RAIRO-Oper. Res. 51 (2017) 211–225
DOI: 10.1051/ro/2016017

RAIRO Operations Research
www.rairo-ro.org

MULTIOBJECTIVE VARIATIONAL PROBLEMS AND GENERALIZED VECTOR VARIATIONAL-TYPE INEQUALITIES *

Anurag Jayswal[1] and Shipra Singh[1]

Abstract. The purpose of this paper is to generalize the vector variational-type inequalities, formulated by Kim [*J. Appl. Math. Comput.* **16** (2004) 279–287], by setting the norms into Minty and Stampacchia forms. We also demonstrate the relationships between these generalized inequalities and multiobjective variational problems, by using the notions of strongly convex functionals. The theoretical developments are illustrated through numerical examples.

Mathematics Subject Classification. 90C30, 90C29, 49J40.

Received July 7, 2015. Accepted February 18, 2016.

1. Introduction

In optimization theory, one uses what is probably one of the most powerful and interesting topic, to establish the necessary and sufficient conditions for optimality. In order to do so, a lot of articles devoted in this direction and concluded that, Kuhn−Tucker conditions are necessary for optimality and when objective functions are convex, then these conditions are sufficient. Various classes of functions, also involving invex functions have been explored for the purpose of weakening this limitations of convexity in mathematical programming. Initially, Hanson [6] spelled out invex function with the aim to extend the validity of sufficiency of the Kuhn−Tucker conditions. Thereafter, for better and accurate results, generalizations of invex functions, such as pseudoinvex, quasiinvex, preinvex, just to name a few, have been defined. Weir and Mond [16], Mohan and Neogy [13] have studied some basic properties of preinvex functions. Other contributions to the invexity were made by [2, 9].

Vector optimization problems constitute an essential and crucial part of study of product and process design, finance, aircraft design, gas industry *etc.* It is a subarea of mathematical optimization, where objective functions of vector optimization problems are vector valued and optimized and also subject to the certain constraints. For extensive developments, Hanson [5] speculated the relationship among mathematical programming and classical calculus of variation. Thus, as a consequence, vector continuous-time programs and multiobjective variational problems arose. This type of problem include variational and optimal control problems. The fundamentals as well as the applications of variational problems have been well documented in [10, 15].

Keywords. Multiobjective variational problem, generalized Minty vector variational-type inequality, generalized Stampacchia vector variational-type inequality, strongly convex functional, strongly preconvex functional.

* *This research is financially supported by SERB-DST, New Delhi, India, through grant no. SR/FTP/MS-007/2011.*

[1] Department of Applied Mathematics, Indian School of Mines, Dhanbad-826 004, Jharkhand, India.
anurag_jais123@yahoo.com; shiprasingh384@gmail.com

Variational inequalities have shown applications to a wide range of problem in many real world problems and in other disciplines: traffic analysis, physics, mechanics, optimization, control transportation and so on because these problems can be transformed into variational inequalities. Initially, this concept was introduced by Giannessi [4]. The growing interest in vector problems, both from a theoretical point of view and as it concerns applications to mathematical problem, asks for various forms of vector variational inequalities, for instance, Stampacchia or Minty vector variational inequalities. Optimization serves an efficient and important theme, the relation of vector variational inequalities with vector optimization problems. Hence, several authors and researchers have intensively contributed in this direction, see [3, 4, 7, 8, 10–12, 14].

Motivated by above research works, we present our paper, in which we generalize the vector variational type-inequalities into Minty and Stampacchia forms. Further, We deduce the relationships of these inequalities with multiobjective variational problems. This paper comprises four sections. In Section 2, we recall some preliminaries, definitions, lemmas and theorems, which are helpful to prove our results. In Section 3, we establish the relationships among Minty and Stampacchia vector variational-like inequalities and multiobjective variational problems. Ultimately, In Section 4, we conclude the paper.

2. Notations and preliminaries

Let $I = [a, b]$ be a real interval and $f : I \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^p$ be a p-dimensional continuously differentiable function with respect to each of its arguments. For notational convenience, we write x and $\dot{x}$ for $x(t)$ and $\dot{x}(t)$, respectively, where $x : I \mapsto \mathbb{R}^n$ is differentiable with derivative $\dot{x}$. We denote the partial derivatives of f with respect to x and $\dot{x}$, respectively, by

$$f_x = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right), \; f_{\dot{x}} = \left(\frac{\partial f}{\partial \dot{x}_1}, \dots, \frac{\partial f}{\partial \dot{x}_n}\right).$$

Let X be a nonempty convex subset of the Banach space $C^1[a, b]$ with the norm $\|x\| = \|x\|_\infty + \|\dot{x}\|_\infty$, for all $x \in X$.

Consider the following multiobjective variational problem:

$$\textbf{(MVP)} \qquad \text{Minimize} \quad \int_a^b f(t, x, \dot{x})\mathrm{d}t = \left(\int_a^b f^1(t, x, \dot{x})\mathrm{d}t, \dots, \int_a^b f^p(t, x, \dot{x})\mathrm{d}t\right)$$

subject to $\quad x \in X$.

First of all, we recall some known concepts of efficiency.

Definition 2.1. A point $y \in X$ is said to be an efficient solution of (MVP), if for all $x \in X$, the following can not hold

$$\int_a^b f^i(t, x, \dot{x})\mathrm{d}t \leq \int_a^b f^i(t, y, \dot{y})\mathrm{d}t,$$

with strict inequality for at least one $i \in P$, that $P = \{1, \dots, p\}$.

Definition 2.2. A point $y \in X$ is said to be a weak efficient solution of (MVP), if for all $x \in X$, the following can not hold

$$\int_a^b f^i(t, x, \dot{x})\mathrm{d}t < \int_a^b f^i(t, y, \dot{y})\mathrm{d}t, \; \forall \; i \in P.$$

Now, we introduce the notions of strong preconvexity and convexity for the functionals, which will be used to prove our results in the sequel of the paper. Let $g : I \times X \times X \mapsto \mathbb{R}$ be differentiable function.

Definition 2.3. A functional $\int_a^b g(t,x,\dot{x})\mathrm{d}t$ is said to be strongly preconvex on X, if there exists a real constant $\alpha > 0$ such that for all $x, y \in X$ and $\lambda \in [0,1]$, one has

$$\int_a^b g(t, y + \lambda(x-y), \dot{y} + \lambda(\dot{x} - \dot{y}))\mathrm{d}t \leq \lambda \int_a^b g(t,x,\dot{x})\mathrm{d}t + (1-\lambda)\int_a^b g(t,y,\dot{y})\mathrm{d}t$$
$$-\alpha\lambda(1-\lambda)\|x-y\|^2.$$

Definition 2.4. A functional $\int_a^b g(t,x,\dot{x})\mathrm{d}t$ is said to be strongly convex on X, if there exists a real constant $\alpha > 0$ such that for all $x, y \in X$, one has

$$\int_a^b \left[g_x(t,y,\dot{y})(x-y) + g_{\dot{x}}(t,y,\dot{y})(\dot{x}-\dot{y})\right]\mathrm{d}t + \alpha\|x-y\|^2 \leq \int_a^b g(t,x,\dot{x})\mathrm{d}t - \int_a^b g(t,y,\dot{y})\mathrm{d}t.$$

It is obvious that, every strong convexity implies convexity but converse is not true. Here, we give an example to deal with its converse.

Example 2.5. Consider the function $g : [0,1] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ defined by $g(t,x,\dot{x}) = x + \dot{x}$.

