{\infty}\frac{\prod_{j=1}^{k}(\mu_0+(n+j)\xi)}{\prod_{i=1}^{n+k}(s+\gamma+\mu_0+i\xi)}\frac{(-\lambda)^k}{\xi^{k-n}k!}\cdot$$

Applying partial fraction in the above equation, we get

$$\frac{{}_1F_1\left(\frac{\mu_0}{\xi}+n+1;\frac{s+\gamma+\mu_0}{\xi}+n+1;-\frac{\lambda}{\xi}\right)}{\prod_{i=1}^{n}\left(\frac{s+\gamma+\mu_0}{\xi}+i\right)} = \sum_{k=0}^{\infty}\frac{\prod_{j=1}^{k}(\mu_0+(n+j)\xi)}{k!}\frac{(-\lambda)^k}{\xi^{2k-1}}$$

$$\sum_{i=1}^{n+k}\frac{(-1)^{i-1}}{(i-1)!(n+k-i)!}\frac{1}{s+\gamma+\mu_0+i\xi}\cdot \tag{B.7}$$

Further,

$${}_1F_1\left(\frac{\mu_0}{\xi}+1;\frac{s+\gamma+\mu_0}{\xi}+1;-\frac{\lambda}{\xi}\right) = \sum_{k=0}^{\infty}\frac{\prod_{j=1}^{k}(\mu_0+j\xi)}{\prod_{i=1}^{k}(s+\gamma+\mu_0+i\xi)}\frac{(-\lambda)^k}{\xi^k k!} = \sum_{k=0}^{\infty}(-\lambda)^k\hat{a}_k(s),$$

where $\hat{a}_k(s) = \frac{\prod_{j=1}^{k}(\mu_0+j\xi)}{\prod_{i=1}^{k}(s+\gamma+\mu_0+i\xi)}\frac{1}{\xi^k k!}$; $\hat{a}_0(s) = 1$. By resolving into partial fractions, we have

$$\hat{a}_k(s) = \frac{1}{\xi^{2k-1}k!}\sum_{r=1}^{\infty}\frac{\prod_{j=1}^{k}(\mu_0+j\xi)\,(-1)^{r-1}}{(r-1)!(k-r)!}\frac{1}{s+\gamma+\mu_0+r\xi},\quad \textit{for } k=1,2,3,\ldots \tag{B.8}$$

Using the identity given in [6], we obtain

$$\left[{}_1F_1\left(\frac{\mu_0}{\xi}+1;\frac{s+\gamma+\mu_0}{\xi}+1;-\frac{\lambda}{\xi}\right)\right]^{-1}=\sum_{k=0}^{\infty}\hat{b}_k(s)(\lambda)^k, \tag{B.9}$$

where $\hat{b}_0(s)=1$ and for $k=1,2,3,\ldots$,

$$\hat{b}_k(s)=\begin{vmatrix} \hat{a}_1(s) & 1 & 0 & \ldots & 0 & 0\\ \hat{a}_2(s) & \hat{a}_1(s) & 1 & \ldots & 0 & 0\\ \hat{a}_3(s) & \hat{a}_2(s) & \hat{a}_1(s) & \ldots & 0 & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ \hat{a}_{k-1}(s) & \hat{a}_{k-2}(s) & \hat{a}_{k-3}(s) & \ldots & \hat{a}_1(s) & 1\\ \hat{a}_k(s) & \hat{a}_{k-1}(s) & \hat{a}_{k-2}(s) & \ldots & \hat{a}_2(s) & \hat{a}_1(s)\end{vmatrix} \tag{B.10}$$

$$=\sum_{i=1}^{k}(-1)^{i-1}\hat{a}_i(s)\hat{b}_{k-i}(s). \tag{B.11}$$

Substitute (B.7) and (B.9) in (B.6), we get

$$\hat{Q}_n(s)=\lambda^n\sum_{k=0}^{\infty}(-\lambda)^k\frac{\prod_{j=1}^{k}(\mu_0+(n+j)\xi)}{\xi^k k!}\hat{a}_{n+k}(s)\sum_{i=1}^{\infty}(\lambda)^i\hat{b}_i(s). \tag{B.12}$$

On Laplace inversion, we get (3.11). Thus we have expressed $P_{n,0}(t)$ in terms of $P_{0,0}(t)$, for $n=1,2,3,\ldots$ and $P_{0,0}(t)$ explicitly. □

Appendix C.

Proof of Theorem 5.1. Define $G_m(z,t)=\sum_{n=1}^{\infty}P_{n,1}(t)z^n$, $G_m(z,0)=0$, where m denotes the multiple vacation case.

From (5.4) and (5.5), we get

$$\frac{\partial G_m(z,t)}{\partial t}=\left[-(\lambda+\mu_1)+\lambda z+\frac{\mu_1}{z}\right]G_m(z,t)-\mu_1P_{1,1}(t)+\gamma\sum_{n=1}^{\infty}P_{n,0}(t)z^n. \tag{C.1}$$

Using the methodology given in Theorem 3.1 of single working vacation model, we obtain (5.6). Thus we have expressed $P_{n,1}(t)$ in terms of $P_{n,0}(t)$, for $n=1,2,3,\ldots$ □

Appendix D.

Proof of Theorem 5.2. Taking Laplace transform on (5.1), we get

$$\hat{P}_{0,0}(s)=\frac{1}{(s+\lambda)-\mu_1\frac{\hat{P}_{1,1}(s)}{\hat{P}_{0,0}(s)}-(\mu_0+\xi)\frac{\hat{P}_{1,0}(s)}{\hat{P}_{0,0}(s)}} \tag{D.1}$$

and from (5.2) and (5.3), we obtain (3.9) and $Q_n(t)$ is as in (3.11). Using (5.5) and (3.9), for $n=1$, in (D.1) and after some algebraic manipulations and taking Laplace inversion, we get (5.7). Thus we have expressed $P_{n,0}(t)$ in terms of $P_{0,0}(t)$, for $n=1,2,3,\ldots$ and $P_{0,0}(t)$ explicitly. □

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