It can be easily shown that the functional $\int_0^1 g(t,x,\dot{x})\mathrm{d}t$ is convex on $\mathbb{R}$. As for all $x, y \in \mathbb{R}$, we have

$$\int_0^1 g(t,x,\dot{x})\mathrm{d}t - \int_0^1 g(t,y,\dot{y})\mathrm{d}t \geq \int_0^1 \left[g_x(t,y,\dot{y})(x-y) + g_{\dot{x}}(t,y,\dot{y})(\dot{x}-\dot{y})\right]\mathrm{d}t.$$

However, the functional $\int_0^1 g(t,x,\dot{x})\mathrm{d}t$ is not strongly convex on $\mathbb{R}$. Since, there does not exist any $\alpha > 0$ such that

$$\int_0^1 \left[g_x(t,y,\dot{y})(x-y) + g_{\dot{x}}(t,y,\dot{y})(\dot{x}-\dot{y})\right]\mathrm{d}t + \alpha\|x-y\|^2 - \left[\int_a^b g(t,x,\dot{x})\mathrm{d}t - \int_a^b g(t,y,\dot{y})\mathrm{d}t\right] = \alpha\|x-y\|^2 \leq 0.$$

Definition 2.6. A functional $\int_a^b g(t,x,\dot{x})\mathrm{d}t$ is said to be strictly strongly convex on X, if there exists a real constant $\alpha > 0$ such that for all $x, y \in X$ and $x \neq y$, one has

$$\int_a^b \left[g_x(t,y,\dot{y})(x-y) + g_{\dot{x}}(t,y,\dot{y})(\dot{x}-\dot{y})\right]\mathrm{d}t + \alpha\|x-y\|^2 < \int_a^b g(t,x,\dot{x})\mathrm{d}t - \int_a^b g(t,y,\dot{y})\mathrm{d}t.$$

On the basis of well known concept of convex sets, we consider the following definitions of the path.

Definition 2.7. Let u and v be two arbitrary points of X. A set P_{uv} is said to be closed path joining the points u and v, if

$$P_{uv} = \{y = u + \lambda(v-u) : \lambda \in [0,1]\}.$$

Analogously, P^0_{uv} is said to be an open path joining the points u and v, if

$$P^0_{uv} = \{y = u + \lambda(v-u) : \lambda \in (0,1)\}.$$

Mean value theorem is a consequence of its applications in applied analysis, including optimization problems, differential equations, approximation and convergence results in numerical analysis. It plays a crucial role in analysis because estimations of function values can be derived from it. Here, we present the mean value theorem for differentiable functionals under the assumption that, considered functionals are defined on a convex set X.

Theorem 2.8. *Let $g : I \times X \times X \mapsto \mathbb{R}$ be differentiable function and P_{xy} be an arbitrary path contained in X. Then for any $x, y \in X$, there exists $x_\circ \in P_{xy}^0$ such that following relation holds*

$$\int_a^b g(t,y,\dot{y})\mathrm{d}t - \int_a^b g(t,x,\dot{x})\mathrm{d}t = \int_a^b \left[g_x(t,x_\circ,\dot{x}_\circ)(y-x) + g_{\dot{x}}(t,x_\circ,\dot{x}_\circ)(\dot{y}-\dot{x})\right]\mathrm{d}t.$$

Proof. Let $h : [0,1] \mapsto \mathbb{R}$ be a real valued function defined by

$$h(\lambda) = \int_a^b g\left(t, x+\lambda(y-x), \dot{x}+\lambda(\dot{y}-\dot{x})\right)\mathrm{d}t - \int_a^b g(t,x,\dot{x})\mathrm{d}t - \lambda\left[\int_a^b g(t,y,\dot{y})\mathrm{d}t - \int_a^b g(t,x,\dot{x})\mathrm{d}t\right]. \quad (2.1)$$

Since, $h(0) = h(1) = 0$, we can apply Rolle's theorem. Then there exists $\overline{\lambda} \in (0,1)$ such that $h'(\overline{\lambda}) = 0$. Now, relation (2.1) yields

$$\begin{aligned} 0 = h'(\overline{\lambda}) = &\int_a^b \Big[g_x\left(t, x+\overline{\lambda}(y-x), \dot{x}+\overline{\lambda}(\dot{y}-\dot{x})\right)(y-x) \\ &+ g_{\dot{x}}\left(t, x+\overline{\lambda}(y-x), \dot{x}+\overline{\lambda}(\dot{y}-\dot{x})\right)(\dot{y}-\dot{x})\Big]\mathrm{d}t \\ &- \int_a^b g\left(t,y,\dot{y}\right)\mathrm{d}t + \int_a^b g(t,x,\dot{x})\mathrm{d}t, \end{aligned}$$

$$\begin{aligned} i.e., \int_a^b g\left(t,y,\dot{y}\right)\mathrm{d}t - \int_a^b g(t,x,\dot{x})\mathrm{d}t = &\int_a^b \Big[g_x\left(t, x+\overline{\lambda}(y-x), \dot{x}+\overline{\lambda}(\dot{y}-\dot{x})\right)(y-x) \\ &+ g_{\dot{x}}\left(t, x+\overline{\lambda}(y-x), \dot{x}+\overline{\lambda}(\dot{y}-\dot{x})\right)(\dot{y}-\dot{x})\Big]\mathrm{d}t. \end{aligned}$$

On putting $x_\circ = x + \overline{\lambda}(y-x)$, above relation implies

$$\int_a^b g\left(t,y,\dot{y}\right)\mathrm{d}t - \int_a^b g(t,x,\dot{x})\mathrm{d}t = \int_a^b \left[g_x\left(t,x_\circ,\dot{x}_\circ\right)(y-x) + g_{\dot{x}}\left(t,x_\circ,\dot{x}_\circ\right)(\dot{y}-\dot{x})\right]\mathrm{d}t.$$

This completes the proof. □

In order to prove our results, we establish the following lemma, that we need in the next section.

Lemma 2.9. *Let $g : I \times X \times X \mapsto \mathbb{R}$ be differentiable function. If the functional $\int_a^b g(t,x,\dot{x})\mathrm{d}t$ is strongly convex, then it is strongly preconvex.*

Proof. Since, X is convex set, then we have

$$\hat{x} = x + \lambda(y-x) \in X, \ \forall\, x, y \in X \text{ and } \lambda \in [0,1].$$

Now, strong convexity of the functional $\int_a^b g(t,x,\dot{x})\mathrm{d}t$ yields, that there exists a real constant $\alpha > 0$ such that for all $\hat{x}, y \in X$

$$\int_a^b \left[g_x(t,\hat{x},\dot{\hat{x}})(y-\hat{x}) + g_{\dot{x}}(t,\hat{x},\dot{\hat{x}})(\dot{y}-\dot{\hat{x}})\right]\mathrm{d}t + \alpha\|y-\hat{x}\|^2 \le \int_a^b g(t,y,\dot{y})\mathrm{d}t - \int_a^b g(t,\hat{x},\dot{\hat{x}})\mathrm{d}t. \quad (2.2)$$

Similarly, strong convexity applied to the pair $\hat{x}, x$, we obtain

$$\int_a^b \left[g_x(t,\hat{x},\dot{\hat{x}})(x-\hat{x}) + g_{\dot{x}}(t,\hat{x},\dot{\hat{x}})(\dot{x}-\dot{\hat{x}})\right]\mathrm{d}t + \alpha\|x-\hat{x}\|^2 \le \int_a^b g(t,x,\dot{x})\mathrm{d}t - \int_a^b g(t,\hat{x},\dot{\hat{x}})\mathrm{d}t. \quad (2.3)$$

Now, we shall multiply inequality (2.2) and (2.3) by λ and $(1-\lambda)$, respectively, and add both inequalities, thereafter, we can say that, there exists a real constant $\alpha > 0$ such that for all $x, y \in X$ and $\lambda \in [0,1]$, one has

$$\int_a^b g(t, x+\lambda(y-x), \dot{x}+\lambda(\dot{y}-\dot{x}))\mathrm{d}t \le \lambda \int_a^b g(t,y,\dot{y})\mathrm{d}t + (1-\lambda)\int_a^b g(t,x,\dot{x})\mathrm{d}t - \alpha\lambda(1-\lambda)\|y-x\|^2.$$

Therefore, $\displaystyle\int_a^b g(t,x,\dot{x})\mathrm{d}t$ is strongly preconvex. This completes the proof. □

Now, by keeping the view of generalization of Minty vector variational-like inequality, given by Oveisiha and Zafarani [14], we introduce the following generalized Minty and Stampacchia vector variational-type inequalities, respectively, and also their weak formulations in the respective manner, which will be used to ensure the efficient solutions of multiobjective variational problems in the sequel of paper.

$(\mathbf{GMVVI})_{\boldsymbol{\gamma}}$. For given real constant γ, find $\overline{x} \in X$ such that there exists no $x \in X$, satisfying

$$\int_a^b \left[f_x^i(t,x,\dot{x})(\overline{x}-x) + f_{\dot{x}}^i(t,x,\dot{x})(\dot{\overline{x}}-\dot{x})\right]\mathrm{d}t + \gamma\|\overline{x}-x\|^2 \ge 0,$$

with strict inequality for at least one $i \in P$.

$(\mathbf{GSVVI})_{\boldsymbol{\gamma}}$. For given real constant γ, find $\overline{x} \in X$ such that there exists no $x \in X$, satisfying

$$\int_a^b \left[f_x^i(t,\overline{x},\dot{\overline{x}})(x-\overline{x}) + f_{\dot{x}}^i(t,\overline{x},\dot{\overline{x}})(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t + \gamma\|x-\overline{x}\|^2 \le 0,$$

with strict inequality for at least one $i \in P$.

$(\mathbf{GWMVVI})_{\boldsymbol{\gamma}}$. For given real constant γ, find $\overline{x} \in X$ such that there exists no $x \in X$, satisfying

$$\int_a^b \left[f_x^i(t,x,\dot{x})(\overline{x}-x) + f_{\dot{x}}^i(t,x,\dot{x})(\dot{\overline{x}}-\dot{x})\right]\mathrm{d}t + \gamma\|\overline{x}-x\|^2 > 0,\ \forall\, i \in P.$$

$(\mathbf{GWSVVI})_{\boldsymbol{\gamma}}$. For given real constant γ, find $\overline{x} \in X$ such that there exists no $x \in X$, satisfying

$$\int_a^b \left[f_x^i(t,\overline{x},\dot{\overline{x}})(x-\overline{x}) + f_{\dot{x}}^i(t,\overline{x},\dot{\overline{x}})(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t + \gamma\|x-\overline{x}\|^2 < 0,\ \forall\, i \in P.$$

Special case. If $\gamma = 0$ in $(\mathrm{GSVVI})_\gamma$ (respectively $(\mathrm{GWSVVI})_\gamma$), then it reduces to Stampacchia (respectively (weak)) vector variational-type inequality that has been formulated by Kim [10].

Remark 2.10. If $\overline{x}$ is either a solution of $(\mathrm{GMVVI})_\gamma$ or its weak formulation with constant γ, then $\overline{x}$ is also their solution for all parameters $\gamma' \le \gamma$.

The following example shows that there exists a solution for $(\mathrm{GMVVI})_\gamma$.

Example 2.11. Consider the function $f : [0,1] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}^2$ defined by

$$\int_0^1 f(t,x,\dot{x})\mathrm{d}t = \left(\int_0^1 f^1(t,x,\dot{x})\mathrm{d}t, \int_0^1 f^2(t,x,\dot{x})\mathrm{d}t\right),$$

where

$$f^1(t,x,\dot{x}) = -x^2\dot{x}^2,\ f^2(t,x,\dot{x}) = -x^4\dot{x}^4$$

and $x : [0,1] \mapsto \mathbb{R}$ is defined by $x(t) = kt,\ \forall\, k \in \mathbb{R}$.

Now, we observe that $\overline{x}=0$ is a solution of $(\text{GMVVI})_\gamma$, as for a constant $\gamma<0$, we have

$$\begin{aligned}&\left(\int_0^1 \left[f_x^1(t,x,\dot{x})(\overline{x}-x)+f_{\dot{x}}^1(t,x,\dot{x})(\dot{\overline{x}}-\dot{x})\right]\mathrm{d}t+\gamma\|\overline{x}-x\|^2,\right.\\ &\qquad\left.\int_0^1 \left[f_x^2(t,x,\dot{x})(\overline{x}-x)+f_{\dot{x}}^2(t,x,\dot{x})(\dot{\overline{x}}-\dot{x})\right]\mathrm{d}t+\gamma\|\overline{x}-x\|^2\right)\\ &\qquad=\left(\frac{4k^4}{3}+4\gamma k^2,\frac{8k^8}{5}+4\gamma k^2\right)\\ &\qquad\ngeq(0,0).\end{aligned}$$

Following example shows that, there exists a solution of $(\text{GSVVI})_\gamma$ but Stampacchia vector variational-type inequality (SVVI), given by Kim [10] is not solvable at that point.

Example 2.12. Consider the function $f:[0,1]\times\mathbb{R}\times\mathbb{R}\mapsto\mathbb{R}^2$ defined by

$$\int_0^1 f(t,x,\dot{x})\mathrm{d}t=\left(\int_0^1 f^1(t,x,\dot{x})\mathrm{d}t,\int_0^1 f^2(t,x,\dot{x})\mathrm{d}t\right),$$

where

$$f^1(t,x,\dot{x})=-x-\dot{x},\ \ f^2(t,x,\dot{x})=-x+\dot{x}^2$$

and $x:[0,1]\mapsto\mathbb{R}$ is defined by $x(t)=t$.

Now, we observe that $\overline{x}=0$ is a solution of $(\text{GSVVI})_\gamma$, as for constant $\gamma=\frac{3}{2}$, we have

$$\begin{aligned}&\left(\int_0^1 \left[f_x^1(t,\overline{x},\dot{\overline{x}})(x-\overline{x})+f_{\dot{x}}^1(t,\overline{x},\dot{\overline{x}})(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t+\gamma\|x-\overline{x}\|^2,\right.\\ &\qquad\left.\int_0^1 \left[f_x^2(t,\overline{x},\dot{\overline{x}})(x-\overline{x})+f_{\dot{x}}^2(t,\overline{x},\dot{\overline{x}})(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t+\gamma\|x-\overline{x}\|^2\right)\\ &\qquad=\left(-\frac{3}{2}+4\gamma,-\frac{1}{2}+4\gamma\right)\\ &\qquad=\left(\frac{9}{2},\frac{11}{2}\right)\\ &\qquad\nleq(0,0).\end{aligned}$$

Further, it can be easily shown that (SVVI) is not solvable at 0.

3. Relationships between generalized vector variational-type inequalities and multiobjective variational problems

In this section, we shall study the relationships between the solutions of generalized Minty, Stampacchia vector variational-type inequalities and multiobjective variational problems (MVP).

Theorem 3.1. *For each $i\in P$, let functional $\int_a^b f^i(t,.,.)\mathrm{d}t$ be strongly convex with constant α_i on X, then $\overline{x}\in X$ is an efficient solution of (MVP), if and only if it is a solution of* $(\text{GMVVI})_\gamma$*, where $\gamma=\min\{\alpha_1,\ldots,\alpha_p\}$.*

Proof. Firstly, we suppose that $\overline{x}$ is an efficient solution of (MVP) but not a solution of $(\mathrm{GMVVI})_\gamma$, then for given real constant γ, there exists $x_\gamma \in X$ such that

$$\int_a^b \left[f_x^i(t, x_\gamma, \dot{x}_\gamma)(\overline{x} - x_\gamma) + f_{\dot{x}}^i(t, x_\gamma, \dot{x}_\gamma)(\dot{\overline{x}} - \dot{x}_\gamma)\right] \mathrm{d}t + \gamma \|\overline{x} - x_\gamma\|^2 \geq 0, \tag{3.1}$$

with strict inequality for at least one $i \in P$.

Now, by using strong convexity of each functional $\int_a^b f^i(t, ., .)\mathrm{d}t$, there exists a real constant $\alpha_i > 0$, such that

$$\begin{aligned}\int_a^b \left[f_x^i(t, x, \dot{x})(\overline{x} - x) + f_{\dot{x}}^i(t, x, \dot{x})(\dot{\overline{x}} - \dot{x})\right] \mathrm{d}t + \alpha_i \|\overline{x} - x\|^2 \\ \leq \int_a^b f^i(t, \overline{x}, \dot{\overline{x}})\mathrm{d}t - \int_a^b f^i(t, x, \dot{x})\mathrm{d}t, \ \forall\, x \in X \text{ and } i \in P.\end{aligned}$$

In particular, for $\gamma = \min\{\alpha_1, \ldots, \alpha_p\}$, the above inequalities yield

$$\begin{aligned}\int_a^b \left[f_x^i(t, x, \dot{x})(\overline{x} - x) + f_{\dot{x}}^i(t, x, \dot{x})(\dot{\overline{x}} - \dot{x})\right] \mathrm{d}t + \gamma \|\overline{x} - x\|^2 \\ \leq \int_a^b f^i(t, \overline{x}, \dot{\overline{x}})\mathrm{d}t - \int_a^b f^i(t, x, \dot{x})\mathrm{d}t, \ \forall\, x \in X \text{ and } i \in P.\end{aligned} \tag{3.2}$$

On combining inequalities (3.1) and (3.2), we have

$$\int_a^b f^i(t, \overline{x}, \dot{\overline{x}})\mathrm{d}t - \int_a^b f^i(t, x_\gamma, \dot{x}_\gamma)\mathrm{d}t \geq 0,$$

which is satisfied as a strict inequality for at least one $i \in P$. This leads to a contradiction, that $\overline{x}$ is an efficient solution of (MVP). Hence, $\overline{x}$ is a solution of $(\mathrm{GMVVI})_\gamma$.

Conversely, let $\overline{x} \in X$ be a solution of $(\mathrm{GMVVI})_\gamma$ with constant γ but it is not an efficient solution of (MVP), then there exists $x \in X$ such that

$$\int_a^b f^i(t, x, \dot{x})\mathrm{d}t \leq \int_a^b f^i(t, \overline{x}, \dot{\overline{x}})\mathrm{d}t, \tag{3.3}$$

with strict inequality for at least one $i \in P$. For the sake of convenience, we set

$$x(\lambda) = \overline{x} + \lambda(x - \overline{x}), \ \forall\, \lambda \in [0, 1].$$

We choose arbitrarily $\lambda' \in (0, 1)$. Now, by using mean value theorem, there exists $\lambda_i \in (0, \lambda']$ for $i \in P$ such that

$$\begin{aligned}\lambda' \int_a^b \Big[f_x^i\left(t, \overline{x} + \lambda_i(x - \overline{x}), \dot{\overline{x}} + \lambda_i(\dot{x} - \dot{\overline{x}})\right)(x - \overline{x}) \\ + f_{\dot{x}}^i\left(t, \overline{x} + \lambda_i(x - \overline{x}), \dot{\overline{x}} + \lambda_i(\dot{x} - \dot{\overline{x}})\right)(\dot{x} - \dot{\overline{x}})\Big] \mathrm{d}t \\ = \int_a^b f^i\left(t, \overline{x} + \lambda'(x - \overline{x}), \dot{\overline{x}} + \lambda'(\dot{x} - \dot{\overline{x}})\right) \mathrm{d}t - \int_a^b f^i(t, \overline{x}, \dot{\overline{x}})\mathrm{d}t.\end{aligned} \tag{3.4}$$

By Lemma 2.1, $\int_a^b f^i(t,.,.)\mathrm{d}t$ is strongly preconvex, then we have

$$\int_a^b f^i\left(t,\overline{x}+\lambda'(x-\overline{x}),\dot{\overline{x}}+\lambda'(\dot{x}-\dot{\overline{x}})\right)\mathrm{d}t-\int_a^b f^i(t,\overline{x},\dot{\overline{x}})\mathrm{d}t$$
$$\leq \lambda'\left[\int_a^b f^i(t,x,\dot{x})\mathrm{d}t-\int_a^b f^i(t,\overline{x},\dot{\overline{x}})\mathrm{d}t\right]-\alpha_i\lambda'(1-\lambda')\|x-\overline{x}\|^2. \tag{3.5}$$

On combining inequalities (3.3),(3.4) and (3.5), we get

$$\int_a^b \left[f_x^i\left(t,\overline{x}+\lambda_i(x-\overline{x}),\dot{\overline{x}}+\lambda_i(\dot{x}-\dot{\overline{x}})\right)(x-\overline{x})+f_{\dot{x}}^i\left(t,\overline{x}+\lambda_i(x-\overline{x}),\dot{\overline{x}}+\lambda_i(\dot{x}-\dot{\overline{x}})\right)(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t$$
$$\leq -\alpha_i(1-\lambda')\|x-\overline{x}\|^2,\ \forall\ i\in\{1,\ldots,p\}, \tag{3.6}$$

with strict inequality for at least one $i \in P$. Since $\lambda_i \in (0,1)$, for $i \in P$, we can choose $\lambda^* \in (0,1)$ such that $\lambda^* < \min\{\lambda_i :\ \forall\ i \in P\}$. Now, for any $i \in P$, it is obvious that

$$x(\lambda^*)-x(\lambda_i)=(\lambda^*-\lambda_i)(x-\overline{x}) \tag{3.7}$$

and

$$x(\lambda_i)-x(\lambda^*)=(\lambda_i-\lambda^*)(x-\overline{x}), \tag{3.8}$$

(3.6) and (3.7) together implies

$$\int_a^b \left[f_x^i\left(t,\overline{x}+\lambda_i(x-\overline{x}),\dot{\overline{x}}+\lambda_i(\dot{x}-\dot{\overline{x}})\right)(x(\lambda^*)-x(\lambda_i))\right.$$
$$\left.+f_{\dot{x}}^i\left(t,\overline{x}+\lambda_i(x-\overline{x}),\dot{\overline{x}}+\lambda_i(\dot{x}-\dot{\overline{x}})\right)(\dot{x}(\lambda^*)-\dot{x}(\lambda_i))\right]\mathrm{d}t$$
$$\geq \alpha_i(1-\lambda')(\lambda_i-\lambda^*)\|x-\overline{x}\|^2.$$

The above inequality can be rewritten as

$$\int_a^b \left[f_x^i(t,x(\lambda_i),\dot{x}(\lambda_i))(x(\lambda^*)-x(\lambda_i))+f_{\dot{x}}^i(t,x(\lambda_i),\dot{x}(\lambda_i))(\dot{x}(\lambda^*)-\dot{x}(\lambda_i))\right]\mathrm{d}t \geq \alpha_i(1-\lambda')(\lambda_i-\lambda^*)\|x-\overline{x}\|^2, \tag{3.9}$$

with strict inequality for at least one $i \in P$. Now, strong convexity of $\int_a^b f^i(t,.,.)\mathrm{d}t$, yields

$$\int_a^b \left[f_x^i(t,x(\lambda_i),\dot{x}(\lambda_i))(x(\lambda^*)-x(\lambda_i))+f_{\dot{x}}^i(t,x(\lambda_i),\dot{x}(\lambda_i))(\dot{x}(\lambda^*)-\dot{x}(\lambda_i))\right]\mathrm{d}t+\alpha_i\|x(\lambda^*)-x(\lambda_i)\|^2$$
$$\leq \int_a^b f^i(t,x(\lambda^*),\dot{x}(\lambda^*))\mathrm{d}t-\int_a^b f^i(t,x(\lambda_i),\dot{x}(\lambda_i))\mathrm{d}t. \tag{3.10}$$

On interchanging $x(\lambda^*)$ and $x(\lambda_i)$ in inequality (3.10), we have

$$\begin{aligned}\int_a^b \left[f_x^i(t,x(\lambda^*),\dot{x}(\lambda^*))(x(\lambda_i)-x(\lambda^*))+f_{\dot{x}}^i(t,x(\lambda^*),\dot{x}(\lambda^*))(\dot{x}(\lambda_i)-\dot{x}(\lambda^*))\right]\mathrm{d}t\\ +\alpha_i\|x(\lambda_i)-x(\lambda^*)\|^2\\ \leq \int_a^b f^i(t,x(\lambda_i),\dot{x}(\lambda_i))\mathrm{d}t-\int_a^b f^i(t,x(\lambda^*),\dot{x}(\lambda^*))\mathrm{d}t.\end{aligned} \tag{3.11}$$

Now, we add inequalities (3.10) and (3.11) and use relations (3.7) and (3.8), we obtain

$$\begin{aligned}\int_a^b \left[f_x^i(t,x(\lambda_i),\dot{x}(\lambda_i))(x(\lambda^*)-x(\lambda_i))+f_{\dot{x}}^i(t,x(\lambda_i),\dot{x}(\lambda_i))(\dot{x}(\lambda^*)-\dot{x}(\lambda_i))\right]\mathrm{d}t\\ +\int_a^b \left[f_x^i(t,x(\lambda^*),\dot{x}(\lambda^*))(x(\lambda_i)-x(\lambda^*))+f_{\dot{x}}^i(t,x(\lambda^*),\dot{x}(\lambda^*))(\dot{x}(\lambda_i)-\dot{x}(\lambda^*))\right]\mathrm{d}t\\ \leq -2\alpha_i(\lambda_i-\lambda^*)^2\|x-\overline{x}\|^2.\end{aligned} \tag{3.12}$$

On combining inequalities (3.9) and (3.12), we have

$$\begin{aligned}\int_a^b \left[f_x^i(t,x(\lambda^*),\dot{x}(\lambda^*))(x(\lambda_i)-x(\lambda^*))+f_{\dot{x}}^i(t,x(\lambda^*),\dot{x}(\lambda^*))(\dot{x}(\lambda_i)-\dot{x}(\lambda^*))\right]\mathrm{d}t\\ \leq -\alpha_i(\lambda_i-\lambda^*)[2(\lambda_i-\lambda^*)+(1-\lambda')]\|x-\overline{x}\|^2,\end{aligned}$$

with strict inequality for at least one $i\in P$. By using, relation (3.8) in the above inequality, we get

$$\begin{aligned}\int_a^b \left[f_x^i(t,x(\lambda^*),\dot{x}(\lambda^*))(\overline{x}-x(\lambda^*))+f_{\dot{x}}^i(t,x(\lambda^*),\dot{x}(\lambda^*))(\dot{\overline{x}}-\dot{x}(\lambda^*))\right]\mathrm{d}t\\ \geq \frac{\alpha_i}{\lambda^*}[2(\lambda_i-\lambda^*)+(1-\lambda')]\|\overline{x}-x(\lambda^*)\|^2.\end{aligned}$$

Set $\gamma_\circ=\min\{\alpha_1,\dots,\alpha_p\}$ and $\lambda_\circ=\min\{\lambda_1,\dots,\lambda_p\}$. Hence, we have

$$\int_a^b \left[f_x^i(t,x(\lambda^*),\dot{x}(\lambda^*))(\overline{x}-x(\lambda^*))+f_{\dot{x}}^i(t,x(\lambda^*),\dot{x}(\lambda^*))(\dot{\overline{x}}-\dot{x}(\lambda^*))\right]\mathrm{d}t+\gamma'\|\overline{x}-x(\lambda^*)\|^2\geq 0,$$

with strict inequality for at least one $i\in P$ and $\gamma'=-\frac{\gamma_\circ}{\lambda^*}[2(\lambda_\circ-\lambda^*)+(1-\lambda')]$. It is clear that, if $\lambda^*\to 0^+$ then $\gamma'\to-\infty$. Therefore, for any chosen $\gamma'\leq\gamma$, $\overline{x}$ is not a solution of $(\text{GMVVI})_\gamma$, which in turn, by Remark 2.1, contradicts the fact that $\overline{x}$ is the solution of $(\text{GMVVI})_\gamma$ with constant γ. Hence the theorem. □

We present the following example to illustrate the result established in the above theorem.

Example 3.2. Consider the function $f:[0,1]\times\mathbb{R}\times\mathbb{R}\mapsto\mathbb{R}^2$, defined by

$$\int_0^1 f(t,x,\dot{x})\mathrm{d}t=\left(\int_0^1 f^1(t,x,\dot{x})\mathrm{d}t,\int_0^1 f^2(t,x,\dot{x})\mathrm{d}t\right),$$

where

$$f^1(t,x,\dot{x})=x^2,\;\; f^2(t,x,\dot{x})=1+x^2.$$

Let $x, y : [0,1] \mapsto \mathbb{R}$ be defined as $x(t) = k_1 t,\ y(t) = k_2 t,\ \forall\ k_1, k_2 \in \mathbb{R}$, respectively. Since, for $x, y \in \mathbb{R}$ and $\alpha_1 = \frac{1}{24}$, we have

$$\begin{aligned}
&\int_0^1 \left[f_x^1(t,y,\dot{y})(x-y) + f_{\dot{x}}^1(t,y,\dot{y})(\dot{x}-\dot{y})\right] \mathrm{d}t + \alpha_1\|x-y\|^2 - \int_0^1 f^1(t,x,\dot{x})\mathrm{d}t + \int_0^1 f^1(t,y,\dot{y})\mathrm{d}t \\
&\qquad = \int_0^1 [2xy - 2y^2]\mathrm{d}t + \alpha_1\|x-y\|^2 - \int_0^1 x^2\mathrm{d}t + \int_0^1 y^2\mathrm{d}t \\
&\qquad = \int_0^1 [2k_1k_2t^2 - 2k_2^2t^2]\mathrm{d}t + \alpha_1\|t(k_1-k_2)\|^2 - \int_0^1 k_1^2t^2\mathrm{d}t + \int_0^1 k_2^2t^2\mathrm{d}t \\
&\qquad = -\frac{(k_1-k_2)^2}{6} \\
&\qquad \leq 0.
\end{aligned}$$

Similarly, for $x, y \in \mathbb{R}$ and $\alpha_2 = \dfrac{1}{24}$, we get

$$\begin{aligned}
&\int_0^1 \left[f_x^2(t,y,\dot{y})(x-y) + f_{\dot{x}}^2(t,y,\dot{y})(\dot{x}-\dot{y})\right] \mathrm{d}t + \alpha_2\|x-y\|^2 - \int_0^1 f^2(t,x,\dot{x})\mathrm{d}t + \int_0^1 f^2(t,y,\dot{y})\mathrm{d}t \\
&\qquad = -\frac{(k_1-k_2)^2}{6} \\
&\qquad \leq 0.
\end{aligned}$$

Therefore, functionals $\displaystyle\int_0^1 f^1(t,x,\dot{x})\mathrm{d}t$ and $\displaystyle\int_0^1 f^2(t,x,\dot{x})\mathrm{d}t$ are strongly convex with constants $\alpha_1 = \frac{1}{24}$ and $\alpha_2 = \frac{1}{24}$, respectively, on $\mathbb{R}$. Further, $\overline{x} = 0$ solves $(\mathrm{GMVVI})_\gamma$, as for a constant $\gamma = \frac{1}{24}$, we have

$$\begin{aligned}
&\Bigg(\int_0^1 \left[f_x^1(t,x,\dot{x})(\overline{x}-x) + f_{\dot{x}}^1(t,x,\dot{x})(\dot{\overline{x}}-\dot{x})\right] \mathrm{d}t + \gamma\|\overline{x}-x\|^2, \\
&\qquad \int_0^1 \left[f_x^2(t,x,\dot{x})(\overline{x}-x) + f_{\dot{x}}^2(t,x,\dot{x})(\dot{\overline{x}}-\dot{x})\right] \mathrm{d}t + \gamma\|\overline{x}-x\|^2\Bigg) \\
&\qquad = \left(k_1^2\left(-\frac{2}{3}+4\gamma\right), k_1^2\left(-\frac{2}{3}+4\gamma\right)\right) \\
&\qquad = \left(-\frac{k_1^2}{2}, -\frac{k_1^2}{2}\right) \\
&\qquad \ngeq (0,0).
\end{aligned}$$

Now, for $\overline{x} = 0$, we obtain

$$\begin{aligned}
&\left(\int_0^1 f^1(t,x,\dot{x})\mathrm{d}t - \int_0^1 f^1(t,\overline{x},\dot{\overline{x}})\mathrm{d}t, \int_0^1 f^2(t,x,\dot{x})\mathrm{d}t - \int_0^1 f^2(t,\overline{x},\dot{\overline{x}})\mathrm{d}t\right) \\
&\qquad = \left(\frac{k_1^2}{3}, \frac{k_1^2}{3}\right) \\
&\qquad \nleq (0,0).
\end{aligned}$$

Therefore, $\overline{x} = 0$ is an efficient solution of (MVP).

Theorem 3.3. *For each $i \in P$, let functional $\int_a^b f^i(t,.,.)\mathrm{d}t$ be strongly convex with constant α_i on X. If $\overline{x} \in X$ is a solution of (GSVVI)$_\gamma$, where $\gamma = \min\{\alpha_1, \dots, \alpha_p\}$, then it is an efficient solution of (MVP).*

Proof. Let $\overline{x}$ be a solution of (GSVVI)γ, then for given real constant γ, there exists no $x \in X$, satisfying

$$\int_a^b \left[f_x^i(t,\overline{x},\dot{\overline{x}})(x-\overline{x}) + f_{\dot{x}}^i(t,\overline{x},\dot{\overline{x}})(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t + \gamma\|x-\overline{x}\|^2 \leq 0, \tag{3.13}$$

with strict inequality for at least one $i \in P$. By using the condition of strong convexity of $\int_a^b f^i(t,.,.)\mathrm{d}t$, there exists a real constant $\alpha_i > 0$ such that

$$\int_a^b \left[f_x^i(t,\overline{x},\dot{\overline{x}})(x-\overline{x}) + f_{\dot{x}}^i(t,\overline{x},\dot{\overline{x}})(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t + \alpha_i\|x-\overline{x}\|^2 \leq \int_a^b f^i(t,x,\dot{x})\mathrm{d}t - \int_a^b f^i(t,\overline{x},\dot{\overline{x}})\mathrm{d}t,\ \forall\, x \in X \text{ and } i \in P.$$

In particular, for $\gamma = \min\{\alpha_1, \dots, \alpha_p\}$, we have

$$\int_a^b \left[f_x^i(t,\overline{x},\dot{\overline{x}})(x-\overline{x}) + f_{\dot{x}}^i(t,\overline{x},\dot{\overline{x}})(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t + \gamma\|x-\overline{x}\|^2 \leq \int_a^b f^i(t,x,\dot{x})\mathrm{d}t - \int_a^b f^i(t,\overline{x},\dot{\overline{x}})\mathrm{d}t,\ \forall\, x \in X \text{ and } i \in P. \tag{3.14}$$

On combining inequalities (3.13) and (3.14), we can say that, there exists no $x \in X$ such that

$$\int_a^b f^i(t,x,\dot{x})\mathrm{d}t \leq \int_a^b f^i(t,\overline{x},\dot{\overline{x}})\mathrm{d}t,$$

with strict inequality for at least one $i \in P$. Therefore, $\overline{x}$ is an efficient solution of (MVP). Hence the theorem. ☐

Example 3.4. Consider the function $f : [0,1] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}^2$, defined by

$$\int_0^1 f(t,x,\dot{x})\mathrm{d}t = \left(\int_0^1 f^1(t,x,\dot{x})\mathrm{d}t, \int_0^1 f^2(t,x,\dot{x})\mathrm{d}t\right),$$

where

$$f^1(t,x,\dot{x}) = -x + \dot{x}^2,\ \ f^2(t,x,\dot{x}) = -x^2 + \dot{x}^3.$$

Let $x, y : [0,1] \mapsto \mathbb{R}$ be defined as $x(t) = t,\ y(t) = t^2$, respectively. Since, for $x, y \in \mathbb{R}$ and $\alpha_1 = \frac{1}{40}$, we have

$$\begin{aligned}
&\int_0^1 \left[f_x^1(t,y,\dot{y})(x-y) + f_{\dot{x}}^1(t,y,\dot{y})(\dot{x}-\dot{y})\right]\mathrm{d}t + \alpha_1\|x-y\|^2 - \int_0^1 f^1(t,x,\dot{x})\mathrm{d}t + \int_0^1 f^1(t,y,\dot{y})\mathrm{d}t \\
&\qquad = \int_0^1 [-x+y+2\dot{x}\dot{y}-2\dot{y}^2]\mathrm{d}t + \alpha_1\|x-y\|^2 - \int_0^1 [-x+\dot{x}^2]\mathrm{d}t + \int_0^1 [-y+\dot{y}^2]\mathrm{d}t \\
&\qquad\quad = \int_0^1 [3t-7t^2]\mathrm{d}t + \alpha_1\|t-t^2\|^2 - \int_0^1 [1-t]\mathrm{d}t + \int_0^1 3t^2\mathrm{d}t \\
&\qquad\quad = -0.294 \\
&\qquad\quad \leq 0.
\end{aligned}$$

Similarly, for $x, y \in \mathbb{R}$ and $\alpha_2 = \frac{1}{40}$, we get

$$\begin{aligned}
&\int_0^1 \left[f_x^2(t,y,\dot{y})(x-y) + f_{\dot{x}}^2(t,y,\dot{y})(\dot{x}-\dot{y})\right]\mathrm{d}t + \alpha_2\|x-y\|^2 - \int_0^1 f^2(t,x,\dot{x})\mathrm{d}t + \int_0^1 f^2(t,y,\dot{y})\mathrm{d}t \\
&\qquad\quad = -0.921 \\
&\qquad\quad \leq 0.
\end{aligned}$$

Therefore, functionals $\int_0^1 f^1(t,x,\dot{x})\mathrm{d}t$ and $\int_0^1 f^2(t,x,\dot{x})\mathrm{d}t$ are strongly convex with constants $\alpha_1 = \frac{1}{40}$ and $\alpha_2 = \frac{1}{40}$, respectively, on $\mathbb{R}$. Further, $\overline{x} = 0$ solves $(\mathrm{GSVVI})_\gamma$, as for a constant $\gamma = \frac{1}{40}$, we have

$$\begin{aligned}\Bigg(\int_0^1 &\left[f_x^1(t,\overline{x},\dot{\overline{x}})(x-\overline{x}) + f_{\dot{x}}^1(t,\overline{x},\dot{\overline{x}})(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t + \gamma\|x-\overline{x}\|^2,\\ &\int_0^1 \left[f_x^2(t,\overline{x},\dot{\overline{x}})(x-\overline{x}) + f_{\dot{x}}^2(t,\overline{x},\dot{\overline{x}})(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t + \gamma\|x-\overline{x}\|^2\Bigg)\\ &= \left(-\frac{1}{2} + 4\gamma, 4\gamma\right)\\ &= \left(-\frac{2}{5}, \frac{1}{10}\right)\\ &\nleq (0,0).\end{aligned}$$

Now, for $\overline{x} = 0$, we obtain

$$\begin{aligned}\Bigg(\int_0^1 &f^1(t,x,\dot{x})\mathrm{d}t - \int_0^1 f^1(t,\overline{x},\dot{\overline{x}})\mathrm{d}t, \int_0^1 f^2(t,x,\dot{x})\mathrm{d}t - \int_0^1 f^2(t,\overline{x},\dot{\overline{x}})\mathrm{d}t\Bigg)\\ &= \left(\frac{1}{2}, \frac{2}{3}\right)\\ &\nleq (0,0).\end{aligned}$$

Therefore, $\overline{x} = 0$ is an efficient solution of (MVP).

Theorem 3.5. *For each $i \in P$, let functional $\int_a^b f^i(t,.,.)\mathrm{d}t$ be strongly convex with constant α_i on X. If $\overline{x} \in X$ is a solution of $(\mathrm{GWSVVI})_\gamma$, then it is a solution of $(\mathrm{GWMVVI})_\gamma$, where $\gamma = \min\{\alpha_1, \ldots, \alpha_p\}$.*

Proof. Let $\overline{x}$ be a solution of $(\mathrm{GWSVVI})_\gamma$, then for given real constant γ, there exists no $x \in X$, satisfying

$$\int_a^b \left[f_x^i(t,\overline{x},\dot{\overline{x}})(x-\overline{x}) + f_{\dot{x}}^i(t,\overline{x},\dot{\overline{x}})(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t + \gamma\|x-\overline{x}\|^2 < 0,\ \forall\ i \in P. \tag{3.15}$$

Now, by using strong convexity of $\int_a^b f^i(t,.,.)\mathrm{d}t$, there exists a real constant $\alpha_i > 0$, such that

$$\begin{aligned}\int_a^b &\left[f_x^i(t,\overline{x},\dot{\overline{x}})(x-\overline{x}) + f_{\dot{x}}^i(t,\overline{x},\dot{\overline{x}})(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t + \alpha_i\|x-\overline{x}\|^2\\ &\leq \int_a^b f^i(t,x,\dot{x})\mathrm{d}t - \int_a^b f^i(t,\overline{x},\dot{\overline{x}})\mathrm{d}t,\ \forall\ x \in X \text{ and } i \in P.\end{aligned}$$

In particular, for $\gamma = \min\{\alpha_1, \ldots, \alpha_p\}$, we have

$$\begin{aligned}\int_a^b &\left[f_x^i(t,\overline{x},\dot{\overline{x}})(x-\overline{x}) + f_{\dot{x}}^i(t,\overline{x},\dot{\overline{x}})(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t + \gamma\|x-\overline{x}\|^2\\ &\leq \int_a^b f^i(t,x,\dot{x})\mathrm{d}t - \int_a^b f^i(t,\overline{x},\dot{\overline{x}})\mathrm{d}t,\ \forall\ x \in X \text{ and } i \in P.\end{aligned} \tag{3.16}$$

On interchanging x and $\overline{x}$ in inequality (3.16), we obtain

$$\int_a^b \left[f_x^i(t,x,\dot{x})(\overline{x}-x)+f_{\dot{x}}^i(t,x,\dot{x})(\dot{\overline{x}}-\dot{x})\right]\mathrm{d}t+\gamma\|\overline{x}-x\|^2 \\ \leq \int_a^b f^i(t,\overline{x},\dot{\overline{x}})\mathrm{d}t-\int_a^b f^i(t,x,\dot{x})\mathrm{d}t,\ \forall\, x\in X \text{ and } i\in P. \tag{3.17}$$

Now, we add inequalities (3.16) and (3.17), we get

$$\int_a^b \left[f_x^i(t,x,\dot{x})(\overline{x}-x)+f_{\dot{x}}^i(t,x,\dot{x})(\dot{\overline{x}}-\dot{x})\right]\mathrm{d}t+\gamma\|\overline{x}-x\|^2 \\ \leq -\left[\int_a^b \left[f_x^i(t,\overline{x},\dot{\overline{x}})(x-\overline{x})+f_{\dot{x}}^i(t,\overline{x},\dot{\overline{x}})(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t\right]-\gamma\|x-\overline{x}\|^2. \tag{3.18}$$

On combining inequalities (3.15) and (3.18), we can say that, for given real constant γ, there exists no $x\in X$, satisfying

$$\int_a^b \left[f_x^i(t,x,\dot{x})(\overline{x}-x)+f_{\dot{x}}^i(t,x,\dot{x})(\dot{\overline{x}}-\dot{x})\right]\mathrm{d}t+\gamma\|\overline{x}-x\|^2>0,\ \forall\, i\in P.$$

Therefore, $\overline{x}$ is a solution of (GWMVVI)$_\gamma$. Hence the theorem. □

Theorem 3.6. *For each $i\in P$, let functional $\int_a^b f^i(t,.,.)\mathrm{d}t$ be strongly convex with constant α_i on X. If $\overline{x}$ is a solution of* (GWSVVI)$_\gamma$*, where $\gamma=\min\{\alpha_1,\dots,\alpha_p\}$, then it is a weak efficient solution of* (MVP).

Proof. Suppose, contrary to the result, that $\overline{x}$ is not a weak efficient solution of (MVP). Then, there exists $x\in X$ such that

$$\int_a^b f^i(t,x,\dot{x})\mathrm{d}t<\int_a^b f^i(t,\overline{x},\dot{\overline{x}})\mathrm{d}t,\ \forall\, i\in P. \tag{3.19}$$

By using strong convexity of $\int_a^b f^i(t,.,.)\mathrm{d}t$, there exists a real constant $\alpha_i>0$, such that

$$\int_a^b \left[f_x^i(t,\overline{x},\dot{\overline{x}})(x-\overline{x})+f_{\dot{x}}^i(t,\overline{x},\dot{\overline{x}})(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t+\alpha_i\|x-\overline{x}\|^2 \\ \leq \int_a^b f^i(t,x,\dot{x})\mathrm{d}t-\int_a^b f^i(t,\overline{x},\dot{\overline{x}})\mathrm{d}t,\ \forall\, x\in X \text{ and } i\in P.$$

In particular, for $\gamma=\min\{\alpha_1,\dots,\alpha_p\}$, we have

$$\int_a^b \left[f_x^i(t,\overline{x},\dot{\overline{x}})(x-\overline{x})+f_{\dot{x}}^i(t,\overline{x},\dot{\overline{x}})(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t+\gamma\|x-\overline{x}\|^2\leq \int_a^b f^i(t,x,\dot{x})\mathrm{d}t-\int_a^b f^i(t,\overline{x},\dot{\overline{x}})\mathrm{d}t,\ \forall\, x\in X \text{ and } i\in P. \tag{3.20}$$

On combining inequalities (3.19) and (3.20), it follows that there exists $x\in X$ such that

$$\int_a^b \left[f_x^i(t,\overline{x},\dot{\overline{x}})(x-\overline{x})+f_{\dot{x}}^i(t,\overline{x},\dot{\overline{x}})(\dot{x}-\dot{\overline{x}})\right]\mathrm{d}t+\gamma\|x-\overline{x}\|^2<0,\ \forall\, i\in P.$$

Therefore, $\overline{x}$ is not a solution of (GWSVVI)$_\gamma$, which leads to a contradiction. Hence, the theorem. □

Theorem 3.7. *For each $i \in P$, let functional $\int_a^b f^i(t,.,.)\mathrm{d}t$ be strictly strongly convex with constant α_i on X. If $\overline{x} \in X$ is a weak efficient solution of* (MVP)*, then it is a solution of* $(\mathrm{GMVVI})_\gamma$*, where $\gamma = \min\{\alpha_1, \ldots, \alpha_p\}$.*

Proof. Suppose, contrary to the result, that $\overline{x}$ is not a solution of $(\mathrm{GMVVI})_\gamma$, then for given real constant γ, there exists $x_\gamma \in X$ such that

$$\int_a^b \left[f_x^i(t,x_\gamma,\dot{x}_\gamma)(\overline{x}-x_\gamma) + f_{\dot{x}}^i(t,x_\gamma,\dot{x}_\gamma)(\dot{\overline{x}}-\dot{x}_\gamma)\right]\mathrm{d}t + \gamma\|\overline{x}-x_\gamma\|^2 \geq 0, \tag{3.21}$$

with strict inequality for at least one $i \in P$. Since, each functional $\int_a^b f^i(t,.,.)\mathrm{d}t$ is strictly strongly convex, then there exists a real constant $\alpha_i > 0$, such that for all $x \in X,\ i \in P$ and $x \neq \overline{x}$, we have

$$\int_a^b \left[f_x^i(t,x,\dot{x})(\overline{x}-x) + f_{\dot{x}}^i(t,x,\dot{x})(\dot{\overline{x}}-\dot{x})\right]\mathrm{d}t + \alpha_i\|\overline{x}-x\|^2 < \int_a^b f^i(t,\overline{x},\dot{\overline{x}})\mathrm{d}t - \int_a^b f^i(t,x,\dot{x})\mathrm{d}t.$$

In particular, for $\gamma = \min\{\alpha_1, \ldots, \alpha_p\}$, we get

$$\int_a^b \left[f_x^i(t,x,\dot{x})(\overline{x}-x) + f_{\dot{x}}^i(t,x,\dot{x})(\dot{\overline{x}}-\dot{x})\right]\mathrm{d}t + \gamma\|\overline{x}-x\|^2 < \int_a^b f^i(t,\overline{x},\dot{\overline{x}})\mathrm{d}t - \int_a^b f^i(t,x,\dot{x})\mathrm{d}t. \tag{3.22}$$

On combining inequalities (3.21) and (3.22), we have

$$\int_a^b f^i(t,\overline{x},\dot{\overline{x}})\mathrm{d}t - \int_a^b f^i(t,x_\gamma,\dot{x}_\gamma)\mathrm{d}t > 0,\ \forall\ i \in P,$$

which leads to a contradiction, that $\overline{x}$ is a weak efficient solution of (MVP). Hence the theorem. □

Theorem 3.8. *For each $i \in P$, let each functional $\int_a^b f^i(t,.,.)\mathrm{d}t$ be strongly convex with constant α_i on X. If $\overline{x}$ is a weak efficient solution of* (MVP)*, then it is a solution of* $(\mathrm{GWMVVI})_\gamma$*, where $\gamma = \min\{\alpha_1, \ldots, \alpha_p\}$.*

Proof. The proof follows in the similar lines of first part of Theorem 3.1 and hence being omitted. □

4. Conclusion

In this paper, we have generalized Minty and Stampacchia vector variational-type inequalities and established the relationships among their solutions and efficient solutions of multiobjective variational problems. Moreover, we have also dealt with weak formulations of generalized Minty and Stampacchia vector variational-type inequalities. In future, we will try to prove the existence of these generalized inequalities.

Acknowledgements. The authors are grateful to reviewer for his/her valuable and constructive suggestions on an early draft.

References

[1] T. Antczak, Mean value in invexity analysis. *Nonlin. Anal.* **60** (2005) 1473–1484.

[2] M. Arana-Jiménez, R. Osuna-Gómez, G. Ruiz-Garzón and M. Rojas-Medar, On variational problems characterization of solutions and duality. *J. Math. Anal. Appl.* **311** (2005) 1–12.

[3] G.P. Crespi, I. Ginchev and M. Rocca, A note on Minty type vector variational inequalities, *RAIRO O.R.* **39** (2005) 253–273.

[4] F. Giannessi, Theorems of the alternative quadratic programs and complementarity problems, in *Variational inequalities and complementarity problems*, edited by R.W. Cottle, F. Giannessi, J.L. Lions. Wiley, Chichester (1980) 151–186.

[5] M.A. Hanson, Bounds for functionally convex optimal control problems. *J. Math. Anal. Appl.* **8** (1964) 84–89.

[6] M.A. Hanson, On sufficiency of the Kuhn−Tucker conditions. *J. Math. Anal. Appl.* **80** (1981) 545–550.

[7] H. Huang, Strict vector variational inequalities and strict pareto efficiency in nonconvex vector optimization. *Positivity* **19** (2015) 95–109.

[8] A. Jayswal, S. Choudhury and R.U. Verma, Exponential type vector variational-like inequalities and vector optimization problems with exponential type invexities. *J. Appl. Math. Comput.* **45** (2014) 87–97.

[9] S. Khurana, Symmetric duality in multiobjective programming involving generalized cone-invex functions. *Eur. J. Oper. Res.* **165** (2005) 592–597.

[10] M.H. Kim, Relations between vector continuous-time program and vector variational-type inequality. *J. Appl. Math. Comput.* **16** (2004) 279–287.

[11] P. Kumam, Strong convergence theorems by an extragradient method for solving variational inequalities and equilibrium problems in a Hilbert space. *Turkish J. Math.* **33** (2009) 85–98.

[12] M. Miholca, On set-valued optimization problems and vector variational-like inequalities. *Optim. Lett.* **8** (2014) 463-476.

[13] S.R. Mohan and S.K. Neogy, On invex sets and preinvex functions. *J. Math. Anal. Appl.* **189** (1995) 901–908.

[14] M. Oveisiha and J. Zafarani, Generalized Minty vector variational-like inequalities and vector optimization problems in Asplund spaces. *Optim. Lett.* **7** (2013) 709–721.

[15] G. Ruiz-Garzón, R. Osuna-Gómez, A. Rufián-Lizana and B. Hernández-Jiménez, Optimality in continuous-time multiobjective optimization and vector variational-like inequalities. *TOP* **23** (2015) 198–219.

[16] T. Weir and B. Mond, Pre-invex functions in multiple objective optimization. *J. Math. Anal. Appl.* **136** (1988) 29–38.